Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13525–13532
DOI: 10.3934/math.2021785
Received: 13 July 2021
Accepted: 13 September 2021
Published: 22 September 2021

Research article

Observability estimate for the parabolic equations with inverse square potential

Guojie Zheng[1,2,*] **and Baolin Ma**[3]

[1] College of Digital Technology and Engineering, Ningbo University of Finance & Economics, Ningbo, 315175, China

[2] College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, China

[3] School of Mathematical Science, Henan Institute of Science and Technology, Xinxiang, 453003, China

* **Correspondence:** Email: guojiezheng@yeah.net.

Abstract: This paper investigates an observability estimate for the parabolic equations with inverse square potential in a C^2 bounded domain $\Omega \subset \mathbb{R}^d$, which contains 0. The observation region is a product set of a subset $E \subset (0, T]$ with positive measure and a non-empty open subset $\omega \subset \Omega$ with $0 \notin \omega$. We build up this estimate by a delicate result in measure theory in [7] and the Lebeau-Robbiano strategy.

Keywords: observability estimate; parabolic equations; spectral inequality; inverse square potential
Mathematics Subject Classification: 35K10, 93B07

1. Introduction

Let Ω be a C^2 bounded domain in $\mathbb{R}^d$ $(d \geq 3)$ such that $0 \in \Omega$. Let $\omega \subset \Omega$ be a non-empty open subset, and $0 \notin \omega$. Let $T > 0$, $E \subset (0, T]$ be a Lebesgue measurable subset with positive measure, and denote by χ_E the characteristic function of E.

In this paper, we study the observability estimate for a parabolic equation with a singular potential term, which is described by

$$\begin{cases} \partial_t y(x,t) - \triangle y(x,t) - \frac{\mu y(x,t)}{|x|^2} = 0, & \text{in } \Omega \times (0,T], \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ y(x,0) = \phi(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the initial datum $\phi(x) \in L^2(\Omega)$, and the parameter μ satisfies

$$\mu < \mu_* := \frac{(d-2)^2}{4}. \tag{1.2}$$

It should be emphasized that condition (1.2) is crucial for Eq (1.1). In [2], it is proved that if the initial datum $\phi(x) \in L^2(\Omega)$ is non-negative, then Eq (1.1) admits a unique global solution under condition (1.2), and when $\mu > \mu_*$, the local solution may not exist. In recent years, the study on Eq (1.1) has been a hot topic in the theory of PDEs due to its important applications in various branches of applied science and engineering. Extensive related references can be found in [4, 10, 11] and the rich works cited therein. In particular, the well-posedness of Eq (1.1) is further discussed from the point of view of semigroup theory in [11]. More specifically, it shows that when $\mu < \mu_*$, for each $\phi \in L^2(\Omega)$, there exists a unique

$$y \in C([0,+\infty); L^2(\Omega)) \cap L^2(0,+\infty; H_0^1(\Omega)),$$

which is a weak solution of the Eq (1.1).

In this paper, we mainly study the observability estimate for Eq (1.1). The main difficulty in this problem lies in the singularity of Eq (1.1) caused by the potential term. It turns out that Carleman inequalities, frequency functions, and the spectral inequality are the main frameworks in the literature on the observability estimate for the parabolic equations in recent decades. The application of Carleman estimates was described systematically in the works [3, 14]. About frequency functions, we would like to mention references [1, 7]. In this article, we apply the spectral inequality for the Schrödinger operator $A = -\triangle - \frac{\mu}{|x|^2}$ to build this estimate.

