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Research article

Multilinear strongly singular integral operators with generalized kernels and applications

Shuhui Yang and Yan Lin*

School of Science, China University of Mining and Technology, Beijing 100083, China

* **Correspondence:** Email: linyan@cumtb.edu.cn.

Abstract: In this paper, the authors study the boundedness properties of a class of multilinear strongly singular integral operator with generalized kernels on product of weighted Lebesgue spaces and product of variable exponent Lebesgue spaces, respectively. Moreover, the types $L^\infty \times \cdots \times L^\infty \to BMO$ and $BMO \times \cdots \times BMO \to BMO$ endpoint estimates are also obtained.

Keywords: multilinear strongly singular integral operator with generalized kernel; sharp maximal function; BMO function; weighted Lebesgue space; variable exponent Lebesgue space
Mathematics Subject Classification: 42B25, 42B35

1. Introduction

For the past few years, more and more attention has been paid to the topic of multilinear singular integrals. The multilinear Calderón-Zygmund theory originated in the works of Coifman and Meyer in [1–3]. Recently, this topic has been studied extensively by many scholars from different perspectives, see for example [4–14].

Lin in [15] introduced the multilinear strongly singular Calderón-Zygmund operator. The kernel does not need any size condition and is more singular near the diagonal than the standard case. It is defined as follows.

Definition 1.1. Suppose $K(y_0, y_1, \dots, y_m)$ is a function defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, where $m \in \mathbb{N}_+$. For some $\varepsilon > 0$ and $0 < \alpha \le 1$, it satisfies

$$|K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)| \le \frac{C|x - x'|^\varepsilon}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon/\alpha}}, \tag{1.1}$$

whenever $|x - x'|^\alpha \le \frac{1}{2} \max_{1 \le j \le m} |x - y_j|$.

T is an m-linear operator with the kernel K given by the integral representation

$$T(f_1,\dots,f_m)(x)=\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}K(x,y_1,\dots,y_m)\prod_{j=1}^{m}f_j(y_j)dy_1\cdots dy_m, \tag{1.2}$$

where $f_j\in C_c^\infty(\mathbb{R}^n)(j=1,\dots,m)$ and $x\notin\cap_{j=1}^{m}\text{supp}f_j$.

For some $1\le r_1,\dots,r_m<\infty$ with $1/r=1/r_1+\cdots+1/r_m$, T is bounded from $L^{r_1}\times\cdots\times L^{r_m}$ to $L^{r,\infty}$, and for some $1\le l_1,\dots,l_m<\infty$ with $1/l=1/l_1+\cdots+1/l_m$, T is bounded from $L^{l_1}\times\cdots\times L^{l_m}$ to $L^{q,\infty}$, where $0<l/q\le\alpha$. Then T is called an m-linear strongly singular Calderón-Zygmund operator.

Lin in [15], Lin-Lu-Lu in [16] and Lin-Han in [17] established the sharp maximal estimates, the boundedness on product of weighted Lebesgue spaces and product of variable exponent Lebesgue spaces for the multilinear strongly singular Calderón-Zygmund operators, their multilinear commutators and multilinear iterated commutators, respectively. Moreover, the types $L^\infty\times\cdots\times L^\infty\to BMO$, $BMO\times\cdots\times BMO\to BMO$ and $LMO\times\cdots\times LMO\to LMO$ endpoint estimates were obtained. In [18], Spanne introduced the function space LMO, which is a subspace of BMO space, equipped with semi-norm:

$$[f]_{LMO}=\sup_{0<r<1}\frac{1+|\ln r|}{|B_r|}\int_{B_r}|f(x)-f_{B_r}|dx+\sup_{r\ge1}\frac{1}{|B_r|}\int_{B_r}|f(x)-f_{B_r}|dx,$$

where B_r denotes by the ball in $\mathbb{R}^n$ with radius r. Lin and Yan in [19] got the boundedness on the product of generalized Morrey spaces and weighted Morrey spaces for multilinear strongly singular Calderón-Zygmund operators, their multilinear commutators and multilinear iterated commutators, respectively. Other related results of this class of operators can be seen in [20] and so on.

In this paper, we will study a class of more general multilinear strongly singular integral operators T defined by (1.2). And instead of (1.1) the kernel K satisfies a weaker condition: there exist $\varepsilon>0$, $0<\alpha\le1$ and $p_0>1$, such that for any $k_1,\cdots,k_h\in\mathbb{N}_+$, $h\in\{1,\dots,m\}$,

$$\begin{aligned}&\Big(\int_{I_{i_1}}\int_{I_{i_2}}\cdots\int_{I_{i_h}}|K(x,y_1,\dots,y_m)-K(x',y_1,\dots,y_m)|^{p_0}dy_{i_1}dy_{i_2}\cdots dy_{i_h}\Big)^{\frac{1}{p_0}}\\&\qquad\le C|x-x'|^{\varepsilon-t[n(m-\frac{h}{p_0})+\frac{\varepsilon}{\alpha}]}\prod_{i=1}^{h}2^{-k_i[n(\frac{m}{h}-\frac{1}{p_0})+\frac{\varepsilon}{h\alpha}]},\end{aligned} \tag{1.3}$$

where $I_{i_s}=\{y_{i_s}:2^{k_s}|x-x'|^t\le|y_{i_s}-x'|\le2^{k_s+1}|x-x'|^t\}$, $s=1,\cdots,h$, and $t=1$ when $|x-x'|\ge1$, $t=\alpha$ when $|x-x'|<1$.

Remark 1.1. It is easy to see that the condition (1.1) implies the condition (1.3) as follows:

$$\begin{aligned}&\Big(\int_{I_{i_1}}\int_{I_{i_2}}\cdots\int_{I_{i_h}}|K(x,y_1,\dots,y_m)-K(x',y_1,\dots,y_m)|^{p_0}dy_{i_1}dy_{i_2}\cdots dy_{i_h}\Big)^{\frac{1}{p_0}}\\&\le\Big(\int_{I_{i_1}}\int_{I_{i_2}}\cdots\int_{I_{i_h}}\Big(\frac{C|x-x'|^\varepsilon}{(|x'-y_1|+\cdots+|x'-y_m|)^{mn+\varepsilon/\alpha}}\Big)^{h\frac{p_0}{h}}dy_{i_1}dy_{i_2}\cdots dy_{i_h}\Big)^{\frac{1}{p_0}}\\&\le C|x-x'|^\varepsilon\Big(\int_{I_{i_1}}(\frac{1}{|x'-y_{i_1}|^{mn+\varepsilon/\alpha}})^{\frac{p_0}{h}}dy_{i_1}\Big)^{\frac{1}{p_0}}\cdots\Big(\int_{I_{i_h}}(\frac{1}{|x'-y_{i_h}|^{mn+\varepsilon/\alpha}})^{\frac{p_0}{h}}dy_{i_h}\Big)^{\frac{1}{p_0}}\end{aligned}$$

$$\leq C|x-x'|^{\varepsilon-t[n(m-\frac{h}{p_0})+\frac{\varepsilon}{\alpha}]}\prod_{i=1}^{h}2^{-k_i[n(\frac{m}{h}-\frac{1}{p_0})+\frac{\varepsilon}{h\alpha}]}.$$

Definition 1.2. Let T be an m-linear operator with kernel K defined by (1.2). T is called an m-linear strongly singular integral operator with generalized kernel if it satisfies the condition (1.3), and for some $1 \leq r_1, \ldots, r_m < \infty$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ to $L^{r,\infty}$, and for some $1 \leq l_1, \ldots, l_m < \infty$ with $1/l = 1/l_1 + \cdots + 1/l_m$, T is bounded from $L^{l_1} \times \cdots \times L^{l_m}$ to $L^{q,\infty}$, where $0 < l/q \leq \alpha$.

In order to establish our main results, we need some notations as follows.

Definition 1.3. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|dy.$$

Denote by $M_s(f) = [M(|f|^s)]^{1/s}$, where $0 < s < \infty$.

Definition 1.4. The sharp maximal function is defined by

$$M^\sharp(f)(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y) - f_B|dy \sim \sup_{B \ni x}\inf_{a \in \mathbb{C}} \frac{1}{|B|}\int_B |f(y) - a|dx,$$

where the supremum is taken over all balls B containing x and $f_B = \frac{1}{|B|}\int_B |f(x)|dx$.

Denote by $M_s^\sharp(f) = [M^\sharp(|f|^s)]^{1/s}$, where $0 < s < \infty$.

