AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13448–13463.
DOI: 10.3934/math.2021779
Received: 14 April 2021
Accepted: 14 September 2021
Published: 18 September 2021

Research article

On the improved thinning risk model under a periodic dividend barrier strategy

Fuyun Sun[1] **and Yuelei Li**[2,*]

[1] School of Mathematics, Tianjin University, Tianjin 300350, China
[2] College of Management and Economics, Tianjin University, Tianjin 300072, China

* **Correspondence:** Email: liyuelei@tju.edu.cn.

Abstract: In this study, we consider a periodic dividend barrier strategy in an improved thinning risk model, which indicates that insurance companies randomly receive premiums and pay dividends. In the improved model, the premium is stochastic, and the claim counting process is a p-thinning process of the premium counting process. The integral equations satisfied by the Gerber-Shiu function and the expected discounted cumulative dividend function are derived. Explicit expressions of those actuarial functions are obtained when the claim and premium sizes are exponentially distributed. We analyze and illustrate the impact of various parameters on them and obtain the optimal barrier. Finally, a conclusion is drawn.

Keywords: thinning model; periodic dividend strategy; Gerber-Shiu function; stochastic premium; expected discounted cumulative dividend function
Mathematics Subject Classification: 91B30, 97M30

1. Introduction

In insurance companies, the premium income is collected through insurance policies at discrete times, and each claim comes from the received policies, where the size of a claim for each policy is random. Therefore, we consider a thinning risk model in which the premium income of an insurance company is not a deterministic function of time (i.e. linear drift, see, e.g., [1, 2]) but a stochastic process.

In resent years, Boikov [3] studied the Cramér-Lundberg model where the premium process is stochastic. References [4, 5] considered a risk model where both the premium process and claim process are compound Poisson processes. Further, Albrecher and Boxma [6] considered a dependent setting between claim sizes and claim intervals in a generalization of the classical risk model. Wang and Yuen [7] studied a thinning dependence structure with $n(n \geq 2)$ dependent classes of insurance business. Moreover, numerous dependence structures have been studied, see e.g. [8–11], and so on. Inspired by the dependence of the premiums and claim amounts in the actual environment, we consider that the premium process is a thinning process of the claim process.

With increases in the standard of living, insurance awareness has improved rapidly, leading to fierce competition among insurance companies. To attract more investors, insurance companies have proposed a dividend plan. Dividend strategy was first introduced into a risk model by reference [12]. The most common dividend payment strategies are the barrier strategy and the threshold strategy [13]. Under the barrier strategy, any excess over a fixed barrier level is paid out immediately. Under the threshold strategy, the dividend is paid out at a constant rate whenever the surplus is above a pre-specified threshold level. Under these strategies, dividend decisions are assumed to be continuous. However, references [10, 14] proposed randomized observations, where dividend decisions are not continuous but are discrete only at "observation time points", which is more realistic. At these points, any excess over a fixed barrier level is paid as a dividend. Some recent papers on periodic dividend strategy can be found in references [15–22].

Motivated by those works above, we consider the model with to combine the thinning process and the periodic dividend strategy into a risk model. Given a probability space $(\Omega, \mathbb{F}, \mathcal{P})$ satisfying the usual conditions and complete filtration $\{\mathcal{F}_t, t \geq 0\}$. Let $u \geq 0$ be the initial capital of an insurance company. Without any dividend strategies, the surplus of the insurance company at time t has the form

$$R(t) = u + P(t) - S(t), u \geq 0, t \geq 0, \tag{1.1}$$

where $P(t) = \sum_{i=1}^{N(t)} Y_i$ represents the total premium amount up to time t, and $S(t) = \sum_{i=1}^{N_p(t)} X_i$ represents the aggregate claim amount up to time t. $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with parameter $\lambda > 0$, which is a counting process that represents the number of premiums up to time t; $\left\{N_p(t), t \geq 0\right\}$ is the thinning process of N with parameter $\lambda p (0 < p < 1)$, which is a counting process that represents the number of claims up to time t; $X_1,\ X_2,\ \ldots$ are positive i.i.d. random variables with common distribution function $F(x)$, where X_i denotes the ith claim size; $Y_1,\ Y_2,\ \ldots$ are positive i.i.d. random variables with common distribution function $G(y)$, where Y_i denotes the ith premium size. We assume that $\{N(t), t \geq 0\}$, $\{X_i\}_{i=1}^{\infty}$, and $\{Y_i\}_{i=1}^{\infty}$ are independent of each other. Without loss of generality, we assume the safety loading condition $\mathbb{E}P(t) > \mathbb{E}S(t)$. Denote the total amount of dividends up to time t by $D(t)$. Then the final risk model is given by

$$U(t) = u + P(t) - S(t) - D(t), t \geq 0, u \geq 0. \tag{1.2}$$

Let $T_{u,b} = \inf\{t > 0 : U(t) < 0\}$ be the time of ruin of risk model (1.2).

For dividends, specifically, we consider such a randomized dividend strategy with a fixed barrier level $b > 0$ and exponential inter-dividend-decision times $\tau_1, \tau_2, \ldots (\tau_1 < \tau_2 < \ldots)$ with the parameter $r > 0$. Without loss of generality, let $\tau_0 = 0$ and note that $\tau_0 = 0$ is not a dividend decision time. Then the total amount of dividends up to time t is

$$D(t) = \int_0^t (U(s-) - b)^+ dN_r(s), t \geq 0,$$

where $\{N_r(t), t \geq 0\}$ is a homogeneous Poisson process with parameter r and it is independent of (X, Y, N, N_p); $(x)^+ = x$ if $x \geq 0$ and otherwise when $x < 0$, $(x)^+ = 0$; $f(x_0\pm)$ means the right (left) hand limit at the point $x = x_0$.

The first question to consider is the expected discounted penalty function, which was first introduced by reference [23]. Because the Gerber-Shiu function provides a comprehensive mathematical tool for studying some related quantities of ruin, such as the ultimate ruin probability, the Laplace transform of the ruin time, the deficit at ruin, the surplus immediately prior to ruin, and so on. Therefore, since it was proposed, it has been studied in various risk models [24–30]. It continues to be a popular topic. Now we define the Gerber-Shiu function as follows:

$$m_b(u) = \mathbb{E}\left[e^{-\delta T_{u,b}}\omega\left(U(T_{u,b}-), \left|U(T_{u,b})\right|\right) I\{T_{u,b} < +\infty\}|U(0) = u\right], \tag{1.3}$$

where $\delta \geq 0$ is a constant discount factor; $\omega(x, y)$ is a non-negative function with $x, y \geq 0$, which can be interpreted as the penalty function of the surplus immediately prior to ruin $U(T_{u,b}-)$ and the deficit at ruin $\left|U(T_{u,b})\right|$; and $I\{A\}$ is the indicator function. For convenience, we use the notation $m_b(u) \triangleq \mathbb{E}_u\left[e^{-\delta T_{u,b}}\omega\left(U(T_{u,b}-), \left|U(T_{u,b})\right|\right)\ I\{T_{u,b} < +\infty\}\right]$ to denote the above conditional expectation. When $\omega(x, y) = 1$, we denote $m_b(u) \triangleq \phi_b(u)$; When $\omega(x, y) = y$, we denote $m_b(u) \triangleq \xi_b(u)$.

