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Research article

On solvability of some p-Laplacian boundary value problems with Caputo fractional derivative

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Abstract: The solvability of some p-Laplace boundary value problems with Caputo fractional derivative are discussed. By using the fixed-point theory and analysis techniques, some existence results of one or three non-negative solutions are obtained. Two examples showed that the conditions used in this paper are somewhat easy to check.

Keywords: solvability; boundary value problem; caputo fractional derivative; fixed point theorem
Mathematics Subject Classification: 47H08, 47H10

1. Introduction

Due to the wide application in many disciplines such as the fields of physics, chemistry, automatic control, signal processing, soft matter research, aerodynamics, etc., the fractional calculus have been widely studied recently. An extensive literature can be found related to the fractional differential equation systems [19,22].The solvability of boundary value problems(BVPs) with fractional derivative are studied with various tools, specially, with topological degree, fixed-point theory, the continuation theorems, and other nonlinear functional analysis method. For example, see fractional two-point BVPs [8], fractional BVPs at resonance, fractional multi-point problems at nonresonance [12,14], fractional initial value problems, fractional impulsive problems [14], fractional integral BVPs [9], fractional p-Laplace problems [4,10,13,15–18], fractional BVPs with the Caputo-Fabrizio derivative [1–3,5], etc.

Zhang [20] has given the solvability results to the BVPs with Caputo fractional derivative

$$\begin{aligned}&{}^{C}D^{\alpha}y(x) = f(x, y), \quad x \in (0, 1), \quad 1 < \alpha \le 2,\\ &y(0) + y'(0) = y(1) + y'(1) = 0,\end{aligned}$$

by using some fixed point Theorems.

Salem [12] has researched the solvability of the following nonlinear m-point fractional BVPs

$$D^{\alpha}y(x) + r(x)f(x, y(x)) = 0,\ a.e.\ x \in [0, 1], n-1 < \alpha \leq n, n \geq 2,$$
$$y(0) = y'(0) = y''(0) = \cdots = y^{(n-2)}(0) = 0,\ y(1) = \sum_{i=1}^{m-2} a_i y(x_i),$$

where $0 < x_1 < x_2 < \cdots < x_{m-2} < 1,\ a_i > 0$ and $\sum_{i=1}^{m-2} a_i x_i^{\alpha-1} < 1$.

Based upon the above research, some authors studied the differential equations with p-Laplacian by the use of the topological method, Leray-Schauder Continuation theorems, fixed-point index theory, etc. For example, in [15], the solvability of BVPs of RL-fractional differential equations with p-Laplacian

$$D^{\gamma}(\phi_p(D^{\alpha}y(x))) = f(x, y(x)), \qquad x \in (0, 1),$$
$$y(0) = D^{\alpha}y(0) = 0,\ D^{\beta}y(1) = aD^{\beta}y(\xi),\ D^{\alpha}y(1) = bD^{\alpha}y(\eta),$$

is discussed, where D^{α} is the RL-fractional differentiation and the function $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous. Using a monotone iterative technique, some solvability results are given. We refer the readers to [4,10,13,15,16] for details.

This work is devoted to the study of the solvability of some p-Laplacian $BVPs$ with Caputo fractional derivative

$$^{C}D^{\beta}(\phi_p(^{C}D^{\alpha}y(x))) = f(x, y(x)),\ \ x \in (0, 1), \tag{1.1}$$
$$y(0) + y'(0) = 0,\ \ y(1) + y'(\xi) = 0,\ \ y''(0) = 0,\ \ ^{C}D^{\alpha}y(0) = 0, \tag{1.2}$$

where $2 < \alpha < 3, 0 < \beta < 1, 0 < \xi < 1, \phi_p(t) = |t|^{p-2}t, p > 1, f \in C([0, 1] \times [0, +\infty))$ is nonnegative, $^{C}D^{q}$ is the Caputo fractional derivative. With the use of some fixed point theorems and properties of the corresponding completely continuous operator, some solvability results of the problem considered are obtained.

2. Preliminaries and lemmas

Definition 2.1. [11] The Caputo fractional derivative of order γ of the function $f : [0, \infty) \to R$ is defined as

$$^{C}D^{\gamma}f(x) = \frac{1}{\Gamma(n-\gamma)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\gamma-n+1}} dt,\ n = [\gamma] + 1,$$

here $[\gamma]$ is the integer part of real number γ.

