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Research article

Prominent interior GE-filters of GE-algebras

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Abstract: The concept of a prominent interior GE-filter (of type 1 and type 2) is introduced, and their properties are investigated. The relationship between a prominent GE-filter and a prominent interior GE-filter and the relationship between an interior GE-filter and a prominent interior GE-filter are discussed. Examples to show that any interior GE-filter is not a prominent interior GE-filter and any prominent GE-filter is not a prominent interior GE-filter are provided. Conditions for an interior GE-filter to be a prominent interior GE-filter are given. Also, conditions under which an internal GE-filter larger than a given internal GE filter can become a prominent internal GE-filter are considered, and an example describing it is given. The relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1 is discussed.

Keywords: (transitive) GE-algebra; GE-filter; interior GE-filter; prominent interior GE-filter (of type 1 and type 2)
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1. Introduction

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [4] proved that Hilbert algebras form a variety which is locally finite. Bandaru et al. introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1, 2, 7–9]). The notion of interior operator is introduced by Vorster [12] in an arbitrary category, and it is used in [3] to study the notions of connectedness and disconnectedness in topology. Interior algebras are a certain type of algebraic structure that encodes the idea of the topological interior of a set, and are a generalization of topological spaces defined by means of topological interior operators. Rachůnek and Svoboda [6] studied interior operators on bounded residuated lattices, and Svrcek [11] studied multiplicative interior operators on GMV-algebras. Lee et al. [5] applied the interior operator theory to GE-algebras, and they introduced the concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras, and investigated their relations and properties. Later, Song et al. [10] introduced the notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter, and investigate their relations and properties. They provided relations between a belligerent interior GE-filter and an interior GE-filter and conditions for an interior GE-filter to be a belligerent interior GE-filter is considered. Given a subset and an element, they established an interior GE-filter, and they considered conditions for a subset to be a belligerent interior GE-filter. They studied the extensibility of the belligerent interior GE-filter and established relationships between weak interior GE-filter and belligerent interior GE-filter of type 1, type 2 and type 3. Rezaei et al. [7] studied prominent GE-filters in GE-algebras. The purpose of this paper is to study by applying interior operator theory to prominent GE-filters in GE-algebras. We introduce the concept of a prominent interior GE-filter, and investigate their properties. We discuss the relationship between a prominent GE-filter and a prominent interior GE-filter and the relationship between an interior GE-filter and a prominent interior GE-filter. We find and provide examples where any interior GE-filter is not a prominent interior GE-filter and any prominent GE-filter is not a prominent interior GE-filter. We provide conditions for an interior GE-filter to be a prominent interior GE-filter. We provide conditions under which an internal GE-filter larger than a given internal GE filter can become a prominent internal GE-filter, and give an example describing it. We also introduce the concept of a prominent interior GE-filter of type 1 and type 2, and investigate their properties. We discuss the relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1. We give examples to show that A and B are independent of each other, where A and B are:

(1) $\begin{cases} \text{A: prominent interior GE-filter of type 1.} \\ \text{B: prominent interior GE-filter of type 2.} \end{cases}$

(2) $\begin{cases} \text{A: prominent interior GE-filter.} \\ \text{B: prominent interior GE-filter of type 2.} \end{cases}$

(3) $\begin{cases} \text{A: interior GE-filter.} \\ \text{B: prominent interior GE-filter of type 1.} \end{cases}$

(4) $\begin{cases} \text{A: interior GE-filter.} \\ \text{B: prominent interior GE-filter of type 2.} \end{cases}$

2. Preliminaries

Definition 2.1. [1] By a *GE-algebra* we mean a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

(GE1) $u * u = 1$,

(GE2) $1 * u = u$,

(GE3) $u * (v * w) = u * (v * (u * w))$

for all $u, v, w \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall x, y \in X)\,(x \leq y \;\Leftrightarrow\; x * y = 1)\,. \tag{2.1}$$

Definition 2.2. [1,2,8] A GE-algebra X is said to be transitive if it satisfies:

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{2.2}$$

Proposition 2.3. *[1] Every GE-algebra X satisfies the following items:*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2.3}$$
$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{2.4}$$
$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{2.5}$$
$$(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,. \tag{2.6}$$
$$(\forall u \in X)\,(1 \leq u \;\Rightarrow\; u = 1)\,. \tag{2.7}$$
$$(\forall u, v \in X)\,(u \leq (v * u) * u)\,. \tag{2.8}$$
$$(\forall u, v \in X)\,(u \leq (u * v) * v)\,. \tag{2.9}$$
$$(\forall u, v, w \in X)\,(u \leq v * w \;\Leftrightarrow\; v \leq u * w)\,. \tag{2.10}$$

