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Research article

A new class of Appell-type Changhee-Euler polynomials and related properties

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Abstract: A remarkably large number of polynomials and their extensions have been presented and studied. In the present paper, we introduce the new type of generating function of Appell-type Changhee-Euler polynomials by combining the Appell-type Changhee polynomials and Euler polynomials and the numbers corresponding to these polynomials are also investigated. Certain relations and identities involving these polynomials are established. Further, the differential equations arising from the generating function of the Appell-type Changhee-Euler polynomials are derived. Also, the graphical representations of the zeros of these polynomials are explored for different values of indices.

Keywords: Euler polynomials; Appell-type Changhee polynomials; Appell-type Changhee-Euler polynomials; differential equation
Mathematics Subject Classification: 11B68, 33B10, 33E20

1. Definition and notations

The special polynomials and numbers play an important role in many branches of mathematics such as combinatorics, computer science, statistics, etc. The generating functions of the special polynomials are used in the investigation of numerous properties satisfied by polynomials. A large number of indispensable special polynomials such as the Bernoulli, Euler and monic Hermite polynomials belong to the Appell class of sequences. These polynomials play crucial roles in many aspects, most importantly, they provide solutions to certain specified problems of physics, engineering and mathematics (see for more details [5–9, 21, 22, 34, 35]).

Throughout this paper, we use the following standard notations: $\mathbb{N} := \{1, 2, 3, \cdots\}, \mathbb{N}_0 := \mathbb{N} \cup \{0\} := \{0, 1, 2, 3, \cdots\}, \mathbb{Z}^- := \{-1, -2, -3, \cdots\}, \mathbb{Z}$ denotes the set of integers and $\mathbb{R}$ denotes the set of real numbers.

The generating function of the Euler polynomials $E_q(v)$ [23], which arise in various classical and numerical analysis (and also of analytic number theory) is defined as:

$$\frac{2}{e^w + 1} e^{vw} = \sum_{q=0}^{\infty} E_q(v) \frac{w^q}{q!}, \qquad (|w| < \pi). \tag{1.1}$$

The corresponding Euler numbers (appear in the Taylor series expansions of the secant and the hyperbolic secant functions) $E_q := E_q(0)$ are given as:

$$\frac{2}{e^w + 1} = \sum_{q=0}^{\infty} E_q \frac{w^q}{q!}, \qquad (|w| < \pi). \tag{1.2}$$

Recently, Kim et al., [31] introduced the Changhee polynomials and numbers and established certain explicit formulae and identities for these polynomials. The Changhee polynomials $Ch_q(v)$ are specified by the following generating function [31]:

$$\frac{2}{2 + w}(1 + w)^v = \sum_{q=0}^{\infty} Ch_q(v) \frac{w^q}{q!}, \tag{1.3}$$

which for $v = 0$ gives the corresponding Changhee numbers $Ch_q := Ch_q(0)$. Thereafter, Lee et al., [32] introduced a new form of Changhee polynomials, called the Appell-type Changhee polynomials $Ch^*_q(v)$, and defined by the following generating equation [32]:

$$\frac{2}{2 + w} e^{vw} = \sum_{q=0}^{\infty} Ch^*_q(v) \frac{w^q}{q!}, \tag{1.4}$$

and have the following explicit relation [32, P3, Theorem 1]:

$$Ch^*_q(v) = \sum_{p=0}^{q} \binom{q}{p} Ch^*_q \, v^{q-p}, \tag{1.5}$$

where $Ch^*_q := Ch^*_q(0)$ are the corresponding Appell-type Changhee numbers:

$$\frac{2}{2 + w} = \sum_{q=0}^{\infty} Ch^*_q \frac{w^q}{q!}. \tag{1.6}$$

The above Appell-type Changhee numbers Ch^*_q satisfy the following relation [32, P4, Theorem 4]:

$$2\, Ch^*_q + q\, Ch^*_{q-1} = 0, \qquad \text{in the case when } q \geq 1, \tag{1.7}$$

with initial condition $Ch^*_0 = 1$.