Definition 2.2. [11] The RL fractional integral of order γ of the function $f : [0, \infty) \to R$ is defined as

$$I^{\gamma}f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f(t) dt,\ \gamma > 0,$$

provided the integral exists.

Lemma 2.1. [11] Let $\gamma > 0, n = [\gamma] + 1$, the equation

$$^{C}D^{\gamma} f(x) = 0$$

has solution $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1},\ c_i \in R,\ i = 0, 1, 2, \cdots, n-1$.

Lemma 2.2. [11] Let $\gamma > 0, n = [\gamma] + 1$, then there holds

$$I^{\gamma C}D^{\gamma} f(x) = f(x) + c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

for $c_i \in R,\ i = 0, 1, 2, \cdots, n-1$.

Lemma 2.3 Given $h \in C[0, 1],\ 2 < \alpha < 3, 0 < \beta < 1, 0 < \xi < 1$. A function y solves the BVPs

$$^{C}D^{\beta}(\phi_p(^{C}D^{\alpha} y(x))) = h(x), \quad x \in (0, 1), \tag{2.1}$$

$$y(0) + y'(0) = 0, \quad y(1) + y'(\xi) = 0, \quad y''(0) = 0, \quad ^{C}D^{\alpha} y(0) = 0, \tag{2.2}$$

iff y solves the following equation

$$y(x) = \int_0^1 G(x, t)\phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} h(s) ds\right) dt,$$

where $\phi_q = (\phi_p)^{-1}, \frac{1}{p} + \frac{1}{q} = 1$, and

$$G(x,t) = \begin{cases} \frac{(x-t)^{\alpha-1}+(1-x)(1-t)^{\alpha-1}+(\alpha-1)(1-x)(\xi-t)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le t \le x \le 1, t \le \xi, \\ \frac{(x-t)^{\alpha-1}+(1-x)(1-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 < \xi \le t \le x \le 1, \\ \frac{(1-x)(1-t)^{\alpha-1}+(\alpha-1)(1-x)(\xi-t)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le x \le t \le \xi < 1, \\ \frac{(1-x)(1-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le x \le t \le 1, \xi \le t. \end{cases} \tag{5}$$

Proof By Lemma 2.2, we have

$$\phi_p(^{C}D^{\alpha} y(x)) = I^{\beta} h(x) - c = \frac{1}{\Gamma(\beta)} \int_0^x (x - t)^{\beta-1} h(t) dt - c$$

for some $c \in R$. Note that $^{C}D^{\alpha} y(0)) = 0$, we have $c = 0$. So, we obtain

$$^{C}D^{\alpha} y(x) = \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^x (x - t)^{\beta-1} h(t) dt\right).$$

By the use of the Lemma 2.2, we holds

$$\begin{aligned} y(x) &= I^{\alpha}\phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^x (x - t)^{\beta-1} h(t) dt\right) - d_1 - d_2 x - d_3 x^2 \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} h(s) ds\right) dt - d_1 - d_2 x - d_3 x^2, \end{aligned}$$

for some $d_1, d_2, d_3 \in R$. From above formula, one has

$$y'(x) = \frac{1}{\Gamma(\alpha-1)}\int_0^x (x-t)^{\alpha-2}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt - d_2 - 2d_3 x,$$

$$y''(x) = \frac{1}{\Gamma(\alpha-2)}\int_0^x (x-t)^{\alpha-3}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt - 2d_3.$$

By $y''(0) = 0$, we obtain $d_3 = 0$, and further the boundary conditions $y(0)+y'(0) = 0$ and $y(1)+y'(\xi) = 0$ yields that

$$d_1 + d_2 = 0,$$

and

$$\begin{aligned} d_1 + 2d_2 &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt \\ &+ \frac{1}{\Gamma(\alpha-1)}\int_0^\xi (\xi-t)^{\alpha-2}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt. \end{aligned}$$

Hence,

$$\begin{aligned} d_1 &= -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt \\ &- \frac{1}{\Gamma(\alpha-1)}\int_0^\xi (\xi-t)^{\alpha-2}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt, \\ d_2 &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt \\ &+ \frac{1}{\Gamma(\alpha-1)}\int_0^\xi (\xi-t)^{\alpha-2}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt. \end{aligned}$$

So, we have

$$\begin{aligned} y(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt \\ &+ \frac{1-x}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt \\ &+ \frac{1-x}{\Gamma(\alpha-1)}\int_0^\xi (\xi-t)^{\alpha-2}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt \\ &= \int_0^1 G(x,t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right)dt. \end{aligned}$$

Thus, the Lemma hold.