We denote $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to be the usual norm and the inner product in $L^2(\Omega)$, respectively. The main result of the paper is presented as follows:

Theorem 1.1. Let $E \subseteq (0,T]$ be a measurable set with a positive measure, and let ω be a non-empty open subset of Ω with $0 \notin \omega$. Suppose that

$$\mu \leq \begin{cases} \frac{7}{2.3^3}, & \text{if } d = 3, \\ \frac{(d-1)(d-3)}{4}, & \text{if } d \geq 4. \end{cases} \tag{1.3}$$

Then, there exists a constant $C = C(\Omega, T, \omega, E)$ such that for each datum $\phi \in L^2(\Omega)$, the solution to Eq (1.1) satisfied

$$\|y(x,T)\|^2_{L^2(\Omega)} \leq C(\Omega,\omega,T,E) \int_E \int_\omega |y(x,t)|^2 dxdt. \tag{1.4}$$

Remark 1.1. *Several notes on Theorem 1.1 are given in order.*

- (a) *Inequality* (1.4) *is called the observability estimate. Observability estimate* (1.4) *in Theorem 1.1 allows estimating the total energy of the solutions to Eq* (1.1) *at time T in terms of the partial energy localized in the observation region $\omega \times E$.*
- (b) *With the aid of observability estimate (1.4), we can study the bang-bang property for the time optimal control problem governed by Eq* (1.1) *with internal control.*
- (c) *From the perspective of control theory, this inequality is equivalent to the null controllability for its adjoint equation. (See [5].) We mention [3, 12, 13] in this direction.*

(d) In [16], a similar problem was investigated for Eq (1.1) when Ω is a convex domain of R^d and $0 \in \omega$, where ω is a non-empty open subset of Ω. It obtains in [16] that there exist two positive numbers $\alpha = \alpha(\Omega, \omega) \in (0, 1)$, $C = C(\Omega, \omega)$ such that for each $T > 0$, the solution of (1.1) satisfies

$$\int_\Omega |y(x,T)|^2 dx \leq Ce^{\frac{C}{T}}(\int_\Omega |\phi(x)|^2 dx)^{1-\alpha}(\int_\omega |y(x,T)|^2 dx)^\alpha, \tag{1.5}$$

where $\phi(x) \in L^2(\Omega)$ is the initial datum. Clearly, (1.5) is a Hölder type interpolation inequality for Eq (1.1), from which and the method in Theorem 4 of [8], we can also get (1.4) in this case.

We organize the paper as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary results

We first introduce certain notations. Let $\{\lambda_i\}_{i=1}^\infty$, with $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_i \to +\infty$ as $i \to \infty$, be the set of eigenvalues of the Schrödinger operator $A := -\triangle - \frac{\mu}{|x|^2}$ with homogeneous Dirichlet boundary conditions, and let $\{e_i(x)\}_{i=1}^\infty$ be the set of corresponding eigenfunctions with $\|e_i(x)\|_{L^2(\Omega)} = 1$, $(i = 1, 2, 3, \cdots)$ which constitutes an orthonormal basis of $L^2(\Omega)$ (e.g., see Theorem 3.1 in [12]). Now, for each $r > 0$, we define two subspaces of $L^2(\Omega)$ as follows

$$X_r := \text{span}\{e_i(x)\}_{\lambda_i \leq r} \text{ and } X_r^\perp := \text{span}\{e_i(x)\}_{\lambda_i > r}.$$

Clearly, we have the following orthogonal direct sum decomposition

$$L^2(\Omega) = X_r \oplus X_r^\perp.$$

Let $S(t)$ be the compact semigroup generated by the operator $-A$ on $L^2(\Omega)$. Then, the unique solution to Eq (1.1) corresponding to the initial datum $\phi(x) \in L^2(\Omega)$ can be written as (e.g., see section 4.1 in [9])

$$y(x,t) = S(t)\phi(x).$$

In addition,

$$S(t)\phi(x) = \sum_{n=1}^\infty e^{-\lambda_n t}\langle \phi, e_n \rangle e_n(x). \tag{2.1}$$

It follows from (2.1) that

$$\|S(t)\phi(x)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t}\|\phi(x)\|_{L^2(\Omega)}. \tag{2.2}$$

Remark 2.1. In fact, it follows from (2.2) that the solution to Eq (1.1) possesses the energy decay property.