If X is transitive, then

$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; w * u \leq w * v,\; v * w \leq u * w)\,. \tag{2.11}$$
$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,. \tag{2.12}$$

Lemma 2.4. *[1] In a GE-algebra X, the following facts are equivalent each other.*

$$(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))\,. \tag{2.13}$$
$$(\forall x, y, z \in X)\,(x * y \leq (y * z) * (x * z))\,. \tag{2.14}$$

Definition 2.5. [1] A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

$$1 \in F, \tag{2.15}$$
$$(\forall x, y \in X)(x * y \in F,\; x \in F \;\Rightarrow\; y \in F). \tag{2.16}$$

Lemma 2.6. *[1] In a GE-algebra X, every filter F of X satisfies:*

$$(\forall x, y \in X)\,(x \leq y,\; x \in F \;\Rightarrow\; y \in F)\,. \tag{2.17}$$

Definition 2.7. [7] A subset F of a GE-algebra X is called a *prominent GE-filter* of X if it satisfies (2.15) and

$$(\forall x, y, z \in X)(x * (y * z) \in F,\; x \in F \;\Rightarrow\; ((z * y) * y) * z \in F). \tag{2.18}$$

Note that every prominent GE-filter is a GE-filter in a GE-algebra (see [7]).

Definition 2.8. [5] By an *interior GE-algebra* we mean a pair (X, f) in which X is a GE-algebra and $f : X \to X$ is a mapping such that

$$(\forall x \in X)(x \le f(x)), \tag{2.19}$$

$$(\forall x \in X)((f \circ f)(x) = f(x)), \tag{2.20}$$

$$(\forall x, y \in X)(x \le y \;\Rightarrow\; f(x) \le f(y)). \tag{2.21}$$

Definition 2.9. [10] Let (X, f) be an interior GE-algebra. A GE-filter F of X is said to be interior if it satisfies:

$$(\forall x \in X)(f(x) \in F \;\Rightarrow\; x \in F). \tag{2.22}$$

3. Prominent interior GE-filters

Definition 3.1. Let (X, f) be an interior GE-algebra. Then a subset F of X is called a prominent interior GE-filter in (X, f) if F is a prominent GE-filter of X which satisfies the condition (2.22).

Example 3.2. Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	4	4
3	1	1	1	5	5
4	1	2	3	1	1
5	1	2	2	1	1

Then X is a GE-algebra. If we define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, 4, 5\}, \\ 2 & \text{if } x \in \{2, 3\}, \end{cases}$$

then (X, f) is an interior GE-algebra and $F = \{1, 4, 5\}$ is a prominent interior GE-filter in (X, f).

It is clear that every prominent interior GE-filter is a prominent GE-filter. But any prominent GE-filter may not be a prominent interior GE-filter in an interior GE-algebra as seen in the following example.

Example 3.3. Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 2,

Table 2. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	1
3	1	2	1	4	5
4	1	2	3	1	5
5	1	1	3	4	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, 2, 3, 5\}, \\ 4 & \text{if } x = 4. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1\}$ is a prominent GE-filter of X. But it is not a prominent interior GE-filter in (X, f) since $f(2) = 1 \in F$ but $2 \notin F$.

We discuss relationship between interior GE-filter and prominent interior GE-filter.

Theorem 3.4. *In an interior GE-algebra, every prominent interior GE-filter is an interior GE-filter.*

Proof. It is straightforward because every prominent GE-filter is a GE-filter in a GE-algebra. □

In the next example, we can see that any interior GE-filter is not a prominent interior GE-filter in general.

Example 3.5. Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 3.

Table 3. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	4	4
3	1	2	1	4	4
4	1	1	3	1	1
5	1	1	1	1	1

Then X is a GE-algebra. If we define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x \in \{2, 4, 5\}, \\ 3 & \text{if } x = 3, \end{cases}$$

then (X, f) is an interior GE-algebra and $F = \{1\}$ is an interior GE-filter in (X, f). But it is not a prominent interior GE-filter in (X, f) since $1 * (2 * 3) = 1 \in F$ but $((3 * 2) * 2) * 3 = 3 \notin F$.