Lemma 2.4 The function $G(x,t)$ satisfy:

(1) $G(x,t)$ is continuous on $C([0,1]\times[0,1])$, and positive for $x,t \in (0,1)$;

(2) $\min\limits_{1/4\le x\le 3/4} G(x,t) \ge \varphi(t),\ \max\limits_{0\le x\le 1} G(x,t) \le \omega(t)$,

where

$$\varphi(t) = \frac{(1-t)^{\alpha-1}}{4\Gamma(\alpha)},\ \omega(t) = \frac{2(1-t)^{\alpha-2}(2-t)}{\Gamma(\alpha)}\ , t \in [0,1]. \tag{6}$$

Proof By the definition of the function $G(x,t)$, the property (1) is clear.
Setting

$$g_1(x,t)=\frac{(x-t)^{\alpha-1}+(1-x)(1-t)^{\alpha-1}+(\alpha-1)(1-x)(\xi-t)^{\alpha-2}}{\Gamma(\alpha)},$$

for $0\leq t\leq x\leq 1, t\leq\xi$;

$$g_2(x,t)=\frac{(x-t)^{\alpha-1}+(1-x)(1-t)^{\alpha-1}}{\Gamma(\alpha)},$$

for $0<\xi\leq t\leq x\leq 1$;

$$g_3(x,t)=\frac{(1-x)(1-t)^{\alpha-1}+(\alpha-1)(1-x)(\xi-t)^{\alpha-2}}{\Gamma(\alpha)},$$

for $0\leq x\leq t\leq\xi<1$;

$$g_4(x,t)=\frac{(1-x)(1-t)^{\alpha-1}}{\Gamma(\alpha)},$$

for $0\leq x\leq t\leq 1$.

By the expression of the $g_i(x,t), i=1,2,3,4$, one can check that

$$\begin{aligned}\min_{1/4\leq x\leq 3/4} g_i(x,t) &\geq \frac{(1-t)^{\alpha-1}}{4\Gamma(\alpha)},\quad i=1,2,3,4,\\ \max_{0\leq x\leq 1} g_i(x,t) &\leq \frac{2(1-t)^{\alpha-1}+2(1-t)^{\alpha-2}}{\Gamma(\alpha)}\\ &= \frac{2(1-t)^{\alpha-2}(2-t)}{\Gamma(\alpha)},\quad i=1,2,3,4.\end{aligned}$$

So, we obtain

$$\begin{aligned}\min_{1/4\leq x\leq 3/4} G(x,t) &\geq \frac{(1-t)^{\alpha-1}}{4\Gamma(\alpha)} := \varphi(t),\ t\in[0,1],\\ \max_{0\leq x\leq 1} G(x,t) &\leq \frac{2(1-t)^{\alpha-2}(2-t)}{\Gamma(\alpha)} := \omega(t),\ t\in[0,1].\end{aligned}$$

The Lemma hold now.

Lemma 2.5. [6] Let X be an order Banach space, $K\subset X$ is a cone, and that Ω_1,Ω_2 are bounded open subsets of X with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $T:K\to K$ be a completely continuous operator such that either

(A_1) $\|Ty\|\leq\|y\|, y\in K\cap\partial\Omega_1$ and $\|Ty\|\geq\|y\|, y\in K\cap\partial\Omega_2$, or

(A_2) $\|Ty\|\geq\|y\|, y\in K\cap\partial\Omega_1$ and $\|Ty\|\leq\|y\|, y\in K\cap\partial\Omega_2$

Then, T has a fixed point in $K\cap\overline{\Omega}_2\backslash\Omega_1$.

Lemma 2.6. [7] Let K be a cone in a real *Banach* space $X, K_c=\{y\in K\mid \|y\|\leq c\}, \theta$ be a nonnegative continuous concave functional on a cone K such that $\theta(y)\leq\|y\|$, for all $y\in\overline{K}_c$, and $K(\theta,b,d)=\{y\in K\mid b\leq\theta(y),\|y\|\leq d\}$. Suppose $T:\overline{K}_c\to\overline{K}_c$ is completely continuous and there exist constants $0<a<b<d\leq c$ such that

(B_1) $\{y\in K(\theta,b,d)\mid\theta(y)>b\}\neq\emptyset$ and $\theta(Ty)>b$ for $y\in K(\theta,b,d)$;

(B_2) $\|Ty\| < a$ for $\|y\| \le a$;

(B_3) $\theta(Ty) > b$ for $y \in K(\theta, b, c)$ with $\|Ty\| > d$.