Definition 1.5. Suppose w is a non-negative measurable function. It belongs to Muckenhoupt class A_p with $1 < p < \infty$ if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\Big)^{p-1} < \infty,$$

where $1/p + 1/p' = 1$.

When $p = 1$, we say that w belongs to A_1, if there exists a constant $C > 0$ such that for any cube Q,

$$\frac{1}{|Q|}\int_Q w(y)dy \leq Cw(x), \quad a.e.\, x \in Q.$$

Denote by $A_\infty = \bigcup_{p \geq 1} A_p$.

Definition 1.6. For the measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \Big\{f \text{ is measurable} : \int_{\mathbb{R}^n}\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}dx < \infty \text{ for some constant } \lambda > 0\Big\}.$$

As well-known that the set $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n}\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}dx \leq 1\Big\}.$$

Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that

$$1 < p_- := \text{essinf}_{x\in\mathbb{R}^n} p(x) \text{ and } p_+ := \text{esssup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

Denote by $\mathcal{B}(\mathbb{R}^n)$ the set of all $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

This paper will be organized as follows. The boundedness properties of multilinear strongly singular integral operators with generalized kernels on product of weighted Lebesgue spaces and product of variable exponent Lebesgue spaces will be obtained, respectively. Moreover, the types $L^\infty\times\cdots\times L^\infty \to BMO$ and $BMO\times\cdots\times BMO \to BMO$ endpoint estimates are also obtained as main results in Section 2. In Section 3, we will provide some necessary lemmas to prove the main results. Finally, in Section 4 we will give the proof details of the main results.

2. Main results

Theorem 2.1. *Suppose T is an m-linear strongly singular integral operator with generalized kernel and $p_0' \geq \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, $1/p_0+1/p_0' = 1$, where p_0, r_j and l_j are given by Definition* 1.2*, $j = 1, \ldots, m$. For any $p_0' < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1+\cdots+1/p_m$, $(w_1, \ldots, w_m) \in (A_{p_1/p_0'}, \ldots, A_{p_m/p_0'})$ and $w = \prod_{j=1}^m w_j^{p/p_j}$, there is*

$$\|T(\vec{f})\|_{L^p(w)} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Theorem 2.2. *Suppose T is an m-linear strongly singular integral operator with generalized kernel and $p_0' \geq \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, $1/p_0 + 1/p_0' = 1$, where p_0, r_j and l_j are given by Definition* 1.2*, $j = 1, \ldots, m$. Suppose $p(\cdot), p_1(\cdot), \ldots, p_m(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $1/p(\cdot) = 1/p_1(\cdot) + \cdots + 1/p_m(\cdot)$. Let q_0^j be given by Lemma* 3.5 *for $p_j(\cdot)$, $j = 1, \ldots, m$. If $p_0' \leq \min_{1\leq j\leq m} q_0^j$, then*

$$\|T(\vec{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

Theorem 2.3. *Suppose T is an m-linear strongly singular integral operator with generalized kernel, p_0, q, r_j, l_j are given by Definition* 1.2*, $j = 1, \ldots, m$ and $q > 1$, $p_0' \geq \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, $1/p_0 + 1/p_0' = 1$. Then*

$$\|T(\vec{f})\|_{BMO} \leq C \prod_{j=1}^m \|f_j\|_\infty.$$

Theorem 2.4. *Suppose T is an m-linear strongly singular integral operator with generalized kernel, p_0, q, r_j, l_j are given by Definition* 1.2*, $j = 1, \ldots, m$ and $q > 1$, $p_0' \geq \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, $1/p_0 + 1/p_0' = 1$, and $T(f_1, \cdots, f_{j-1}, 1, f_{j+1}, \cdots, f_m) = 0\,(j = 1, \ldots, m)$. Then*

$$\|T(\vec{f})\|_{BMO} \leq C \prod_{j=1}^m \|f_j\|_{BMO}.$$

3. Necessary lemmas

Lemma 3.1. [9, 21] Let $0 < p < r < \infty$, then there exists a constant $C = C_{p,r} > 0$ such that for any measurable function f there has

$$|Q|^{-1/p}\|f\|_{L^p(Q)} \leq C|Q|^{-1/r}\|f\|_{L^{r,\infty}(Q)}.$$

Lemma 3.2. Let T be an m-linear strongly singular integral operator with generalized kernel and $p_0' \geq \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, $1/p_0 + 1/p_0' = 1$, where p_0, r_j and l_j are given by Definition 1.2, $j = 1, \ldots, m$. If $0 < \delta < 1/m$, then

$$M_\delta^\sharp(T(\vec{f}))(x) \leq C\prod_{j=1}^m M_{p_0'}(f_j)(x),$$

for all m-tuples $\vec{f} = (f_1, \ldots, f_m)$ of bounded measurable functions with compact support.

Proof of Lemma 3.2. Let us just consider the case $m = 2$ in order to simplify the proof. In fact, a similar procedure can be applied to all other situations.

Let f_1, f_2 be bounded measurable functions with compact support. Then for any ball $B = B(x_0, r_B)$ containing x with $r_B > 0$, we will take into account two cases, respectively.

Case 1: $r_B \geq \frac{1}{4}$.

Write

$$f_1 = f_1\chi_{32B} + f_1\chi_{(32B)^c} := f_1^1 + f_1^2, \quad f_2 = f_2\chi_{32B} + f_2\chi_{(32B)^c} := f_2^1 + f_2^2.$$

Choose a $z_0 \in 6B \setminus 5B$ and take a $c_0 = T(f_1^2, f_2^1)(z_0) + T(f_1^1, f_2^2)(z_0) + T(f_1^2, f_2^2)(z_0)$, then

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B\Big||T(f_1,f_2)(z)|^\delta - |c_0|^\delta\Big|dz\Big)^{1/\delta}\\
&\leq \Big(\frac{1}{|B|}\int_B|T(f_1,f_2)(z) - c_0|^\delta dz\Big)^{1/\delta}\\
&\leq C\Big(\frac{1}{|B|}\int_B|T(f_1^1,f_2^1)(z)|^\delta dz\Big)^{1/\delta} + C\Big(\frac{1}{|B|}\int_B|T(f_1^2,f_2^1)(z) - T(f_1^2,f_2^1)(z_0)|^\delta dz\Big)^{1/\delta}\\
&+C\Big(\frac{1}{|B|}\int_B|T(f_1^1,f_2^2)(z) - T(f_1^1,f_2^2)(z_0)|^\delta dz\Big)^{1/\delta} + C\Big(\frac{1}{|B|}\int_B|T(f_1^2,f_2^2)(z) - T(f_1^2,f_2^2)(z_0)|^\delta dz\Big)^{1/\delta}\\
&:= \sum_{j=1}^4 I_j.
\end{aligned}$$

Notice that $0 < \delta < r < \infty$, where r is given as in Definition 1.2. By Lemma 3.1 and the boundedness of T from $L^{r_1} \times L^{r_2}$ to $L^{r,\infty}$, we have

$$\begin{aligned}
I_1 &\leq C|B|^{-1/\delta}\|T(f_1^1,f_2^1)\|_{L^\delta(B)}\\
&\leq C|B|^{-1/r}\|T(f_1^1,f_2^1)\|_{L^{r,\infty}(B)}\\
&\leq C\Big(\frac{1}{|32B|}\int_{32B}|f_1(y_1)|^{r_1}dy_1\Big)^{\frac{1}{r_1}}\Big(\frac{1}{|32B|}\int_{32B}|f_2(y_2)|^{r_2}dy_2\Big)^{\frac{1}{r_2}}
\end{aligned}$$

$$\begin{aligned}&\leq CM_{r_1}(f_1)(x)M_{r_2}(f_2)(x)\\&\leq CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x).\end{aligned}$$