As a direct consequence of (2.1) and (2.2), we have

Lemma 2.1. For each positive number $r > 0$ and each $\eta \in X_r^\perp$, the following conclusion is true:

$$\|S(t)\eta\|_{L^2(\Omega)} \leq e^{-rt}\|\eta\|_{L^2(\Omega)}, \quad \text{for } t \geq 0. \tag{2.3}$$

Next, the following result is brought from [6] (see Theorem 1.2 in [6]).

Lemma 2.2. Let ω be a non-empty open subset of Ω with $0 \notin \omega$. Suppose (1.3) holds. Then, there exist two positive constants $C_1 > 1$ and $C_2 > 0$, which only depend on Ω, ω, such that

$$\sum_{\lambda_i \le r} |a_i|^2 \le C_1 e^{C_2 \sqrt{r}} \int_\omega \Big| \sum_{\lambda_i \le r} a_i e_i(x) \Big|^2 dx \tag{2.4}$$

for each positive number $r > 0$ and each choice of the coefficients $\{a_i\}_{\lambda_i \le r}$ with $a_i \in \mathbb{R}$.

Remark 2.2. (i) Inequality (2.4) is called the *spectral inequality* for the Schrödinger operator $A = -\triangle - \frac{\mu}{|x|^2}$.

(ii) By Lemma 2.2, we have that for each positive number $r > 0$ and each $\psi \in X_r$,

$$||\psi||_{L^2(\Omega)} \le C_1 e^{C_2 \sqrt{r}} ||\psi||_{L^2(\omega)}. \tag{2.5}$$

Throughout the rest of this paper, the following notation will be used. For each measurable set $A \subseteq \mathbb{R}$, we denote by $|A|$ its Lebesgue measure in $\mathbb{R}$. The following lemma is quoted from [7, 15] (e.g., see Proposition 2.1 in [7]).

Lemma 2.3. Let $E \subseteq [0, T]$ be a measurable set with a positive measure, and let l be a density point of E. Then, for each $z > 1$, there exists $l_1 \in (l, T)$ such that the sequence $\{l_i\}_{i=1}^\infty$ given by

$$l_{i+1} := l + \frac{1}{z^i}(l_1 - l), \tag{2.6}$$

satisfies

$$|E \cap (l_{i+1}, l_i)| \ge \frac{1}{2}(l_i - l_{i+1}). \tag{2.7}$$

3. The proof of the main result

Proof. The proof will be organized in three steps.

***Step 1**. We first construct a subset sequence of* $(0, T]$.

Let l be a density point for E. We arbitrarily fix a positive number $z > 1$. By Lemma 2.3, there exists $l_1 \in (l, T]$ and a sequence $\{l_i\}_{i=1}^\infty$ satisfying (2.6) and (2.7). We now define a subset sequence of $(0, T]$ as follows:

$$E_i := E \cap (l_{i+1} + \frac{l_i - l_{i+1}}{4}, l_i). \tag{3.1}$$

It is obvious that

$$|E_i| = \Big| E \cap [(l_{i+1}, l_i) \setminus (l_{i+1}, l_{i+1} + \frac{l_i - l_{i+1}}{4})] \Big|.$$

Then, it follow from (2.7) that

$$|E_i| \ge \frac{l_i - l_{i+1}}{4}. \tag{3.2}$$

It follows from (2.6) that

$$l_i - l_{i+1} = \frac{1}{z^i}(z-1)(l_1-1), \;\; i = 1,2,3,\cdots . \tag{3.3}$$

Thus,

$$\frac{l_i - l_{i+1}}{l_{i+1} - l_{i+2}} = z, \;\; i = 1,2,3,\cdots . \tag{3.4}$$