Then T has at least three fixed points y_1, y_2, y_3 satisfying

$\|y_1\| < a, \quad b < \theta(y_2)$ and $\|y_3\| > a$ with $\theta(y_3) < b$.

3. Existence of positive solutions

Let $X = C[0,1]$ with the norm $\|y\| = \max\limits_{0 \le x \le 1} |y(x)|$. The cone $K \subset X$ is defined as $K = \{y \in X \mid y(x) \ge 0, 0 \le x \le 1\}$ and the continuous concave functional θ on the K defined as

$$\theta(y) = \min_{\frac{1}{4} \le x \le \frac{3}{4}} |y(x)|.$$

Lemma 3.1 Suppose $T : K \to X$ be defined by

$$Ty(x) = \int_0^1 G(x,t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, y(s))ds\right)dt.$$

Then $T : K \to K$ is completely continuous.

Proof Taking into account that two functions $G(x,t)$ and $f(x,y)$ are all nonnegative and continuity in their domain, one has the operator $T : K \to K$ and it is continuous. Furthermore, by the use of Lebesgue dominated convergence Theorem and Ascoli-Arzela Theorem, a standard argument can show that $T : K \to K$ is completely continuous operator (see, for example, [21]).

For the convenience , the following notations are introduced.

$$\chi(h) = \max\{f(x,y), (x,y) \in [0,1] \times [0,h]\},$$

$$\psi(h) = \min\{f(x,y), (x,y) \in [0,1] \times [0,h]\},$$

$$M^{-1} = \int_0^1 \omega(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} ds\right)dt,$$

$$N^{-1} = \int_0^1 \varphi(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} ds\right)dt.$$

where $\varphi(t), \omega(t)$ are defined as (6) and $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

Theorem 3.1 Suppose that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function, and there exists two constants $0 < k < d$ such that

$$\chi(d) \le \phi_p(dM), \psi(k) \ge \phi_p(kN).$$

Then BVPs (1), (2) have at least one solution $y \in K$ satisfying $k \le \|y\| \le d$.

Proof We will use Lemma 2.5 to obtain the results. The proof is separated into two steps.

Step 1 Let $\Omega_d = \{y \in K \mid \|y\| < d\}$. For any $y \in \partial\Omega_d$, we have $\|y\| = d$ and $f(x, y(x)) \le \chi(d) \le \phi_p(dM)$ for $(x, y) \in [0, 1] \times [0, d]$. Hence,

$$\begin{aligned}
\|Ty\| &= \max_{0\le x\le 1} |Ty(x)| \\
&= \max_{0\le x\le 1} \int_0^1 G(x,t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, y(s))ds\right)dt \\
&\le \int_0^1 \omega(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_p(dM)ds\right)dt \\
&= dM\int_0^1 \omega(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds\right)dt \\
&= d.
\end{aligned}$$

That is to say that,

$$\|Ty\| \le \|y\|, \quad \forall y \in \partial\Omega_d.$$

Step 2 Let $\Omega_k = \{y \in K \mid \|y\| < k\}$. For any $y \in \partial\Omega_k$, we have $\|y\| = k$ and $f(x, y(t)) \ge \psi(k) \ge \phi_p(kN)$ for $(x, y) \in [0, 1] \times [0, k]$. Hence,

$$\begin{aligned}
\|Ty\| &= \max_{0\le x\le 1} |Ty(x)| \\
&\ge \min_{1/4\le x\le 3/4} |Ty(x)| \\
&= \min_{1/4\le x\le 3/4} \int_0^1 G(x,t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, y(s))ds\right)dt \\
&\ge \int_0^1 \varphi(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_p(kN)ds\right)dt \\
&\ge kN\int_0^1 \varphi(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds\right)dt \\
&= k.
\end{aligned}$$

That is to say that,

$$\|Ty\| \ge \|y\|, \quad \forall y \in \partial\Omega_k.$$

Combining step 1, step 2 and Lemma 2.5, one obtains the results that the operator T has at least one fixed point $y \in K \cap \overline{\Omega}_d \backslash \Omega_k$. Consequently, the BVPs (1),(2) have at least one solution y satisfying $k \le \|y\| \le d$.

