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Research article

Further results on permutation polynomials and complete permutation polynomials over finite fields

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Abstract: In this paper, by employing the AGW criterion and determining the number of solutions to some equations over finite fields, we further investigate nine classes of permutation polynomials over $\mathbb{F}_{p^n}$ with the form $(x^{p^m} - x + \delta)^{s_1} + (x^{p^m} - x + \delta)^{s_2} + x$ and propose five classes of complete permutation polynomials over $\mathbb{F}_{p^{2m}}$ with the form $ax^{p^m} + bx + h(x^{p^m} - x)$.

Keywords: finite field; AGW criterion; permutation polynomial; complete permutation polynomial
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1. Introduction

For a prime power q, let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{F}_q^*$ denote its multiplicative group. A polynomial $f(x) \in \mathbb{F}_q[x]$ is called a *permutation polynomial* (PP) if its associated polynomial mapping $f : c \mapsto f(c)$ from $\mathbb{F}_q$ into itself is a bijection [9]. Furthermore, it is called a *complete permutation polynomial* (CPP) if both $f(x)$ and $f(x) + x$ are bijections over $\mathbb{F}_q$. PPs and CPPs over finite fields have been an active topic of study for many years due to their significant applications areas such as cryptography [3, 14], combinatorial designs [2], design theory [9, 13], coding theory [5], and other areas of mathematics and engineering [9, 11]. Information about properties, constructions, and applications of PPs and CPPs can be found in [9, 12]. Discovering new constructions of PPs is of tremendous interest in both theoretical and applied aspects. Some recent progresses on PPs can be referred to [4, 10, 15, 16, 21–23]. Meanwhile, more significant improvements had been obtained in finding new CPPs, see [17–19] for example.

Very recently, eight classes of PPs with the form

$$(x^{2^i} + x + \delta)^{s_1} + (x^{2^i} + x + \delta)^{s_2} + x, \tag{1.1}$$

over finite fields of characteristic 2 were presented in [20], where i, n, s_1, s_2 are positive integers and $\delta \in \mathbb{F}_{2^n}$. Particularly, by finding a link between two classes of PPs over finite fields with even characteristic, four classes of PPs with the form $(x^2 + x + \delta)^{s_1} + (x^2 + x + \delta)^{s_2} + x$ over $\mathbb{F}_{2^n}$ were obtained in [7], where s_1, s_2 and δ satisfying some conditions. Later, according to the AGW criterion and determination of the number of solutions to certain equations over finite fields, several classes of PPs with the form

$$(x^{p^m} - x + \delta)^{s_1} + (x^{p^m} - x + \delta)^{s_2} + x, \tag{1.2}$$

over $\mathbb{F}_{p^n}$ were proposed in [8], where p is a prime, m, n, s_1, s_2 are positive integers with $m|n$ and $\delta \in \mathbb{F}_{p^n}$. Furthermore, for $f(x) = ax^{p^m} + bx + h(x^{p^m} - x)$, where p is a prime, $a, b \in \mathbb{F}^*_{p^{2m}}$ and $h(x) \in \mathbb{F}_{p^{2m}}$, the authors of [6] established a link between the complete permutation property of $f(x)$ and the bijection property of some polynomials defined on subsets of $\mathbb{F}_{p^{2m}}$.

In this paper, following the lines of the works done in [6, 8], we further study several PPs and CPPs over $\mathbb{F}_{p^n}$. More precisely, nine classes of PPs over $\mathbb{F}_{p^n}$ with the form $(x^{p^m} - x + \delta)^{s_1} + (x^{p^m} - x + \delta)^{s_2} + x$ are considered and five classes of CPPs over $\mathbb{F}_{p^{2m}}$ with the form $ax^{p^m} + bx + h(x^{p^m} - x)$ are proposed. Our approach is based on the AGW criterion and a method to decide the number of solutions of certain equations over finite fields.

The remainder of this paper is organized as follows. In Section 2, some basic concepts and related results are presented. In Section 3, nine classes of PPs with the form $(x^{p^m} - x + \delta)^{s_1} + (x^{p^m} - x + \delta)^{s_2} + x$ over $\mathbb{F}_{p^n}$ are given. In Section 4, five classes of CPPs with the form $ax^{p^m} + bx + h(x^{p^m} - x)$ over $\mathbb{F}_{p^{2m}}$ are proposed. Finally, we give some conclusions in Section 5.

2. Preliminaries

In this section, some notations and useful lemmas are introduced. We always let $\mathbb{F}_{p^n}$ be a finite field with p^n elements. For two positive integers m and n with $m|n$, we use $Tr^n_m(\cdot)$ to denote the trace function from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^m}$, i.e.,

$$Tr^n_m(x) = x + x^{p^m} + x^{p^{2m}} + \cdots + x^{p^{(\frac{n}{m}-1)m}}.$$

Define $J = \{\gamma^{p^m} - \gamma : \gamma \in \mathbb{F}_{p^n}\} = \{\alpha \in \mathbb{F}_{p^n} : Tr^n_m(\alpha) = 0\}$ and we will use it frequently in the following.

First of all, we give the definition of q-polynomial as follows.

Definition 1. *([9]) For a fixed prime power q, a polynomial of the form*

$$L(x) = \sum_{i=0}^{t} a_i x^{q^i},$$

with coefficients in an extension field $\mathbb{F}_{q^t}$ of $\mathbb{F}_q$ is called a q-polynomial or a linearized polynomial.

Next, we recall three useful lemmas needed in the subsequent section.

Lemma 1. *([1]) For positive integers m, n with $m|n$, let $\varphi(x)$ be a p^m-polynomial over $\mathbb{F}_{p^m}$, $h(x) \in \mathbb{F}_{q^n}[x]$ be any polynomial such that $h(x^{p^m} - x) \in \mathbb{F}_{p^m}\backslash\{0\}$ for all $x \in \mathbb{F}_{p^n}$, and let $g(x) \in \mathbb{F}_{p^n}[x]$ be any polynomial. Then $h(x^{p^m} - x)\varphi(x) + g(x^{p^m} - x)$ is a permutation of $\mathbb{F}_{p^n}$ if and only if the following two conditions hold:*

(i) $\varphi(1) \neq 0$;

(ii) $h(x)\varphi(x) + g(x)^{p^m} - g(x)$ permutes J.

Taking $g(x) = \sum_{j=1}^{t}(x+\delta)^{s_j}$, $h(x) = 1$ and $\varphi(x) = x$, we obtain the following lemma.