The second question we consider is the expected discounted cumulative dividend function, which is given by

$$V_b(u) = \mathbb{E}_u\left[\sum_{k=1}^{\infty} e^{-\delta T_k}(U(T_k) - b)^+ I\{T_k < T_{u,b}\}\right], \quad u \geq 0, \tag{1.4}$$

where $T_k = \sum_{i=0}^{k} \tau_i$, $k \geq 1$, represents the kth dividend decision time.

The rest of the paper is organized as follows: We obtain the integral equations satisfied by $m_b(u)$ and $V_b(u)$, and prove the continuity of $m_b(u)$ and $V_b(u)$ in Section 2. When the claim and premium sizes are exponentially distributed, the explicit expressions for $m_b(u)$ and $V_b(u)$ are derived in Section 3. Furthermore, we conclude that in Section 3 both $m_b'(u)$ and $V_b'(u)$ are discontinuous at $u = b$. In Section 4 we present the graphs of the Laplace transform of the deficit at ruin $\xi_b(u)$, the Laplace transform of the time of ruin $\phi_b(u)$, and $V_b(u)$. Finally, Section 5 provides the conclusions.

2. Main results

Theorem 1. *We denote $m_b(u) = m_{b1}(u)$ for $0 \leq u < b$, and $m_b(u) = m_{b2}(u)$ for $u \geq b$. Then the Gerber-Shiu function $m_b(u)$, $u > 0$, satisfies the following integral equations:*

$$\begin{aligned}
&(\lambda + \delta)m_{b1}(u) \\
&- \lambda p \int_0^{b-u}\int_0^{u+y} m_{b1}(u + y - x)dF(x)dG(y) \\
&- \lambda p \int_{b-u}^{\infty}\left(\int_{u+y-b}^{u+y} m_{b1}(u + y - x)dF(x) + \int_0^{u+y-b} m_{b2}(u + y - x)dF(x)\right)dG(y) \\
&- \lambda(1 - p)\left(\int_0^{b-u} m_{b1}(u + y)dG(y) + \int_{b-u}^{\infty} m_{b2}(u + y)dG(y)\right) \\
&- \lambda p \int_0^{\infty}\int_{u+y}^{\infty} \omega(u + y, x - u - y)dF(x)dG(y) \\
&= 0,\ 0 \leq u < b,
\end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&(\lambda+\delta+r)m_{b2}(u)\\
&-\lambda(1-p)\int_0^\infty m_{b2}(u+y)dG(y)-rm_{b2}(b)\\
&-\lambda p\int_0^\infty\left(\int_{u+y-b}^{u+y} m_{b1}(u+y-x)dF(x)+\int_0^{u+y-b} m_{b2}(u+y-x)dF(x)\right)dG(y)\\
&-\lambda p\int_0^\infty\int_{u+y}^\infty \omega(u+y,x-u-y)dF(x)G(y)\\
&=0,\ u\geq b,
\end{aligned}
\tag{2.2}
$$

with the continuity condition

$$
m_{b1}(b-)=m_{b2}(b). \tag{2.3}
$$

Proof of Theorem 1. Note that it is intuitive that the surplus process U well defined in (1.2) has the strong Markov property, though the formal verification of this fact may prove tedious. The prove is omitted here. For convenience, we denote $H_{u,b} = e^{-\delta T_{u,b}}\omega\left(U(T_{u,b}-),\left|U(T_{u,b})\right|\right)I\{T_{u,b} < +\infty\}$, then $m_b(u)=\mathbb{E}_u\left[H_{u,b}\right]$.

Let l_i denote the ith premium arrival time and let m_i denote the ith claim arrival time, $i\geq 1, i\in N$. In an infinitesimal time interval $[0,t]$, the event $\{l_1 > t\}$ denotes that there is no income in the time interval $[0,t]$, the event $\{m_1 > t\}$ denotes that there is no claim, the event $\{\tau_1 > t\}$ denotes that there is no observation, that means there is no paid, similarly. According to the assumption of independence of $\{N(t), t\geq 0\}$, $\{N_r(t), t\geq 0\}$, $\{Y_i\}_{i=1}^\infty$, and $\{X_i\}_{i=1}^\infty$ and dependence of $\{N(t), t\geq 0\}$ and $\{N_p(t), t\geq 0\}$, we consider four possible events in time interval $[0,t]$:

1. No income, no claim, and no observation;
2. No income, no claim, but an observation time occurs;
3. One-time income, no claim, and no observation;
4. One-time income, one-time claim, and no observation.

The probability of other events is $o(t)$, which is equal to 0 as t tends to 0.

Using the total probability formula, we have

$$
\begin{aligned}
m_b(u)=&\mathbb{E}_u[H_{u,b}, l_1 > t, m_1 > t, \tau_1 > t]+\mathbb{E}_u[H_{u,b}, l_1 > t, m_1 > t, \tau_1 < t < \tau_2]\\
&+\mathbb{E}_u[H_{u,b}, l_1 < t < l_2, m_1 > t, \tau_1 > t]\\
&+\mathbb{E}_u[H_{u,b}, l_1 < t < l_2, m_1 < t < m_2, \tau_1 > t]+o(t)\\
\triangleq& I_1+I_2+I_3+I_4+o(t).
\end{aligned}
\tag{2.4}
$$

From the double conditional expectation theorem, the above assumptions, and the strong Markov property, we obtain

$$
\begin{aligned}
I_1&=\mathbb{E}[\mathbb{E}_u[H_{u,b}, l_1 > t, m_1 > t, \tau_1 > t|F_t]]\\
&=\mathbb{E}[l_1 > t, m_1 > t, \tau_1 > t, e^{-\delta t}m_b(U(t))]\\
&=\mathbb{P}(l_1 > t, m_1 > t)\mathbb{P}(\tau_1 > t)e^{-\delta t}m_{b1}(u)\\
&=e^{-(\lambda+r+\delta)t}m_{b1}(u),
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \mathbb{E}[\mathbb{E}_u[H_{u,b}, l_1 > t, m_1 > t, \tau_1 < t < \tau_2 | F_t]] \\
&= \mathbb{E}[l_1 > t, m_1 > t, \tau_1 < t < \tau_2, e^{-\delta t} m_b(u)] \\
&= \mathbb{P}(l_1 > t, m_1 > t)\mathbb{P}(\tau_1 < t < \tau_2) m_{b1}(u) \\
&= e^{-(\lambda+\delta)t}(1 - e^{-rt}) m_{b1}(u), \\
I_3 &= \mathbb{E}[\mathbb{E}_u[H_{u,b}, l_1 < t < l_2, m_1 > t, \tau_1 > t | F_t]] \\
&= \mathbb{E}[l_1 < t < l_2, m_1 > t, \tau_1 > t, e^{-\delta t} m_b(u + Y)] \\
&= \mathbb{P}(\tau_1 > t)\mathbb{P}(l_1 < t < l_2, m_1 > t) e^{-\delta t}\mathbb{E}[m_b(u + Y)] \\
&= e^{-(r+\delta)t} \int_0^t \mathbb{P}(m_1 > l | l_1 = l)\lambda e^{-\lambda l} dl \mathbb{E}[m_b(u + Y)] \\
&= e^{-(r+\delta)t}(1 - p)(1 - e^{-\lambda t}) \left[\int_0^{b-u} m_{b1}(u + y) dG(y) + \int_{b-u}^{\infty} m_{b2}(u + y) dG(y) \right], \\
I_4 &= \mathbb{E}[\mathbb{E}_u[H_{u,b}, l_1 < t < l_2, m_1 < t < m_2, \tau_1 > t | F_t]] \\
&= \mathbb{E}[l_1 < t < l_2, m_1 < t < m_2, \tau_1 > t, e^{-\delta t} m_b(u + Y - X)] \\
&= e^{-\delta t}\mathbb{P}(\tau_1 > t)\mathbb{P}(l_1 < t < l_2, m_1 < t < m_2)\mathbb{E}[m_b(u + Y - X)] \\
&= e^{-(r+\delta)t} \int_0^t \mathbb{P}(m_1 = l | l_1 = l)\lambda e^{-\lambda l} dl \, \mathbb{E}[m_b(u + Y - X)] \\
&= e^{-(r+\delta)t} p(1 - e^{-\lambda t}) \left[\int_0^{b-u} \int_0^{u+y} m_{b1}(u + y - x) dF(x) dG(y) \right. \\
&+ \int_{b-u}^{\infty} \left(\int_{u+y-b}^{u+y} m_{b1}(u + y - x) dF(x) + \int_0^{u+y-b} m_{b2}(u + y - x) dF(x) \right) dG(y) \\
&+ \left. \int_0^{\infty} \int_{u+y}^{\infty} \omega(u + y, x - u - y) dF(x) dG(y) \right].
\end{aligned}
$$