Let $b > 1$ be a positive number satisfying

$$\left(\frac{b}{z}\right)^i (z-1)(l_1-l) > 4C_2 b^{\frac{i+1}{2}} + 4\ln(8C_1 z), \;\; i = 2,3,4\cdots , \tag{3.5}$$

where C_1 and C_2 are the positive numbers given in Lemma 2.2. Taking $r_i = b^i$ $(i = 2,3,4\cdots)$, and using (3.3) and (3.5) , we obtain that

$$r_i \frac{l_i - l_{i+1}}{4} - C_2\sqrt{r_{i+1}} > \ln(8C_1 z), \;\; i = 2,3,4,\cdots .$$

It, together with (3.4), indicates

$$\frac{l_{i+1} - l_{i+2}}{4C_1 e^{C_2\sqrt{r_{i+1}}}} > 2(l_i - l_{i+1})e^{-r_i\frac{l_i - l_{i+1}}{4}}, \; i = 2,3,4,\cdots . \tag{3.6}$$

Step 2. *For each $i \in \{2,3,4,\cdots\}$, we estimate the term $\frac{l_i-l_{i+1}}{4}\|S(l_i)\phi\|_{L^2(\Omega)}$, where $\phi \in L^2(\Omega)$.*

From (3.2) and (2.2), it follows that for each $\varphi \in X_{r_i}$ $(i \in \{2,3,4\cdots\})$,

$$\frac{l_i - l_{i+1}}{4}\|S(l_i)\varphi\|_{L^2(\Omega)} \leq \int_{l_{i+1}+\frac{l_i-l_{i+1}}{4}}^{l_i} \chi_{E_i}(t)\|S(t)\varphi\|_{L^2(\Omega)}dt. \tag{3.7}$$

For each initial datum $\phi \in L^2(\Omega)$, there exists a unique decomposition

$$\phi = \phi_1 + \phi_2, \tag{3.8}$$

where $\phi_1 \in X_{r_i}$ and $\phi_2 \in X_{r_i}^{\perp}$. Taking $\varphi = \phi_1$ in (3.7), and then using (2.5), we obtain

$$\begin{aligned} \frac{l_i - l_{i+1}}{4}\|S(l_i)\phi_1\|_{L^2(\Omega)} &\leq \int_{l_{i+1}+\frac{l_i-l_{i+1}}{4}}^{l_i} \chi_{E_i}(t)\|S(t)\phi_1\|_{L^2(\Omega)}dt \\ &\leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}+\frac{l_i-l_{i+1}}{4}}^{l_i} \chi_{E_i}(t)\|S(t)\phi_1\|_{L^2(\omega)}dt. \end{aligned}$$

By the triangle inequality and (3.1), we deduce

$$\frac{l_i - l_{i+1}}{4}\|S(l_i)\phi_1\|_{L^2(\Omega)} \leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}+\frac{l_i-l_{i+1}}{4}}^{l_i} \chi_E(t)\left(\|S(t)\phi\|_{L^2(\omega)} + \|S(t)\phi_2\|_{L^2(\Omega)}\right)dt.$$

Together with (2.2), we have

$$\begin{aligned}\frac{l_i - l_{i+1}}{4}\|S(l_i)\phi_1\|_{L^2(\Omega)} &\leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}+\frac{l_i - l_{i+1}}{4}}^{l_i} \chi_E(t)\|S(t)\phi\|_{L^2(\omega)}dt \\ &+ C_1 e^{C_2\sqrt{r_i}}(l_i - l_{i+1})\|S(l_{i+1} + \frac{l_i - l_{i+1}}{4})\phi_2\|_{L^2(\Omega)}.\end{aligned}$$

This, along with Lemma 2.1, indicates that

$$\begin{aligned}\frac{l_i - l_{i+1}}{4}\|S(l_i)\phi_1\|_{L^2(\Omega)} &\leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_i} \chi_E(t)\|S(t)\phi\|_{L^2(\omega)}dt \\ &+ C_1 e^{C_2\sqrt{r_i}}(l_i - l_{i+1})e^{-r_i\frac{l_i - l_{i+1}}{4}}\|S(l_{i+1})\phi_2\|_{L^2(\Omega)}.\end{aligned} \tag{3.9}$$