Proposition 3.6. *Every prominent interior GE-filter F in an interior GE-algebra (X, f) satisfies:*

$$(\forall x, y \in X)\,(f(x * y) \in F \;\Rightarrow ((y * x) * x) * y \in F)\,. \tag{3.1}$$

Proof. Let F be a prominent interior GE-filter in (X, f). Let $x, y \in X$ be such that $f(x * y) \in F$. Then $x * y \in F$ by (2.22), and so $1 * (x * y) = x * y \in F$ by (GE2). Since $1 \in F$, it follows from (2.18) that $((y * x) * x) * y \in F$. □

Corollary 3.7. *Every prominent interior GE-filter F in an interior GE-algebra (X, f) satisfies:*

$$(\forall x, y \in X)\,(x * y \in F \;\Rightarrow ((y * x) * x) * y \in F)\,. \tag{3.2}$$

Proof. Let F be a prominent interior GE-filter in (X, f). Then F is an interior GE-filter in (X, f) by Theorem 3.4. Let $x, y \in X$ be such that $x * y \in F$. Since $x * y \leq f(x * y)$ by (2.19), it follows from Lemma 2.6 that $f(x * y) \in F$. Hence $((y * x) * x) * y \in F$ by Proposition 3.6. □

Corollary 3.8. *Every prominent interior GE-filter F in an interior GE-algebra* (X, f) *satisfies:*

$$(\forall x, y \in X)\,(x * y \in F \;\Rightarrow\; f(((y * x) * x) * y) \in F)\,.$$

Proof. Straightforward. □

Corollary 3.9. *Every prominent interior GE-filter F in an interior GE-algebra* (X, f) *satisfies:*

$$(\forall x, y \in X)\,(f(x * y) \in F \;\Rightarrow\; f(((y * x) * x) * y) \in F)\,.$$

Proof. Straightforward. □

In the following example, we can see that any interior GE-filter F in an interior GE-algebra (X, f) does not satisfy the conditions (3.1) and (3.2).

Example 3.10. Consider the interior GE-algebra (X, f) in Example 3.5. The interior GE-filter $F := \{1\}$ does not satisfy conditions (3.1) and (3.2) since $f(2 * 3) = f(1) = 1 \in F$ and $2 * 3 = 1 \in F$ but $((3 * 2) * 2) * 3 = 3 \notin F$.

We provide conditions for an interior GE-filter to be a prominent interior GE-filter.

Theorem 3.11. *If an interior GE-filter F in an interior GE-algebra* (X, f) *satisfies the condition* (3.1)*, then F is a prominent interior GE-filter in* (X, f)*.*

Proof. Let F be an interior GE-filter in (X, f) that satisfies the condition (3.1). Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x \in F$. Then $y * z \in F$. Since $y * z \leq f(y * z)$ by (2.19), it follows from Lemma 2.6 that $f(y * z) \in F$. Hence $((z * y) * y) * z \in F$ by (3.1), and therefore F is a prominent interior GE-filter in (X, f). □

Lemma 3.12. *[10] In an interior GE-algebra, the intersection of interior GE-filters is also an interior GE-filter.*

Theorem 3.13. *In an interior GE-algebra, the intersection of prominent interior GE-filters is also a prominent interior GE-filter.*

Proof. Let $\{F_i \mid i \in \Lambda\}$ be a set of prominent interior GE-filters in an interior GE-algebra (X, f) where Λ is an index set. Then $\{F_i \mid i \in \Lambda\}$ is a set of interior GE-filters in (X, f), and so $\cap\{F_i \mid i \in \Lambda\}$ is an interior GE-filter in (X, f) by Lemma 3.12. Let $x, y \in X$ be such that $f(x * y) \in \cap\{F_i \mid i \in \Lambda\}$. Then $f(x * y) \in F_i$ for all $i \in \Lambda$. It follows from Proposition 3.6 that $((y * x) * x) * y \in F_i$ for all $i \in \Lambda$. Hence $((y * x) * x) * y \in \cap\{F_i \mid i \in \Lambda\}$ and therefore $\cap\{F_i \mid i \in \Lambda\}$ is a prominent interior GE-filter in (X, f) by Theorem 3.11. □

Theorem 3.14. *If an interior GE-filter F in an interior GE-algebra* (X, f) *satisfies the condition* (3.2)*, then F is a prominent interior GE-filter in* (X, f)*.*

Proof. Let F be an interior GE-filter in (X, f) that satisfies the condition (3.2). Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x \in F$. Then $y * z \in F$ and thus $((z * y) * y) * z \in F$. Therefore F is a prominent interior GE-filter in (X, f). □

Given an interior GE-filter F in an interior GE-algebra (X, f), we consider an interior GE-filter G which is greater than F in (X, f), that is, we take two interior GE-filters F and G such that $F \subseteq G$ in an interior GE-algebra (X, f). We are now trying to find the condition that G can be a prominent interior GE-filter in (X, f).