For $z \in B$ and $y_1 \in (32B)^c$, there are $2|z - z_0| \leq |y_1 - z_0|$, $4r_B \leq |z - z_0| \leq 7r_B$ and $|z - z_0| \geq 1$. By Hölder's inequality and the condition (1.3) of the kernel, we have

$$\begin{aligned}I_2 &\leq C\frac{1}{|B|}\int_B |T(f_1^2, f_2^1)(z) - T(f_1^2, f_2^1)(z_0)|dz\\&\leq C\frac{1}{|B|}\int_B \sum_{k=1}^{\infty}\int_{2^k|z-z_0|\leq|y_1-z_0|\leq 2^{k+1}|z-z_0|}\int_{32B}|K(z,y_1,y_2) - K(z_0,y_1,y_2)||f_1(y_1)||f_2(y_2)|dy_2dy_1dz\\&\leq C\frac{1}{|B|}\int_B \sum_{k=1}^{\infty}\Big(\int_{32B}\Big(\int_{2^k|z-z_0|\leq|y_1-z_0|\leq 2^{k+1}|z-z_0|}|K(z,y_1,y_2) - K(z_0,y_1,y_2)|^{p_0}dy_1\Big)^{\frac{1}{p_0}}\\&\quad\times\Big(\int_{2^k|z-z_0|\leq|y_1-z_0|\leq 2^{k+1}|z-z_0|}|f_1(y_1)|^{p_0'}dy_1\Big)^{\frac{1}{p_0'}}|f_2(y_2)|dy_2\Big)dz\\&\leq C\frac{1}{|B|}\int_B \sum_{k=1}^{\infty}|B|2^{-k(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}|z-z_0|^{\varepsilon-(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}(2^{k+1}|z-z_0|)^{\frac{n}{p_0'}}dzM_{p_0'}(f_1)(x)M(f_2)(x)\\&\leq C\sum_{k=1}^{\infty}2^{-k(n+\frac{\varepsilon}{\alpha})}\int_B|z-z_0|^{-n-\frac{\varepsilon}{\alpha}+\varepsilon}dzM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x)\\&\leq C\sum_{k=1}^{\infty}2^{-k(n+\frac{\varepsilon}{\alpha})}r_B^{\varepsilon-\frac{\varepsilon}{\alpha}}M_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x)\\&\leq CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x).\end{aligned}$$

Similarly we can also get that

$$I_3 \leq CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x).$$

For $z \in B$ and $y_1, y_2 \in (32B)^c$, there are $2|z - z_0| \leq |y_1 - z_0|$, $2|z - z_0| \leq |y_2 - z_0|$, $4r_B \leq |z - z_0| \leq 7r_B$ and $|z - z_0| \geq 1$. By Hölder's inequality and the condition (1.3) of the kernel, we have

$$\begin{aligned}I_4 &\leq C\frac{1}{|B|}\int_B |T(f_1^2, f_2^2)(z) - T(f_1^2, f_2^2)(z_0)|dz\\&\leq C\frac{1}{|B|}\int_B \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|z-z_0|}\int_{2^{k_2}|z-z_0|\leq|y_2-z_0|\leq 2^{k_2+1}|z-z_0|}|K(z,y_1,y_2) - K(z_0,y_1,y_2)|\\&\quad\times|f_1(y_1)||f_2(y_2)|dy_2dy_1dz\\&\leq C\frac{1}{|B|}\int_B \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|z-z_0|}\int_{2^{k_2}|z-z_0|\leq|y_2-z_0|\leq 2^{k_2+1}|z-z_0|}|K(z,y_1,y_2) - K(z_0,y_1,y_2)|^{p_0}\\&\quad dy_2dy_1\Big)^{\frac{1}{p_0}}\Big(\int_{2^{k_1}|z-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|z-z_0|}|f_1(y_1)|^{p_0'}dy_1\Big)^{\frac{1}{p_0'}}\Big(\int_{2^{k_2}|z-z_0|\leq|y_2-z_0|\leq 2^{k_2+1}|z-z_0|}|f_2(y_2)|^{p_0'}dy_2\Big)^{\frac{1}{p_0'}}dz\\&\leq C\frac{1}{|B|}\int_B \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{-k_i(n(1-\frac{1}{p_0})+\frac{\varepsilon}{2\alpha})}|z-z_0|^{\varepsilon-(n(2-\frac{2}{p_0})+\frac{\varepsilon}{\alpha})}\prod_{i=1}^{2}(2^{k_i+1}|z-z_0|)^{\frac{n}{p_0'}}dzM_{p_0'}(f_1)(x)\end{aligned}$$

$$
\begin{aligned}
&\times M_{p_0'}(f_2)(x)\\
\le\;& C\frac{1}{|B|}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{\frac{-k_i\varepsilon}{2\alpha}}\int_B |z-z_0|^{\varepsilon-\frac{\varepsilon}{\alpha}}dz M_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x)\\
\le\;& C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{\frac{-k_i\varepsilon}{2\alpha}}r_B^{\varepsilon-\frac{\varepsilon}{\alpha}}M_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x)\\
\le\;& CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x).
\end{aligned}
$$

Case 2: $0 < r_B < \frac{1}{4}$.
Denote by $\tilde{B} = B(x_0, r_B^{\alpha})$. Write

$$
f_1 = f_1\chi_{16\tilde{B}} + f_1\chi_{(16\tilde{B})^c} := \tilde{f}_1^1 + \tilde{f}_1^2, \quad f_2 = f_2\chi_{16\tilde{B}} + f_2\chi_{(16\tilde{B})^c} := \tilde{f}_2^1 + \tilde{f}_2^2.
$$

Choose a $\tilde{z}_0 \in 3B \setminus 2B$ and take a $\tilde{c}_0 = T(\tilde{f}_1^2, \tilde{f}_2^1)(\tilde{z}_0) + T(\tilde{f}_1^1, \tilde{f}_2^2)(\tilde{z}_0) + T(\tilde{f}_1^2, \tilde{f}_2^2)(\tilde{z}_0)$, then

$$
\begin{aligned}
&\Big(\frac{1}{|B|}\int_B\Big||T(f_1,f_2)(z)|^{\delta}-|\tilde{c}_0|^{\delta}\Big|dz\Big)^{1/\delta}\\
&\le C\Big(\frac{1}{|B|}\int_B |T(\tilde{f}_1^1,\tilde{f}_2^1)(z)|^{\delta}dz\Big)^{1/\delta} + C\Big(\frac{1}{|B|}\int_B |T(\tilde{f}_1^2,\tilde{f}_2^1)(z)-T(\tilde{f}_1^2,\tilde{f}_2^1)(\tilde{z}_0)|^{\delta}dz\Big)^{1/\delta}\\
&+C\Big(\frac{1}{|B|}\int_B |T(\tilde{f}_1^1,\tilde{f}_2^2)(z)-T(\tilde{f}_1^1,\tilde{f}_2^2)(\tilde{z}_0)|^{\delta}dz\Big)^{1/\delta} + C\Big(\frac{1}{|B|}\int_B |T(\tilde{f}_1^2,\tilde{f}_2^2)(z)-T(\tilde{f}_1^2,\tilde{f}_2^2)(\tilde{z}_0)|^{\delta}dz\Big)^{1/\delta}\\
&:= \sum_{j=1}^{4}\tilde{I}_j.
\end{aligned}
$$

Notice that $0 < \delta < q < \infty$ and $0 < l/q \le \alpha$, where l and q are given as in Definition 1.2. By Lemma 3.1, we have

$$
\begin{aligned}
\tilde{I}_1 \le\;& C|B|^{-1/\delta}\|T(\tilde{f}_1^1,\tilde{f}_2^1)\|_{L^{\delta}(B)}\\
\le\;& C|B|^{-1/q}\|T(\tilde{f}_1^1,\tilde{f}_2^1)\|_{L^{q,\infty}(B)}\\
\le\;& C|B|^{-1/q}|\tilde{B}|^{1/l}\Big(\frac{1}{|16\tilde{B}|}\int_{16\tilde{B}}|f_1(y_1)|^{l_1}dy_1\Big)^{\frac{1}{l_1}}\Big(\frac{1}{|16\tilde{B}|}\int_{16\tilde{B}}|f_2(y_2)|^{l_2}dy_2\Big)^{\frac{1}{l_2}}\\
\le\;& Cr_B^{n(\frac{\alpha}{l}-\frac{1}{q})}M_{l_1}(f_1)(x)M_{l_2}(f_2)(x)\\
\le\;& CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x).
\end{aligned}
$$