Lemma 2. *([8]) For given positive integers m, n, t with $m|n$, nonnegative integers s_j for $1 \leq j \leq t$, and a fixed $\delta \in \mathbb{F}_{p^n}$, the polynomial*

$$f(x) = \sum_{j=1}^{t}(x^{p^m} - x + \delta)^{s_j} + x,$$

permutes $\mathbb{F}_{p^n}$ if and only if

$$\sum_{j=1}^{t}((x+\delta)^{p^m s_j} - (x+\delta)^{s_j}) + x,$$

permutes J.

Lemma 3. *([8]) Let p be a prime, two positive integers m and n satisfying $mp \nmid n$ and $m|n$, and two nonnegative integers i, j satisfying $\gcd(i-j, n) = 1$ and $\gcd(i-j, p-1) = 1$. If the element $\alpha \in \mathbb{F}_{p^n}$ is a $(p-1)$-th power in $\mathbb{F}_{p^m}$, then the equation $x^{p^i} - \alpha x^{p^j} + \beta = 0$ has at most one solution in J, where $\beta \in \mathbb{F}_{p^n}$.*

3. New PPs of the form $(x^{p^m} - x + \delta)^{s_1} + (x^{p^m} - x + \delta)^{s_2} + x$ over $\mathbb{F}_{p^n}$

In this section, we investigate the permutation behavior of $(x^{p^m} - x + \delta)^{s_1} + (x^{p^m} - x + \delta)^{s_2} + x$ over $\mathbb{F}_{p^n}$, where exponents s_1 and s_2 are positive integers, $m|n$ and $\delta \in \mathbb{F}_{p^n}$. Furthermore, using an approach introduced by [8], we can obtain nine classes of PPs with this form over $\mathbb{F}_{p^{2m}}$ and $\mathbb{F}_{p^{3m}}$ as below.

Theorem 1. *For an odd prime p and a positive integer m. Let δ be an element of $\mathbb{F}_{p^{2m}}$ such that $Tr_m^{2m}(\delta) = 0$ or $\frac{(Tr_m^{2m}(\delta)-1)^p - 2Tr_m^{2m}(\delta)}{Tr_m^{2m}(\delta)}$ is a $(p-1)$-th power in $\mathbb{F}_{p^m}^*$. Then the polynomial*

$$f_1(x) = (x^{p^m} - x + \delta)^{2p^m} + (x^{p^m} - x + \delta)^{p^{m+1}+1} + x,$$

permutes $\mathbb{F}_{p^{2m}}$.

Proof. According to Lemma 2, we need to show that for each $d \in J$, the equation

$$(x+\delta)^{p^m(2p^m)} - (x+\delta)^{2p^m} + (x+\delta)^{p^m(p^{m+1}+1)} - (x+\delta)^{p^{m+1}+1} + x = d, \tag{3.1}$$

has at most one solution in J.

Notice that $x + x^{p^m} = 0$ since $x \in J$. Then the left-hand side of (3.1) can be written as

$$\begin{aligned} & (x+\delta)^{p^m(2p^m)} - (x+\delta)^{2p^m} + (x+\delta)^{p^m(p^{m+1}+1)} - (x+\delta)^{p^{m+1}+1} + x \\ = \ & (x+\delta)^2 - (-x+\delta^{p^m})^2 + (x+\delta)^{p^m+p} - (x+\delta)^{p^{m+1}+1} + x \end{aligned}$$

$$
\begin{aligned}
&= (x^2+2\delta x+\delta^2)-(x^2-2\delta^{p^m}x+\delta^{2p^m})+(-x+\delta^{p^m})(x^p+\delta^p)-(-x^p+\delta^{p^{m+1}})(x+\delta)+x\\
&= 2Tr_m^{2m}(\delta)x-\delta^{2p^m}+\delta^2+(-x^{p+1}+\delta^{p^m}x^p-\delta^p x+\delta^{p^m+p})\\
&\quad -(-x^{p+1}-\delta x^p+\delta^{p^{m+1}}x+\delta^{p^{m+1}+1})+x\\
&= (\delta^{p^m}+\delta)x^p+[2Tr_m^{2m}(\delta)-(\delta^{p^m}+\delta-1)^p]x+\delta^{p^m+p}-\delta^{p^{m+1}+1}-\delta^{2p^m}+\delta^2\\
&= Tr_m^{2m}(\delta)x^p-[(Tr_m^{2m}(\delta)-1)^p-2Tr_m^{2m}(\delta)]x+\delta^{p^m+p}-\delta^{p^{m+1}+1}+\delta^2-\delta^{2p^m}.
\end{aligned}
$$

Thus we have

$$
Tr_m^{2m}(\delta)x^p-[(Tr_m^{2m}(\delta)-1)^p-2Tr_m^{2m}(\delta)]x=\delta^{p^{m+1}+1}-\delta^{p^m+p}+\delta^{2p^m}-\delta^2+d. \tag{3.2}
$$

If $Tr_m^{2m}(\delta)=0$, $\delta^{p^{m+1}+1}-\delta^{p^m+p}=(-\delta)^p\delta-\delta^p(-\delta)=0$ and then $x=\delta^{2p^m}-\delta^2+d$ is the unique solution of (3.1) in J.

If $Tr_m^{2m}(\delta)\neq 0$, then (3.2) becomes

$$
x^p-\frac{(Tr_m^{2m}(\delta)-1)^p-2Tr_m^{2m}(\delta)}{Tr_m^{2m}(\delta)}x=\frac{\delta^{p^{m+1}+1}-\delta^{p^m+p}+\delta^{2p^m}-\delta^2+d}{Tr_m^{2m}(\delta)}. \tag{3.3}
$$

Note that $\frac{(Tr_m^{2m}(\delta)-1)^p-2Tr_m^{2m}(\delta)}{Tr_m^{2m}(\delta)}$ is a $(p-1)$-th power in $\mathbb{F}_{p^m}^*$, and $mp\nmid 2m$ since p is odd. Therefore, we deduce that (3.3) has at most one solution in J, which follows from Lemma 3. Furthermore, we conclude that (3.1) has at most one solution in J and $f_1(x)=(x^{p^m}-x+\delta)^{2p^m}+(x^{p^m}-x+\delta)^{p^{m+1}+1}+x$ permutes $\mathbb{F}_{p^{2m}}$. □

With the same method as in Theorem 1, we propose two classes of PPs with the form $(x^{3^m}-x+\delta)^{s_1}+(x^{3^m}-x+\delta)^{s_2}+x$ over $\mathbb{F}_{3^{2m}}$, and five classes of PPs with the form $(x^{2^m}+x+\delta)^{s_1}+(x^{2^m}+x+\delta)^{s_2}+x$ over $\mathbb{F}_{2^{3m}}$, respectively. The results can be similarly proved and we omit the details here.

