



*Research article*

## Final dynamics of systems of nonlinear parabolic equations on the circle

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**Abstract:** We consider the class of dissipative reaction-diffusion-convection systems on the circle and obtain conditions under which the final (at large times) phase dynamics of a system can be described by an ODE with Lipschitz vector field in  $\mathbb{R}^N$ . Precisely in this class, the first example of a parabolic problem of mathematical physics without the indicated property was recently constructed.

**Keywords:** reaction-diffusion-convection equations; finite-dimensional dynamics on attractor; inertial manifold

**Mathematics Subject Classification:** 35B41, 35K57, 35K40, 35K90, 35K91

### 1. Introduction

The problem of describing the final (at large times) dynamics of dissipative semilinear parabolic equations (SPE)

$$\partial_t u = G(u) \tag{*}$$

(see [1]) with a Hilbert phase space  $\mathcal{X}$  by an ordinary differential equation (ODE) in  $\mathbb{R}^N$  has been attracting researcher's attraction for a long time. In fact, it is required to separate finitely many "determining" degrees of freedom of an infinite-dimensional dynamical system. In this case, the key geometric object is the so-called (global) attractor [2–4], i.e., the connected compact invariant set  $\mathcal{A} \subset \mathcal{X}$  that uniformly attracts bounded subsets  $\mathcal{X}$  as  $t \rightarrow +\infty$ .

The required ODE can sometimes be implemented as an inertial form [3–5] obtained by restricting the initial equation to an inertial manifold, i.e, a finite-dimensional invariant  $C^1$ -surface  $\mathcal{M} \subset \mathcal{X}$  containing the attractor and exponentially attracting (with asymptotic phase) all trajectories of (\*) as  $t \rightarrow +\infty$ . The theory of inertial manifolds originally encountered systematic difficulties, and several alternative concepts of finite-dimensional reduction of SPE have therefore been developed starting from [6–9]. Following [8], we will say that the dynamics of (\*) on the attractor (final dynamics) is finite-dimensional if there exists an ODE in  $\mathbb{R}^N$  with Lipschitz vector field, resolving flow  $\{\Theta_t\}_{t \in \mathbb{R}}$ ,

and invariant compact set  $\mathcal{K} \subset \mathbb{R}^N$  such that the phase semiflows  $\{\Phi_t\}_{t \geq 0}$  of equation (\*) on  $\mathcal{A}$  and  $\{\Theta_t\}_{t \geq 0}$  are Lipschitz conjugate on  $\mathcal{K}$ . The latter means that there is a Lipschitz embedding  $g : \mathcal{A} \rightarrow \mathbb{R}^N$ ,  $g\mathcal{A} = \mathcal{K}$ , such that  $g\Phi_t u = \Theta_t g u$  for  $u \in \mathcal{A}$  and  $t \geq 0$ . The existence of the inertial manifold implies that the dynamics is finite-dimensional on the attractor and, in general, looks like a more attractive property. Indeed, in the first case, the inertial form provides an exponential asymptotics of any solution of the equation at large times, and in the second case, we have an ODE reproducing the original dynamics only on the attractor itself. Nevertheless, the fact that the dynamics is finite-dimensional on  $\mathcal{A}$  means that the structure of limit regimes of SPE with infinitely many degrees of freedom is no more complicated than the structure of similar regimes of an ODE with Lipschitz vector field in  $\mathbb{R}^N$ .

An alternative approach to the problem of finite-dimensional reduction of SPEs with a parameter was developed by J. Hale, A. N. Carvalho and many others (see, for example, [10] and references therein). For diffusion equations, this approach is based on the concept of “large diffusion”.

In this paper, we consider the problem of whether the final dynamics is finite-dimensional for one-dimensional systems of reaction-diffusion-convection equations

$$\partial_t u = D \partial_{xx} u - u + f(x, u) \partial_x u + g(x, u), \quad (1.1)$$