Motivated by the work of Kim et al., [24, 26–31] and Lee et al., [31, 32], in this paper we introduce the generating function of the Appell-type Changhee-Euler polynomials (ATCEP) and derive its properties.

2. Appell-type Changhee-Euler polynomials

In this section, the Appell-type Changhee-Euler polynomials (ATCEP), denoted by $ChE^*_q(v)$ are introduced and certain properties of these polynomials are established.

Definition 1. The Appell-type Changhee-Euler polynomials $ChE^*_q(v)$ are defined by the following generating relation:

$$\frac{4}{(e^w+1)(2+w)}e^{vw} = \sum_{q=0}^{\infty} ChE^*_q(v)\frac{w^q}{q!}. \tag{2.1}$$

Utilizing Eqs (1.1) and (1.6) in the left hand side of generating function (2.1) and then using Cauchy-product rule in the left hand side of the obtained equation, we get

$$\sum_{q=0}^{\infty}\sum_{l=0}^{q}\binom{q}{l}E_l(v)\, Ch^*_{q-l}\frac{w^q}{q!} = \sum_{q=0}^{\infty} ChE^*_q(v)\frac{w^q}{q!}.$$

Comparing the coefficients of identical powers of w in both sides of the foregoing equation, we get the following series expansion for Appell-type Changhee-Euler polynomials $ChE^*_q(v)$:

$$\sum_{l=0}^{q}\binom{q}{l}E_l(v)\, Ch^*_{q-l} = ChE^*_q(v). \tag{2.2}$$

Motivated by the importance of the Euler numbers [36, 38] in various branches of mathematics specially in number theory and combinatorics, we introduce the numbers related to the polynomial families introduced here. Taking $v = 0$ in series definition (2.2) of the ATCEP $ChE^*_q(v)$ and using notation:

$$ChE^*_q := ChE^*_q(0), \tag{2.3}$$

in the left hand side, we get the following series form for Appell-type Changhee-Euler number (ATCEN) ChE^*_q:

$$\sum_{l=0}^{q}\binom{q}{l}E_l\, Ch^*_{q-l} = ChE^*_q. \tag{2.4}$$

Also, we find the following generating functions for ATCEN ChE^*_q:

$$\frac{4}{(e^w+1)(2+w)} = \sum_{q=0}^{\infty} ChE^*_q\frac{w^q}{q!}. \tag{2.5}$$

Theorem 2.1. *Let $u, v \in \mathbb{C}$ and $q \in \mathbb{N}_0$, then the following addition formulas hold:*

$$ChE^*_q(u+v) = \sum_{p=0}^{q}\binom{q}{p}E_p(u)\, Ch^*_{q-p}(v). \tag{2.6}$$

$$ChE^*_q(u+v) = \sum_{p=0}^{q}\binom{q}{p}u^{q-p}\, ChE^*_p(v). \tag{2.7}$$

Proof. Replacing v by $u+v$ in Eq (2.1), we get

$$\left(\frac{2}{e^w+1}\right)\left(\frac{2}{2+w}\right)e^{(v+u)w} = \sum_{q=0}^{\infty} ChE_q^*(v+u)\frac{w^q}{q!}, \tag{2.8}$$

which on using Eqs (1.4) and (1.1) in the l.h.s. and then on comparing the coefficients of the identical powers of w in both sides of the obtained relation, we get assertion (2.6).

Equation (2.8) can be written as

$$\sum_{q=0}^{\infty} ChE_q^*(v+u)\frac{w^q}{q!} = \frac{4}{(e^w+1)(2+w)}e^{vw}\, e^{uw}, \tag{2.9}$$

which on using Eq (2.1) and expanding the second exponential of the right side gives

$$\sum_{q=0}^{\infty} ChE_q^*(v+u)\frac{w^q}{q!} = \sum_{q=0}^{\infty}\frac{(uw)^q}{q!}\ \sum_{p=0}^{\infty} ChE_p^*(v)\frac{w^p}{p!}. \tag{2.10}$$

Using the Cauchy product rule on the right side of Eq (2.10) and on comparing the coefficients of w, we are lead to assertion (2.7).