Theorem 3.2 Suppose that $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is a continuous function and there exist constants $0 < a < b < c$ such that the following three assumptions hold

(C_1) $f(x, y) < \phi_p(Ma)$, for $(x, y) \in [0, 1] \times [0, a]$;

(C_2) $f(x, y) > \phi_p(Nb)$, for $(x, y) \in [\frac{1}{4}, \frac{3}{4}] \times [b, c]$;

(C_3) $f(x, y) \le \phi_p(Mc)$, for $(x, y) \in [0, 1] \times [0, c]$.

Then the BVPs (1), (2) possess three non-negative solutions y_1, y_2 and y_3 such that

$$0 \le y_1(x) \le \max_{0\le x\le 1} |y_1(x)| < a,$$

$$b < \min_{1/4\le x\le 3/4} |y_2(x)| < \max_{0\le x\le 1} |y_2(x)| \le c,$$

$$a < \max_{0\le x\le 1} |y_3(x)| \le c, \quad \min_{1/4\le x\le 3/4} |y_3(x)| < b.$$

Proof We will use Lemma 2.6 to obtain the multiplicity results.

Step 1. Firstly, for any $y \in \overline{K}_c$, there is $\|y\| \le c$. Then the assumption (C_3) yields that $f(x, y(x)) \le \phi_p(Mc)$ for $0 \le x \le 1$. So,

$$\begin{aligned}
\|Ty\| &= \max_{0\le x\le 1}\left|\int_0^1 G(x,t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))ds\right)dt\right| \\
&\le \int_0^1 \omega(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_p(Mc)ds\right)dt \\
&= Mc\int_0^1 \omega(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds\right)dt \\
&= c.
\end{aligned}$$

Hence the operator $T : \overline{K}_c \to \overline{K}_c$.

Step 2 Secondly, We claim that the condition (B_1) of Lemma 2.6 hold. Choosing $y_0(x) = (b + c)/2$ aconstant function. It is clear that $y_0(x) \in K(\theta, b, c), \theta(y_0) > b$. That is to say that the set $\{y \in K(\theta, b, c) \mid \theta(y) > b\}$ is not empty. Moreover, for $y \in K(\theta, b, c)$, there holds $b \le y(x) \le c$ for $\frac{1}{4} \le x \le \frac{3}{4}$. Then, the assumption (C_2) yields that

$$f(x, y(x)) > \phi_p(Nb), \text{ for } y \in K(\theta, b, c)$$

So

$$\begin{aligned}
\theta(Ty) &= \min_{\frac{1}{4}\le x\le \frac{3}{4}} |(Ty)(x)| \\
&= \min_{1/4\le x\le 3/4}\int_0^1 G(x,t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))ds\right)dt \\
&> \int_0^1 \varphi(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_p(Nb)ds\right)dt \\
&= bN\int_0^1 \varphi(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds\right)dt \\
&= b.
\end{aligned}$$

Then, we have

$$\theta(Ty) > b, \quad \forall y \in K(\theta, b, c).$$

The above arguments implies that the condition (B_1) of Lemma 2.6 holds.

Step 3 Thirdly, for $y \in \overline{K}_a$, there holds $\|y\| \leq a$. Then, the Assumption (C_1) implies that $f(x, y(x)) < \phi_p(Ma)$ for all $0 \leq x \leq 1$. Thus

$$\begin{aligned}
\|Ty\| &= \max_{0\leq x\leq 1}\left|\int_0^1 G(x,t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,y(s))ds\right)dt\right| \\
&< \int_0^1 \omega(t)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_p(Ma)ds\right)dt \\
&= Ma\int_0^1 \omega(t)\phi_q\left(\int_0^1 (t-s)^{\beta-1}ds\right)dt \\
&= a.
\end{aligned}$$

That is to say that

$\|Ty\| < a$, for all $y \in \overline{K}_a$,

i.e., the condition (B_2) of Lemma 2.6 is also satisfied.

Step 4 At last, we claim that the condition (B_3) of Lemma 2.6 holds too. If $y \in K(\theta, b, c)$, taking into account *Step 2* , there holds $\theta(Ty) > b$. Consequently, the condition (B_3) of Lemma 2.6 holds.