For $z \in B$ and $y_1 \in (16\tilde{B})^c$, there are $|z-\tilde{z}_0|^{\alpha} \le (4r_B)^{\alpha} \le \dfrac{1}{2}|y_1-\tilde{z}_0|$, $r_B \le |z-\tilde{z}_0| \le 4r_B$ and $|z-\tilde{z}_0| < 1$. By Hölder's inequality and the condition (1.3) of the kernel, we have

$$\begin{aligned}
\tilde{I}_2 &\leq C\frac{1}{|B|}\int_B |T(\tilde{f}_1^2,\tilde{f}_2^1)(z)-T(\tilde{f}_1^2,\tilde{f}_2^1)(\tilde{z}_0)|dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\int_{2^k|z-\tilde{z}_0|^\alpha\leq|y_1-\tilde{z}_0|\leq 2^{k+1}|z-\tilde{z}_0|^\alpha}\int_{16\tilde{B}}|K(z,y_1,y_2)-K(\tilde{z}_0,y_1,y_2)||f_1(y_1)||f_2(y_2)|dy_2dy_1dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\int_{16\tilde{B}}\Big(\int_{2^k|z-\tilde{z}_0|^\alpha\leq|y_1-\tilde{z}_0|\leq 2^{k+1}|z-\tilde{z}_0|^\alpha}|K(z,y_1,y_2)-K(\tilde{z}_0,y_1,y_2)|^{p_0}dy_1\Big)^{\frac{1}{p_0}}\\
&\quad\times\Big(\int_{2^k|z-\tilde{z}_0|^\alpha\leq|y_1-\tilde{z}_0|\leq 2^{k+1}|z-\tilde{z}_0|^\alpha}|f_1(y_1)|^{p_0'}dy_1\Big)^{\frac{1}{p_0'}}|f_2(y_2)|dy_2dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}|\tilde{B}|2^{-k(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}|z-\tilde{z}_0|^{\varepsilon-\alpha(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}(2^{k+1}|z-\tilde{z}_0|^\alpha)^{\frac{n}{p_0'}}dzM_{p_0'}(f_1)(x)M(f_2)(x)\\
&\leq C\frac{1}{|B|}\sum_{k=1}^{\infty}|\tilde{B}|2^{-k(n+\frac{\varepsilon}{\alpha})}\int_B|z-\tilde{z}_0|^{-\alpha n}dzM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x)\\
&\leq CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x).
\end{aligned}$$

Similarly we can also get that

$$\tilde{I}_3\leq CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x).$$

For $z\in B$ and $y_1,y_2\in(16\tilde{B})^c$, there are $2|z-\tilde{z}_0|^\alpha\leq|y_1-\tilde{z}_0|$, $2|z-\tilde{z}_0|^\alpha\leq|y_2-\tilde{z}_0|$, $r_B\leq|z-\tilde{z}_0|\leq 4r_B$ and $|z-\tilde{z}_0|<1$. By Hölder's inequality and the condition (1.3) of the kernel, we have

$$\begin{aligned}
\tilde{I}_4 &\leq C\frac{1}{|B|}\int_B|T(\tilde{f}_1^2,\tilde{f}_2^2)(z)-T(\tilde{f}_1^2,\tilde{f}_2^2)(\tilde{z}_0)|dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\int_{2^{k_1}|z-\tilde{z}_0|^\alpha\leq|y_1-\tilde{z}_0|\leq 2^{k_1+1}|z-\tilde{z}_0|^\alpha}\int_{2^{k_2}|z-\tilde{z}_0|^\alpha\leq|y_2-\tilde{z}_0|\leq 2^{k_2+1}|z-\tilde{z}_0|^\alpha}\\
&\quad|K(z,y_1,y_2)-K(\tilde{z}_0,y_1,y_2)|^{p_0}dy_2dy_1\Big)^{\frac{1}{p_0}}\Big(\int_{2^{k_1}|z-\tilde{z}_0|^\alpha\leq|y_1-\tilde{z}_0|\leq 2^{k_1+1}|z-\tilde{z}_0|^\alpha}|f_1(y_1)|^{p_0'}dy_1\Big)^{\frac{1}{p_0'}}\\
&\quad\times\Big(\int_{2^{k_2}|z-\tilde{z}_0|^\alpha\leq|y_2-\tilde{z}_0|\leq 2^{k_2+1}|z-\tilde{z}_0|^\alpha}|f_2(y_2)|^{p_0'}dy_2\Big)^{\frac{1}{p_0'}}dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{-k_i(n(1-\frac{1}{p_0})+\frac{\varepsilon}{2\alpha})}|z-\tilde{z}_0|^{\varepsilon-\alpha(n(2-\frac{2}{p_0})+\frac{\varepsilon}{\alpha})}\prod_{i=1}^{2}(2^{k_i+1}|z-\tilde{z}_0|^\alpha)^{\frac{n}{p_0'}}dz\\
&\quad\times M_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x)\\
&\leq C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{\frac{-k_i\varepsilon}{2\alpha}}M_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x)\\
&\leq CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x).
\end{aligned}$$

Thus, according to the estimates in both cases, there is

$$M_\delta^\sharp(T(f_1,f_2))(x) \sim \sup_{B\ni x}\inf_{a\in\mathbb{C}}\Big(\frac{1}{|B|}\int_B\Big||T(f_1,f_2)(z)|^\delta - a\Big|dz\Big)^{1/\delta}$$
$$\leq CM_{p_0'}(f_1)(x)M_{p_0'}(f_2)(x),$$

which completes the proof of the lemma.

Lemma 3.3. [9] *Let $0 < p, \delta < \infty$ and $w \in A_\infty$. Then there exists a constant $C > 0$ depending only on the A_∞ constant of w such that*

$$\int_{\mathbb{R}^n}[M_\delta(f)(x)]^p w(x)dx \leq C\int_{\mathbb{R}^n}[M_\delta^\sharp(f)(x)]^p w(x)dx,$$

for every function f such that the left-hand side is finite.

Lemma 3.4. [22] *For $(w_1,\ldots,w_m) \in (A_{p_1},\ldots,A_{p_m})$ with $1 \leq p_1,\ldots,p_m < \infty$, and for $0 < \theta_1,\ldots,\theta_m < 1$ such that $\theta_1+\cdots+\theta_m = 1$, we have $w_1^{\theta_1}\cdots w_m^{\theta_m} \in A_{\max\{p_1,\ldots,p_m\}}$.*

Lemma 3.5. [23] *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then M_{q_0} is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for some $1 < q_0 < \infty$ if and only if M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Lemma 3.6. [24] *Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose for some fixed $0 < \tilde{p_0} < \infty$, every $(f,g) \in \mathcal{F}$ and every $w \in A_1$,*

$$\int_{\mathbb{R}^n}|f(x)|^{\tilde{p_0}}w(x)dx \leq C_0\int_{\mathbb{R}^n}|g(x)|^{\tilde{p_0}}w(x)dx.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $\tilde{p_0} \leq p_-$. If $(\frac{p(\cdot)}{\tilde{p_0}})' \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$, $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 3.7. [25] *Let $p(\cdot), p_1(\cdot),\ldots,p_m(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $1/p(x) = 1/p_1(x)+\cdots+1/p_m(x)$. Then for any $f_j \in L^{p_j(\cdot)}(\mathbb{R}^n)$, $j = 1,\ldots,m$, there has*

$$\Big\|\prod_{j=1}^m f_j\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2^{m-1}\prod_{j=1}^m\|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

Lemma 3.8. [23] *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following conditions are equivalent.*

(i) *$p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*
(ii) *$p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*
(iii) *$\frac{p(\cdot)}{\tilde{p_0}} \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < \tilde{p_0} < p_-$.*
(iv) *$(\frac{p(\cdot)}{\tilde{p_0}})' \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < \tilde{p_0} < p_-$.*

Lemma 3.9. [24] *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$.*

Lemma 3.10. [26] *Let f be a function in BMO. Suppose $1 \leq p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}|f(y) - f_{B(x,r_2)}|^p dy\Big)^{1/p} \leq C\Big(1+\Big|\ln\frac{r_1}{r_2}\Big|\Big)\|f\|_{BMO},$$

where $C > 0$ is independent of f, x, r_1 and r_2.

4. Proof of main results

Proof of Theorem 2.1. From the fact that $w_j \in A_{p_j/p_0'}$ and $p_j > p_0'$ $(j = 1, \dots, m)$, it follows that the Hardy-Littlewood maximal operator M is bounded on $L^{p_j/p_0'}(w_j)$. By Lemma 3.4, we obtain $w \in A_{\max\{p_1/p_0', \dots, p_m/p_0'\}} \subset A_\infty$.

Take a δ such that $0 < \delta < 1/m$, then by Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}
&\|T(\vec{f})\|_{L^p(w)} \le \|M_\delta(T(\vec{f}))\|_{L^p(w)} \le C\|M_\delta^\sharp(T(\vec{f}))\|_{L^p(w)}\\
&\le C\Big\|\prod_{j=1}^m M_{p_0'}(f_j)\Big\|_{L^p(w)} \le C\prod_{j=1}^m \|M_{p_0'}(f_j)\|_{L^{p_j}(w_j)}\\
&= C\prod_{j=1}^m \|M(|f_j|^{p_0'})\|_{L^{p_j/p_0'}(w_j)}^{1/p_0'} \le C\prod_{j=1}^m \big\||f_j|^{p_0'}\big\|_{L^{p_j/p_0'}(w_j)}^{1/p_0'}\\
&= C\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.
\end{aligned}$$

This completes the proof of the theorem.