Theorem 2.2. *Let $q \in \mathbb{N}_0$ and $v \in \mathbb{C}$, then the following identities hold true:*

$$2\ E_q(v) = 2\ Ch^*E_q(v) + q\ ChE_{q-1}^*(v). \tag{2.11}$$

$$2\ Ch_q^*(v) = Ch^*E_q(v) + \sum_{p=o}^{q}\binom{q}{p}ChE_{q-p}^*(v). \tag{2.12}$$

$$ChE_q^*(v) = \sum_{p=0}^{q}\binom{q}{p}\frac{(-1)^p p!\ E_{q-p}(v)}{2^p}. \tag{2.13}$$

$$ChE_q^*(v) = \sum_{p=0}^{q}\binom{q}{p}ChE_{q-p}^*(v)^p. \tag{2.14}$$

$$ChE_q^*(v) = \sum_{p=0}^{q}\binom{q}{p}Ch^*{}_{q-p}\left(\frac{v}{2}\right)E_p\left(\frac{v}{2}\right). \tag{2.15}$$

$$\frac{d}{dv}ChE_q^*(v) = q\ ChE_{q-1}^*(v). \tag{2.16}$$

$$\frac{d^p}{dv^p}ChE_q^*(v) = \frac{q!}{(q-p)!}ChE_{q-p}^*(v). \tag{2.17}$$

$$\int_a^v ChE_{q-1}^*(u)du = \frac{1}{q}\ \left(ChE_q^*(v) - ChE_q^*(a)\right). \tag{2.18}$$

$$ChE_q^* = \sum_{p=0}^{q}\binom{q}{p}(-v)^p\ ChE_{q-p}^*(v). \tag{2.19}$$

$$2\ Ch_q^* = \sum_{p=0}^{q}\binom{q}{p}ChE_{q-p}^*(v)(-v)^p\left(1+\left(1-\frac{1}{v}\right)^p\right). \tag{2.20}$$

Proof. These identities are easily derivable with the help of generating Eqs (1.1), (1.4) and (2.1) by manipulating double series. Now, rewriting the generating Eq (2.1) as

$$\frac{4}{(e^w+1)(2+w)} = \sum_{q=0}^{\infty} ChE_q^*(v)\frac{w^q}{q!}\left(e^{-vw}\right), \tag{2.21}$$

which on using generating relation (2.5) in the l.h.s. and expanding the exponential on the r.h.s. and then on equating the coefficients of identical powers of w in both sides, we are led to assertion (2.19).

From Eq (2.21), we have

$$\frac{4}{2+w} = \sum_{q=0}^{\infty} ChE_q^*(v)\frac{w^q}{q!}\left(e^{-vw}(e^w+1)\right), \tag{2.22}$$

which on using generating relation (1.6) in l.h.s. and expanding the exponentials on the r.h.s. and then on equating the coefficients of identical powers of w in both sides of the obtained result, yields assertion (2.20).

Many researchers have established the determinant form of certain special and q-special polynomials [4, 12, 16, 18–20, 33]. Costabile [10, 11, 13] has given several approaches to Bernoulli polynomials. An important approach based on a determinant definition was given in [10]. This approach is further extended to provide determinant definitions of the Appell and Sheffer polynomial sequences by Costabile and Longo in [12] and [14] respectively. Motivated by the recent work on the determinant approaches, here we give the determinant representation of ATCEP $ChE_q^*(v)$. We recall the determinant form of the Euler polynomials $E_q(v)$:

The Euler polynomials $E_q(v)$ of degree $q = 0, 1, 2, \cdots$ are defined by

$$E_0(v) = 1,$$

$$E_q(v) = (-1)^q \begin{vmatrix} 1 & v & v^2 & \cdots & v^{q-1} & v^q \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1} & \cdots & \frac{1}{2}\binom{q-1}{1} & \frac{1}{2}\binom{q}{1} \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\binom{q-1}{2} & \frac{1}{2}\binom{q}{2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{q}{q-1} \end{vmatrix}, \quad q = 1, 2, ... \tag{2.23}$$