where  $u = (u_1, \dots, u_m)$  and  $f$  and  $g$  are sufficiently smooth matrix and vector functions. We assume that  $x \in J$ , where  $J$  is a circle of length 1. The matrix of diffusion coefficients  $D$  is assumed to be diagonal,  $D = \text{diag}\{d_j\}$ ,  $d_j > 0$ . As the phase space we choose an appropriate space  $X \subset C^1(J, \mathbb{R}^m)$  in the Hilbert semiscale  $\{X^\alpha\}_{\alpha \geq 0}$  generated by a linear positive definite operator  $u \rightarrow u - Du_{xx}$  in  $X = L^2(J, \mathbb{R}^m)$ . We postulate that evolution Eq (1.1) is dissipative in  $X$  and there exists the attractor  $\mathcal{A} \subset X$  consisting of functions  $u = u(x)$ ,  $u \in C^1(J, \mathbb{R}^m)$ . The algebraic structure of the “convection matrix”  $f = f(x, u)$ ,  $f = \{f_{ij}\}$ ,  $i, j \in \overline{1, m}$ , on the convex hull  $\text{co } \mathcal{A} \subset X$  plays an important role. We will highlight the case of the scalar diffusion matrix  $D = dE$ , where  $d = \text{const}$  and  $E$  is the identity matrix.

For scalar equations of the form (1.1), the fact that the dynamics is finite-dimensional on the attractor was established in [9]. The problem of finite-dimensional reduction (1.1) becomes much more complicated when passing to the systems. In the vector case, the final dynamics of systems (1.1) with scalar diffusion matrix  $D$  and spatially homogeneous nonlinearity  $f(u) \partial_x u + g(u)$  was studied in [11], and the second restriction seems to be technical. The existence of an inertial manifold was proved in [11] for the scalar equation ( $m = 1$ ), and for  $m > 1$ , it was proved under the assumption that the function matrix  $f(u)$  is diagonal with a unique nonzero element in a convex neighborhood of the attractor. The results obtained in [11] are based on a non-local change of the phase variable  $u$  which “decreases” the dependence of the nonlinear part (1.1) on  $\partial_x u$  and allows using the well-known “spectral gap condition”. Note that in the absence of convection ( $f \equiv 0$ ), the existence of inertial manifold is a classical fact [4].

Generalizing and developing the approach in [9], we study whether the dynamics is finite-dimensional on the attractor, but we do not consider the problem of existence of an inertial manifold for systems of periodic Eq (1.1). At the same time, we here consider the case of *nonscalar* diffusion matrix  $D$  and spatially nonhomogeneous nonlinearity with  $f = f(x, u)$ ,  $g = g(x, u)$ . We prove that the limit dynamics is finite-dimensional for wide classes of systems (1.1). Now, omitting the details related to the choice of phase space and dissipativity conditions, we formulate the main results of the

paper are as follows:

The phase dynamics on the attractor of system (1.1) is finite-dimensional if any of the following three conditions are satisfied.

(A) The convection matrix  $f = \text{diag}$  on  $\text{co } \mathcal{A}$  (Theorem 4.3).

(B) The diffusion matrix  $D$  is scalar. For all  $(x, u) \in J \times \text{co } \mathcal{A}$ , the numerical matrices  $f(x, u(x))$  have  $m$  distinct real eigenvalues and commute with each other (Theorem 4.5).

(C) The diffusion matrix  $D$  is scalar. For all  $(x, u) \in J \times \text{co } \mathcal{A}$ , the matrices  $f(x, u)$  are symmetric and commute with each other (Theorem 4.6).

In the case (A), we have  $Df = fD$  on  $\text{co } \mathcal{A}$ . The assumptions that the matrices are commutative can conditionally be formulated as the consistency of convection with diffusion and the self-consistency of convection on the convex hull of the attractor. Usually, the attractor  $\mathcal{A}$  of system (1.1) can be localized in a ball  $B \subset \mathcal{X}$  centered at zero. Since the embedding  $\mathcal{X} \rightarrow C(J, \mathbb{R}^m)$  is continuous, it is actually sufficient to verify the conditions on  $f = f(x, u)$  in assertions (A), (B), and (C) for  $x \in J$ ,  $u \in \mathbb{R}^m : |u| < r$  with an appropriate  $r > 0$ .

In the class of one-dimensional systems (1.1), was constructed [11, Theorem 1.2] the first example of semilinear parabolic equation of mathematical physics (actually, a system of eight equations with scalar diffusion) that does not demonstrate any finite-dimensional dynamics on the attractor. This class seems to be a good testing ground for understanding where the finite-dimensional final dynamics of semilinear parabolic equations terminates and the infinite-dimensional final dynamics begins.