Theorem 3.15. *Let (X, f) be an interior GE-algebra in which X is transitive. Let F and G be interior GE-filters in (X, f). If $F \subseteq G$ and F is a prominent interior GE-filter in (X, f), then G is also a prominent interior GE-filter in (X, f).*

Proof. Assume that F is a prominent interior GE-filter in (X, f). Then it is an interior GE-filter in (X, f) by Theorem 3.4. Let $x, y \in X$ be such that $f(x * y) \in G$. Then $x * y \in G$ by (2.22), and so $1 = (x * y) * (x * y) \leq x * ((x * y) * y)$ by (GE1) and (2.6). Since $1 \in F$, it follows from Lemma 2.6 that $x * ((x * y) * y) \in F$. Hence $((((x * y) * y) * x) * x) * ((x * y) * y) \in F \subseteq G$ by Corollary 3.7. Since

$$((((x * y) * y) * x) * x) * ((x * y) * y) \leq (x * y) * (((((x * y) * y) * x) * x) * y)$$

by (2.6), we have $(x * y) * (((((x * y) * y) * x) * x) * y) \in G$ by Lemma 2.6. Hence

$$((((x * y) * y) * x) * x) * y \in G.$$

Since $y \leq (x * y) * y$, it follows from (2.11) that

$$((((x * y) * y) * x) * x) * y \leq ((y * x) * x) * y.$$

Thus $((y * x) * x) * y \in G$ by Lemma 2.6. Therefore G is a prominent interior GE-filter in (X, f). by Theorem 3.11. □

The following example describes Theorem 3.15.

Example 3.16. Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 4,

Table 4. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	5	5
3	1	1	1	5	5
4	1	3	3	1	1
5	1	3	3	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 3 & \text{if } x \in \{2, 3\}, \\ 5 & \text{if } x \in \{4, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra in which X is transitive, and $F := \{1\}$ and $G := \{1, 4, 5\}$ are interior GE-filters in (X, f) with $F \subseteq G$. Also we can observe that F and G are prominent interior GE-filters in (X, f).

In Theorem 3.15, if F is an interior GE-filter which is not prominent, then G is also not a prominent interior GE-filter in (X, f) as shown in the next example.

Example 3.17. Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 5,

Table 5. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	4	1
3	1	5	1	4	5
4	1	1	1	1	1
5	1	1	1	4	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 3 & \text{if } x = 3, \\ 4 & \text{if } x = 4, \\ 2 & \text{if } x \in \{2, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra in which X is transitive, and $F := \{1\}$ and $G := \{1, 3\}$ are interior GE-filters in (X, f) with $F \subseteq G$. We can observe that F and G are not prominent interior GE-filters in (X, f) since $2 * 3 = 1 \in F$ but $((3 * 2) * 2) * 3 = (5 * 2) * 3 = 1 * 3 = 3 \notin F$, and $4 * 2 = 1 \in G$ but $((2 * 4) * 4) * 2 = (4 * 4) * 2 = 1 * 2 = 2 \notin G$.

In Theorem 3.15, if X is not transitive, then Theorem 3.15 is false as seen in the following example.

Example 3.18. Let $X = \{1, 2, 3, 4, 5, 6\}$ be a set with the Cayley table which is given in Table 6.

Table 6. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	1	6	6	6
3	1	1	1	5	5	5
4	1	1	3	1	1	1
5	1	2	3	2	1	1
6	1	2	3	2	1	1

If we define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 4 & \text{if } x = 4, \\ 5 & \text{if } x = 5, \\ 6 & \text{if } x = 6, \\ 2 & \text{if } x \in \{2, 3\}, \end{cases}$$

then (X, f) is an interior GE-algebra in which X is not transitive. Let $F := \{1\}$ and $G := \{1, 5, 6\}$. Then F is a prominent interior GE-filter in (X, f) and G is an interior GE-filter in (X, f) with $F \subseteq G$. But G is not prominent interior GE-filter since $5 * (3 * 4) = 5 * 5 = 1 \in G$ and $5 \in G$ but $((4 * 3) * 3) * 4 = (3 * 3) * 4 = 1 * 4 = 4 \notin G$.