Definition 2. The Appell-type Changhee-Euler polynomials $ChE_q^*(v)$ of degree q are defined by

$$ChE_0^*(v) = 1,$$
$$ChE_q^*(v) = (-1)^q \begin{vmatrix} 1 & Ch_1^*(v) & Ch_2^*(v) & \cdots & Ch_{q-1}^*(v) & Ch_q^*(v) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1} & \cdots & \frac{1}{2}\binom{q-1}{1} & \frac{1}{2}\binom{q}{1} \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\binom{q-1}{2} & \frac{1}{2}\binom{q}{2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{q}{q-1} \end{vmatrix}, \quad q = 1, 2, ... \tag{2.24}$$

where the Appell-type Changhee polynomials $Ch_q^*(v)$ ($q = 0, 1, 2, ...$) are given by Eq (1.4).

In the forthcoming section, a family of linear differential equations arising from the generating function of ATCEN ChE_q^* is derived by following the approach presented in [26].

3. Differential equations

Differential equations, besides playing important role in pure mathematics, constitute fundamental part of mathematical description of physical processes. Thus, obtaining the solutions for differential equations is of paramount importance. Few types of differential equations allow explicit and straightforward analytical solutions. Many researchers obtained the differential equations for special polynomials and numbers utilizing their generating functions [17, 25, 30].

Theorem 3.1. For $n \in \mathbb{N}$ the following family of differential equations for Appell-type Changhee-Euler polynomials $ChE_q^*(v)$ holds true:

$$\sum_{j=1}^{n} \beta_j(n) e^w G^{(j-1)}(w) + (e^w + 1) G^{(n)}(w) = (-1)^n \frac{\beta_0(n)}{(2 + w)^{(n+1)}}, \tag{3.1}$$

where $G^{(n)}(w) := (\frac{d}{dw})^n G(w)$,

$$G = G(w) = \frac{4}{(e^w + 1)(2 + w)}, \tag{3.2}$$

$$\beta_0(n) = 4\, n!, \qquad \beta_1(n) = 1, \qquad \beta_n(n) = n, \tag{3.3}$$

and

$$\beta_i(n + 1) = 1 + \sum_{j_1=0}^{n-i+1} \beta_{i-1}(n - j_1), \qquad (2 \le i \le n). \tag{3.4}$$

Proof. Let

$$G = G(w) = \frac{4}{(e^w + 1)(2 + w)},$$

which can be rewritten as:

$$(e^{w}+1)G(w)=\frac{4}{2+w}. \tag{3.5}$$

Taking the derivative of Eq (3.5) with respect to w, it follows that:

$$(e^{w}+1)G^{(1)}(w)+e^{w}G(w)=(-1)\,\frac{4}{(2+w)^{2}}, \tag{3.6}$$

Again, by taking the derivative with respect to w of Eq (3.6), we find

$$(e^{w}+1)G^{(2)}(w)+2e^{w}G^{(1)}(w)+e^{w}G(w)=(-1)^{2}\,\frac{8}{(2+w)^{3}}, \tag{3.7}$$

and consequently

$$(e^{w}+1)G^{(3)}(w)+3e^{w}G^{(2)}(w)+3e^{w}G^{(1)}(w)+e^{w}G(w)=(-1)^{3}\,\frac{24}{(2+w)^{4}}. \tag{3.8}$$

Continuing this process, we find

$$\sum_{j=1}^{n}\beta_{j}(n)e^{w}G^{(j-1)}(w)+(e^{w}+1)G^{(n)}(w)=(-1)^{n}\,\frac{\beta_{0}(n)}{(2+w)^{(n+1)}}. \tag{3.9}$$