By Lemma 2.2 and $\phi_2 \in X_{r_i}^{\perp}$, we have

$$\frac{l_i - l_{i+1}}{4}\|S(l_i)\phi_2\|_{L^2(\Omega)} \leq \frac{l_i - l_{i+1}}{4}e^{-r_i(l_i - l_{i+1})}\|S(l_{i+1})\phi_2\|_{L^2(\Omega)}. \tag{3.10}$$

Applying (3.8), and the triangle inequality, we obtain

$$\frac{l_i - l_{i+1}}{4}\|S(l_i)\phi\|_{L^2(\Omega)} \leq \frac{l_i - l_{i+1}}{4}\|S(l_i)\phi_1\|_{L^2(\Omega)} + \frac{l_i - l_{i+1}}{4}\|S(l_i)\phi_2\|_{L^2(\Omega)}.$$

Then, it, together with (3.9), and (3.10), indicates

$$\begin{aligned}&\frac{l_i - l_{i+1}}{4}\|S(l_i)\phi\|_{L^2(\Omega)} \leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_i} \chi_E(t)\|S(t)\phi\|_{L^2(\omega)}dt \\ &+ C_1 e^{C_2\sqrt{r_i}}(l_i - l_{i+1})e^{-r_i\frac{l_i - l_{i+1}}{4}}\|S(l_{i+1})\phi_2\|_{L^2(\Omega)} + \frac{l_i - l_{i+1}}{4}e^{-r_i(l_i - l_{i+1})}\|S(l_{i+1})\phi_2\|_{L^2(\Omega)},\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{l_i - l_{i+1}}{4}\|S(l_i)\phi\|_{L^2(\Omega)} \leq C_1 e^{C_2\sqrt{r_i}} \int_{l_{i+1}}^{l_i} \chi_E(t)\|S(t)\phi\|_{L^2(\omega)}dt \\ &+ (l_i - l_{i+1})e^{-r_i\frac{l_i - l_{i+1}}{4}}\left(C_1 e^{C_2\sqrt{r_i}} + 1\right)\|S(l_{i+1})\phi\|_{L^2(\Omega)},\end{aligned}$$

which leads to

$$\begin{aligned}&\frac{l_i - l_{i+1}}{4C_1 e^{C_2\sqrt{r_i}}}\|S(l_i)\phi\|_{L^2(\Omega)} - \frac{C_1 e^{C_2\sqrt{r_i}} + 1}{C_1 e^{C_2\sqrt{r_i}}}(l_i - l_{i+1})e^{-r_i\frac{l_i - l_{i+1}}{4}}\|S(l_{i+1})\phi\|_{L^2(\Omega)} \\ \leq\ & \int_{l_{i+1}}^{l_i} \chi_E(t)\|S(t)\phi\|_{L^2(\omega)}dt.\end{aligned} \tag{3.11}$$

Step 3. *We prove* (1.4).

Summing (3.11) from 2 to ∞, it follows that

$$\frac{l_2 - l_3}{4C_1 e^{C_2\sqrt{r_2}}}\|S(l_2)\phi\|_{L^2(\Omega)} + \sum_{i=2}^{\infty} k_i\|S(l_{i+1})\phi\|_{L^2(\Omega)} \leq \int_0^T \chi_E(t)\|S(t)\phi\|_{L^2(\omega)}dt, \tag{3.12}$$

where

$$k_i = \frac{l_{i+1} - l_{i+2}}{4C_1 e^{C_2\sqrt{r_{i+1}}}} - \frac{C_1 e^{C_2\sqrt{r_i}} + 1}{C_1 e^{C_2\sqrt{r_i}}}(l_i - l_{i+1})e^{-r_i\frac{l_i - l_{i+1}}{4}} \quad (i = 2, 3, 4, \cdots).$$

