Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13423–13431.
DOI: 10.3934/math.2021777
Received: 07 June 2021
Accepted: 10 September 2021
Published: 17 September 2021

*Research article*

# Time decay rates for the coupled modified Navier-Stokes and Maxwell equations on a half space

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**Abstract:** This paper is concerned with time decay rates of the strong solutions of an incompressible the coupled modified Navier-Stokes and Maxwell equations in a half space $\mathbb{R}^3_+$. With the use of the spectral decomposition of the Stokes operator and $L^p - L^q$ estimates developed by Borchers and Miyakawa [2], we study the $L^2$-decay rate of strong solutions.

**Keywords:** non-Newtonian fluid; MHD equations; decay rate
**Mathematics Subject Classification:** 35Q30, 76A05, 35B35

## 1. Introduction

In this paper, we study the non-Newtonian fluids associated with Maxwell equations:

$$\begin{cases} u_t - \nabla \cdot S(Du) + (u \cdot \nabla)u + \nabla \pi - (b \cdot \nabla)b = 0, \\ b_t - \triangle b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0. \\ \operatorname{div} u = 0 \text{ and } \operatorname{div} b = 0, \end{cases} \quad \text{in } Q_T := \mathbb{R}^3_+ \times (0, T), \tag{1.1}$$

Here $u : Q_T \times (0, T) \to \mathbb{R}^3$ is the flow velocity vector, $b : Q_T \times (0, T) \to \mathbb{R}^3$ is the magnetic vector, $\pi : Q_T \times (0, T) \to \mathbb{R}$ is the total pressure and $Du$ is the symmetric part of the velocity gradient, i.e.

$$Du = D_{ij}u := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \ i, j = 1, 2, 3.$$

To motivate the conditions on the stress tensor $S$, we recall the following examples of constitutive laws

$$S(Du) = (\mu_0 + \mu_1|Du|^{p-2})Du, \tag{1.2}$$

where $\mu_0 \geq 0$ and $\mu_1 > 0$ are constants (see e.g. [1, 16]). We consider the initial-boundary value problem of (1.1), which requires initial conditions

$$u(x, 0) = u_0(x) \quad \text{and} \quad b(x, 0) = b_0(x) \qquad x \in \mathbb{R}^3_+, \tag{1.3}$$

together with the boundary conditions defined as follows:

$$u = 0 \qquad \text{and} \qquad b \cdot n = 0,\ (\nabla \times b) \times n = 0, \tag{1.4}$$

where $n$ is the outward unit normal vector along boundary $\partial\mathbb{R}^3_+$.

The fluids such as coal-water, faint, soaps, etc., is not followed linear relationship between rate of strain and shear stress. In languages of mathematics, commonly, the standard Navier-Stokes equations refers to the equations of motion of an incompressible fluid type of $S = \mu_0 Du$, where $\mu_0 > 0$ is constant. On the other hand, one class of non-Newtonian fluids is defined by $S = \mu(|Du|)Du$ ($\mu(\cdot)$ a positive nonlinear function). For example, we note that

$$S(Du) = (\mu_0 + \mu_1 |Du|^{q-2})Du, \quad 1 < q < \infty, \quad \mu_0 > 0,\ \mu_1 > 0. \tag{1.5}$$

For the existence of solutions to the model with the stress tensor (1.5), we make a brief comment. After the pioneering study done by Ladyzhenskaya [10], the global-in-time existence of strong solutions is proved for $q > \frac{11}{5}$ in [11]. On the other hand, they also established the small data global-in-time existence of strong solution for $\frac{5}{3} < q < \infty$ in three dimensional space (see also [3] for weak solutions)