Proof of Theorem 2.2. Since $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then by Lemma 3.8, there exists a $\tilde{p_0}$ such that $1 < \tilde{p_0} < p_-$ and $(\frac{p(\cdot)}{\tilde{p_0}})' \in \mathcal{B}(\mathbb{R}^n)$. Take a δ such that $0 < \delta < 1/m$. For any $w \in A_1$, by Lemma 3.2 and Lemma 3.3, we can get that

$$\begin{aligned}
\int_{\mathbb{R}^n} |T(\vec{f})(x)|^{\tilde{p_0}} w(x)dx &\le \int_{\mathbb{R}^n} [M_\delta(T(\vec{f}))(x)]^{\tilde{p_0}} w(x)dx\\
&\le C\int_{\mathbb{R}^n} [M_\delta^\sharp(T(\vec{f}))(x)]^{\tilde{p_0}} w(x)dx\\
&\le C\int_{\mathbb{R}^n} \Big[\prod_{j=1}^m M_{p_0'}(f_j)(x)\Big]^{\tilde{p_0}} w(x)dx\\
&\le C\int_{\mathbb{R}^n} \Big[\prod_{j=1}^m M_{q_0^j}(f_j)(x)\Big]^{\tilde{p_0}} w(x)dx
\end{aligned}$$

holds for all m-tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded measurable functions with compact support.

By Lemma 3.9, we can get that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p_j(\cdot)}(\mathbb{R}^n)(j = 1, 2, \cdots, m)$. For $f_j \in C_c^\infty(\mathbb{R}^n)$, applying Lemma 3.6 to the pair $(T(\vec{f}), \prod_{j=1}^m M_{q_0^j}(f_j))$, we have

$$\|T(\vec{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\Big\|\prod_{j=1}^m M_{q_0^j}(f_j)\Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Then by Lemma 3.5 and Lemma 3.7, we have

$$\|T(\vec{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\prod_{j=1}^m \|M_{q_0^j}(f_j)\|_{L^{p_j(\cdot)}(\mathbb{R}^n)} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

Since the denseness of $C_c^\infty(\mathbb{R}^n)$ in $L^{p_j(\cdot)}(\mathbb{R}^n)(j=1,2,\cdots,m)$, this completes the proof of the theorem.

Proof of Theorem 2.3. To simplify the proof, we only give the proof when $m=2$ since their similarities.

Let $f_1, f_2 \in L^\infty$, then for any ball $B=B(x_0,r_B)$ with $r_B>0$, we will think about two cases, respectively.

Case 1: $r_B \geq \frac{1}{4}$.

Write

$$f_1 = f_1\chi_{32B} + f_1\chi_{(32B)^c} := f_1^1 + f_1^2, \quad f_2 = f_2\chi_{32B} + f_2\chi_{(32B)^c} := f_2^1 + f_2^2.$$

Choose a $z_0 \in 6B \setminus 5B$ and take a $c_0 = T(f_1^2, f_2^1)(z_0) + T(f_1^1, f_2^2)(z_0) + T(f_1^2, f_2^2)(z_0)$, then

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T(f_1,f_2)(x)-c_0|dx\\
&\leq \frac{1}{|B|}\int_B |T(f_1^1,f_2^1)(x)|dx + \frac{1}{|B|}\int_B |T(f_1^2,f_2^1)(x)-T(f_1^2,f_2^1)(z_0)|dx\\
&\quad+\frac{1}{|B|}\int_B |T(f_1^1,f_2^2)(x)-T(f_1^1,f_2^2)(z_0)|dx + \frac{1}{|B|}\int_B |T(f_1^2,f_2^2)(x)-T(f_1^2,f_2^2)(z_0)|dx\\
&:= \sum_{j=1}^4 J_j.
\end{aligned}$$

Take $p_1,\ldots,p_m$ such that $\max\{p_0', m\} < p_1,\ldots,p_m < \infty$. Let $1/p = 1/p_1 + \cdots + 1/p_m$, then $1<p<\infty$. By Theorem 2.1, we obtain that T is bounded from $L^{p_1}\times\cdots\times L^{p_m}$ into L^p.

By Hölder's inequality and the $L^{p_1}\times L^{p_2} \to L^p$ boundedness of T, we have

$$\begin{aligned}
J_1 &\leq \left(\frac{1}{|B|}\int_B |T(f_1^1,f_2^1)(x)|^p dx\right)^{\frac{1}{p}}\\
&\leq C|B|^{-1/p}\|f_1^1\|_{L^{p_1}}\|f_2^1\|_{L^{p_2}}\\
&\leq C\|f_1\|_\infty\|f_2\|_\infty.
\end{aligned}$$

For $x\in B$ and $y_1 \in (32B)^c$, there are $2|x-z_0| \leq |y_1-z_0|$, $4r_B \leq |x-z_0| \leq 7r_B$ and $|x-z_0|\geq 1$. By Hölder's inequality and the condition (1.3) of the kernel, we have

$$\begin{aligned}
J_2 &\leq \frac{1}{|B|}\int_B\int_{(32B)^c}\int_{32B} |K(x,y_1,y_2)-K(z_0,y_1,y_2)||f_1(y_1)||f_2(y_2)|dy_2dy_1dx\\
&\leq \|f_1\|_\infty\|f_2\|_\infty\frac{1}{|B|}\int_B\sum_{k=1}^\infty\int_{2^k|x-z_0|\leq|y_1-z_0|\leq 2^{k+1}|x-z_0|}\int_{32B}|K(x,y_1,y_2)-K(z_0,y_1,y_2)|dy_2dy_1dx\\
&\leq \frac{1}{|B|}\int_B\sum_{k=1}^\infty\int_{32B}\Big(\int_{2^k|x-z_0|\leq|y_1-z_0|\leq 2^{k+1}|x-z_0|}|K(x,y_1,y_2)-K(z_0,y_1,y_2)|^{p_0}dy_1\Big)^{\frac{1}{p_0}}\\
&\quad\times(2^{k+1}|x-z_0|)^{\frac{n}{p_0'}}dy_2dx\|f_1\|_\infty\|f_2\|_\infty\\
&\leq C\|f_1\|_\infty\|f_2\|_\infty\frac{1}{|B|}\int_B\sum_{k=1}^\infty|B|2^{-k(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}|x-z_0|^{\varepsilon-(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}(2^{k+1}|x-z_0|)^{\frac{n}{p_0'}}dx\\
&\leq Cr_B^{-\frac{\varepsilon}{\alpha}+\varepsilon}\|f_1\|_\infty\|f_2\|_\infty
\end{aligned}$$

$$\leq \quad C\|f_1\|_\infty\|f_2\|_\infty.$$

Similarly we can also get that

$$J_3 \leq C\|f_1\|_\infty\|f_2\|_\infty.$$

For $x \in B$ and $y_1, y_2 \in (32B)^c$, there are $2|x-z_0| \leq |y_1 - z_0|$, $2|x-z_0| \leq |y_2 - z_0|$, $4r_B \leq |x - z_0| \leq 7r_B$ and $|x - z_0| \geq 1$. By Hölder's inequality and the condition (1.3) of the kernel, we have

$$\begin{aligned}
J_4 &\leq \frac{1}{|B|}\int_B\int_{(32B)^c}\int_{(32B)^c}|K(x,y_1,y_2)-K(z_0,y_1,y_2)||f_1(y_1)||f_2(y_2)|dy_2dy_1dx \\
&\leq \|f_1\|_\infty\|f_2\|_\infty\frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\int_{2^{k_1}|x-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|x-z_0|}\int_{2^{k_2}|x-z_0|\leq|y_2-z_0|\leq 2^{k_2+1}|x-z_0|} \\
&\quad |K(x,y_1,y_2)-K(z_0,y_1,y_2)|^{p_0}dy_2dy_1\Big)^{\frac{1}{p_0}}\prod_{i=1}^{2}(2^{k_i+1}|x-z_0|)^{\frac{n}{p_0}}dx \\
&\leq C\frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{-k_i(n(1-\frac{1}{p_0})+\frac{\varepsilon}{2\alpha})}|x-z_0|^{\varepsilon-(n(2-\frac{2}{p_0})+\frac{\varepsilon}{\alpha})}\prod_{i=1}^{2}(2^{k_i+1}|x-z_0|)^{\frac{n}{p_0}}dx\|f_1\|_\infty\|f_2\|_\infty \\
&\leq C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{\frac{-k_i\varepsilon}{2\alpha}}\frac{1}{|B|}\int_B|x-z_0|^{\varepsilon-\frac{\varepsilon}{\alpha}}dx\|f_1\|_\infty\|f_2\|_\infty \\
&\leq C\|f_1\|_\infty\|f_2\|_\infty.
\end{aligned}$$