By (3.6), we obtain

$$k_i > 0, \quad \text{for } i = 2, 3, 4, \cdots.$$

This, together with (3.12) and (2.2), indicates that

$$\|S(T)\phi\|_{L^2(\Omega)} \leq \|S(l_2)\phi\|_{L^2(\Omega)} \leq \frac{4C_1 e^{C_2\sqrt{r_2}}}{l_2 - l_3}\int_E \|S(t)\phi\|_{L^2(\omega)}dt.$$

That is

$$\|y(x,T)\|_{L^2(\Omega)} \leq \frac{4C_1 e^{C_2\sqrt{r_2}}}{l_2 - l_3}\int_E \|y(x,t)\|_{L^2(\omega)}dt,$$

from which and Cauchy's inequality, it follows (1.4). This completes the proof. □

4. Conclusions

Using the Lebeau-Robbiano strategy, we obtain the observation estimate (1.4) for Eq (1.1) in this paper. From (1.4), we have that if the solution $y = 0$ over $\omega \times E$, then $y(\cdot, T) = 0$ over Ω. It, together with (2.1), shows that $y \equiv 0$ over $\Omega \times [0, T]$. From the application point of view, we can recover the initial state and the evolution history, according to the observation of y in $\omega \times E$.

Acknowledgments

The authors would like to express their sincere thanks to the referees for their valuable suggestions. This work was partially supported by the Natural Science Foundation of Henan Province (No. 202300410248), and by the Natural Science Foundation of Henan Normal University (No. 2019PL15).

Conflict of interest

The authors declare that they have no competing interests in this paper.

References

1. J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, Observability inequalities and measurable sets, *J. Eur. Math. Soc.,* **16** (2014), 2433–2475.
2. P. Baras, J. A. Goldstein, The heat equation with a singular potential, *Trans. Amer. Math. Soc.,* **284** (1984), 121–139.
3. S. Ervedoza, Control and stabilization properties for a singular heat equation with an inverse square potential, *Commun. Part. Diff. Eq.,* **33** (2008), 1996–2019.

4. J. A. Goldstein, Q. S. Zhang, Linear parabolic equations with strong singular potentials, *Trans. Amer. Math. Soc.,* **355** (2003), 197–211.

5. J. L. Lions, Exact controllability, stabilization and perturbations for distributed system, *SIAM Rev.,* **30** (1988), 1–68.

6. K. D. Phung, Carleman commutator approach in logarithmic convexity for parabolic equations, *Math. Control Relat. F.,* **8** (2018), 899–933.

7. K. D. Phung, G. Wang, An observability estimate for the parabolic equations from a measurable set in time and its applications, *J. Eur. Math. Soc.,* **15** (2013), 681–703.

8. K. D. Phung, L. Wang, C. Zhang, Bang-bang property for time optimal control of semilinear heat equation, *Ann. I. H. Poincare-An.,* **31** (2014), 477–499.

9. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

10. I. Peral, J. L. Vazquez, On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term, *Arch. Ration. Mech. An.,* **129** (1995), 201–224.

11. J. L. Vazquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, *J. Funct. Anal.,* **173** (2000), 103–153.

12. J. Vancostenoble, E. Zuazua, Null controllability for the heat equation with singular inverse square potentials, *J. Funct. Anal.,* **254** (2008), 1864–1902.

13. G. Wang, L^∞-null controllability for the heat equation and its consequences for the time optimal control problem, *SIAM J. Control Optim.,* **47** (2008), 1701–1720.

14. G. Wang, L. Wang, The Carleman inequality and its application to periodic optimal control governed by semilinear parabolic differential equations, *J. Optimz. Theory App.,* **118** (2003), 429–461.

15. C. Zhang, An observability estimate for the heat equation from a product of two measurable sets, *J. Math. Anal. Appl.,* **396** (2012), 7–12.

16. G. Zheng, K. Li, Y. Zhang, Quantitative unique continuation for the heat equations with inverse square potential, *J. Inequal. Appl.,* **1** (2018), 1–17.

AIMS Press