The Eqs (1.1), which are the generalized incompressible magnetohydrodynamics equations is regarded as one of the simplest model describing the dynamics of electrically conducting liquid with involved rheological structure in a magnetic field. For the model (1.1), Gunzburger et al. in [6] considered (1.1)–(1.4) for the case of bounded or periodic domains, and they established unique solvability of the initial-boundary value problem. More specifically, assuming that $u_0 \in H^2(\Omega)$ and $b_0 \in H^1(\Omega)$ with the following boundary conditions (1.4) for a bounded domain, it was shown in [6] that if $\frac{5}{2} < q \leq 6$, a generalized solution exists, and moreover, it satisfies

$$\begin{gathered} u \in L^\infty(0, T; L^2(\Omega) \cap H^1(\Omega)), \qquad \nabla u \in L^\infty(0, T; L^q(\Omega)), \\ b \in L^\infty(0, T; L^2(\Omega) \cap H^1(\Omega)), \qquad \nabla b \in L^\infty(0, T; L^2(\Omega)), \qquad b \in L^2(0, T; H^2(\Omega)). \\ u_t \in L^2(\Omega \times (0, T)), \qquad b_t \in L^2(\Omega \times (0, T)). \end{gathered} \tag{1.6}$$

Furthermore, they shown the uniqueness of solutions. Here strong solutions means that solutions satisfy (1.1) pointwise a.e. and the energy equality holds. Recently, the authors in [8] establish global unique solvability to (1.1)–(1.4) for $u_0 \in (W^{1,2} \cap W^{1,p})$ and $b_0 \in W^{1,2}$, $\frac{5}{2} \leq p$ in the same class above (see [12] for weak solutions). For a half space, the proof in [6] is also held. And thus, we will not comment further on the existence and uniqueness of strong solutions to (1.1) and (1.2).

For the asymptotic behavior of strong solutions to (1.1) and (1.2), the author in [9] recently examined the $L^2$-algebraic decay in the whole space $\mathbb{R}^3$ with respect to the monopolar shear thickening fluids using Fourier splitting method in [13]. Precisely, he shown that

$$\|(u, b)(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{4}}, \quad \forall t > 0.$$

We also refer to [5, 7] for Navier-Stokes equations of non-Newtonian type.

On the other hand, in the case of a half space $\mathbb{R}^n_+$, Dong and Chen [4] obtain that the weak solution $u(t)$ of the Naiver-Stokes equations of non-Newtonian type, that is $b \equiv 0$ in (1.1) enjoys the optimal algebraic decay estimates

$$\begin{cases} \|u(t)\|^2_{L^2} \leq c(1 + t)^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2})}, \quad \text{if} \quad 1 \leq r < \infty, \\ \|(u, b)(t)\|^2_{L^2} \leq c(1 + t)^{-\frac{3}{2}(\frac{1}{r} - \frac{1}{2}) - \frac{1}{2}}, \quad \displaystyle\int_{\mathbb{R}^3_+} |x_3 u_0(x)|^r\, dx < \infty, \quad \text{if} \quad 1 \leq r < 2. \end{cases} \tag{1.7}$$

by using the $L^p - L^q$ estimates based on the explicit solution formula to the Stokes equation given by Ukai [15] and the spectral decomposition method and fractional powers of the Stokes operator derived by Borchers and Miyakawa [2].

For a domain, we briefly some comments. For a whole space, we obtain the same result like as Theorem 1.1 by the fourier splitting method (see [8]). However, in $\mathbb{R}^3_+$ case with boundary conditions, we needs to be handled the solution form due to the boundary effects. To effectively deal with boundary effect, we use the well-known spectral method, we would like to obtain temporal rate of the strong solutions. So far, there are few known results on the time decay problem to (1.1) in $\mathbb{R}^3_+$. In this direction, our main results are as follows:

**Theorem 1.1.** *Suppose that $(u, b)$ is a strong solution of* (1.1)–(1.4) *with $p \geq 5/2$. Then*

*(A) $lim_{t\to\infty}\|(u,b)(t)\|_{L^2} = 0$, whenever $u_0, b_0 \in L^2_\sigma(\mathbb{R}^3_+)$,*
*(B) $\|(u,b)(t)\|_{L^2} \leq c(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})}$, whenever $u_0, b_0 \in (L^2_\sigma \cap L^r)(\mathbb{R}^3_+)$ $(1 \leq r < 2)$.*