Theorem 2. *For a positive integer m, let δ be an element of $\mathbb{F}_{3^{2m}}$ such that $Tr_m^{2m}(\delta)=1$ or $2Tr_m^{2m}(\delta)+1$ is a square element in $\mathbb{F}_{3^m}^*$. Then the polynomial*

$$
f_2(x)=(x^{3^m}-x+\delta)^{2\cdot 3^m}+(x^{3^m}-x+\delta)^{3^{2m-1}+2\cdot 3^{m-1}}+x,
$$

permutes $\mathbb{F}_{3^{2m}}$.

Theorem 3. *For a positive integer m and an element $\delta\in\mathbb{F}_{3^{2m}}$, the polynomial*

$$
f_3(x)=(x^{3^m}-x+\delta)^{3^{2m-1}+2\cdot 3^{m-1}}+(x^{3^m}-x+\delta)^{2\cdot 3^{2m-1}+3^{m-1}}+x,
$$

permutes $\mathbb{F}_{3^{2m}}$.

Theorem 4. *For two positive integers m and s satisfying $s\equiv 0 \pmod{1+2^m+2^{2m}}$, let δ be an element of $\mathbb{F}_{2^{3m}}$. Then the polynomial*

$$
f_4(x)=(x^{2^m}+x+\delta)^{2^{2m}+1}+(x^{2^m}+x+\delta)^{s}+x,
$$

permutes $\mathbb{F}_{2^{3m}}$.

Theorem 5. *For a positive integer m and an element $\delta \in \mathbb{F}_{2^{3m}}$, the polynomial*

$$f_5(x) = (x^{2^m} + x + \delta)^{2^{2m}+1} + (x^{2^m} + x + \delta)^{2^m} + x,$$

permutes $\mathbb{F}_{2^{3m}}$.

Theorem 6. *For a positive integer m and an element $\delta \in \mathbb{F}_{2^{3m}}$ with $Tr_m^{3m}(\delta) \neq 0$, the polynomial*

$$f_6(x) = (x^{2^m} + x + \delta)^{2^{2m-1}+2^{m-1}} + (x^{2^m} + x + \delta)^{2^{2m}} + x,$$

permutes $\mathbb{F}_{2^{3m}}$.

Theorem 7. *For a positive integer m and an element $\delta \in \mathbb{F}_{2^{3m}}$, the polynomial*

$$f_7(x) = (x^{2^m} + x + \delta)^{2^{2m-1}+2^{m-1}} + (x^{2^m} + x + \delta)^{2^m} + x,$$

is a permutation of $\mathbb{F}_{2^{3m}}$.

Theorem 8. *For a positive integer m with $m \not\equiv -1 \pmod 3$ and an element $\delta \in \mathbb{F}_{2^{3m}}$, the polynomial*

$$f_8(x) = (x^{2^m} + x + \delta)^{2^{2m-1}+2^{m-1}} + (x^{2^m} + x + \delta) + x,$$

permutes $\mathbb{F}_{2^{3m}}$.

In the above considerations, we mainly investigate several PPs over $\mathbb{F}_{3^{2m}}$ and $\mathbb{F}_{2^{3m}}$, respectively. Below we analyzes the permutation behavior of the polynomial $f_9(x)$ over $\mathbb{F}_{3^{3m}}$ for certain elements $\delta \in \mathbb{F}_{3^{3m}}$ such that $Tr_m^{3m}(\delta) = 0$ and integers s satisfying $s(3^m - 1) \equiv 0 \pmod{3^{3m} - 1}$.

Theorem 9. *For two positive integers m and s satisfying $s \equiv 0 \pmod{1 + 3^m + 3^{2m}}$, let δ be an element of $\mathbb{F}_{3^{3m}}$ with $Tr_m^{3m}(\delta) = 0$. Then the polynomial*

$$f_9(x) = (x^{3^m} - x + \delta)^{\frac{3^{3m}-1}{2}+1} + (x^{3^m} - x + \delta)^s + x,$$

permutes $\mathbb{F}_{3^{3m}}$.

Proof. Applying Lemma 2, to prove that $f_9(x)$ permutes $\mathbb{F}_{3^{3m}}$, it is sufficient to show that for any $d \in J$, the equation

$$(x + \delta)^{3^m(\frac{3^{3m}-1}{2}+1)} - (x + \delta)^{\frac{3^{3m}-1}{2}+1} + (x + \delta)^{3^m s} - (x + \delta)^s + x = d, \tag{3.4}$$

has a unique solution in J.

If $x + \delta = 0$, then $x = d = -\delta$.

If $x + \delta \neq 0$, note that $x + x^{3^m} + x^{3^{2m}} = 0$ for $x \in J$, we have $(x + \delta)^{\frac{3^{3m}-1}{2}} = \pm 1$. Then the solutions of (3.4) are divided into the following two cases.

Case 1: $(x + \delta)^{\frac{3^{3m}-1}{2}} = -1$. In this case, (3.4) turns to $-(x + \delta)^{3^m} + (x + \delta) + x = d$, that is

$$x^{3^{2m}} = d - \delta^{3^m} + \delta^{3^{2m}}. \tag{3.5}$$

Taking the 3^m-th power on both sides of (3.5) yields

$$x = d^{3^m} - \delta^{3^{2m}} + \delta,$$

which implies that

$$(d^{3^m} - \delta^{3^{2m}} - \delta)^{\frac{3^{3m}-1}{2}} = -1. \tag{3.6}$$

Case 2: $(x+\delta)^{\frac{3^{3m}-1}{2}} = 1$. In this case, (3.4) becomes $(x+\delta)^{3^m} - (x+\delta) + x = d$, we calculate

$$x^{3^m} = \delta - \delta^{3^m} + d. \tag{3.7}$$

Raising both sides of (3.7) to the power 3^{2m} leads to

$$x = d^{3^{2m}} + \delta^{3^{2m}} - \delta,$$

which means that

$$(d^{3^{2m}} - \delta^{3^m} - \delta)^{\frac{3^{3m}-1}{2}} = 1. \tag{3.8}$$

When $d = -\delta$, we have $d^{3^m} - \delta^{3^{2m}} + \delta = d^{3^{2m}} + \delta^{3^{2m}} - \delta = -\delta$. Thus, in this case, all three possible solutions, $d^{3^m} - \delta^{3^{2m}} + \delta$, $d^{3^{2m}} + \delta^{3^{2m}} - \delta$ and $-\delta$ are the same and hence (3.4) has a unique solution in J.