Using Taylor's theorem, we derive

$$I_1 = [1 - (\lambda + r + \delta)t] m_{b1}(u) + o(t), \tag{2.5}$$

$$I_2 = rt m_{b1}(u) + o(t), \tag{2.6}$$

$$I_3 = \lambda(1 - p)t \left[\int_0^{b-u} m_{b1}(u + y) dG(y) + \int_{b-u}^{\infty} m_{b2}(u + y) dG(y) \right] + o(t), \tag{2.7}$$

$$
\begin{aligned}
I_4 = \lambda pt &\left[\int_0^{b-u} \int_0^{u+y} m_{b1}(u + y - x) dF(x) dG(y) + \int_0^{\infty} \int_{u+y}^{\infty} \omega(u + y, x - u - y) dF(x) dG(y) \right. \\
&+ \left. \int_{b-u}^{\infty} \left(\int_{u+y-b}^{u+y} m_{b1}(u + y - x) dF(x) + \int_0^{u+y-b} m_{b2}(u + y - x) dF(x) \right) dG(y) \right] + o(t).
\end{aligned} \tag{2.8}
$$

Thus,

$$
\begin{aligned}
&\lim_{t\to 0} \frac{I_1 - m_{b1}(u)}{t} = -(\lambda + r + \delta) m_{b1}(u), \\
&\lim_{t\to 0} \frac{I_2}{t} = r m_{b1}(u),
\end{aligned}
$$

$$\lim_{t\to 0}\frac{I_3}{t} = \lambda(1-p)\left[\int_0^{b-u} m_{b1}(u+y)dG(y) + \int_{b-u}^{\infty} m_{b2}(u+y)dG(y)\right],$$

$$\lim_{t\to 0}\frac{I_4}{t} = \lambda p\left[\int_0^{b-u}\int_0^{u+y} m_{b1}(u+y-x)dF(x)dG(y) + \int_0^{\infty}\int_{u+y}^{\infty} \omega(u+y, x-u-y)dF(x)dG(y)\right.$$
$$\left. + \int_{b-u}^{\infty}\left(\int_{u+y-b}^{u+y} m_{b1}(u+y-x)dF(x) + \int_0^{u+y-b} m_{b2}(u+y-x)dF(x)\right)dG(y)\right].$$

Substituting Eqs (2.5)–(2.8) into Eq (2.4), and rewriting it in a more visible way. Thus Eq (2.1) can be obtained by dividing t both sides as well as considering that t tends to 0.

When $u \geq b$, we can obtain Eq (2.2) by using similar methods.

The continuity condition (2.11) can be obtained easily by letting u tend to b in Eqs (2.1) and (2.2). □

Theorem 2. *We denote $V_b(u) = V_{b1}(u)$ for $0 \leq u < b$, and $V_b(u) = V_{b2}(u)$ for $u \geq b$. Then the expected discounted cumulative dividend function $V_b(u)$, $u > 0$, satisfies the following integral equations:*

$$\begin{aligned}
&(\lambda+\delta)V_{b1}(u) \\
&- \lambda p\int_0^{b-u}\int_0^{u+y} V_{b1}(u+y-x)dF(x)dG(y) \\
&- \lambda p\int_{b-u}^{\infty}\left(\int_{u+y-b}^{u+y} V_{b1}(u+y-x)dF(x) + \int_0^{u+y-b} V_{b2}(u+y-x)dF(x)\right)dG(y) \\
&- \lambda(1-p)\left(\int_0^{b-u} V_{b1}(u+y)dG(y) + \int_{b-u}^{\infty} V_{b2}(u+y)dG(y)\right) \\
&= 0,\ 0 \leq u < b,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&(\lambda+\delta+r)V_{b2}(u) \\
&- \lambda(1-p)\int_0^{\infty} V_{b2}(u+y)dG(y) - r(u-b+V_{b2}(b)) \\
&- \lambda p\int_0^{\infty}\left(\int_{u+y-b}^{u+y} V_{b1}(u+y-x)dF(x) + \int_0^{u+y-b} V_{b2}(u+y-x)dF(x)\right)dG(y) \\
&= 0,\ u \geq b,
\end{aligned} \tag{2.10}$$

with the continuity condition

$$V_{b1}(b-) = V_{b2}(b). \tag{2.11}$$

Proof of Theorem 2. The method of this proof is similar to Theorem 1, which is omitted here. □

Remark 3. *When $r \to \infty$ (i.e. periodic dividend strategy evolved into continuous dividend strategy), our results are consistent with [8].*

Remark 4. *When $r \to 0$, which means the expectation of the first dividend decision time tends to infinity, then in this case $V_b(u) \to 0$ (i.e. no dividends).*

3. Closed-form expressions

Since it is difficult to solve Eqs (2.1), (2.2), (2.9) and (2.10), in this section, both claim sizes and premium sizes are assumed to be independent, exponentially distributed random variables. Specifically, let $F(x) = 1 - e^{-ax}$, $G(y) = 1 - e^{-my}$, $a > 0$, $m > 0$, and the net profit condition $a > mp$. We give the closed-form expressions of Gerber-Shiu function $m_b(u)$, $u > 0$ and the expected discounted cumulative dividend function $V_b(u)$, $u > 0$.

3.1. Laplace transform of the deficit at ruin

In this subsection, we assume that $\omega(x, y) = y$. Then in this case $m_b(u) \triangleq \xi_b(u)$ denotes the Laplace transform of the deficit at ruin. Its explicit expressions under these assumptions are given as follows.