Definition 3.19. Let (X, f) be an interior GE-algebra and let F be a subset of X which satisfies (2.15). Then F is called:

- A *prominent interior GE-filter of type 1* in (X, f) if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * f(z)) \in F,\ f(x) \in F \ \Rightarrow ((f(z) * y) * y) * f(z) \in F)\,. \tag{3.3}$$

- A *prominent interior GE-filter of type 2* in (X, f) if it satisfies:

$$(\forall x, y, z \in X)\,(x * (y * f(z)) \in F,\ f(x) \in F \ \Rightarrow ((z * f(y)) * f(y)) * z \in F)\,. \tag{3.4}$$

Example 3.20. (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 7,

Table 7. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	2	1	2	2
4	1	1	1	1	1
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, 3\} \\ 2 & \text{if } x = 2, \\ 4 & \text{if } x = 4, \\ 5 & \text{if } x = 5. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, 3\}$ is a prominent interior GE-filter of type 1 in (X, f).

(2). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 8,

Table 8. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	1	1	4	1
4	1	1	1	1	5
5	1	1	3	4	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x \in \{2, 3, 4, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, 3\}$ is a prominent interior GE-filter of type 2 in (X, f).

Theorem 3.21. *In an interior GE-algebra, every prominent interior GE-filter is of type 1.*

Proof. Let F be a prominent interior GE-filter in an interior GE-algebra (X, f). Let $x, y, z \in X$ be such that $x * (y * f(z)) \in F$ and $f(x) \in F$. Then $x \in F$ by (2.22). It follows from (2.18) that $((f(z) * y) * y) * f(z) \in F$. Hence F is a prominent interior GE-filter of type 1 in (X, f). □

The following example shows that the converse of Theorem 3.21 may not be true.

Example 3.22. Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 9,

Table 9. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	1	1	1	5
4	1	1	3	1	1
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x \in \{2, 3\}, \\ 5 & \text{if } x \in \{4, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1\}$ is a prominent interior GE-filter of type 1 in (X, f). But it is not a prominent interior GE-filter in (X, f) since $1 * (3 * 4) = 1 \in F$ but $(4 * 3) * 3) * 4 = 4 \notin F$.

The following example shows that prominent interior GE-filter and prominent interior GE-filter of type 2 are independent of each other, i.e., prominent interior GE-filter is not prominent interior GE-filter of type 2 and neither is the inverse.

Example 3.23. (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 10,

Table 10. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	5	1	1	5
4	1	1	1	1	1
5	1	3	3	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 4 & \text{if } x \in \{3, 4\} \\ 5 & \text{if } x \in \{2, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1\}$ F is a prominent interior GE-filter in (X, f). But it is not a prominent interior GE-filter of type 2 since $1 * (5 * f(2)) = 5 * 5 = 1 \in F$ and $f(1) = 1 \in F$ but $((2 * f(5)) * f(5)) * 2 = ((2 * 5) * 5) * 2 = (1 * 5) * 2 = 5 * 2 = 3 \notin F$.

(2). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 11,

Table 11. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	2	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 5 & \text{if } x \in \{2, 3, 4, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1\}$ is a prominent interior GE-filter of type2 in (X, f). But it is not a prominent interior GE-filter in (X, f) since $1 * (2 * 3) = 1 * 1 = 1 \in F$ and $1 \in F$ but $((3 * 2) * 2) * 3 = (2 * 2) * 3 = 1 * 3 = 3 \notin F$.

The following example shows that prominent interior GE-filter of type 1 and prominent interior GE-filter of type 2 are independent of each other.

Example 3.24. (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 12,

Table 12. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	5	5
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 3 & \text{if } x \in \{2, 3\}, \\ 5 & \text{if } x \in \{4, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, 2, 4\}$ is a prominent interior GE-filter of type 1 in (X, f). But it is not a prominent interior GE-filter of type 2 since $1*(5*f(2)) = 1*(5*3) = 1*1 = 1 \in F$ and $f(1) = 1 \in F$ but $((2 * f(5)) * f(5)) * 2 = ((2 * 5) * 5) * 2 = (5 * 5) * 2 = 1 * 2 = 2 \notin F$.