**Theorem 1.2.** *Suppose that $(u, b)$ is a strong solution of* (1.1)–(1.4) *with $p \geq 5/2$. under $u_0, b_0 \in (L^2_\sigma \cap L^r)(\mathbb{R}^3_+)$ for $1 < r \leq 2$*

$$\int_{\mathbb{R}^3_+} |x_3 u_0(x)|^r + |x_3 b_0(x)|^r \, dx < \infty. \tag{1.8}$$

*Then we have*

$$\|(u,b)(t)\|_{L^2} \leq c(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\frac{1}{2}}.$$

**Corollary 1.3.** *Suppose that u is a strong solution of Naiver-Stokes equations of non-Newtonian type with $p \geq 11/5$, namely $b = 0$ in* (1.1)–(1.4). *Then*

*(A) $lim_{t\to\infty}\|u(t)\|_{L^2} = 0$, whenever $u_0 \in L^2_\sigma$,*
*(B) $\|u(t)\|_{L^2} \leq c(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})}$, whenever $u_0 \in L^2_\sigma \cap L^r$ $(1 \leq r < 2)$.*

**Remark 1.4.** *Comparing to [4], since the result of Corollary 1.3 is about the time decay rate for strong solution to Naiver-Stokes equations of non-Newtonian type, the restriction to the range of p can be slightly relaxed.*

Let us rewrite the abstract formulation of (1.1),

$$\begin{cases} u_t + A_s u + B_1(u,b) = 0, u(0) = u_0, \\ b_t + Au + B_2(u,b) = 0, b(0) = b_0, \end{cases} \tag{1.9}$$

where the Stokes operators $A_s$ and Laplacian operator $A$ to be specially considered the boundary conditions (1.4) are defined as follows:

$$A_s u = -P\Delta u, \quad u \in D(A) := W^{2,q}(\mathbb{R}^3_+) \cap W^{1,q}_{0,\sigma}(\mathbb{R}^3_+),$$

and

$$Ab = \nabla \times (\nabla \times b), \quad b \in D(B_q) = \{b \in W^{2,q}(\mathbb{R}^3_+) \cap W^{1,2}_{0,\sigma}(\mathbb{R}^3_+) \mid (\nabla \times b) \times n = 0 \text{ on } \partial\mathbb{R}^3_+\}.$$

(see [14, page 262]). And also the bilinear operator $B_1$ and $B_2$ is defined as follows:

$$B_1(u,b) := (u \cdot \nabla)u - (b \cdot \nabla)b - \nabla \cdot (|Du|^{q-2}Du),$$

and

$$B_2(u,b) := (u\cdot\nabla)b - (b\cdot\nabla)u.$$

Here $P$ is the orthogonal projection mapping $L^q(\mathbb{R}^3_+)$ onto $L^q_\sigma(\mathbb{R}^3_+)$ (see, e.g. Ukai [15]).

**Lemma 1.5.** *([2, Theorem 3.6]). Let either* $1 < r \le q < \infty$ *or* $1 \le r < q \le \infty$ *and* $v \in L^2_\sigma \cap L^r$*. Then we have the* $L^r$*–*$L^q$ *estimate*

$$\|\nabla^k e^{-At}v\|_{L^q} \le ct^{-\frac{k}{2}(\frac{1}{r}-\frac{1}{q})}\|v\|_{L^r}, k \ge 0,$$

*and*

$$\|e^{-At}v\|_{L^q} \le ct^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}}\Big(\int_{\mathbb{R}^3_+}|x_n u(x)|^r\Big)^{1/r},$$

*where* $e^{-At}$ *denotes the analytic semigroup generated by the Stokes operator* $A$*. Here* $P$ *is the orthogonal projection mapping* $L^q(\mathbb{R}^3_+)$ *onto* $L^q_\sigma(\mathbb{R}^3_+)$ *(see, for example, Ukai [12]).*