Let $z = u + y$ in Eqs (2.1) and (2.2), so that they can be simplified as

$$\begin{aligned}
&(\lambda+\delta)\xi_{b1}(u)\\
&= \lambda(1-p)me^{mu}\left(\int_u^b \xi_{b1}(z)e^{-mz}dz + \int_b^\infty \xi_{b2}(z)e^{-mz}dz\right)\\
&+ \lambda pame^{mu}\int_b^\infty \left(\int_{z-b}^z \xi_{b1}(z-x)e^{-(ax+mz)}dx + \int_0^{z-b}\xi_{b2}(z-x)e^{-(ax+mz)}dx\right)dz\\
&+ \lambda pame^{mu}\int_u^b\int_0^z \xi_{b1}(z-x)e^{-(ax+mz)}dxdz + \frac{\lambda mpe^{-au}}{a(a+m)},\ 0 \le u < b,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&(\lambda+\delta+r)\xi_{b2}(u)\\
&= \lambda(1-p)me^{mu}\int_u^\infty \xi_{b2}(z)e^{-mz}dz + \frac{\lambda mpe^{-au}}{a(a+m)} + r\xi_{b2}(b+)\\
&+ \lambda pame^{mu}\int_b^\infty \left(\int_{z-b}^z \xi_{b1}(z-x)e^{-(ax+mz)}dx + \int_0^{z-b}\xi_{b2}(z-x)e^{-(ax+mz)}dx\right)dz,\ u \ge b.
\end{aligned} \tag{3.2}$$

By differentiating both sides of (3.1) and (3.2) with respect to u, we obtain

$$\begin{aligned}
&(\lambda+\delta)\xi_{b1}'(u)\\
&= \lambda(1-p)m^2e^{mu}\left(\int_u^b \xi_{b1}(z)e^{-mz}dz + \int_b^\infty \xi_{b2}(z)e^{-mz}dz\right)\\
&+ \lambda pam^2e^{mu}\int_b^\infty \left(\int_{z-b}^z \xi_{b1}(z-x)e^{-(ax+mz)}dx + \int_0^{z-b}\xi_{b2}(z-x)e^{-(ax+mz)}dx\right)dz\\
&+ \lambda pam^2e^{mu}\int_u^b\int_0^z \xi_{b1}(z-x)e^{-(ax+mz)}dxdz - \frac{\lambda mpe^{-au}}{a+m}\\
&- \lambda(1-p)m\xi_{b1}(u) - \lambda pma\int_0^u \xi_{b1}(u-x)e^{-ax}dx,\ 0 \le u < b,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&(\lambda+\delta+r)\xi_{b2}'(u)\\
&= \lambda(1-p)m^2e^{mu}\int_u^\infty \xi_{b2}(z)e^{-mz}dz - \frac{\lambda mpe^{-au}}{a+m}
\end{aligned}$$

$$+\lambda pam^2e^{mu}\int_u^{\infty}\left(\int_{z-b}^{z}\xi_{b1}(z-x)e^{-(ax+mz)}dx+\int_0^{z-b}\xi_{b2}(z-x)e^{-(ax+mz)}dx\right)dz$$
$$-\lambda(1-p)m\xi_{b2}(u),\ u\geq b. \tag{3.4}$$

By differentiating both sides of (3.3) and (3.4) with respect to u, we have

$$\begin{aligned}&(\lambda+\delta)\xi_{b1}''(u)\\&=\lambda(1-p)m^3e^{mu}\left(\int_u^b\xi_{b1}(z)e^{-mz}dz+\int_b^{\infty}\xi_{b2}(z)e^{-mz}dz\right)\\&+\lambda pam^3e^{mu}\int_u^b\int_0^z\xi_{b1}(z-x)e^{-(ax+mz)}dxdz+\frac{\lambda pame^{-au}}{a+m}\\&+\lambda pam^3e^{mu}\int_b^{\infty}\left(\int_{z-b}^{z}\xi_{b1}(z-x)e^{-(ax+mz)}dx+\int_0^{z-b}\xi_{b2}(z-x)e^{-(ax+mz)}dx\right)dz\\&+(a-m)\lambda pma\int_0^u\xi_{b1}(u-x)e^{-ax}dx-\lambda(1-p)m\xi_{b1}'(u)\\&-\lambda(1-p)m^2\xi_{b1}(u)-\lambda pam\xi_{b1}(u),\ 0\leq u<b,\end{aligned}\tag{3.5}$$

$$\begin{aligned}&(\lambda+\delta+r)\xi_{b2}''(u)\\&=\lambda(1-p)m^3e^{mu}\int_u^{\infty}\xi_{b2}(z)e^{-mz}dz+\frac{a\lambda mpe^{-au}}{a+m}\\&+\lambda pam^3e^{mu}\int_b^{\infty}\left(\int_{z-b}^{z}\xi_{b1}(z-x)e^{-(ax+mz)}dx+\int_0^{z-b}\xi_{b2}(z-x)e^{-(ax+mz)}dx\right)dz\\&+(a-m)\lambda pma\left(\int_{u-b}^{u}\xi_{b1}(u-x)e^{-ax}dx+\int_0^{u-b}\xi_{b2}(u-x)e^{-ax}dx\right)\\&-\lambda(1-p)m^2\xi_{b2}(u)-\lambda(1-p)m\xi_{b2}'(u)-\lambda pam\xi_{b2}(u),\ u\geq b.\end{aligned}\tag{3.6}$$

Calculating (3.3) − $m\times$(3.1) and (3.5) − $m\times$(3.3), we obtain

$$(\lambda+\delta)\xi_{b1}'(u)-m(\lambda p+\delta)\xi_{b1}(u)+\lambda pm\left(e^{-au}+a\int_0^u\xi_{b1}(u-x)e^{-ax}dx\right)=0. \tag{3.7}$$

$$(\lambda+\delta)\xi_{b1}''(u)-m(\lambda p+\delta)\xi_{b1}'(u)+\lambda pam\xi_{b1}(u)-\lambda pma\left(e^{-au}+a\int_0^u\xi_{b1}(u-x)e^{-ax}dx\right)=0. \tag{3.8}$$

Calculating (3.8) + $a\times$(3.7) and rearranging, the following second-order homogeneous ordinary differential equation (ODE) is derived:

$$(\lambda+\delta)\xi_{b1}''(u)+[a(\lambda+\delta)-m(\lambda p+\delta)]\xi_{b1}'(u)-am\delta\xi_{b1}(u)=0. \tag{3.9}$$

Calculating (3.4) − $m\times$(3.2) and (3.6) − $m\times$(3.4) yields

$$\begin{aligned}&(\lambda+\delta+r)\xi_{b2}'(u)-m(\lambda p+\delta+r)\xi_{b2}(u)+\lambda pme^{-au}+mr\xi_{b2}(b+)\\&\quad+\lambda pma\left(\int_{u-b}^{u}\xi_{b1}(u-x)e^{-ax}dx+\int_0^{u-b}\xi_{b2}(u-x)e^{-ax}dx\right)=0.\end{aligned}\tag{3.10}$$

$$(\lambda+\delta+r)\xi_{b2}^{''}(u)-m(\lambda p+\delta+r)\xi_{b2}^{'}(u)-\lambda pmae^{-au}+\lambda pam\xi_{b2}(u)$$
$$-\lambda pma^{2}\left(\int_{u-b}^{u}\xi_{b1}(u-x)e^{-ax}dx+\int_{0}^{u-b}\xi_{b2}(u-x)e^{-ax}dx\right)=0. \quad (3.11)$$

Calculating (3.11)+$a\times$(3.10) and rearranging, we obtain the following second-order nonhomogeneous ODE:

$$(\lambda+\delta+r)\xi_{b2}^{''}(u)+[a(\lambda+\delta+r)-m(\lambda p+\delta+r)]\xi_{b2}^{'}(u)-am(\delta+r)\xi_{b2}(u)+amr\xi_{b2}(b+)=0. \quad (3.12)$$

The general solution to Eq (3.9) is obtained as follows:

$$\xi_{b1}(u)=A_1e^{s_1u}+A_2e^{s_2u},0\le u<b, \quad (3.13)$$

where $s_1<0$, $s_2\ge 0$ are the roots of the characteristic equation $(\lambda+\delta)s^2+[a(\lambda+\delta)-m(\lambda p+\delta)]s-am\delta=0$; and A_1, A_2 are the undetermined coefficients.