Our results can be generalized to systems on the circle of the form

$$\partial_t u = D \partial_{xx} u + f(x, u, \partial_x u) \quad (1.2)$$

with a smooth vector function  $f = (f_1, \dots, f_m)$ . Such systems with various boundary conditions can be reduced (see [11, 12]) to the form (1.1) by the termwise differentiation and an appropriate change of the variable. The fact that the final dynamics is finite-dimensional for scalar Eq (1.2) was already proved in [9].

We here do not consider the Dirichlet and Neumann boundary conditions for systems of the form (1.1) on  $(0, 1)$ , this can be studied in a subsequent publication. The existence of an inertial manifold is proved in a similar situation in [12] for systems of general form (1.2) with  $f = f(u, u_x)$  and a scalar diffusion matrix. Surprisingly, the dynamics in the Dirichlet–Neumann cases looks simpler than in the periodic case. The original methods of recent works [13, 14] allowed to establish the finite-dimensionality of the final dynamics and the existence of an inertial manifold for 3D periodic Ginzburg–Landau equations.

So, in this paper, the possibility of finite-dimensional reduction systems of 1D periodic reaction-diffusion-convection Eq (1.1) is established in terms of the algebraic properties of the diffusion matrix  $D$ , the convection matrix  $f$ , and the relations between them.

The paper is organized as follows. Section 2 contains necessary information about abstract SPEs and the conditions for their final dynamics to be finite-dimensional. In Section 3, it is shown how these conditions can be applied to parabolic systems (1.1). The main results are obtained in Section 4. In the short Section 5, we present several examples of system (1.1) which admit a finite-dimensional final dynamics. Finally, in Section 6, we discuss alternative approaches to the problem of finite-dimensional reduction of systems (1.1).

## 2. General information

First, we consider the abstract dissipative SPE

$$\partial_t u = -Au + F(u) \quad (2.1)$$

in a real separable Hilbert space  $X$  with scalar product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|$ . We assume that the unbounded self-adjoint positive definite linear operator  $A$  with domain of definition  $\mathcal{D}(A) \subset X$  has a compact resolvent. We assume that  $X^\alpha = \mathcal{D}(A^\alpha)$  with  $\alpha \geq 0$ . Then  $\|u\|_\alpha = \|A^\alpha u\|$ ,  $X^0 = X$ , and  $X^1 = \mathcal{D}(A)$ . For arbitrary Banach spaces  $Y_1$  and  $Y_2$ , we let  $BC^\nu(Y_1, Y_2)$ ,  $\nu \in \mathbb{N}_0$ , denote the class of  $C^\nu$ -smooth mappings  $Y_1 \rightarrow Y_2$  that are bounded on balls. We assume that the nonlinear function  $F$  belongs to  $BC^2(X^\alpha, X)$  for some  $\alpha \in [0, 1)$  and equation (2.1) is dissipative, i.e., generates a resolving semiflow  $\{\Phi_t\}_{t \geq 0}$  in the phase space  $X^\alpha$  and there exists an absorbing ball  $\mathcal{B}_a = \{u \in X^\alpha : \|u\|_\alpha < a\}$  such that  $\Phi_t \mathcal{B}_r \subset \mathcal{B}_a$  for any ball  $\mathcal{B}_r : \|u\|_\alpha < r$  for  $t > t^*(r)$ . In this case, the semiflow  $\{\Phi_t\}$  inherits [1] the  $C^2$ -smoothness, and there exists the compact attractor  $\mathcal{A} \subset \mathcal{B}_a$  consisting of all bounded complete trajectories  $\{u(t)\}_{t \in \mathbb{R}} \subset X^\alpha$  and uniformly attracting balls  $X^\alpha$  as  $t \rightarrow +\infty$ . In fact,  $\mathcal{A} \subset X^1$  due to the smoothing action of the parabolic equation [1, Section 3.5].