The above four steps show that all the conditions By Lemma 2.6 hold. Thus, the BVPs (1),(2) possess three non-negative solutions y_1, y_2 and y_3 such that

$$0 \leq y_1(x) \leq \max_{0\leq x\leq 1}|y_1(x)| < a,$$

$$b < \min_{\frac{1}{4}\leq x\leq \frac{3}{4}}|y_2(x)| < \max_{0\leq x\leq 1}|y_2(x)| \leq c,$$

$$a < \max_{0\leq x\leq 1}|y_3(x)| \leq c, \quad \min_{\frac{1}{4}\leq x\leq \frac{3}{4}}|y_3(x)| < b.$$

The Theorem is proven.

4. Some examples

In the section, we give two examples to illustrate the main results obtained for BVPs (1),(2) in the section 3.

Let $p = \frac{3}{2},\ \alpha = \frac{5}{2},\ \beta = \frac{1}{2},\ \xi = \frac{1}{2}$. By simple computation, we have

$$M \approx 1.3703, \quad N \approx 36.5412.$$

Example 4.1 Consider the following BVPs:

$$\begin{cases} {}^{C}D^{\frac{1}{2}}(\phi_{\frac{3}{2}}({}^{C}D^{\frac{5}{2}}y(x))) = \frac{y^2}{9} + \frac{x^{\frac{3}{2}}}{39} + \frac{38}{39}, \quad 0 < x < 1, \\ y(0) + y'(0) = 0, \quad y(1) + y'(\frac{1}{2}) = 0, \quad y''(0) = 0, \quad {}^{C}D^{\alpha}y(0)) = 0. \end{cases} \tag{4.1}$$

It is clear that the real function

$$f(x, y) = \frac{y^2}{9} + \frac{x^{\frac{3}{2}}}{39} + \frac{38}{39},$$

is nonnegative continuous on $[0,1]\times[0,+\infty)$. Choosing $d=3,\ k=\frac{1}{39}$, one can check that $\chi(d)\leq 2<\phi_p(dM)\approx 2.0275$ and $\psi(k)\geq\frac{38}{39}\approx 0.9744>\phi_p(kN)\approx 0.9680$. By theorem 3.1, the BVPs (4.1) possess one positive solution y such that $\frac{1}{39}\leq\|y\|\leq 3$.

Example 4.2 Consider the following BVPs:

$$\begin{cases} {}^{C}D^{\frac{1}{2}}(\phi_{\frac{3}{2}}({}^{C}D^{\frac{5}{2}}y(x)))=f(x,y(x)), & 0<x<1,\\ y(0)+y'(0)=0, \quad y(1)+y'(\frac{1}{2})=0, \ y''(0)=0, \ {}^{C}D^{\alpha}y(0))=0, \end{cases} \tag{4.2}$$

where

$$f(x,y)=\begin{cases} \frac{1}{6}x^2+7y^2, & y<1,\\ \frac{1}{6}x^2+\frac{1}{60}y+\frac{419}{60}, & y\geq 1. \end{cases}$$

Choosing $a=\frac{1}{36},\ b=1,\ c=60$, then the function $f(x,y)$ satisfies the following three conditions
(1) $f(x,y)=\frac{1}{6}x^2+7y^2<0.1721<\phi_p(Ma)\approx 0.1951,\quad \text{for } (x,y)\in[0,1]\times[0,\frac{1}{36}]$;
(2) $f(x,y)=\frac{1}{6}x^2+\frac{1}{60}y+\frac{419}{60}>7.0104>\phi_p(Nb)\approx 6.0449,\quad \text{for } (x,y)\in[\frac{1}{4},\frac{3}{4}]\times[1,60]$;
(3) $f(x,y)=\frac{1}{6}x^2+\frac{1}{60}y+\frac{419}{60}<8.1500<\phi_p(Mc)\approx 9.0674,\quad \text{for } (x,y)\in[0,1]\times[0,60]$.