When $d \neq -\delta$, we claim that $x = d^{3^m} - \delta^{3^{2m}} + \delta$ and $x = d^{3^{2m}} + \delta - \delta^{3^m}$ can not hold simultaneously. Otherwise, combining (3.6) and (3.8), we obtain

$$1 = (d^{3^{2m}} - \delta^{3^m} - \delta)^{\frac{3^{3m}-1}{2}} = ((d^{3^m} - \delta - \delta^{3^{2m}})^{\frac{3^{3m}-1}{2}})^{3^m} = -1.$$

This contradicts the assumption that $(d^{3^{2m}} - \delta^{3^m} - \delta)^{\frac{3^{3m}-1}{2}} = 1$. Consequently, we know that (3.4) has only a unique solution in J.

Summarizing the discussions of the above two cases, we conclude that the polynomial $f_9(x) = (x^{3^m} - x + \delta)^{\frac{3^{3m}-1}{2}+1} + (x^{3^m} - x + \delta)^s + x$ permutes $\mathbb{F}_{3^{3m}}$. □

4. New CPPs of the form $ax^{p^m} + bx + h(x^{p^m} - x)$ over $\mathbb{F}_{p^{2m}}$

In this section, we consider five classes of CPPs with the form $ax^{p^m} + bx + h(x^{p^m} - x)$ over $\mathbb{F}_{p^{2m}}$ in detail when $h(x) = \sum_{j=1}^{t} (x+\delta)^{s_j}$ for $\delta \in \mathbb{F}_{p^{2m}}$.

Lemma 4. *([6]) For a prime p and a positive integer m, let $a, b \in \mathbb{F}_{p^{2m}}^*$ with $a+b, a+b+1 \in \mathbb{F}_{p^m}^*$, and $h(x) \in \mathbb{F}_{p^{2m}}[x]$. Then $F(x) = ax^{p^m} + bx + h(x^{p^m} - x)$ is a* CPP over $\mathbb{F}_{p^{2m}}$ if and only if both $h(x)^{p^m} - h(x) + (b - a^{p^m})x$ and $h(x)^{p^m} - h(x) + (b - a^{p^m} + 1)x$ are bijective on J.

Theorem 10. *For an odd prime p and a positive integer m, let $a, b \in \mathbb{F}_{p^{2m}}^*$ with $a+b, a+b+1 \in \mathbb{F}_{p^m}^*$, and $b - a^{p^m} \neq 0, -1$. Let $h(x) = (x+\delta)^{p^{m+1}+1}$ with $\delta \in \mathbb{F}_{p^{2m}}$. If $Tr_m^{2m}(\delta) = 0$ or $\frac{(Tr_m^{2m}(\delta))^p + a^{p^m} - b}{Tr_m^{2m}(\delta)}$, $\frac{(Tr_m^{2m}(\delta))^p + a^{p^m} - b - 1}{Tr_m^{2m}(\delta)}$ are $(p-1)$-th power in $\mathbb{F}_{p^m}^*$, then $F(x) = ax^{p^m} + bx + h(x^{p^m} - x)$ is a* CPP over $\mathbb{F}_{p^{2m}}$.

Proof. Based on Lemma 4, in order to prove that $F(x)$ is a CPP over $\mathbb{F}_{p^{2m}}$, we only need to consider that both $g(x) = (x+\delta)^{p^m(p^{m+1}+1)} - (x+\delta)^{p^{m+1}+1} + (b - a^{p^m})x$ and $g(x) + x = (x+\delta)^{p^m(p^{m+1}+1)} - (x+\delta)^{p^{m+1}+1} + (b - a^{p^m} + 1)x$ are bijective on J.

Firstly, we claim to show that $g(x)$ permutes J is equivalent to show that for any $d \in J$, the equation

$$(x+\delta)^{p^m(p^{m+1}+1)} - (x+\delta)^{p^{m+1}+1} + (b - a^{p^m})x = d, \tag{4.1}$$

has exactly one solution in J.

For $x \in J$, it can be verified that $x^{p^m} + x = 0$, then the left-hand side of (4.1) becomes

$$\begin{aligned}
& (x+\delta)^{p^m+p} - (x+\delta)^{p^{m+1}+1} + (b - a^{p^m})x \\
= \ & (-x+\delta^{p^m})(x^p+\delta^p) - (-x^p+\delta^{p^{m+1}})(x+\delta) + (b-a^{p^m})x \\
= \ & (-x^{p+1} + \delta^{p^m}x^p - \delta^p x + \delta^{p^m+p}) - (-x^{p+1} - \delta x^p + \delta^{p^{m+1}}x + \delta^{p^{m+1}+1}) + (b-a^{p^m})x \\
= \ & (\delta^{p^m}+\delta)x^p - (\delta^{p^{m+1}}+\delta^p)x + (b-a^{p^m})x + \delta^{p^m+p} - \delta^{p^{m+1}+1} \\
= \ & Tr_m^{2m}(\delta)x^p - ((Tr_m^{2m}(\delta))^p + a^{p^m} - b)x + \delta^{p^m+p} - \delta^{p^{m+1}+1}.
\end{aligned}$$

Then (4.1) can be rewritten as

$$Tr_m^{2m}(\delta)x^p - ((Tr_m^{2m}(\delta))^p + a^{p^m} - b)x = d - \delta^{p^m+p} + \delta^{p^{m+1}+1}. \tag{4.2}$$

When $Tr_m^{2m}(\delta) = 0$, $\delta^{p^m+p} - \delta^{p^{m+1}+1} = (-\delta)\delta^p - (-\delta)^p\delta = 0$ and then $x = \frac{d}{b-a^{p^m}}$ is the unique solution of (4.2) in J. When $Tr_m^{2m}(\delta) \neq 0$, (4.2) turns to

$$x^p - \frac{(Tr_m^{2m}(\delta))^p + a^{p^m} - b}{Tr_m^{2m}(\delta)}x = \frac{d - \delta^{p^m+p} + \delta^{p^{m+1}+1}}{Tr_m^{2m}(\delta)}. \tag{4.3}$$

Since $\frac{(Tr_m^{2m}(\delta))^p + a^{p^m} - b}{Tr_m^{2m}(\delta)}$ is a $(p-1)$-th power in $\mathbb{F}_{p^m}^*$, it then follows from Lemma 3 that (4.3) has at most one solution in J. Therefore, we conclude that (4.1) has only one solution in J and $g(x) = (x+\delta)^{p^m(p^m+1)} - (x+\delta)^{p^{m+1}+1} + (b-a^{p^m})x$ permutes on J.