## 2. Proof of Theorems

**Lemma 2.1.** *Suppose that* $(u,b)$ *is a strong solution of* (1.1)–(1.4)*. Then*

$$\|E(\lambda)B_1(u,b)\| \le c(\|u\|^2_{L^2} + \|b\|^2_{L^2} + \|\nabla u\|^{p-1}_{L^{p-1}})\lambda^{\frac{5}{4}},$$

*and*

$$\|E(\lambda)B_2(u,b)\| \le c(\|u\|^2_{L^2} + \|b\|^2_{L^2})\lambda^{\frac{5}{4}}, \quad \forall\ \lambda > 0.$$

*Proof.* Following [2, Theorem 3.6], we note that for $1 < r < \infty$, $\alpha > 0$ and $0 < \frac{1}{q} := \frac{1}{r} - 2\frac{\alpha}{3} \le 1$

$$\sum_{i,j=1}^{3}\|\partial_{x_i}\partial_{x_j}v\|_{L^r} \le C\|A_s v\|_{L^r}, \|\nabla v\|_{L^r} \le C\|A_s^{1/2}v\|_{L^r}, \|v\|_{L^q} \le C\|A_s^{1/2}v\|_{L^r}. \tag{2.1}$$

The proof is almost same to that in [4, Lemma 2.3]. For the convenience of readers, we give a proof. Indeed, a proof is based on (2.1) and Gagliardo-Nirenberg inequality as follows:

$$\begin{aligned}
|\langle E(\lambda)B_1(u,b), v\rangle| &= |\langle E(\lambda)\Big(P(u\cdot\nabla)u - P(b\cdot\nabla)b - P\nabla\cdot|e(u)|^{p-2}e(u)\Big), v\rangle| \\
&= |\langle\Big(P(u\cdot\nabla)u - P(b\cdot\nabla)b - P\nabla\cdot|e(u)|^{p-2}e(u)\Big), E(\lambda)v\rangle| \\
&\le |\langle u, (u\cdot\nabla)E(\lambda)v\rangle| + |\langle b, (b\cdot\nabla)E(\lambda)v\rangle| + |\langle |e(u)|^{p-2}e(u), \nabla E(\lambda)v\rangle| \\
&\le c\|u\|^2_{L^2}\|\nabla E(\lambda)v\|_{L^\infty} + c\|b\|^2_{L^2}\|\nabla E(\lambda)v\|_{L^\infty} + c\|\nabla u\|^{p-1}_{L^{p-1}}\|\nabla E(\lambda)v\|_{L^\infty} \\
&\le c(\|u\|^2_{L^2} + \|b\|^2_{L^2} + \|\nabla u\|^{p-1}_{L^{p-1}})\|\nabla E(\lambda)v\|^{1/2}_{L^6}\|\nabla^2 E(\lambda)v\|^{1/2}_{L^6} \\
&\le c(\|u\|^2_{L^2} + \|b\|^2_{L^2} + \|\nabla u\|^{p-1}_{L^{p-1}})\|A_s^{1/2}E(\lambda)v\|^{1/2}_{L^6}\|A_s E(\lambda)v\|^{1/2}_{L^6} \\
&\le c(\|u\|^2_{L^2} + \|b\|^2_{L^2} + \|\nabla u\|^{p-1}_{L^{p-1}})\|A_s^{\frac{1}{2}+\frac{2}{4}}E(\lambda)v\|^{1/2}_{L^2}\|A_s^{1+\frac{2}{4}}E(\lambda)v\|^{1/2}_{L^2} \\
&= c(\|u\|^2_{L^2} + \|b\|^2_{L^2} + \|\nabla u\|^{p-1}_{L^{p-1}})\lambda^{\frac{5}{4}}\|v\|_{L^2}
\end{aligned} \tag{2.2}$$