The embeddings  $X^\sigma \subset X^\alpha$  with  $\alpha < \sigma < 1$  are dense and compact, and  $\|u\|_\alpha \leq c\|u\|_\sigma$ ,  $c = c(\alpha, \sigma)$ , for  $u \in X^\sigma$ . Moreover, the proof of Theorem 3.3.6 in [1] can be used to derive the estimate  $\|\Phi_1 u\|_\sigma \leq L(r)\|u\|_\alpha$  on the balls  $\mathcal{B}_r \subset X^\alpha$ . This implies that  $F \in BC^\nu(X^\sigma, X)$  if  $F \in BC^\nu(X^\alpha, X)$  and the  $X^\alpha$ -dissipativity implies the  $X^\sigma$ -dissipativity. Thus, in all constructions related to SPE (2.1), one can replace the nonlinearity index  $\alpha$  with any value  $\sigma \in (\alpha, 1)$ . The linear operator  $A : X^{\theta+1} \rightarrow X^\theta$  is positive definite in  $X^\theta$  with  $\theta > 0$ . If  $F \in BC^2(X^{\theta+\alpha}, X^\theta)$ , then one can consider (2.1) in the pair of spaces  $(X^\theta, X^{\theta+\alpha})$  instead of  $(X, X^\alpha)$ . In this case, the phase dynamics preserves all its properties listed above.

We say that the phase dynamics of (2.1) is asymptotically finite-dimensional if there exists an inertial manifold, i.e., a smooth finite-dimensional invariant surface  $\mathcal{M} \subset X^\alpha$  containing the attractor and exponentially attracting (with asymptotic phase) all solutions  $u(t)$  at large times. Such a manifold is usually [3–5] a Lipschitz graph over the lowest modes of the operator  $A$ . The restriction of SPE (2.1) to  $\mathcal{M}$  is an ODE in  $\mathbb{R}^N$ ,  $N = \dim \mathcal{M}$  which completely describes the final dynamics of the original evolution system.

A less rigorous approach to the problem of finite-dimensional limit dynamic of SPE was proposed in [8, 9]. So the dynamics of (2.1) on the attractor is finite-dimensional if, for some ODE  $\partial_t x = h(x)$  in  $\mathbb{R}^N$  with  $h \in \text{Lip}(\mathbb{R}^N, \mathbb{R}^N)$  and resolving flow  $\{\Theta_t\}_{t \in \mathbb{R}}$ , there exists an invariant compact set  $\mathcal{K} \subset \mathbb{R}^N$  such that the dynamical systems  $\{\Phi_t\}$  on  $\mathcal{A}$  and  $\{\Theta_t\}$  on  $\mathcal{K}$  are Lipschitz conjugate for  $t \geq 0$ . The properties of the dynamics to be asymptotically finite-dimensional and to be finite-dimensional on the attractor have not yet been separated; there is a hypothesis [5] that they are equivalent.

Here are two criteria [8] for the dynamics to be finite-dimensional on the attractor under the assumption that  $F \in BC^2(X^\alpha, X)$ .

(F1) The phase semiflow on  $\mathcal{A}$  can be extended to the Lipschitz flow:

$$\|\Phi_t u - \Phi_t v\|_\alpha \leq M \|u - v\|_\alpha e^{\kappa|t|}, \quad t \in \mathbb{R},$$

where  $M > 0$  and  $\kappa \geq 0$  depend only on  $\mathcal{A}$ .

(GrF) The attractor is a Lipschitz graph over the lowest Fourier modes:

$$\|Pu - Pv\|_\alpha \geq M \|u - v\|_\alpha, \quad M = M(\mathcal{A}),$$

for some finite-dimensional spectral projection  $P \in \mathcal{L}(X^\alpha)$  of the operator  $A$  and all  $u, v \in \mathcal{A}$ .

Property (GrF) was established for scalar Eq (1.2) in [15] independently of the results obtained in [8, 9]. We shall further use other sufficient conditions for the dynamics to be finite-dimensional on the attractor, which were obtained in [9] (by misunderstanding, an important assumption that  $X$  is real was not mentioned in [9]). Assume that  $G(u) = F(u) - Au$  is the vector field of (2.1),  $\mathcal{N} = \mathcal{A} \times \mathcal{A} \subset X^\alpha \times X^\alpha$  is a compact set, and  $Y$  is a Banach space.

**Definition 2.1** ([9]). A continuous field  $\Pi : \mathcal{N} \rightarrow Y$  is said to be regular if, for any  $u, v \in \mathcal{A}$ , the function  $\Pi(\Phi_t u, \Phi_t v) : [0, +\infty) \rightarrow Y$  belongs to the class  $C^1$  and its derivative  $\partial_t \Pi(u, v)$  at zero is bounded uniformly with respect to  $(u, v) \in \mathcal{N}$ .