In a similar way, we can prove that $g(x) + x = (x+\delta)^{p^m(p^m+1)} - (x+\delta)^{p^{m+1}+1} + (b - a^{p^m} + 1)x$ also permutes on J.

To summarize, we conclude that $F(x) = ax^{p^m} + bx + h(x^{p^m} - x)$ is a CPP over $\mathbb{F}_{p^{2m}}$. □

Example 1. *Take $p = 3$, $m = 2$, then $h(x) = (x+\delta)^{28}$. It can be verified that there are 3852 different triples $(a, b, \delta) \in \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that $a, b \in \mathbb{F}_{3^4}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^2}^*$, $b - a^9 \neq 0, -1$ and $Tr_2^4(\delta) = 0$, and 23112 different triples $(a, b, \delta) \in \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that $a, b \in \mathbb{F}_{3^4}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^2}^*$, $b - a^9 \neq 0, -1$, $\frac{(Tr_2^4(\delta))^3 + a^9 - b}{Tr_2^4(\delta)}$ and $\frac{(Tr_2^4(\delta))^3 + a^9 - b - 1}{Tr_2^4(\delta)}$ are square elements in $\mathbb{F}_{3^2}^*$. These (a, b, δ) are exactly all 26964 triples in $\mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ that make*

$$F(x) = ax^9 + bx + (x^9 - x + \delta)^{28},$$

a CPP over $\mathbb{F}_{3^4}$.

In the sequel, four classes of CPPs with different conditions on a, b, δ and the polynomials $h_i(x)$ for $i = 1, 2, 3, 4$ are given. The discussions are similar to that in Theorem 10, so we omit the proofs.

Theorem 11. *For an odd prime p and two positive integers i and even m, let $a, b \in \mathbb{F}_{p^{2m}}^*$ with $a+b, a+b+1 \in \mathbb{F}_{p^m}^*$, and $b - a^{p^m} \neq 0, -1$. Let $h_1(x) = (x+\delta)^{\frac{i(p^{2m}-1)}{p^2-1}+1} + (x+\delta)^{\frac{i(p^{2m}-1)}{p^2-1}+p^m}$ with $\delta \in \mathbb{F}_{p^{2m}}$. Then $F_1(x) = ax^{p^m} + bx + h_1(x^{p^m} - x)$ is a* CPP over $\mathbb{F}_{p^{2m}}$.

Example 2. *Take $p = 3$, $m = 2$ and $i = 1$, then $h_1(x) = (x+\delta)^{11} + (x+\delta)^{19}$. It can be verified that there are 34668 different triples $(a, b, \delta) \in \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that $a, b \in \mathbb{F}_{3^4}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^2}^*$, $b - a^9 \neq 0, -1$. These (a, b, δ) are exactly all triples in $\mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that*

$$F_1(x) = ax^9 + bx + (x^9 - x + \delta)^{11} + (x^9 - x + \delta)^{19},$$

is a CPP over $\mathbb{F}_{3^4}$.

Theorem 12. *For an odd prime p and a positive integer m, let $a, b \in \mathbb{F}_{p^{2m}}^*$ with $a+b, a+b+1 \in \mathbb{F}_{p^m}^*$, and $b - a^{p^m} \neq 0, -1$. Let $h_2(x) = (x+\delta)^{2p^m} + (x+\delta)^{p^{m+1}+1}$ with $\delta \in \mathbb{F}_{p^{2m}}$. If $Tr_m^{2m}(\delta) = 0$ or $\frac{(Tr_m^{2m}(\delta))^p - b + a^{p^m} - 2Tr_m^{2m}(\delta)}{Tr_m^{2m}(\delta)}$, $\frac{(Tr_m^{2m}(\delta))^p - b + a^{p^m} - 1 - 2Tr_m^{2m}(\delta)}{Tr_m^{2m}(\delta)}$ are $(p-1)$-th power in $\mathbb{F}_{p^m}^*$, then $F_2(x) = ax^{p^m} + bx + h_2(x^{p^m} - x)$ is a* CPP over $\mathbb{F}_{p^{2m}}$.

Example 3. *Take $p = 3$ and $m = 2$, then $h_2(x) = (x+\delta)^{18} + (x+\delta)^{28}$. It can be verified that there are 3852 different $(a, b, \delta) \in \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that $a, b \in \mathbb{F}_{3^4}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^2}^*$, $b - a^9 \neq 0, -1$ and $Tr_2^4(\delta) = 0$, and 27468 different triples $(a, b, \delta) \in \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that $a, b \in \mathbb{F}_{3^4}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^2}^*$, $b - a^9 \neq 0, -1$, $\frac{(Tr_2^4(\delta))^3 - b + a^9 - 2Tr_2^4(\delta)}{Tr_2^4(\delta)}$ and $\frac{(Tr_2^4(\delta))^3 - b + a^9 - 1 - 2Tr_2^4(\delta)}{Tr_2^4(\delta)}$ are square elements in $\mathbb{F}_{3^2}^*$. These (a, b, δ) are exactly all 31320 triples in $\mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ that make*

$$F_2(x) = ax^9 + bx + (x^9 - x + \delta)^{18} + (x^9 - x + \delta)^{28},$$

a CPP over $\mathbb{F}_{3^4}$.

Theorem 13. *For a positive integer m and $a, b \in \mathbb{F}_{3^{2m}}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^m}^*$. Let $h_3(x) = (x+\delta)^{2\cdot 3^m} + (x+\delta)^{3^{2m-1}+2\cdot 3^{m-1}}$ with $\delta \in \mathbb{F}_{3^{2m}}$. Then $F_3(x) = ax^{3^m} + bx + h_3(x^{3^m} - x)$ is a* CPP over $\mathbb{F}_{3^{2m}}$ if one of the following conditions are satisfied:

(i) $Tr_m^{2m}(\delta) = 0$, $a^{3^m} - b \neq \pm 1$;

(ii) $2Tr_m^{2m}(\delta) - a^{3^m} + b + 1$ and $2Tr_m^{2m}(\delta) - a^{3^m} + b - 1$ are square elements in $\mathbb{F}_{3^m}^*$.