The general solution of Eq (3.12) has the form

$$\xi_{b2}(u)=A_3e^{s_3u}+A_4e^{s_4u}+\frac{r}{r+\delta}\xi_{b2}(b+),u\ge b, \quad (3.14)$$

where $s_3<0$, $s_4\ge 0$ are the roots of the characteristic equation $(\lambda+\delta+r)s^2+[a(\lambda+\delta+r)-m(\lambda p+\delta+r)]s-am(\delta+r)=0$; A_3, A_4 are the undetermined coefficients; and $\frac{r}{r+\delta}\xi_{b2}(b+)$ is a special solution of Eq (3.12).

By substituting (3.13) into (3.7), we see that (3.13) satisfies (3.7) only if

$$a^2(s_2+a)A_1+a^2(s_1+a)A_2=(s_1+a)(s_2+a). \quad (3.15)$$

From the continuity condition (2.11), we have $\xi_{b1}(b-)=\xi_{b2}(b+)$. From the formulae (3.3) and (3.4), we have $(\lambda+\delta)\xi_{b1}^{'}(b-)=(\lambda+\delta+r)\xi_{b2}^{'}(b+)$. Then

$$\delta e^{s_1b}A_1+\delta e^{s_2b}A_2-(\delta+r)e^{s_3b}A_3-(\delta+r)e^{s_4b}A_4=0, \quad (3.16)$$
$$(\lambda+\delta)(e^{s_1b}s_1A_1+e^{s_2b}s_2A_2)-(\lambda+\delta+r)(e^{s_3b}s_3A_3+e^{s_4b}s_4A_4)=0. \quad (3.17)$$

By substituting (3.13) and (3.14) into (3.1), we see that (3.13) and (3.14) satisfy (3.1) only if

$$l_1e^{s_1b}A_1+l_2e^{s_2b}A_2-l_3e^{s_3b}A_3-l_4e^{s_4b}A_4=0, \quad (3.18)$$

where

$$l_1=[\frac{r(1-p)}{r+\delta}+\frac{pa}{a+m}\left(\frac{m}{a+s_1}+\frac{r}{r+\delta}\right)-\frac{m}{m-s_1}\left(1-p+\frac{pa}{a+s_1}\right),$$
$$l_2=\frac{r(1-p)}{r+\delta}+\frac{pa}{a+m}\left(\frac{m}{a+s_2}+\frac{r}{r+\delta}\right)-\frac{m}{m-s_2}\left(1-p+\frac{pa}{a+s_2}\right),$$
$$l_3=\frac{m}{m-s_3}\left(1-p+\frac{pa}{a+m}\right),$$
$$l_4=\frac{m}{m-s_4}\left(1-p+\frac{pa}{a+m}\right).$$

Let

$$w_1 = \delta(\lambda+\delta+r)(l_4s_3 - l_3s_4) + (\delta+r)[(\lambda+\delta+r)(s_4-s_3)l_2 - (\lambda+\delta)(l_4-l_3)s_2],$$
$$w_2 = \delta(\lambda+\delta+r)(l_3s_4 - l_4s_3) + (\delta+r)[(\lambda+\delta)(l_4-l_1)s_1 - (\lambda+\delta+r)(s_4-s_3)l_1],$$
$$w_3 = \delta[(\lambda+\delta)(s_1-s_2)l_4 + (\lambda+\delta+r)(l_2-l_1)s_4] - (\delta+r)(\lambda+\delta)(l_2s_1 - l_1s_2),$$
$$w_4 = \delta[(\lambda+\delta)(s_1-s_2)l_3 + (\lambda+\delta+r)(l_2-l_1)s_3] - (\delta+r)(\lambda+\delta)(l_2s_1 - l_1s_2),$$
$$w_5 = (\lambda+\delta+r)(s_4-s_3)(l_2-l_1) + (\lambda+\delta)[(l_4-l_1)s_1 - (l_4-l_3)s_2].$$

Solving the system of Eqs (3.15)−(3.18), we can obtain A_1–A_4. Then we have

$$\xi_b(u) = \begin{cases} \frac{(s_1+a)(s_2+a)[w_1e^{(s_2b+s_1u)}+w_2e^{(s_1b+s_2u)}]}{a^2[(s_2+a)w_1e^{s_2b}+(s_1+a)w_2e^{s_1b}]}, & 0 \le u < b, \\ \frac{(s_1+a)(s_2+a)e^{(s_1+s_2)b}[rw_5+w_3e^{(s_4b+s_3u)}-w_4e^{(s_3b+s_4u)}]}{a^2e^{(s_3+s_4)b}[(s_2+a)w_1e^{s_2b}+(s_1+a)w_2e^{s_1b}]}, & u \ge b. \end{cases}$$

3.2. Laplace transform of the ruin time

In this subsection, we assume $\omega(x,y) = 1$. Then in this case $m_b(u) \triangleq \phi_b(u)$ denotes the Laplace transform of the ruin time. The explicit expressions of $\phi_b(u)$ under the above assumptions are given as follows:

Using the same method as in the previous section, we obtain

$$(\lambda+\delta)\phi''_{b1}(u) + [a(\lambda+\delta) - m(\lambda p+\delta)]\phi'_{b1}(u) - am\delta\phi_{b1}(u) = 0, \tag{3.19}$$

and

$$(\lambda+\delta+r)\phi''_{b2}(u) + [a(\lambda+\delta+r) - m(\lambda p+\delta+r)]\phi'_{b2}(u) - am(\delta+r)\phi_{b2}(u) + amr\phi_{b2}(b+) = 0. \tag{3.20}$$

The explicit expressions for the Laplace transform of the ruin time can be obtained from the following equations:

$$a(s_2+a)B_1 + a(s_1+a)B_2 = (s_1+a)(s_2+a), \tag{3.21}$$
$$\delta e^{s_1b}B_1 + \delta e^{s_2b}B_2 - (\delta+r)e^{s_3b}B_3 - (\delta+r)e^{s_4b}B_4 = 0, \tag{3.22}$$
$$(\lambda+\delta)(e^{s_1b}s_1B_1 + e^{s_2b}s_2B_2) - (\lambda+\delta+r)(e^{s_3b}s_3B_3 + e^{s_4b}s_4B_4) = 0, \tag{3.23}$$

and

$$l_1e^{s_1b}B_1 + l_2e^{s_2b}B_2 - l_3e^{s_3b}B_3 - l_4e^{s_4b}B_4 = 0, \tag{3.24}$$

where l_i, s_i, $i = 1,2,3,4$ are the same as in Section 3.1; and the undetermined coefficients B_1–B_4 can be easily obtained from (3.21)−(3.24). Then we have

$$\phi_b(u) = \begin{cases} \frac{(s_1+a)(s_2+a)[w_1e^{(s_2b+s_1u)}+w_2e^{(s_1b+s_2u)}]}{a[(s_2+a)w_1e^{s_2b}+(s_1+a)w_2e^{s_1b}]}, & 0 \le u < b, \\ \frac{(s_1+a)(s_2+a)e^{(s_1+s_2)b}[rw_5+w_3e^{(s_4b+s_3u)}-w_4e^{(s_3b+s_4u)}]}{ae^{(s_3+s_4)b}[(s_2+a)w_1e^{s_2b}+(s_1+a)w_2e^{s_1b}]}, & u \ge b. \end{cases}$$