Similarly, we have

$$|\langle E(\lambda)B_2(u,b), v\rangle| = |\langle E(\lambda)\Big((u\cdot\nabla)b - (b\cdot\nabla)u\Big), v\rangle| \le c(\|u\|^2_{L^2} + \|b\|^2_{L^2})\lambda^{\frac{5}{4}}\|v\|_{L^2},$$

which is complete of the proof. □

**Proof of Theorem 1.** Following the argument in [4], we will prove Theorem. From the energy inequality, we know

$$\frac{d}{dt}\|(u,b)(t)\|^2_{L^2(\mathbb{R}^3_+)} + 2\|A_s^{1/2}u\|^2_{L^2(\mathbb{R}^3_+)} + \|B^{1/2}b\|^2_{L^2(\mathbb{R}^3_+)} \leq 0. \tag{2.3}$$

To obtain a lower bound of the second term in (2.3), we get for $\rho > 0$,

$$\|A_s^{1/2}z\|^2 = \int_0^\infty \lambda d\|E(\lambda)z(t)\|^2 \geq \int_\rho^\infty \lambda d\|E(\lambda)z(t)\|^2 \geq \rho\int_\rho^\infty \lambda d\|E(\lambda)z(t)\|^2 \geq \frac{\rho}{2}(\|z(t)\|^2 - \|E(\rho)z(t)\|^2),$$

and

$$\|B^{1/2}z\|^2 \geq \frac{\rho}{2}(\|z(t)\|^2 - \|E(\rho)z(t)\|^2).$$

And thus, we have

$$\frac{d}{dt}\|(u,b)(t)\|^2_{L^2} + \rho\|(u,b)(t)\|^2 \leq \rho\|E(\rho)u(t)\|^2_{L^2} + \rho\|E(\rho)b(t)\|^2_{L^2} \tag{2.4}$$

To estimate the right-hand side of (2.4), we consider the integral form of (1.9),

$$u(t) = e^{-tA_s}u_0 + \int_0^t e^{-(t-s)A_s}B_1(u,b)\,ds,$$

and

$$b(t) = e^{-tA_s}u_0 + \int_0^t e^{-(t-s)B}B_2(u,b)\,ds.$$

Applying the operator $E(\rho)$ to the both sides of this integral equation and integrating by parts, we obtain

$$E(\rho)u(t) = E(\rho)e^{-tA_s}u_0 + \int_0^t\int_0^\rho e^{-\lambda(t-s)}d(E(\lambda)B_1(u,b))ds$$

$$= E(\rho)e^{-tA_s}u_0 + \int_0^t e^{-\rho(t-s)}(E(\rho)B(u))ds + \int_0^t(t-s)\Big(e^{-\lambda(t-s)}(E(\lambda)B_1(u,b))d\lambda\Big)ds.$$

This together with Lemma 2.1 implies

$$\|E(\rho)u(t) - E(\rho)e^{-tA_s}u_0\| \leq c\rho^{\frac{5}{4}}\int_0^t e^{-\rho(t-s)}\Big(\|u\|^2 + \|\nabla u\|^{p-1}_{p-1}\Big) + c\rho^{\frac{5}{4}}\int_0^t(t-s)\Big(e^{-\lambda(t-s)}(\Big(\|u\|^2 + \|\nabla u\|^{p-1}_{p-1}\Big))d\lambda\Big)ds$$

$$\leq c\rho^{\frac{5}{4}}\int_0^t e^{-\rho(t-s)}\Big(\|u\|^2 + \|\nabla u\|^{p-1}_{p-1}\Big).$$

where we use Korn's inequality and the following inequality in the last inequality: For $\frac{11}{5} \leq p < 3$,

$$\begin{aligned}\int_0^t \|\nabla z(s)\|^{p-1}_{L^{p-1}}ds &\leq \int_0^t \|z(s)\|^{\frac{7-p}{4}}_{L^2}\|\nabla^2 z(s)\|^{\frac{5p-11}{4}}_{L^2}\,ds\\ &\leq C\Big(\int_0^t \|z(s)\|^{\frac{14-2p}{19-5p}}ds\Big)^{\frac{19-5p}{8}}\Big(\int_0^\infty \|\nabla^2 z(t)\|^2_{L^2}dt\Big)^{\frac{5q-11}{8}}\\ &\leq C\Big(\int_0^t \|z(s)\|^{\frac{14-2p}{19-5p}}_{L^2}ds\Big)^{\frac{19-5p}{8}},\end{aligned} \tag{2.5}$$