(2). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 13,

Table 13. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	4	4	5
3	1	1	1	1	1
4	1	2	2	1	5
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x = 2, \\ 4 & \text{if } x = 4, \\ 3 & \text{if } x \in \{3, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1\}$ is a prominent interior GE-filter of type 2 in (X, f). But it is not a prominent interior GE-filter of type 1 in (X, f) since $1*(5*f(2)) = 1*(5*2) = 1*1 = 1 \in F$ and $f(1) \in F$ but $((f(2) * 5) * 5) * f(2) = ((2 * 5) * 5) * 2 = (5 * 5) * 2 = 1 * 2 = 2 \notin F$.

The following example shows that interior GE-filter and prominent interior GE-filter of type 1 are independent of each other.

Example 3.25. (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 14,

Table 14. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	5	5	5
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x = 2, \\ 5 & \text{if } x \in \{3, 4, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1\}$ is an interior GE-filter in (X, f). But F is not prominent interior GE-filter of type 1 since $1 * (5 * f(2)) = 1 * (5 * 2) = 1 * 1 = 1 \in F$ and $f(1) = 1 \in F$ but $((f(2) * 5) * 5) * 2 = ((2 * 5) * 5) * 2 = (5 * 5) * 2 = 1 * 2 = 2 \notin F$.

(2). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 15,

Table 15. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	5	1	5
3	1	2	1	1	1
4	1	1	3	1	5
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, 2, 4\}, \\ 5 & \text{if } x \in \{3, 5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, 2\}$ is a prominent interior GE-filter of type 1 in (X, f). But it is not an interior GE-filter in (X, f) since $2 * 4 = 1$ and $2 \in F$ but $4 \notin F$.

The following example shows that interior GE-filter and prominent interior GE-filter of type 2 are independent of each other.

Example 3.26. (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 16,

Table 16. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	2	1	1	2
4	1	2	3	1	5
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1,4\} \\ 2 & \text{if } x = 2, \\ 3 & \text{if } x = 3, \\ 5 & \text{if } x = 5. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, 4\}$ is an interior GE-filter in (X, f). But F is not prominent interior GE-filter of type 2 since $4 * (2 * f(3)) = 4 * (2 * 3) = 4 * 1 = 1 \in F$ and $f(4) = 1 \in F$ but $((3 * f(2)) * f(2)) * 3 = ((3 * 2) * 2) * 3 = (2 * 2) * 3 = 1 * 3 = 3 \notin F$.

(2). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 17,

Table 17. Cayley table for the binary operation "$*$".

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	5
3	1	1	1	1	1
4	1	1	1	1	5
5	1	1	1	1	1

and define a mapping f on X as follows:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 3 & \text{if } x \in \{2,3,4,5\}. \end{cases}$$

Then (X, f) is an interior GE-algebra and $F := \{1, 2, 5\}$ is a prominent interior GE-filter of type 2 in (X, f). But it is not an interior GE-filter in (X, f) since $5 * 4 = 1$ and $5 \in F$ but $4 \notin F$.

Before we conclude this paper, we raise the following question.

Question. *Let (X, f) be an interior GE-algebra. Let F and G be interior GE-filters in (X, f). If $F \subseteq G$ and F is a prominent interior GE-filter of type 1 (resp., type 2) in (X, f), then is G also a prominent interior GE-filter of type 1 (resp., type 2) in (X, f)?*

4. Conclusions

We have introduced the concept of a prominent interior GE-filter (of type 1 and type 2), and have investigated their properties. We have discussed the relationship between a prominent GE-filter and a prominent interior GE-filter and the relationship between an interior GE-filter and a prominent interior GE-filter. We have found and provide examples where any interior GE-filter is not a prominent interior GE-filter and any prominent GE-filter is not a prominent interior GE-filter. We have provided conditions for an interior GE-filter to be a prominent interior GE-filter. We have given conditions under which an internal GE-filter larger than a given internal GE filter can become a prominent internal GE-filter, and have provided an example describing it. We have considered the relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1. We have found and provide examples to verify that a prominent interior GE-filter of type 1 and a prominent interior GE-filter of type 2, a prominent interior GE-filter and a prominent interior GE-filter of type 2, an interior GE-filter and a prominent interior GE-filter of type 1, and an interior GE-filter and a prominent interior GE-filter of type 2 are independent each other. In future, we will study the prime and maximal prominent interior GE-filters and their topological properties. Moreover, based on the ideas and results obtained in this paper, we will study the interior operator theory in related algebraic systems such as MV-algebra, BL-algebra, EQ-algebra, etc. It will also be used for pseudo algebra systems and further to conduct research related to the very true operator theory.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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