Example 4. *Take $m = 2$, then $h_3(x) = (x+\delta)^{18} + (x+\delta)^{33}$. It can be verified that there are 3861 different $(a, b, \delta) \in \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that $a, b \in \mathbb{F}_{3^4}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^2}^*$, $Tr_2^4(\delta) = 0$ and $a^9 - b \neq \pm 1$, and 34839 different triples $(a, b, \delta) \in \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that $a, b \in \mathbb{F}_{3^4}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^2}^*$, $2Tr_2^4(\delta) - a^9 + b + 1$ and $2Tr_2^4(\delta) - a^9 + b - 1$ are square elements in $\mathbb{F}_{3^2}^*$. These (a, b, δ) are exactly all 38700 triples in $\mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}^* \times \mathbb{F}_{3^4}$ such that*

$$F_3(x) = ax^9 + bx + (x^9 - x + \delta)^{18} + (x^9 - x + \delta)^{33},$$

is a CPP over $\mathbb{F}_{3^4}$.

Theorem 14. *For a positive integer m and an element $\delta \in \mathbb{F}_{3^{2m}}$, let $a, b \in \mathbb{F}_{3^{2m}}^*$ with $a+b, a+b+1 \in \mathbb{F}_{3^m}^*$, and $b - a^{3^m} \neq 0, -1$. Let $h_4(x) = (x+\delta)^{3^{2m-1}+2\cdot 3^{m-1}} + (x+\delta)^{2\cdot 3^{2m-1}+3^{m-1}}$ with $\delta \in \mathbb{F}_{3^{2m}}$. Then $F_4(x) = ax^{3^m} + bx + h_4(x^{3^m} - x)$ is a* CPP over $\mathbb{F}_{3^{2m}}$.

Example 5. *Take $m = 1$, then $h_4(x) = (x+\delta)^5 + (x+\delta)^7$. It can be verified that there are 18 different $(a, b, \delta) \in \mathbb{F}_{3^2}^* \times \mathbb{F}_{3^2}^* \times \mathbb{F}_{3^2}$ such that $a, b \in \mathbb{F}_{3^2}^*$ with $a+b, a+b+1 \in \mathbb{F}_3^*$, $b - a^3 \neq 0, -1$. These (a, b, δ) are exactly all triples in $\mathbb{F}_{3^2}^* \times \mathbb{F}_{3^2}^* \times \mathbb{F}_{3^2}$ that make*

$$F_4(x) = ax^3 + bx + (x^3 - x + \delta)^5 + (x^3 - x + \delta)^7,$$

a CPP over $\mathbb{F}_{3^2}$.

Table 1. Known PPs of the form $(x^{2^m}+x+\delta)^{s_1}+(x^{2^m}+x+\delta)^{s_2}+x$ over $\mathbb{F}_{2^n}$.

Values of m and n	(s_1, s_2)	Condition on δ	Reference
$n = 2m$	$(2^m+1, 2^{m+1}-1)$	all δ	[20]
$n = 2m$	$(2-2^m, 3-2^{m+1})$	all δ	[20]
$n = 3m$, $m = 2$	$(2^{2m}+1, 2^m+1)$	$Tr_1^n(\delta) = 1$	[20]
n even, $m = 2$	$(-4, -2)$	$Tr_1^n(\delta) = 1$	[20]
$n \equiv 2 \pmod 4$, $m = 2$	$(\frac{2^n+1}{5}, \frac{3\cdot 2^n-2}{5})$	all δ	[20]
$n \equiv 1 \pmod 3$ and n even, $m = 2$	$(\frac{2^{n+2}-1}{7}, \frac{6\cdot 2^n-5}{7})$	all δ	[20]
n even, $m = 2$	$(\frac{2^{n-1}+1}{3}, \frac{5\cdot 2^{n-1}-1}{3})$	all δ	[20]
$n = 2m$	$(s_1, 2^m s_1)$	all δ	[8]
$n = 2m$	$(\frac{i(2^{2m}-1)}{3}+1, \frac{i(2^{2m}-1)}{3}+2^m)$	all δ	[8]
$n = 2m$	$(2^{m+i}+2^j, 2^{m+i}+2^{m+j})$	all δ	[8]
$n = 2m$ and m even	$(2^{\frac{jm}{2^i}}+2^{\frac{jm}{2^i}+\frac{km}{2}}, 2^{(\frac{j}{2^i}+\frac{1}{2})m}+2^{\frac{jm}{2^i}+\frac{(k+1)m}{2}})$	all δ	[8]
$n = 2m$	$(2^{\frac{im}{3}}+2^{\frac{km}{3}}, 2^{\frac{2im}{3}}+2^{\frac{(k+i)m}{3}})$	all δ	[8]
$n = 3m$	$(2^m+1, 2^{2m}+2^m)$	all δ	[8]
$n = 3m$ and $\gcd(m+1, 3m) = 1$	$(2^{2m-1}+2^{m-1}, 2^{3m-1}+2^{2m-1})$	all δ	[8]
$n = 3m$ and $\gcd(2m+1, 3m) = 1$	$(2^{3m-1}+2^{2m-1}, 2^{4m-1}+2^{3m-1})$	all δ	[8]
$n = 3m$, $\gcd(jm+i, 3m) = 1$ and $j \in \{0, 1, 2\}$	$(2^{jm+i}+2^{2m}, 2^{(j+2)m+i}+1)$	all δ	[8]
n even, $m \mid n$ and $j \in \{0, \frac{n}{m}-1\}$	$(\frac{2^n-1}{3}+2^{jm}, \frac{2(2^n-1)}{3}+2^{jm})$	all δ	[8]
$n = 3m$	$(2^{2m+1}+2^m, 2^{2m}+2^{m+1})$	$Tr_m^{3m}(\delta) = 0$	[8]
$m = 1, n = 2s$	$(2^{n-2}-2^{s-2}, 2^{n-1})$	$Tr_1^n(\delta) = 1$	[7]
$m = 1, n \equiv 2 \pmod 3$	$(\frac{2^{n+1}-1}{7}, \frac{3(2^{n+1}-1)}{7})$	all δ	[7]
$m = 1, n \equiv 0 \pmod 3$	$(\frac{2^{n+1}+2}{9}, \frac{2^{n+1}-1}{3})$	all δ	[7]
$m = 1, n \equiv 1 \pmod 6$	$(\frac{5\cdot 2^n-1}{9}, \frac{2^{n+1}-1}{3})$	all δ	[7]
$n = 3m$ and $s \equiv 0 \pmod{1+2^m+2^{2m}}$	$(2^{2m}+1, s)$	all δ	This paper
$n = 3m$	$(2^{2m}+1, 2^m)$	all δ	This paper
$n = 3m$	$(2^{2m-1}+2^{m-1}, 2^{2m})$	$Tr_m^{3m}(\delta) \neq 0$	This paper
$n = 3m$	$(2^{2m-1}+2^{m-1}, 2^m)$	all δ	This paper
$n = 3m$ and $m \not\equiv -1 \pmod 3$	$(2^{2m-1}+2^{m-1}, 1)$	all δ	This paper

Table 2. Known PPs of the form $(x^{p^m}-x+\delta)^{s_1}+(x^{p^m}-x+\delta)^{s_2}+x$ over $\mathbb{F}_{p^n}$ with odd prime p.