and for $p \geq 3$

$$\int_0^t \|\nabla z(s)\|_{L^{q-1}}^{p-1} ds \leq C \int_0^t \|\nabla z(s)\|_{L^2}^{\frac{2}{q-2}} \|\nabla z\|_{L^p}^{\frac{p(p-3)}{p-2}} ds \leq C \|\nabla z\|_{L^2((0,t);L^2)}^{\frac{2}{q-2}} \|\nabla z\|_{L^q((0,t);L^q)}^{\frac{q(q-3)}{q-2}} < \infty. \tag{2.6}$$

Hence, we have

$$\|E(\rho)u(t)\|_{L^2} \leq \|e^{-tA_s}u_0\|_{L^2} + c\rho^{\frac{5}{4}} \int_0^t \|u(t)\|_{L^2}^2 ds + C\rho^{\frac{5}{4}} \Big( \int_0^t \|z(s)\|_{L^2}^{\frac{14-2p}{19-5p}} ds \Big)^{\frac{19-5p}{8}} + c\rho^{\frac{5}{4}}. \tag{2.7}$$

In the same way, we get

$$\|E(\rho)b(t)\|_{L^2} \leq \|e^{-tB}b_0\|_{L^2} + c\rho^{\frac{5}{4}} \int_0^t \|(u,b)(t)\|_{L^2}^2 ds + c\rho^{\frac{5}{4}}. \tag{2.8}$$

Using (2.7), (2.8) and the energy inequality, it implies

$$\begin{aligned} \frac{d}{dt}\|(u,b)(t)\|_{L^2}^2 + \rho\|(u,b)(t)\|_{L^2}^2 &\leq \|e^{-tA_s}u_0\|_{L^2}^2 + \|e^{-tB}b_0\|_{L^2}^2 \\ &+ c\rho^{\frac{5}{2}} \Big( \int_0^t \|u(t)\|_{L^2}^2 ds \Big)^2 + c\rho^{\frac{5}{4}} \Big[ \Big( \int_0^t \|(u,b)(s)\|_{L^2}^{\frac{14-2p}{19-5p}} ds \Big)^{\frac{19-5p}{8}} \Big]^2 + c\rho^{\frac{5}{2}} \\ &\leq \|e^{-tA_s}u_0\|_{L^2}^2 + \|e^{-tB}b_0\|_{L^2}^2 + c\rho^{\frac{5}{2}} t^2 + c\rho^{\frac{5}{2}}, \quad \forall \rho > 0. \end{aligned}$$

Here, we use the following estimate

$$\begin{aligned} \Big[ \Big( \int_0^t \|z(s)\|_{L^2}^{\frac{14-2p}{19-5p}} ds \Big)^{\frac{19-5p}{8}} \Big]^2 &\leq \Big( \int_0^t \|z(s)\|_{L^2}^{\frac{2(14-2p)}{19-5p}} dst \Big)^{\frac{19-5p}{8}} \\ \leq \int_0^t \|z(s)\|_{L^2}^{\frac{2(14-2p)}{19-5p}} dst + C &\leq \int_0^t \|z(s)\|_{L^2}^2 dst + C \leq t^2 + C. \end{aligned}$$

Now let $\rho = 3(1+t)^{-1}$ and multiply both sides above by $(1+t)^3$ to obtain

$$\begin{aligned} \frac{d}{dt}\Big( (1+t)^3 \|(u,b)(t)\|_{L^2}^2 \Big) &\leq c(1+t)^2 \Big( \alpha\|e^{-tA_s}u_0\|_{L^2}^2 + \alpha\|e^{-tB}b_0\|_{L^2}^2 + (1+t)^{1-\frac{3}{2}} + (1+t)^{-\frac{5}{2}} \Big) \\ &\leq c(1+t)^2\|e^{-tA_s}u_0\|_{L^2}^2 + c(1+t)^2\|e^{-tB}b_0\|_{L^2}^2 + (1+t)^{\frac{3}{2}} + (1+t)^{-\frac{1}{2}}. \end{aligned}$$