Case 2: $0 < r_B < \frac{1}{4}$.
Denote by $\tilde{B} = B(x_0, r_B^\alpha)$. Write

$$f_1 = f_1\chi_{16\tilde{B}} + f_1\chi_{(16\tilde{B})^c} := \tilde{f}_1^1 + \tilde{f}_1^2, \quad f_2 = f_2\chi_{16\tilde{B}} + f_2\chi_{(16\tilde{B})^c} := \tilde{f}_2^1 + \tilde{f}_2^2.$$

Choose a $\tilde{z}_0 \in 3B \setminus 2B$ and take a $\tilde{c}_0 = T(\tilde{f}_1^2, \tilde{f}_2^1)(\tilde{z}_0) + T(\tilde{f}_1^1, \tilde{f}_2^2)(\tilde{z}_0) + T(\tilde{f}_1^2, \tilde{f}_2^2)(\tilde{z}_0)$, then

$$\begin{aligned}
&\frac{1}{|B|}\int_B|T(f_1,f_2)(x)-\tilde{c}_0|dx \\
&\leq \frac{1}{|B|}\int_B|T(\tilde{f}_1^1,\tilde{f}_2^1)(x)|dx + \frac{1}{|B|}\int_B|T(\tilde{f}_1^2,\tilde{f}_2^1)(x)-T(\tilde{f}_1^2,\tilde{f}_2^1)(\tilde{z}_0)|dx \\
&\quad + \frac{1}{|B|}\int_B|T(\tilde{f}_1^1,\tilde{f}_2^2)(x)-T(\tilde{f}_1^1,\tilde{f}_2^2)(\tilde{z}_0)|dx + \frac{1}{|B|}\int_B|T(\tilde{f}_1^2,\tilde{f}_2^2)(x)-T(\tilde{f}_1^2,\tilde{f}_2^2)(\tilde{z}_0)|dx \\
&:= \sum_{j=1}^{4}\tilde{J}_j.
\end{aligned}$$

Notice that $1 < q < \infty$ and $0 < l/q \leq \alpha$, where l is given as in Definition 1.2. By Lemma 3.1, we have

$$\begin{aligned}
\tilde{J}_1 &\leq C|B|^{-1}\|T(\tilde{f}_1^1,\tilde{f}_2^1)\|_{L^1(B)} \\
&\leq C|B|^{-1/q}\|T(\tilde{f}_1^1,\tilde{f}_2^1)\|_{L^{q,\infty}(B)} \\
&\leq C|B|^{-1/q}\|\tilde{f}_1^1\|_{L^{l_1}}\|\tilde{f}_2^1\|_{L^{l_2}}
\end{aligned}$$

$$\leq C\|f_1\|_\infty\|f_2\|_\infty r_B^{n(\frac{\alpha}{l}-\frac{1}{q})}$$
$$\leq C\|f_1\|_\infty\|f_2\|_\infty.$$

For $x \in B$ and $y_1 \in (16\tilde{B})^c$, there are $2|x-\tilde{z}_0|^\alpha \leq |y_1-\tilde{z}_0|$, $r_B \leq |x-\tilde{z}_0| \leq 4r_B$ and $|x-\tilde{z}_0| < 1$. By Hölder's inequality and the condition (1.3) of the kernel, we have

$$\begin{aligned}
\tilde{J}_2 &\leq \frac{1}{|B|}\int_B\int_{(16\tilde{B})^c}\int_{16\tilde{B}}|K(x,y_1,y_2)-K(\tilde{z}_0,y_1,y_2)|f_1(y_1)||f_2(y_2)|dy_2dy_1dx \\
&\leq \frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\int_{16\tilde{B}}\Big(\int_{2^k|x-\tilde{z}_0|^\alpha\leq|y_1-\tilde{z}_0|\leq 2^{k+1}|x-\tilde{z}_0|^\alpha}|K(x,y_1,y_2)-K(x_0,y_1,y_2)|^{p_0}dy_1\Big)^{\frac{1}{p_0}} \\
&\quad \times(2^{k+1}|x-\tilde{z}_0|^\alpha)^{\frac{n}{p_0'}}dy_2dx\|f_1\|_\infty\|f_2\|_\infty \\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}|\tilde{B}|2^{-k(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}|x-\tilde{z}_0|^{\varepsilon-\alpha(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}(2^{k+1}|x-\tilde{z}_0|^\alpha)^{\frac{n}{p_0'}}dx\|f_1\|_\infty\|f_2\|_\infty \\
&\leq C\sum_{k=1}^{\infty}|\tilde{B}|2^{-k(n+\frac{\varepsilon}{\alpha})}\frac{1}{|B|}\int_B|x-\tilde{z}_0|^{-\alpha n}dx\|f_1\|_\infty\|f_2\|_\infty \\
&\leq C\|f_1\|_\infty\|f_2\|_\infty.
\end{aligned}$$

Similarly we can also get that

$$\tilde{J}_3 \leq C\|f_1\|_\infty\|f_2\|_\infty.$$

For $x \in B$ and $y_1, y_2 \in (16\tilde{B})^c$, there are $|x-\tilde{z}_0|^\alpha \leq \frac{1}{2}|y_1-\tilde{z}_0|$, $|x-\tilde{z}_0|^\alpha \leq \frac{1}{2}|y_2-\tilde{z}_0|$, $r_B \leq |x-\tilde{z}_0| \leq 4r_B$ and $|x-\tilde{z}_0| < 1$. By Hölder's inequality and the condition (1.3) of the kernel, we have

$$\begin{aligned}
\tilde{J}_4 &\leq \frac{1}{|B|}\int_B\int_{(16\tilde{B})^c}\int_{(16\tilde{B})^c}|K(x,y_1,y_2)-K(\tilde{z}_0,y_1,y_2)||f_1(y_1)||f_2(y_2)|dy_2dy_1dx \\
&\leq \frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\int_{2^{k_1}|x-\tilde{z}_0|^\alpha\leq|y_1-\tilde{z}_0|\leq 2^{k_1+1}|x-\tilde{z}_0|^\alpha}\int_{2^{k_2}|x-\tilde{z}_0|^\alpha\leq|y_2-\tilde{z}_0|\leq 2^{k_2+1}|x-\tilde{z}_0|^\alpha} \\
&\quad |K(x,y_1,y_2)-K(\tilde{z}_0,y_1,y_2)|^{p_0}dy_2dy_1\Big)^{\frac{1}{p_0}}\prod_{i=1}^{2}(2^{k_i+1}|x-\tilde{z}_0|^\alpha)^{\frac{n}{p_0'}}dx\|f_1\|_\infty\|f_2\|_\infty \\
&\leq C\frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{-k_i(n(1-\frac{1}{p_0})+\frac{\varepsilon}{2\alpha}))}|x-\tilde{z}_0|^{\varepsilon-\alpha(n(2-\frac{2}{p_0})+\frac{\varepsilon}{\alpha})}\prod_{i=1}^{2}(2^{k_i+1}|x-\tilde{z}_0|^\alpha)^{\frac{n}{p_0'}}dx\|f_1\|_\infty\|f_2\|_\infty \\
&\leq C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{\frac{-k_i\varepsilon}{2\alpha}}\|f_1\|_\infty\|f_2\|_\infty \\
&\leq C\|f_1\|_\infty\|f_2\|_\infty.
\end{aligned}$$

Thus, according to the estimates in both cases, there is

$$\|T(f_1,f_2)\|_{BMO} \sim \sup_B\inf_{a\in\mathbb{C}}\frac{1}{|B|}\int_B|T(f_1,f_2)(x)-a|dx \leq C\|f_1\|_\infty\|f_2\|_\infty,$$

which completes the proof of the theorem.

Proof of Theorem 2.4. We only give the proof when $m = 2$.

Let $f_1, f_2 \in BMO$, then for any ball $B = B(x_0, r_B)$ with $r_B > 0$, we will consider two cases, respectively.