That is to say that all conditions of Theorem 3.2 hold. Then, Theorem 3.2 yields that the BVPs (4.2) possess three non-negative solutions y_1,y_2 and y_3 such that

$$0\leq y_1(x)\leq\max_{0\leq x\leq 1}|y_1(x)|<\frac{1}{36},\quad 1<\min_{1/4\leq x\leq 3/4}|y_2(x)|<\max_{0\leq x\leq 1}|y_2(x)|\leq 60,$$

$$\frac{1}{36}<\max_{0\leq x\leq 1}|y_3(x)|\leq 60,\quad \min_{1/4\leq x\leq 3/4}|y_3(x)|<1.$$

5. Conclusions

The present paper concentrated the solvability of some p-Laplace boundary value problems with Caputo fractional derivative. In this work, we obtained some existence results of one or three non-negative solutions by using the fixed-point theory. At the last we yielded two examples which fulfills our findings.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. A. Atangana, D. Baleanu, Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer, *J. Eng. Mech.,* **143** (2017), D4016005.
2. D. Baleanu, A. Mousalou, S. Rezapour, A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo-Fabrizio derivative, *Adv. Diff. Eq.,* **2017** (2017), 52.
3. D. Baleanu, A. Mousalou, S. Rezapour, On the existence of solutions for some infinite coefficient-symmetric Caputo-Fabrizio fractional integro-differential equations, *Bound. Value Probl.,* **2017** (2017), 145.
4. C. Chen, H. Song, H. Yang, Liouville type theorems for stable solutions of p-Laplace equation in Rn, *Nonlinear Anal. Theor.,* **160** (2017), 44–52.
5. F. Chen, D. Baleanu, G. Wu, Existence results of fractional differential equations with Riesz-Caputo derivative, *Eur. Phys. J. Spec. Top.,* **226** (2017), 3411–3425.
6. M. Krasnoselskii, *Positive Solutions of Operator Equations*, Groningen, 1964.
7. R. Leggett, L. Williams, Multiple positive fixed points of nonliear operators on ordered Banach spaces, *Indiana Univ. Math. J.,* **28** (1979), 673–688.
8. Z. Li, Z. Bai, Existence of solutions for some two-point fractional boundary value problems under barrier strip conditions, *Bound. Value Probl.,* **2019** (2019), 192.
9. X. Liu, M. Jia, Solvability and numerical simulations for BVPs of fractional coupled systems involving left and right fractional derivatives, *Appl. Math. Comput.,* **353** (2019), 230–242.
10. X. Li, M. He, Monotone iterative method for fractional p-Laplacian differential equations with four-point boundary conditions, *Adv. Differ. Equ.,* **2020** (2020), 686.
11. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
12. H. Salem, On the fractional order m-point boundary value problem in reflexive Banach spaces and weak topologies, *J. Comp. Appl. Math.,* **224** (2009), 567–572.
13. K. Sheng, W. Zhang, Problems with p-Laplacian on time scales, *Bound. Value Probl.,* 2018. DOI: 10.1186/s13661-018-0990-2.
14. Y. Tian, Z. Bai, Existence results for the three-point impulsive boundary value problem involving fractional differential equations, *Comp. Math. Appl.,* **59** (2010), 2601–2609.
15. Y. Tian, S. Sun, Z. Bai, Positive solution of fractional differential equations with p-Laplacian, *J. Funct. Space,* 2017, Article ID 3187492, 9 pages. Available from: `https://doi.org/10.1155/2017/3187492`.
16. Y. Tian, Y. Wei, S. Sun, Multiplicity for fractional differential equations with p-Laplacian, *Bound. Value Probl.,* **127** (2018), 1–14.
17. L. Guo, L. Liu, Y. Feng, Uniqueness of iterative positive solutions for the singular infinite-point p-Laplacian fractional differential system via sequential technique, *Nonlinear Anal. Model.* **25** (2020), 786–805.

18. L. Guo, L. Liu, Maximal and minimal iterative positive solutions for singular infinite point p-Laplacian fractional differential equations, *Nonlinear Anal. Model.,* **23** (2018), 851–865.

19. B. Ahmad, J. Henderson, R. Luca, *Boundary value problems for fractional differential equations and systems*, Trends in Abstract and Applied Analysis, 9. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2021.

20. S. Zhang, Positive solutions for boundary value problem of nonlinear fractional differential equations, *Electron. J. Differ. Eq.,* **2006** (2006), 1–12.

21. F. Yan, M. Zuo, X. Hao, Positive solution for a fractional singular boundary value problem with p-Laplacian operator, *Bound. Value Probl.,* **2018** (2018), 51.

22. M. Galewski, G. Bisci, Existence results for one-dimensional fractional equations, *Math. Method. Appl. Sci.,* **39** (2016), 1480–1492.

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