And thus, we get

$$\|(u,b)(t)\|_{L^2}^2 \leq c + c(1+t)^{-\frac{3}{2}} + c(1+t)^{-\frac{7}{2}}. \tag{2.9}$$

Since $\|e^{-tA_s}u_0\|_{L^2}^2 \to 0$ and $\|e^{-tB}b_0\|_{L^2}^2 \to 0$ as $t \to \infty$, we conclude that $u(t) \to 0$ and $b(t) \to 0$ as $t \to \infty$ and thus it complete the proof of first part in Theorem 1.1.

For second part, it follows from (2.9) and Lemma 2.2 that

$$\|(u,b)(t)\|_{L^2}^2 \leq c\Big( t^{-(\frac{3}{r}-\frac{3}{2})} \|(u_0,b_0)\|_{L^r}^2 + (1+t)^{-\frac{3}{2}} + (1+t)^{-\frac{7}{2}} \Big). \tag{2.10}$$

Since $\frac{3}{r} - \frac{3}{2} < \frac{5}{2}$, the desired assertion is obtained if $\frac{3}{r} - \frac{3}{2} \leq \frac{3}{2}$. It remains to consider the case $\frac{3}{r} - \frac{3}{2} > \frac{3}{2}$ Hence (2.10) implies that

$$\||(u,b)(t)\|_{L^2}^2 \leq (1+t)^{-3/2}. \tag{2.11}$$

Using (2.11), through the same method as before, we have

$$\frac{d}{dt}\|(u,b)(t)\|_{L^2}^2+\rho\|(u,b)(t)\|_{L^2}^2\le \rho t^{-(\frac{3}{r}-\frac{3}{2})}\|(u_0,b_0)\|_{L^r}^2+c\rho^{\frac{7}{2}}t+c\rho^{\frac{7}{2}}.$$

Let $\rho=3t^{-1}$ and then multiply the both sides of this equation by $t^3$ to get

$$\frac{d}{dt}(t^3\|(u,b)(t)\|\|_{L^2}^2)\le \rho t^{2-\frac{3}{r}-\frac{3}{2}}+c\rho^{\frac{1}{2}}+c\rho^{\frac{-1}{2}}.$$

Since $1\le r<2$ implies $\frac{3}{r}-\frac{3}{2}\le\frac{3}{2}$ , we have

$$\|(u,b)(t)\|_{L^2}^2\le c(1+t)^{-(\frac{3}{r}-\frac{3}{2})}\quad \text{for}\quad t\ge 1,$$

and complete the proof of Theorem 3.1.

**Proof of Theorem 2.**

$$\frac{d}{dt}\|(u,b)(t)\|_{L^2}^2+\rho\|(u,b)(t)\|_{L^2}^2\le \|e^{-tA_s}b_0\|_{L^2}^2+\|e^{-tB}u_0\|_{L^2}^2$$

$$+c\rho^{\frac{7}{2}}\Big[\Big(\int_0^t\|(u,b)(t)\|^2\,ds\Big)^2+1\Big]+c\rho^{\frac{7}{2}}\Big[\Big(\int_0^t\|u(s)\|_{L^2}^{\frac{14-2p}{19-5p}}ds\Big)^{\frac{19-5p}{8}}\Big]^2.$$