3.3. Expected discounted cumulative dividend function

In this subsection, we provide the explicit expression of $V_b(u),\ u > 0$, by using the same method as in Section 3.1. Similar to (3.19)–(3.24), we have

$$(\lambda+\delta)V_{b1}''(u) + [a(\lambda+\delta) - m(\delta+\lambda+mp)]V_{b1}'(u) - am\delta V_{b1}(u) = 0, \tag{3.25}$$

and

$$\begin{aligned}(\lambda+\delta+r)V_{b2}''(u) + [a(r+\lambda+\delta) - m(r+\delta+\lambda p)]V_{b2}'(u)\\ -am(\delta+r)V_{b2}(u) + amr(u-b+V_{b2}(b+)) = 0,\end{aligned} \tag{3.26}$$

The explicit expressions can be obtained from the following equations:

$$\frac{1}{s_1+a}C_1 + \frac{1}{s_2+a}C_2 = 0, \tag{3.27}$$

$$\delta e^{s_1 b}C_1 + \delta e^{s_2 b}C_2 - (\delta+r)e^{s_3 b}C_3 = \frac{r[a(r+\lambda+\delta) - m(r+\delta+\lambda p)]}{am(r+\delta)}, \tag{3.28}$$

$$(\lambda+\delta)(e^{s_1 b}s_1C_1 + e^{s_2 b}s_2C_2) - (\lambda+\delta+r)e^{s_3 b}s_3 = \frac{r\lambda}{r+\delta}, \tag{3.29}$$

where s_1–s_3 are the same as in Section 3.1; and the undetermined coefficients C_1–C_3 can be easily obtained from (3.27)–(3.29).

Let

$$\begin{aligned}
w_6 &= [a(r+\lambda+\delta) - m(r+\delta+\lambda p)](r+\lambda+\delta)s_3 - \lambda am(r+\delta),\\
w_7 &= (s_2+a)[(\lambda+\delta)(r+\delta)s_2 - \delta(r+\lambda+\delta)s_3]e^{s_2 b} - (s_1+a)[(\lambda+\delta)(r+\delta)s_1 - \delta(r+\lambda+\delta)s_3]e^{s_1 b},\\
w_8 &= \lambda amr\delta(r+\delta)[(s_2+a)e^{s_2 b} + (s_1+a)e^{s_1 b}]\\
&\quad - r(\lambda+\delta)[a(r+\lambda+\delta) - m(r+\delta+\lambda p)][(s_2+a)s_2e^{s_2 b} + (s_1+a)s_1e^{s_1 b}].
\end{aligned}$$

Then we have

$$V_b(u) = \begin{cases} \frac{rw_6[(s_1+a)e^{s_1 u}+(s_2+a)e^{s_2 u}]}{am(r+\delta)w_7}, & 0 \le u < b,\\ \frac{r^2w_6[(s_1+a)e^{s_1 b}+(s_2+a)e^{s_2 b}]+(r+\delta)w_8e^{s_3(u-b)}}{am(r+\delta)^2w_7} + \frac{r[am(r+\delta)(u-b)+a(r+\lambda+\delta)-m(r+\delta+\lambda p)]}{am(r+\delta)^2}, & u \ge b.\end{cases}$$

4. Numerical analysis

In this section, we respectively reveal the impact of various parameters on the Laplace transform of the deficit at ruin, the Laplace transform of the ruin time, and the expected discounted cumulative dividend function. In order to investigate that, in the following analysis, unless otherwise specified, the basic parameter settings are as follows: $\lambda = 1$, $p = 0.2$, $b = 2$, $u = 1$, $r = 0.05$, $a = 1$, $m = 1$, $\delta = 0.05$.

4.1. Laplace transform of the deficit at ruin

In this subsection, we examine the impact of each parameter on the Laplace transform of the deficit at ruin $\xi_b(u)$ to study its sensitivity. According to the sensitivity, we can control the deficit at ruin by adjusting the parameters of different insurance products.

In Figure 1, we respectively present the graphs of the Laplace transform of the deficit at ruin $\xi_b(u)$ for three different values of λ, p, a, m, r, and δ.