Case 1: $r_B \geq \frac{1}{4}$.

Write

$$f_1 = (f_1)_{32B} + (f_1 - (f_1)_{32B})\chi_{32B} + (f_1 - (f_1)_{32B})\chi_{(32B)^c} := f_1^1 + f_1^2 + f_1^3,$$
$$f_2 = (f_2)_{32B} + (f_2 - (f_2)_{32B})\chi_{32B} + (f_2 - (f_2)_{32B})\chi_{(32B)^c} := f_2^1 + f_2^2 + f_2^3.$$

It follows from the hypothesis of the theorem that

$$T(f_1, f_2) = T(f_1^2, f_2^2) + T(f_1^2, f_2^3) + T(f_1^3, f_2^2) + T(f_1^3, f_2^3).$$

Choose a $z_0 \in 6B \setminus 5B$ and take a $d_0 = T(f_1^2, f_2^3)(z_0) + T(f_1^3, f_2^2)(z_0) + T(f_1^3, f_2^3)(z_0)$, then

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T(f_1, f_2)(x) - d_0| dx \\
&\leq \frac{1}{|B|}\int_B |T(f_1^2, f_2^2)(x)| dx + \frac{1}{|B|}\int_B |T(f_1^2, f_2^3)(x) - T(f_1^2, f_2^3)(z_0)| dx \\
&\quad + \frac{1}{|B|}\int_B |T(f_1^3, f_2^2)(x) - T(f_1^3, f_2^2)(z_0)| dx + \frac{1}{|B|}\int_B |T(f_1^3, f_2^3)(x) - T(f_1^3, f_2^3)(z_0)| dx \\
&:= \sum_{j=1}^{4} L_j.
\end{aligned}$$

Take $p_1, \ldots, p_m$ such that $\max\{p_0', m\} < p_1, \ldots, p_m < \infty$. Let $1/p = 1/p_1 + \cdots + 1/p_m$, then $1 < p < \infty$. By Theorem 2.1, we obtain the boundedness of T from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p. By Hölder's inequality, we have

$$\begin{aligned}
L_1 &\leq \Big(\frac{1}{|B|}\int_B |T(f_1^2, f_2^2)(x)|^p dx\Big)^{\frac{1}{p}} \\
&\leq C\Big(\frac{1}{|32B|}\int_{32B} |f_1(y_1) - (f_1)_{32B}|^{p_1} dy_1\Big)^{\frac{1}{p_1}}\Big(\frac{1}{|32B|}\int_{32B} |f_2(y_2) - (f_2)_{32B}|^{p_2} dy_2\Big)^{\frac{1}{p_2}} \\
&\leq C\|f_1\|_{BMO}\|f_2\|_{BMO}.
\end{aligned}$$

For $x \in B$ and $y_2 \in (32B)^c$, there are $2|x - z_0| \leq |y_2 - z_0|$, $4r_B \leq |x - z_0| \leq 7r_B$ and $|x - z_0| \geq 1$. By Hölder's inequality, Lemma 3.10 and the condition (1.3) of the kernel, we have

$$\begin{aligned}
L_2 &\leq \frac{1}{|B|}\int_B\int_{(32B)^c}\int_{32B} |K(x, y_1, y_2) - K(z_0, y_1, y_2)||f_1(y_1) - (f_1)_{32B}||f_2(y_2) - (f_2)_{32B}| dy_1 dy_2 dx \\
&\leq \frac{1}{|B|}\int_B \sum_{k=1}^{\infty}\int_{32B}\Big(\int_{2^k|x-z_0|\leq|y_2-z_0|\leq 2^{k+1}|x-z_0|} |K(x, y_1, y_2) - K(z_0, y_1, y_2)|^{p_0} dy_2\Big)^{\frac{1}{p_0}} \\
&\quad \times\Big(\int_{2^k|x-z_0|\leq|y_2-z_0|\leq 2^{k+1}|x-z_0|} |f_2(y_2) - (f_2)_{32B}|^{p_0'} dy_2\Big)^{\frac{1}{p_0'}} |f_1(y_1) - (f_1)_{32B}| dy_1 dx \\
&\leq C\frac{1}{|B|}\int_B \sum_{k=1}^{\infty} |B| 2^{-k(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})} |x - z_0|^{\varepsilon-(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})} dx\Big(\int_{2^{k+5}B} |f_2(y_2) - (f_2)_{32B}|^{p_0'} dy_2\Big)^{\frac{1}{p_0'}} \|f_1\|_{BMO}
\end{aligned}$$

$$
\begin{aligned}
&\leq C r_B^{\frac{n}{p_0'}} \sum_{k=1}^{\infty} 2^{-k(n+\frac{\varepsilon}{\alpha})}\Big(1+\ln\frac{2^{k+5}r_B}{32r_B}\Big)\int_B |x-z_0|^{-2n-\frac{\varepsilon}{\alpha}+\varepsilon+\frac{n}{p_0}}dx\|f_1\|_{BMO}\|f_2\|_{BMO}\\
&\leq C\sum_{k=1}^{\infty} k2^{-k(n+\frac{\varepsilon}{\alpha})} r_B^{-\frac{\varepsilon}{\alpha}+\varepsilon}\|f_1\|_{BMO}\|f_2\|_{BMO}\\
&\leq C\|f_1\|_{BMO}\|f_2\|_{BMO}.
\end{aligned}
$$

Similarly we also can get that

$$L_3 \leq C\|f_1\|_{BMO}\|f_2\|_{BMO}.$$

For $x\in B$ and $y_1, y_2 \in (32B)^c$, there are $2|x-z_0| \leq |y_1-z_0|$, $2|x-z_0|\leq |y_2-z_0|$, $4r_B \leq |x-z_0| \leq 7r_B$ and $|x-z_0|\geq 1$. By Hölder's inequality, Lemma 3.10 and the condition (1.3) of the kernel, we have

$$
\begin{aligned}
L_4 &\leq \frac{1}{|B|}\int_B\int_{(32B)^c}\int_{(32B)^c}|K(x,y_1,y_2)-K(z_0,y_1,y_2)||f_1(y_1)-(f_1)_{32B}|\\
&\quad\times|f_2(y_2)-(f_2)_{32B}|dy_2dy_1dx\\
&\leq \frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\int_{2^{k_1}|x-z_0|\leq|y_1-z_0|\leq 2^{k_1+1}|x-z_0|}\int_{2^{k_2}|x-z_0|\leq|y_2-z_0|\leq 2^{k_2+1}|x-z_0|}\\
&\quad|K(x,y_1,y_2)-K(z_0,y_1,y_2)|^{p_0}dy_2dy_1\Big)^{\frac{1}{p_0}}\Big(\int_{2^{k_1+5}B}|f_1(y_1)-(f_1)_{32B}|^{p_0'}dy_1\Big)^{\frac{1}{p_0'}}\\
&\quad\times\Big(\int_{2^{k_2+5}B}|f_2(y_2)-(f_2)_{32B}|^{p_0'}dy_2\Big)^{\frac{1}{p_0'}}dx\\
&\leq C\frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{-k_i(n(1-\frac{1}{p_0})+\frac{\varepsilon}{2\alpha})}|x-z_0|^{\varepsilon-(n(2-\frac{2}{p_0})+\frac{\varepsilon}{\alpha})}\prod_{i=1}^{2}\Big((2^{k_i+5}r_B)^{\frac{n}{p_0'}}\\
&\quad\times\Big(1+\ln\frac{2^{k_i+5}r_B}{32r_B}\Big)\Big)dx\|f_1\|_{BMO}\|f_2\|_{BMO}\\
&\leq C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}k_i2^{\frac{-k_i\varepsilon}{2\alpha}}r_B^{-\frac{\varepsilon}{\alpha}+\varepsilon}\|f_1\|_{BMO}\|f_2\|_{BMO}\\
&\leq C\|f_1\|_{BMO}\|f_2\|_{BMO}.
\end{aligned}
$$

Case 2: $0<r_B<\frac{1}{4}$.
Denote by $\tilde{B}=B(x_0, r_B^{\alpha})$. Write

$$f_1 = (f_1)_{16\tilde{B}} + (f_1-(f_1)_{16\tilde{B}})\chi_{16\tilde{B}} + (f_1-(f_1)_{16\tilde{B}})\chi_{(16\tilde{B})^c} := \tilde{f}_1^1+\tilde{f}_1^2+\tilde{f}_1^3,$$

$$f_2 = (f_2)_{16\tilde{B}} + (f_2-(f_2)_{16\tilde{B}})\chi_{16\tilde{B}} + (f_2-(f_2)_{16\tilde{B}})\chi_{(16\tilde{B})^c} := \tilde{f}_2^1+\tilde{f}_2^2+\tilde{f}_2^3.$$