Further, taking the derivative with respect to w of Eq (3.9), we get

$$\begin{aligned}\beta_{1}(n)e^{w}G(w)+\sum_{j=2}^{n}e^{w}(\beta_{j}(n)+\beta_{j-1}(n))G^{(j-1)}(w)+(\beta_{n}(n)+1)e^{w}G^{(n)}(w)\\+(e^{w}+1)G^{(n+1)}(w)=(-1)^{n+1}(n+1)\beta_{0}(n)(2+w)^{-(n+2)}.\end{aligned} \tag{3.10}$$

Replacing n by $n+1$, Eq (3.9) may be rewritten as:

$$\begin{aligned}\sum_{j=1}^{n+1}\beta_{j}(n+1)e^{w}G^{(j-1)}(w)+(e^{w}+1)G^{(n+1)}(w)\\=(-1)^{n+1}\,\frac{\beta_{0}(n+1)}{(2+w)^{(n+2)}}.\end{aligned} \tag{3.11}$$

Comparison of the coefficients of $G^{(i)}$'s on both sides of Eqs (3.10) and (3.11), yields the following recursive formulas:

$$\beta_{0}(n+1)=(n+1)\beta_{0}(n), \tag{3.12}$$

$$\beta_{1}(n+1)=\beta_{1}(n), \tag{3.13}$$

$$\beta_{j}(n+1)=\beta_{j}(n)+\beta_{j-1}(n), \qquad (2\le j\le n) \tag{3.14}$$

and

$$\beta_{n+1}(n+1)=\beta_{n}(n)+1. \tag{3.15}$$

From Eqs (3.6) and (3.9), we have

$$\beta_0(1) = 4, \qquad \beta_1(1) = 1. \tag{3.16}$$

Again, from Eq (3.12), we note that

$$\beta_0(n+1) = (n+1)\beta_0(n) = (n+1)n\beta_0(n) = \ldots = (n+1)n(n-1)\ldots 2.1\,\beta_0(1),$$

that is

$$\beta_0(n+1) = 4\,(n+1)!\,. \tag{3.17}$$

In view of Eq (3.13), we have

$$\beta_1(n+1) = \beta_1(n) = \beta_1(n-1) = \ldots = \beta_1(1) = 1. \tag{3.18}$$

Further, from Eq (3.15), it follows that:

$$\beta_{n+1}(n+1) = \beta_n(n) + 1 = \beta_{n-1}(n-1) + 2 = \ldots = \beta_1(1) + n,$$

that is

$$\beta_{n+1}(n+1) = 1 + n. \tag{3.19}$$

Taking $j = 2$ in Eq (3.14), we find

$$\begin{aligned}\beta_2(n+1) &= \beta_2(n) + \beta_1(n) = \beta_2(n-1) + \beta_1(n-1) + \beta_1(n)\\ &= \ldots \quad = \sum_{j=0}^{n-2}\beta_1(n-j) + \beta_2(2),\end{aligned}$$

that is

$$\beta_2(n+1) = 1 + \sum_{j=0}^{n-1}\beta_1(n-j). \tag{3.20}$$

Similarly, for $j = 3$ Eq (3.14) gives

$$\beta_3(n+1) = 1 + \sum_{j=0}^{n-2}\beta_2(n-j), \tag{3.21}$$

consequently for $j = 4$, we have

$$\beta_4(n+1) = 1 + \sum_{j=0}^{n-3}\beta_3(n-j). \tag{3.22}$$

Continuing this process, we deduce that

$$\beta_i(n+1) = 1 + \sum_{j_1=0}^{n-i+1}\beta_{i-1}(n-j_1), \qquad (2 \le j \le n). \tag{3.23}$$

Equation (3.9) together with Eqs (3.17), (3.18) and (3.23) completes the proof of Theorem 3.1. □

Theorem 3.2. For $i, n \in \mathbb{N}$, $p \in \mathbb{N}_0$ the following identity for the Appell-type Changhee-Euler numbers ChE_q^* holds true:

$$\begin{aligned} \sum_{j=1}^{n}\sum_{p=0}^{i}\binom{i}{p}\beta_j(n)\, ChE^*_{p+j-1} + \sum_{p=0}^{i}\binom{i}{p} ChE^*_{p+n} + ChE^*_{i+n} \\ = (-1)^{n+i}(n+1)_i\left(\frac{\beta_0(n)}{2^{n+i+1}}\right), \end{aligned} \tag{3.24}$$

where $(n+1)_i$ is the pochhammer symbol [37, P21] and $\beta_j(n)$, $(0 \leq j \leq n)$ defined in Eqs (3.3) and (3.4).