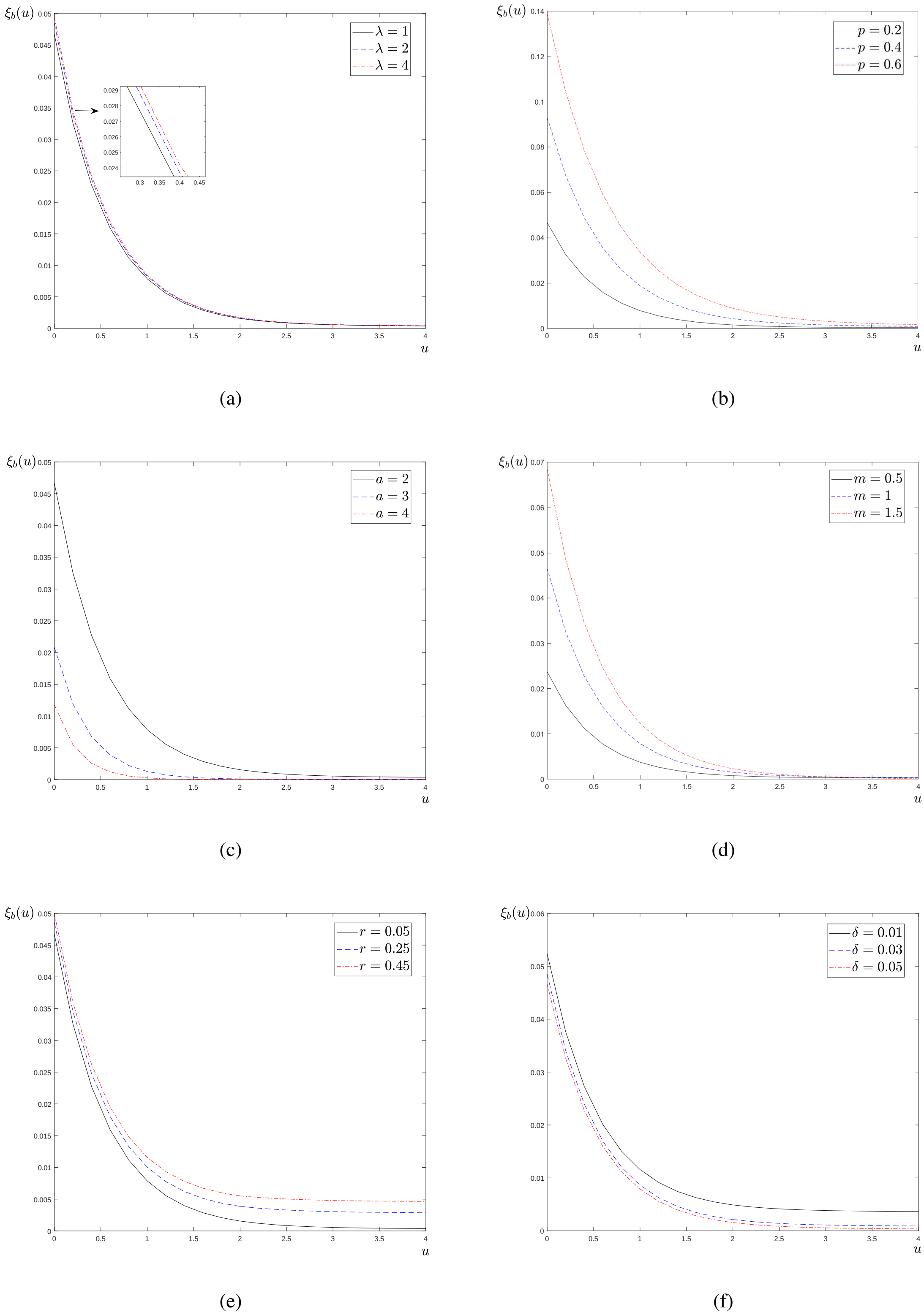


Figure 1. The curves of Laplace transform of the deficit at ruin $\xi_b(u)$ as a function of u when: (a) $\lambda = 1, 2, 4$; (b) $p = 0.2, 0.4, 0.6$; (c) $a = 2, 3, 4$; (d) $m = 0.5, 1, 1.5$; (e) $r = 0.05, 0.25, 0.45$; (f) $\delta = 0.01, 0.03, 0.05$.

From Figure 1, we can see the following conclusions:

(1) Figure 1(a) shows that the Laplace transform of the deficit at ruin $\xi_b(u)$ increases with respect to the Poisson parameter λ of the premium counting process $\{N(t), t > 0\}$. Note that parameter λ denotes the average incidence of random events per unit time. An increase of λ means that the number of premium policies and claims occurred increases in the unit area. Figure 1(a) shows that the more premiums collected per unit of time, the greater the liabilities arising from bankruptcy, which is consistent with our intuitive understanding.

(2) From Figure 1(b) we see that as p increases, $\xi_b(u)$ increases. This phenomenon is because that the number of claims generated increases when p increases. For insurance companies, different values of p can represent different insurance products.

(3) $\xi_b(u)$ is decreasing with respect to a and δ, respectively, see Figure 1(c),(f). The parameter δ is the discount rate, which is easy to understand. An increase in a leads to a decrease in cost for the insurance company, which makes bankruptcy happen earlier.

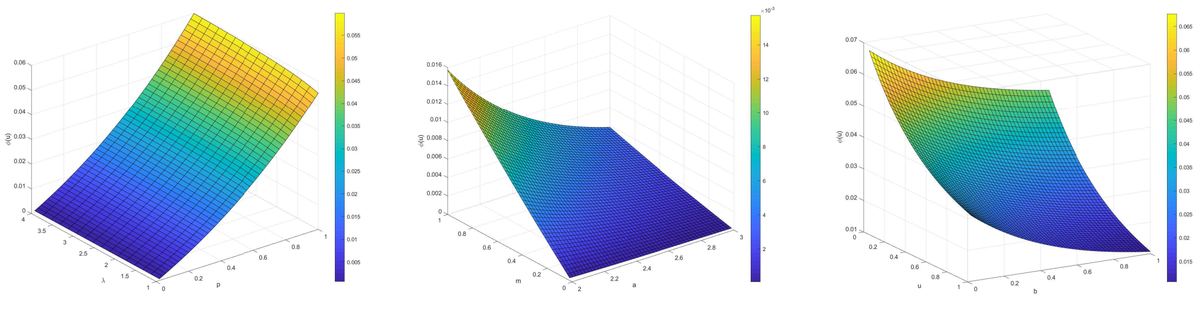
(4) As is seen from Figure 1(d), $\xi_b(u)$ is a monotonic increasing function of m. Increased m causes an increase in the insurance company's revenue, which makes bankruptcy occur later.

(5) Figure 1(e) shows that $\xi_b(u)$ is increasing with respect to r. Dividends are equivalent to the expenses of the insurance company. The larger r is, the more dividends paid out per unit of time, which means the higher costs for insurance companies. We can control the deficit by appropriately reducing the value of r.

4.2. Laplace transform of the ruin time

In this subsection, we depict the effects of various parameters on the Laplace transform of ruin time $\phi_b(u)$. In applications, it is reasonable that the shareholders of the company are interested in $\phi_b(u)$ for that they can avoid ruin by adjusting the values of parameters.

In Figure 2, we present the graphs of the Laplace transform of the ruin time $\phi_b(u)$ as the functions of $(\lambda, p) \in [1, 4] \times (0, 1)$; $(m, a) \in (0, 1] \times [2, 3]$; $(u, b) \in [0, 1] \times [0, 1]$, respectively.



(a) (b) (c)

Figure 2. Laplace transform of the time of ruin $\phi_b(u)$ as functions of (a) λ and p; (b) m and a; (c) u and b.

From these graphs, we obtain some results as follows:

(1) $\phi_b(u)$ is decreasing in u, and b, respectively. The initial capital u has a significant impact on the ruin time: A high initial capital can curb bankruptcy. An increase of b means a reduction in the

payment of dividends per time, that is, a lower cost of insurance companies, which leads to that ruin occurs later.

(2) $\phi_b(u)$ decreases with respect to a and increases with respect to λ, p, and m, respectively. The interpretation is similar to that of the previous one. We omitted here.

4.3. *Expected discounted cumulative dividend function*

In this subsection, we depict the effects of the parameters u and b on the expected discounted cumulative dividend function $V_b(u)$.

From Figure 3(a), we can see that $V_b(u)$ is increasing in the initial capital u, which is obvious and easy to understand. However, from Figure 3(b), it can be seen that $V_b(u)$ is decreasing in the dividend barrier b. For fixed initial capital u, the maximum value is $V_0(u)$, then we have that the optimal barrier $b^* = 0$. We also conclude that the optimal barrier b^* is independent of the initial capital u. From Figure 3(c), we have $V_b(u) = 0$ when $r \to 0$. This verifies the result of Remark 4.

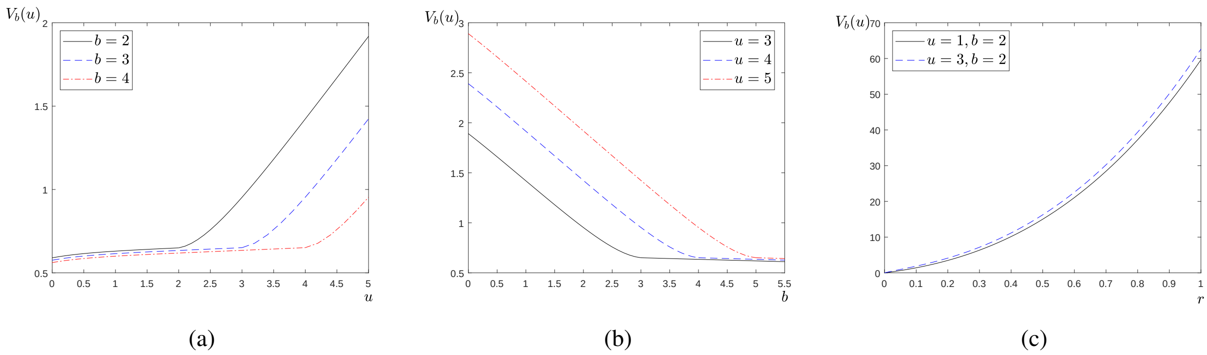


Figure 3. The value of $V_b(u)$ as functions of (a) u when $b = 2,\ 3, 4$ and (b) b when $u = 3,\ 4,\ 5$ and (c) r when $u = 1,\ b = 2$ and $u = 3,\ b = 2$, respectively.