Let $\rho=3t^{-1}$ and multiply both sides above by $t^3$ to obtain

$$\frac{d}{dt}(t^3\|(u,b)(t)\|_{L^2}^2)\le ct^3\Big(t^{-(\frac{3}{r}-\frac{3}{2})-2}+t^{-\frac{7}{2}}\Big(\int_0^t\|(u,b)(t)\|_{L^2}^2\,ds\Big)^2+t^{-\frac{7}{2}}\Big[\Big(\int_0^t\|u(s)\|_{L^2}^{\frac{14-2p}{19-5p}}ds\Big)^{\frac{19-5p}{8}}\Big]^2+t^{-\frac{7}{2}}\Big)$$

$$\le c\Big(t^{-(\frac{3}{r}-\frac{3}{2})+1}+t^{-\frac{1}{2}}\Big(\int_0^t\|(u,b)(t)\|_{L^2}^2\,ds\Big)^2+t^{-\frac{1}{2}}\Big[\Big(\int_0^t\|z(s)\|_{L^2}^{\frac{14-2p}{19-5p}}ds\Big)^{\frac{19-5p}{8}}\Big]^2+t^{-\frac{1}{2}}\Big),$$

where we have used Lemma 1.5 and the assumption (1.8). Integrating with respect to $t$, we get

$$\|(u,b)(t)\|_{L^2}^2\le ct^{-(\frac{3}{r}-\frac{3}{2})-1}+t^{-\frac{5}{2}}\Big(\int_0^t\|u(t)\|^2\,ds\Big)^2+t^{-\frac{5}{2}}\Big[\Big(\int_0^t\|z(s)\|_{L^2}^{\frac{14-2p}{19-5p}}ds\Big)^{\frac{19-5p}{8}}\Big]^2+t^{-\frac{5}{2}}$$

$$\le ct^{-(\frac{3}{r}-\frac{3}{2})-1}+t^{-\frac{5}{2}}\Big(\int_0^t\|u(t)\|_{L^2}^2\,ds\Big)^2+t^{-\frac{3}{2}}\Big(\int_0^t\|u(s)\|_{L^2}^2ds\Big)+t^{-\frac{5}{2}}.$$

From Theorem 3.1(ii), we know

$$\|(u,b)(t)\|_{L^2}^2\le c(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})}. \tag{2.12}$$

A. If $\frac{3}{2}(\frac{1}{r}-\frac{1}{2})>\frac{1}{2}$, we have

$$\int_0^t\|(u,b)(s)\|_{L^2}^2<\infty. \tag{2.13}$$

Substituting this equation into (2.13) yields

$$\|(u,b)(t)\|_{L^2}^2\le ct^{-(\frac{3}{r}-\frac{3}{2})-1}+t^{-\frac{5}{2}}+t^{-\frac{3}{2}}\le ct^{-(\frac{3}{r}-\frac{3}{2})-1}.$$

B. if $\frac{3}{2}(\frac{1}{r}-\frac{1}{2})=\frac{1}{2}$, then

$$\|(u,b)(t)\|_{L^2}^2 \le c(1+t)^{-(\frac{3}{r}-\frac{3}{2})-1} + (1+t)^{-\frac{5}{2}}(ln(1+t)^2)$$
$$+(1+t)^{-\frac{3}{2}}ln(1+t) + t^{-\frac{5}{2}} \le c(1+t)^{-(\frac{3}{r}-\frac{3}{2})-1}.$$

C. if $\frac{3}{2}(\frac{1}{r}-\frac{1}{2})<\frac{1}{2}$, we have

$$\|(u,b)(t)\|_{L^2}^2 \le c(1+t)^{-(\frac{3}{r}-\frac{3}{2})-1} + c(1+t)^{(-\frac{6}{r}+\frac{3}{2})} + c(1+t)^{-\frac{3}{r}} + c(1+t)^{-\frac{5}{2}} \le c(1+t)^{-(\frac{3}{r}-\frac{3}{2})-1}.$$

The proof is complete.

## Acknowledgments

The author thank the knowledgeable referee for his/her valuable comments and helpful suggestions. Jae-Myoung Kim was supported by National Research Foundation of Korea Grant funded by the Korean Government (NRF-2020R1C1C1A01006521).

## Conflict of interest

The authors declare that they have no competing interests.

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