Proof. Making use of Eq (2.5) in the first term of the left hand side of Eq (3.1), we find

$$\sum_{j=1}^{n}\beta_j(n)e^w G^{(j-1)}(w) = \sum_{j=1}^{n}\beta_j(n)\sum_{l=0}^{\infty}\frac{w^l}{l!}\sum_{p=0}^{\infty} ChE^*_{p+j-1}\frac{w^p}{p!},$$

or equivalently

$$\sum_{j=1}^{n}\beta_j(n)e^w G^{(j-1)}(w) = \sum_{i=0}^{\infty}\left(\sum_{j=1}^{n}\sum_{p=0}^{i}\binom{i}{p}\beta_j(n)\, ChE^*_{p+j-1}\right)\frac{w^i}{i!}. \tag{3.25}$$

Again, using Eq (2.5) in the second term of the left hand side of Eq (3.1), we find

$$(e^w+1)G^{(n)}(w) = \left(\sum_{p=0}^{\infty}\frac{w^p}{p!}+1\right)\left(\left(\frac{d}{dw}\right)^n\sum_{l=0}^{\infty} ChE^*_l\frac{w^l}{l!}\right),$$

that is

$$(e^w+1)G^{(n)}(w) = \sum_{i=0}^{\infty}\left(\sum_{p=0}^{i}\binom{i}{p} ChE^*_{p+n} + ChE^*_{i+n}\right)\frac{w^i}{i!}. \tag{3.26}$$

Finally, from the right hand side of Eq (3.1), we have

$$\beta_0(n)(-1)^n(2+w)^{-(n+1)} = \beta_0(n)(-1)^n 2^{-(n+1)}(1+w/2)^{-(n+1)},$$

which on using the following identity [37, P34]:

$$(1-z)^{-a} = \sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!},$$

gives

$$\beta_0(n)(-1)^n(2+w)^{-(n+1)} = \sum_{i=0}^{\infty}\left(\frac{(-1)^{n+i}\beta_0(n)(n+1)_i}{2^{n+i+1}}\right)\frac{w^i}{i!}. \tag{3.27}$$

Use of Eqs (3.25)–(3.27) in Eq (3.1), yields assertion (3.24). □

In the next section, the graphical representations of the ATCEP $ChE_q^*(v)$ and distribution of their zeros are discussed.

4. Graphical approach

Over the years, there has been increasing interest in solving mathematical problems with the aid of computers. The software Mathematica is used to show the behaviour of these newly introduced polynomials using the graphs plotted for the special values of indices. The manual computation of the zeros is too complicated, therefore, we use Mathematica to investigate the zeros of the ATCEP $ChE_q^*(v)$. By using numerical investigations and computer experiments, we find the real and complex zeros for certain values of index q. The investigation in this direction will lead to a new approach employing numerical methods in the field of the hybrid special polynomials to appear in mathematics and physics, see for example [18, 19].

With the help of Mathematica and using Eqs (1.5) and (1.7) in relation (2.2), Figures 1 and 2 are drawn for odd values of index q:

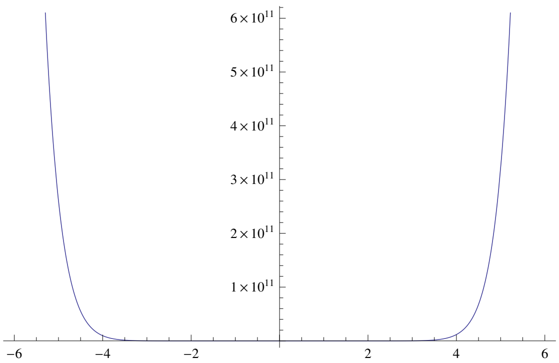

Figure 1. Curve of $ChE_{15}^*(v)$.