It follows from the hypothesis of the theorem that

$$T(f_1,f_2)=T(\tilde{f}_1^2,\tilde{f}_2^2)+T(\tilde{f}_1^2,\tilde{f}_2^3)+T(\tilde{f}_1^3,\tilde{f}_2^2)+T(\tilde{f}_1^3,\tilde{f}_2^3).$$

Choose a $\tilde{z}_0\in 3B\setminus 2B$ and take a $\tilde{d}_0 = T(\tilde{f}_1^2,\tilde{f}_2^3)(\tilde{z}_0)+T(\tilde{f}_1^3,\tilde{f}_2^2)(\tilde{z}_0)+T(\tilde{f}_1^3,\tilde{f}_2^3)(\tilde{z}_0)$, then

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T(f_1,f_2)(x)-\tilde{d}_0|dx\\
&\leq \frac{1}{|B|}\int_B |T(\tilde{f}_1^2,\tilde{f}_2^2)(x)|dx+\frac{1}{|B|}\int_B |T(\tilde{f}_1^2,\tilde{f}_2^3)(x)-T(\tilde{f}_1^2,\tilde{f}_2^3)(\tilde{z}_0)|dx\\
&\quad+\frac{1}{|B|}\int_B |T(\tilde{f}_1^3,\tilde{f}_2^2)(x)-T(\tilde{f}_1^3,\tilde{f}_2^2)(\tilde{z}_0)|dx+\frac{1}{|B|}\int_B |T(\tilde{f}_1^3,\tilde{f}_2^3)(x)-T(\tilde{f}_1^3,\tilde{f}_2^3)(\tilde{z}_0)|dx\\
&:=\sum_{j=1}^{4}\tilde{L}_j.
\end{aligned}$$

Notice that $1 < q < \infty$ and $0 < l/q \leq \alpha$, where l is given as in Definition 1.2. By Lemma 3.1, we have

$$\begin{aligned}
\tilde{L}_1 &\leq |B|^{-1}\|T(\tilde{f}_1^2,\tilde{f}_2^2)\|_{L^1(B)}\\
&\leq C|B|^{-1/q}\|T(\tilde{f}_1^2,\tilde{f}_2^2)\|_{L^{q,\infty}(B)}\\
&\leq C|B|^{-1/q}|\tilde{B}|^{1/l}\Big(\frac{1}{|16\tilde{B}|}\int_{16\tilde{B}}|f_1(y_1)-(f_1)_{16\tilde{B}}|^{l_1}dy_1\Big)^{\frac{1}{l_1}}\Big(\frac{1}{|16\tilde{B}|}\int_{16\tilde{B}}|f_2(y_2)-(f_2)_{16\tilde{B}}|^{l_2}dy_2\Big)^{\frac{1}{l_2}}\\
&\leq Cr_B^{n(\frac{\alpha}{l}-\frac{1}{q})}\|f_1\|_{BMO}\|f_2\|_{BMO}\\
&\leq C\|f_1\|_{BMO}\|f_2\|_{BMO}.
\end{aligned}$$

For $x \in B$ and $y_2 \in (16\tilde{B})^c$, there is $2|x-\tilde{z}_0|^\alpha \leq |y_2-\tilde{z}_0|$, $r_B \leq |x-\tilde{z}_0| \leq 4r_B$ and $|x-\tilde{z}_0| < 1$. By Hölder's inequality, Lemma 3.10 and the condition (1.3) of the kernel, we have

$$\begin{aligned}
\tilde{L}_2 &\leq \frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\int_{16\tilde{B}}\Big(\int_{2^k|x-\tilde{z}_0|^\alpha\leq|y_2-\tilde{z}_0|\leq 2^{k+1}|x-\tilde{z}_0|^\alpha}|K(x,y_1,y_2)-K(\tilde{z}_0,y_1,y_2)|^{p_0}dy_2\Big)^{\frac{1}{p_0}}\\
&\quad\times\Big(\int_{2^{k+4}\tilde{B}}|f_2(y_2)-(f_2)_{16\tilde{B}}|^{p_0'}dy_2\Big)^{\frac{1}{p_0'}}|f_1(y_1)-(f_1)_{16\tilde{B}}|dy_1dx\\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}|\tilde{B}|2^{-k(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}|x-\tilde{z}_0|^{\varepsilon-\alpha(n(2-\frac{1}{p_0})+\frac{\varepsilon}{\alpha})}dx(2^{k+4}r_B^\alpha)^{\frac{n}{p_0'}}\Big(1+\ln\frac{2^{k+4}r_B^\alpha}{16r_B^\alpha}\Big)\|f_1\|_{BMO}\|f_2\|_{BMO}\\
&\leq C\sum_{k=1}^{\infty}k2^{-k(n+\frac{\varepsilon}{\alpha})}\|f_1\|_{BMO}\|f_2\|_{BMO}\\
&\leq C\|f_1\|_{BMO}\|f_2\|_{BMO}.
\end{aligned}$$

Similarly we can also get that

$$\tilde{L}_3 \leq C\|f_1\|_{BMO}\|f_2\|_{BMO}.$$

For $x \in B$ and $y_1, y_2 \in (16\tilde{B})^c$, there are $2|x-\tilde{z}_0|^\alpha \leq |y_1-\tilde{z}_0|$, $2|x-\tilde{z}_0|^\alpha \leq |y_2-\tilde{z}_0|$, $r_B \leq |x-\tilde{z}_0| \leq 4r_B$ and $|x-\tilde{z}_0| < 1$. By Hölder's inequality, Lemma 3.10 and the condition (1.3) of the kernel, we have

$$
\begin{aligned}
\tilde{L}_4 \;\leq\;& \frac{1}{|B|}\int_B \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big(\int_{2^{k_1}|x-\tilde{z}_0|^{\alpha}\leq|y_1-\tilde{z}_0|\leq 2^{k_1+1}|x-\tilde{z}_0|^{\alpha}}\int_{2^{k_2}|x-\tilde{z}_0|^{\alpha}\leq|y_2-\tilde{z}_0|\leq 2^{k_2+1}|x-\tilde{z}_0|^{\alpha}}\\
& |K(x,y_1,y_2)-K(\tilde{z}_0,y_1,y_2)|^{p_0}dy_2dy_1\Big)^{\frac{1}{p_0}}\Big(\int_{2^{k+4}\tilde{B}}|f_1(y_1)-(f_1)_{16\tilde{B}}|^{p_0'}dy_1\Big)^{\frac{1}{p_0'}}\\
& \times\Big(\int_{2^{k+4}\tilde{B}}|f_2(y_2)-(f_2)_{16\tilde{B}}|^{p_0'}dy_2\Big)^{\frac{1}{p_0'}}dx\\
\leq\;& C\frac{1}{|B|}\int_B\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}2^{-k_i(n(1-\frac{1}{p_0})+\frac{\varepsilon}{2\alpha})}|x-\tilde{z}_0|^{\varepsilon-\alpha(n(2-\frac{2}{p_0})+\frac{\varepsilon}{\alpha})}dx\prod_{i=1}^{2}\Big((2^{k_i+4}r_B^{\alpha})^{\frac{n}{p_0}}(1+\ln\frac{2^{k_i+4}r_B^{\alpha}}{16r_B^{\alpha}})\Big)\\
& \times\|f_1\|_{BMO}\|f_2\|_{BMO}\\
\leq\;& C\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\prod_{i=1}^{2}k_i 2^{\frac{-k_i\varepsilon}{2\alpha}}\|f_1\|_{BMO}\|f_2\|_{BMO}\\
\leq\;& C\|f_1\|_{BMO}\|f_2\|_{BMO}.
\end{aligned}
$$

Thus, according to the estimates in both cases, there is

$$
\|T(f_1,f_2)\|_{BMO}\sim\sup_B\inf_{a\in\mathbb{C}}\frac{1}{|B|}\int_B|T(f_1,f_2)(x)-a|dx\leq C\|f_1\|_{BMO}\|f_2\|_{BMO},
$$

which completes the proof of the theorem.

5. Conclusions

Compared with the multilinear strongly singular Calderón-Zygmund operator in [15], the kernel function of the multilinear strongly singular integral operator with generalized kernel in the present article satisfies the weaker smoothness condition. Our results further generalize the corresponding results in [15].

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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