Values of m and n	(s_1, s_2)	Condition on δ	Reference
$n = 2m$	$(s_1, p^m s_1)$	all δ	[8]
$n = 2m$ and m even	$(\frac{i(p^{2m}-1)}{p^2-1}+1, \frac{i(p^{2m}-1)}{p^2-1}+p^m)$	all δ	[8]
$n = 2m$ and $p = 3$	$(2\cdot 3^{m+i}+3^j, 2\cdot 3^{m+i}+3^{m+j})$	all δ	[8]
$n = 2m$ and $p = 3$	$(\frac{3^{2m}+1}{2}, \frac{3^{2m}+3^m}{2})$	$Tr_m^{3m}(\delta) \neq 0$	[8]
$n = 2m$ and $p = 3$	$(3^m+4, 5)$	$1-(Tr_m^{2m}(\delta))^4$ is a square of $\mathbb{F}_{3^m}$	[8]
$n = 3m$, $p = 3$ and $i \in \{0, 1\}$	$(3^m+2, 2\cdot 3^{im})$	all δ	[8]
$n = 3m$	$(2p^{2m}+p^m, p^{2m}+2p^m)$	$Tr_m^{3m}(\delta) = 0$	[8]
$n = 3m$ and $p = 3$	$(\frac{3^{3m}-1}{2}+1, \frac{3^{3m}-1}{2}+3^m)$	$Tr_m^{3m}(\delta) = 0$	[8]
$n = 3m$ and $p = 3$	$(\frac{3^{3m}-1}{2}+3^m, \frac{3^{3m}-1}{2}+3^{2m})$	$Tr_m^{3m}(\delta) = 0$	[8]
$n = 3m$	$(\frac{p^{3m}-1}{2}+p^{2m}+p^m-1, \frac{p^{3m}-1}{2}+2p^m-1)$	$Tr_m^{3m}(\delta) = 0$	[8]
$n = 2m$	$(2p^m, p^{m+1}+1)$	$Tr_m^{2m}(\delta) = 0$ or $\frac{(Tr_m^{2m}(\delta)-1)^p-2Tr_m^{2m}(\delta)}{Tr_m^{2m}(\delta)}$ is a $(p-1)$-th power in $\mathbb{F}_{p^m}^*$	This paper
$n = 2m$ and $p = 3$	$(2\cdot 3^m, 3^{2m-1}+2\cdot 3^{m-1})$	$Tr_m^{2m}(\delta) = 1$ or $2Tr_m^{2m}(\delta)+1$ is a square element in $\mathbb{F}_{3^m}^*$	This paper
$n = 2m$ and $p = 3$	$(3^{2m-1}+2\cdot 3^{m-1}, 2\cdot 3^{2m-1}+3^{m-1})$	all δ	This paper
$n = 3m$, $p = 3$ and $s \equiv 0 \pmod{1+3^m+3^{2m}}$	$(\frac{3^{3m}-1}{2}+1, s)$	$Tr_m^{3m}(\delta) = 0$	This paper

5. Conclusions

In this paper, nine classes of PPs with the form $(x^{p^m} - x + \delta)^{s_1} + (x^{p^m} - x + \delta)^{s_2} + x$ over $\mathbb{F}_{p^n}$ and five classes of CPPs with the form $ax^{p^m} + bx + h(x^{p^m} - x)$ over $\mathbb{F}_{p^{2m}}$ were obtained by using the AGW criterion and some techniques in solving equations over finite fields. It was a continuation of some previous works [6, 8]. All known classes of PPs of the form $(x^{2^m} + x + \delta)^{s_1} + (x^{2^m} + x + \delta)^{s_2} + x$ over $\mathbb{F}_{2^n}$ and of the form $(x^{p^m} - x + \delta)^{s_1} + (x^{p^m} - x + \delta)^{s_2} + x$ over $\mathbb{F}_{p^n}$ (p an odd prime) were summarized in Tables 1 and 2, respectively. Moreover, we listed the known CPPs of the form $ax^{p^m} + bx + h(x^{p^m} - x)$ over $\mathbb{F}_{p^{2m}}$ in Table 3. It would be interesting to find new ideas to derive more PPs and CPPs over finite fields in the future work.

Table 3. Known CPPs of the form $ax^{p^m} + bx + h(x^{p^m} - x)$ over $\mathbb{F}_{p^{2m}}$.