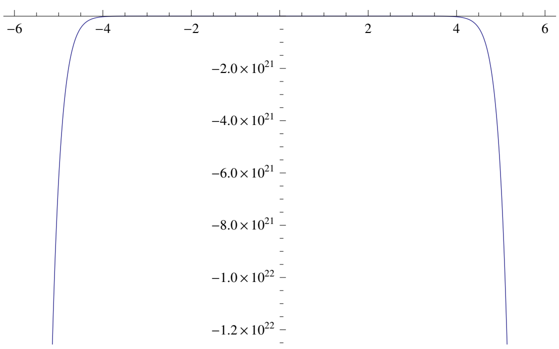

Figure 2. Curve of $ChE_{25}^*(v)$.

Similarly, for even values of index q, Figures 3 and 4 are drawn:

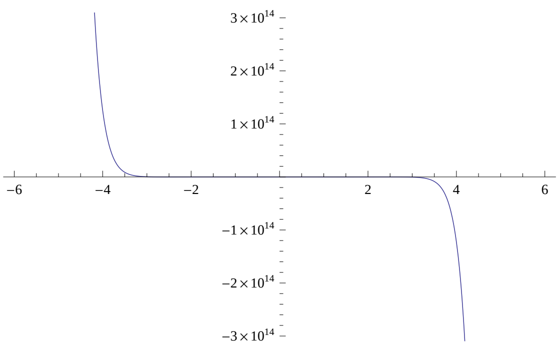

Figure 3. Curve of $ChE_{20}^*(v)$.

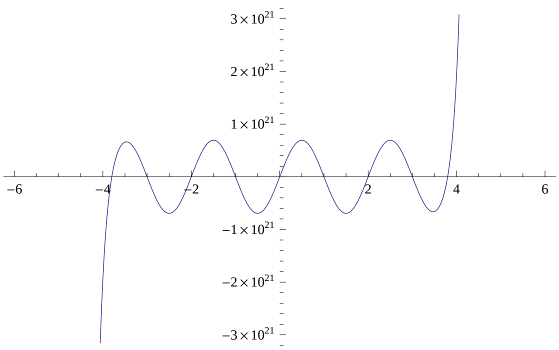

Figure 4. Curve of $ChE_{30}^*(v)$.

Next, a remarkably regular structure of the zeros of ATCEP $ChE_q^*(v)$ are explored with the help of Mathematica in Figures 5–8.

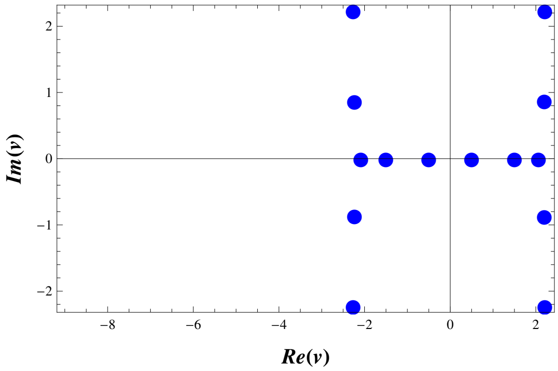

Figure 5. Zeros of $ChE^{*}_{15}(v)$.

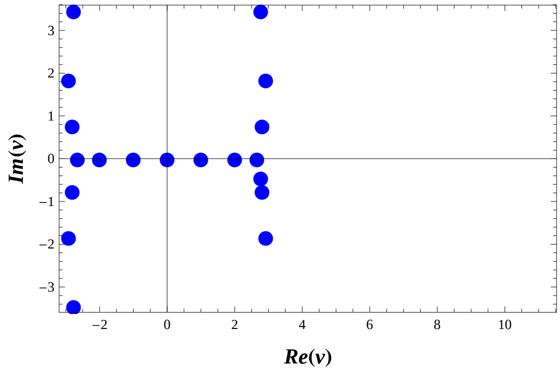

Figure 6. Zeros of $ChE^{*}_{20}(v)$.