5. Conclusions

In this paper, we consider an improved thinning risk model with a periodic barrier strategy. This improved risk model is of great practical significance since it is much closer to the actual operate model of insurance companies. We examined the expected discounted penalty function $m_b(u)$ and the expected discounted cumulative dividend function $V_b(u)$ under the assumption that inter-dividend-decision times is subject to exponential distribution. Not only the integral equation satisfied by them are obtained, but the explicit expressions for them are derived by means of the integral and differential method when the claim amount and premium sizes are exponentially distributed. Finally, by some numerical analysis, we conclude some results that can be used to risk management of insurance companies. In the end, we find $V_b(u)$ is decreasing in b and that the optimal barrier $b^* = 0$.

For the further research, diffusion could be considered in this thinning model. In addition, we can also consider the inter-dividend-decision times following Erlang(n) distribution.

Acknowledgments

The authors would like to thank to Profs. Yuhua Lv, Chuancun Yin, and Zhanjie Song for their useful suggestions, which improved an earlier version of the paper. The authors thank as well the two anonymous referees and editor for their helpful comments and suggestions. The research was supported by the National Natural Science Foundation of China (No.12071251).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. W. Yu, P. Guo, Q. Wang, G. Guan, Y. Huang, X. Yu, Randomized observation periods for compound Poisson risk model with capital injection and barrier dividend, *Adv. Differ. Equ.*, **220** (2021), 220.
2. W. Yu, P. Guo, Q. Wang, G. Guan, Q. Yang, Y. Huang, et al., On a periodic capital injection and barrier dividend strategy in the compound Poisson risk model, *Mathematics*, **8** (2020), 511.
3. A. V. Boikov, The Cramér-Lundberg model with stochastic premium process, *Theory Probab. Appl.*, **47** (2003), 489–493.
4. C. Labbé, K. P. Sendova, The expected discounted penalty function under a risk model with stochastic income, *Appl. Math. Comput.*, **215** (2009), 1852–1867.
5. T. A. Belkina, N. B. Konyukhova, S. V. Kurochkin, Singular boundary value problem for the integro-differential equation in an insurance model with stochastic premiums: Analysis and numerical solution, *Comput. Math. Math. Phys.*, **52** (2012), 1384–1416.
6. H. Albrecher, O. J. Boxma, A ruin model with dependence between claim sizes and claim intervals, *Insur. Math. Econ.*, **35** (2004), 245–254.
7. G. Wang, K. C Yuen, On a correlated aggregate claims model with thinning-dependence structure, *Insur. Math. Econ.*, **36** (2005), 456–468.
8. J. Pan, G. Wang, Expected discounted penalty function for a thinning risk model, *Chin. J. Appl. Probab. Stat.*, **25** (2009), 544–552.
9. M. Boudreault, H. Cossette, D. Landriault, E. Marceau, On risk model with dependence between interclaim arrivals and claim sizes, *Scand. Actuar. J.*, **5** (2006), 265–285.
10. H. Albrecher, C. Constantinescu, S. Loisel, Explicit ruin formulas for models with dependence among risks, *Insur. Math. Econ.*, **48** (2011), 265–270.
11. M. Vidmar, Ruin under stochastic dependence between premium and claim arrivals, *Scand. Actuar. J.*, **2018** (2018), 505–513.
12. B. De Finetti, Su un'impostazione alternativa della teoria collettiva del rischio, *Transactions of The XVth International Congress of Actuaries*, **2** (1957), 433–443.
13. C. Yin, K. C. Yuen, Optimality of the threshold dividend strategy for the compound Poisson model, *Stat. Probab. Lett.*, **81** (2011), 1841–1846.

14. H. Albrecher, E. C. Cheung, S. Thonhauser, Randomized observation periods for the compound poisson risk model: Dividends, *ASTIN Bull.*, **41** (2011), 645–672.

15. K. Noba, J. L. Pérez, K. Yamazaki, K. Yano, On optimal periodic dividend strategies for Lévy risk processes, *Insur. Math. Econ.*, **80** (2018), 29–44.

16. Z. Zhang, E. C. Cheung, H. Yang, On the compound poisson risk model with periodic capital injections, *ASTIN Bull.*, **48** (2018), 435–477.

17. B. Avanzi, E. C. Cheung, B. Wong, J. K. Woo, On a periodic dividend barrier strategy in the dual model with continuous monitoring of solvency, *Insur. Math. Econ.*, **52** (2013), 98–113.

18. J. Song, F. Sun, The dual risk model under a mixed ratcheting and periodic dividend strategy, *Commun. Stat. Theory Methods*, in press.

19. D. Peng, D. Liu, Z. Liu, Dividend problems in the dual risk model with exponentially distributed observation time, *Stat. Probab. Lett.*, **83** (2013), 841–849.

20. M. C. Choi, E. C. Cheung, On the expected discounted dividends in the Cramér-Lundberg risk model with more frequent ruin monitoring than dividend decisions, *Insur. Math. Econ.*, **59** (2014), 121–132.

21. B. Avanzi, V. Tu, B. Wong, On optimal periodic dividend strategies in the dual model with diffusion, *Insur. Math. Econ.*, **55** (2014), 210–224.

22. B. Avanzi, V. Tu, B. Wong, Optimal dividends under Erlang(2) inter-dividend decision times, *Insur. Math. Econ.*, **79** (2018), 225–242.

23. H. U. Gerber, E. S. Shiu, On the time value of ruin, *N. Am. Actuar. J.*, **2** (1998), 48–72.

24. D. C. Dickson, C. Hipp, On the time to ruin for Erlang(2) risk processes, *Insur. Math. Econ.*, **29** (2001), 333–344.

25. S. Li, J. Garrido, On a general class of renewal risk process: Analysis of the Gerber-Shiu function, *Adv. Appl. Probab.*, **37** (2005), 836–856.

26. E. J. Baurdoux, J. C. Pardo, J. L. Pérez, J. F. Renaud, Gerber-Shiu distribution at Parisian ruin for Lévy insurance risk processes, *J. Appl. Probab.*, **53** (2016), 572–584.

27. E. C. Cheung, D. Landriault, G. E. Willmot, J. K. Woo, Structural properties of Gerber-Shiu functions in dependent Sparre Andersen models, *Insur. Math. Econ.*, **46** (2010), 117–126.

28. S. Li, Y. Lu, K. P. Sendova, The expected discounted penalty function: From infinite time to finite time, *Scand. Actuar. J.*, **2019** (2019), 336–354.

29. W. Wang, Z. Zhang, Computing the Gerber-Shiu function by frame duality projection, *Scand. Actuar. J.*, **219** (2019), 291–307.

30. L. Zhang, The Erlang(n) risk model with two-sided jumps and a constant dividend barrier, *Commun. Stat. Theory Methods*, **2020** (2020), 1–19.