Value of p	$h(x)$	Conditions on a, b, δ	Reference
p	$u(x)^{p^m s} + u(x)^s$, $u(x) \in \mathbb{F}_{p^{2m}}[x]$	$(a,b) \in \mathbb{F}^*_{p^{2m}} \times \mathbb{F}^*_{p^{2m}} \mid a+b \in \mathbb{F}^*_{p^m}$, $(a+b+1)(b-a^{p^m})(b-a^{p^m}+1) \neq 0$	[6]
p	$c(x+\delta)^{i(p^m+1)}$	$(a,b) \in \mathbb{F}^*_{p^{2m}} \times \mathbb{F}^*_{p^{2m}} \mid a+b \in \mathbb{F}^*_{p^m}$, $(a+b+1)(b-a^{p^m})(b-a^{p^m}+1) \neq 0$, $c \in \mathbb{F}_{p^m}$, $\delta \in \mathbb{F}_{p^{2m}}$	[6]
p	$c(x+\delta)^s$	even s, $c \in \mathbb{F}_{p^m}$ and $\delta \in S$ or odd s, $c, \delta \in S$, where $S = \{\gamma^{p^m} - \gamma \mid \gamma \in \mathbb{F}_{p^{2m}}\}$	[6]
p	$(x+\delta)^{2p^m}$	$\delta \in \mathbb{F}_{p^{2m}}$ and $2Tr_m^{2m}(\delta) + b - a^{p^m} \neq 0, -1$	[6]
2	$(x+\delta)^{2^{m+k}+2^l} + (x+\delta)^{2^{m+k}+2^{m+l}}$	k, l are nonnegative integers, $\delta \in \mathbb{F}_{p^{2m}}$ and $b - a^{2^m} \neq 0, 1$	[6]
3	$(x+\delta)^{3^{2m-1}+2\cdot 3^{m-1}}$	$\delta \in \mathbb{F}_{p^{2m}}$, $b - a^{3^m} + 1$ and $b - a^{3^m} + 2$ are square elements in $\mathbb{F}^*_{3^m}$	[6]
3	$(x+\delta)^{2\cdot 3^{m+k}+3^l} + (x+\delta)^{2\cdot 3^{m+k}+3^{m+l}}$	k, l are nonnegative integers with $\gcd(k, 2m) = 1$, $\delta \in \mathbb{F}_{p^{2m}}$, $b - a^{3^m}$ and $b - a^{3^m} + 1$ are square elements in $\mathbb{F}^*_{3^m}$	[6]
3	$(x+\delta)^{3^m+4} + (x+\delta)^5$	$b - a^{3^m} \neq 0$, $\delta \in \mathbb{F}_{p^{2m}}$, $b - a^{3^m} - [Tr_m^{2m}(\delta)]^4$ and $b - a^{3^m} - [Tr_m^{2m}(\delta)]^4 + 1$ are square elements in $\mathbb{F}^*_{3^m}$	[6]
3	$(x+\delta)^{3^m+2} + (x+\delta)^{2\cdot 3^{im}}$	$i \in \{0, 1\}$, $\delta \in \mathbb{F}_{p^{2m}}$, $[Tr_m^{2m}(\delta) - (i+1)]^2 + b - a^{3^m}$ and $[Tr_m^{2m}(\delta) - (i+1)]^2 + b - a^{3^m} - 1$ are square elements in $\mathbb{F}^*_{3^m}$	[6]
p	$(x+\delta)^{p^i+p^j}$	$(a,b) \in \mathbb{F}^*_{p^{2m}} \times \mathbb{F}^*_{p^{2m}} \mid a+b \in \mathbb{F}^*_{p^m}$, $(a+b+1)(b-a^{p^m})(b-a^{p^m}+1) \neq 0$, two positive integers $i, j < 2m$ and $\delta \in \mathbb{F}_{p^{2m}}$, $Tr_m^{2m}(\delta) = 0$ or $Tr_m^{2m}(\delta) \neq 0$ and $\frac{b-a^{p^m}}{[Tr_m^{2m}(\delta)]^{p^i+p^j-1}}, \frac{b-a^{p^m}+1}{[Tr_m^{2m}(\delta)]^{p^i+p^j-1}} \notin \{\alpha^{p^i-1} + \alpha^{p^j-1} \mid \alpha \in S\}$, where $S = \{\gamma^{p^m} - \gamma \mid \gamma \in \mathbb{F}_{p^{2m}}\}$	[6]
odd p	$(x+\delta)^{p^{m+1}+1}$	$a, b \in \mathbb{F}^*_{p^{2m}}$ with $a+b, a+b+1 \in \mathbb{F}^*_{p^m}$, and $b - a^{p^m} \neq 0, -1$, $\delta \in \mathbb{F}_{p^{2m}}$, $Tr_m^{2m}(\delta) = 0$ or $\frac{(Tr_m^{2m}(\delta))^p + a^{p^m} - b}{Tr_m^{2m}(\delta)}, \frac{(Tr_m^{2m}(\delta))^p + a^{p^m} - b - 1}{Tr_m^{2m}(\delta)}$ are $(p-1)$-th power in $\mathbb{F}^*_{p^m}$	This paper
odd p	$(x+\delta)^{\frac{i(p^{2m}-1)}{p^2-1}+1} + (x+\delta)^{\frac{i(p^{2m}-1)}{p^2-1}+p^m}$	two positive integers i and even m, $a, b \in \mathbb{F}^*_{p^{2m}}$ with $a+b, a+b+1 \in \mathbb{F}^*_{p^m}$, and $b - a^{p^m} \neq 0, -1$, $\delta \in \mathbb{F}_{p^{2m}}$	This paper
odd p	$(x+\delta)^{2p^m} + (x+\delta)^{p^{m+1}+1}$	a positive integer m, $a, b \in \mathbb{F}^*_{p^{2m}}$ with $a+b, a+b+1 \in \mathbb{F}^*_{p^m}$, and $b - a^{p^m} \neq 0, -1$, $\delta \in \mathbb{F}_{p^{2m}}$, $Tr_m^{2m}(\delta) = 0$ or $\frac{(Tr_m^{2m}(\delta))^p - b + a^{p^m} - 2Tr_m^{2m}(\delta)}{Tr_m^{2m}(\delta)}$, $\frac{(Tr_m^{2m}(\delta))^p - b + a^{p^m} - 1 - 2Tr_m^{2m}(\delta)}{Tr_m^{2m}(\delta)}$ are $(p-1)$-th power in $\mathbb{F}^*_{p^m}$	This paper
3	$(x+\delta)^{2\cdot 3^m} + (x+\delta)^{3^{2m-1}+2\cdot 3^{m-1}}$	a positive integer m, $a, b \in \mathbb{F}^*_{3^{2m}}$ with $a+b, a+b+1 \in \mathbb{F}^*_{3^m}$, $\delta \in \mathbb{F}_{3^{2m}}$, $a^{3^m} - b \neq \pm 1$ and $Tr_m^{2m}(\delta) = 0$ or $2Tr_m^{2m}(\delta) - a^{3^m} + b + 1$ and $2Tr_m^{2m}(\delta) - a^{3^m} + b - 1$ are square elements in $\mathbb{F}^*_{3^m}$	This paper
3	$(x+\delta)^{3^{2m-1}+2\cdot 3^{m-1}} + (x+\delta)^{2\cdot 3^{2m-1}+3^{m-1}}$	a positive integer m and $\delta \in \mathbb{F}_{3^{2m}}$, $a, b \in \mathbb{F}^*_{3^{2m}}$ with $a+b, a+b+1 \in \mathbb{F}^*_{3^m}$, and $b - a^{3^m} \neq 0, -1$	This paper

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Conflict of interest

The authors declare there is no conflict of interests.

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