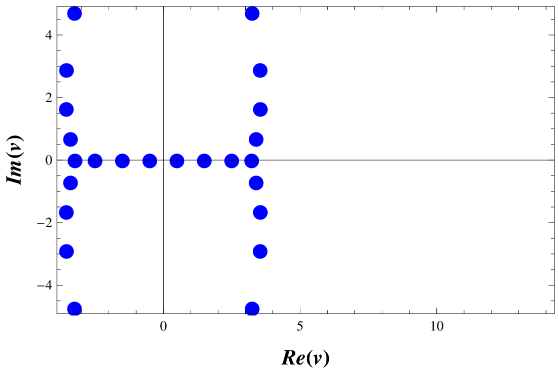

Figure 7. Zeros of $ChE^{*}_{25}(v)$.

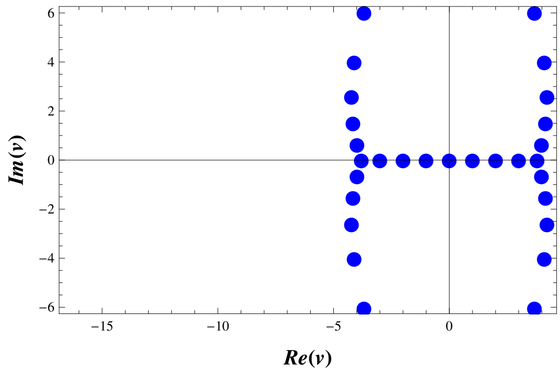

Figure 8. Zeros of $ChE^{*}_{30}(v)$.

Further properties, applications and theoretical investigation of scattering of zeros of the ATCEP $ChE^{*}_{q}(v)$ introduced here are left to the authors and the interested researchers, for future study.

5. Concluding remarks

Exclusive role have been played by special polynomials in applied mathematics. It is not astonishing when new classes of special polynomials are established as the issues associated with special polynomials are too immense. In this paper, by incorporating the Appell-type Changhee polynomials $Ch^{*}_{q}(v)$ and the Euler polynomials $E_q(v)$ in a natural way, the so-called Appell-type Changhee-Euler polynomials $ChE^{*}_{q}(v)$ are introduced. Then, we investigate certain properties and identities for these new polynomials such as series representation, determinant form and some novel identities. We also present non-homogeneous linear differential equations for the Appell-type Changhee-Euler numbers ChE^{*}_{q}. Further we discuss the shape and zero distributions of these polynomials by observing their graphs drawn by Mathematica. The ATCEP $ChE^{*}_{q}(v)$ were found to be the member of well-known Appell family and therefore many important properties and identities for this new polynomials were easily established by employing those in the well-developed theory of Appell polynomials. A unifying tool for studying polynomial sequences, namely the representation of Appell polynomials in matrix form has been studied in [1] and extended in [3] to the hypercomplex case. After that, Aceto and Caçāo [2] extends this approach to find the matrix representation of the Sheffer polynomials. The matrix which represents the Sheffer polynomial coefficients can be factorized into two matrices, one associated to Appell polynomials and the other linked to the binomial type polynomial sequences. This approach can be further extended to find the matrix representations of mixed special polynomials related to the Appell polynomials.

Posing a problem

If we consider the Genocchi polynomials [15] in place of Euler polynomials in Eq (2.1), we get the following generating function for the Appell-type Changhee-Genocchi polynomials (ATCGP) $ChG_q^*(v)$:

$$\frac{4w}{(e^w+1)(2+w)}e^{vw} = \sum_{q=0}^{\infty} ChG_q^*(v)\frac{w^q}{q!}. \tag{5.1}$$

We are believed to be able to establish the corresponding results for the Appell-type Changhee-Genocchi polynomials $ChG_q^*(v)$ as those established in this paper. This posed problem is left to the interested researchers and authors for further investigation.

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Conflict of interest

No conflict of interest was declared by the authors.

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