The smoothness of the semiflow  $\{\Phi_t\}$  and the invariance of the compact set  $\mathcal{A} \subset X^\alpha$  imply the regularity of the identical embedding  $\mathcal{N} \rightarrow X^\alpha \times X^\alpha$  and hence the regularity of any field  $\Pi : \mathcal{N} \rightarrow Y$  that can be continued to a  $C^1$ -mapping into the  $(X^\alpha \times X^\alpha)$ -neighborhood of the set  $\mathcal{N}$ . In this situation,  $\partial_t \Pi(u, v) = D\Pi(u, v)(G(u), G(v))$ , where  $D$  is Fréchet differentiation. The regular fields  $\Pi : \mathcal{N} \rightarrow Y$  form a linear structure which is also multiplicative if  $Y$  is a Banach algebra. In the last case, if the elements of  $\Pi(u, v) \in Y$  are invertible, then the field  $\Pi^{-1}$  is also regular, and  $\partial_t \Pi^{-1} = -\Pi^{-1}(\partial_t \Pi)\Pi^{-1}$  for  $(u, v) \in \mathcal{N}$ . We start from the decomposition

$$G(u) - G(v) = (T_0(u, v) - T(u, v))(u - v), \quad (u, v) \in \mathcal{N}, \quad (2.2)$$

of the vector field  $G(u)$  on  $\mathcal{A}$ , where  $T_0 \in \mathcal{L}(X^\alpha)$  and  $T \in \mathcal{L}(X^1, X)$  are unbounded linear sectorial operators in  $X$  similar to normal ones. We write

$$\Gamma_a = \{z \in \mathbb{C} : \operatorname{Re} z = a\}, \quad \Gamma(a, \xi) = \{z \in \mathbb{C} : a - \xi \leq \operatorname{Re} z \leq a + \xi\}$$

for  $a > \xi > 0$  and assume that, for some  $c > 0$ ,  $\theta \in [0, 1]$ , the total spectrum

$$\Sigma_T = \bigcup_{u, v \in \mathcal{A}} \operatorname{spec} T(u, v)$$

is localized in the domain

$$\Omega(c, \theta) = \{x + iy \in \mathbb{C} : |y| < cx^\theta\}, \quad x > 0. \quad (2.3)$$

Let  $\beta = \alpha/2$  for  $0 \leq \theta \leq \alpha/2$ , and let  $\beta = (\alpha + \theta)/3$  for  $\alpha/2 < \theta \leq 1$ . Assume that the set  $\mathbb{C} \setminus \Sigma_T$  contains strips  $\Gamma(a_k, \xi_k)$ ,  $k \in \mathbb{N}$ , with  $a_k, \xi_k \rightarrow +\infty$  as  $k \rightarrow \infty$ .

**Theorem 2.2** (see [9, Theorem 2.8]). Assume that

$$T(u, v) = S^{-1}(u, v)H(u, v)S(u, v) \quad (2.4)$$

on  $\mathcal{N}$ , where the unbounded linear sectorial operators  $H(u, v)$  are normal in  $X$ , the fields  $S, S^{-1} : \mathcal{N} \rightarrow \mathcal{L}(X)$  and  $T_0 : \mathcal{N} \rightarrow \mathcal{L}(X^\alpha, X)$  are regular, and the field  $T_0 : \mathcal{N} \rightarrow \mathcal{L}(X^\alpha)$  is bounded. In this case, if

$$d_k^\beta = o(\xi_k), \quad k \rightarrow \infty, \quad (2.5)$$

then the dynamics of Eq (2.1) is finite-dimensional on the attractor.

### 3. Parabolic systems

Now we consider the system of Eq (1.1) on  $J = \mathbb{R} | \text{mod } \mathbb{Z}$  with  $u = (u_1, \dots, u_m)$ . We assume that the matrix function  $f = f(x, u)$  and the vector function  $g = g(x, u)$  belong to the smoothness class  $C^\infty$  on  $J \times \mathbb{R}^m$  and write system (1.1) in the abstract form (2.1) with  $X = L^2(J, \mathbb{R}^m)$ , positive definite operator  $Au = u - Du_{xx}$ , and nonlinearity  $F : u \rightarrow f(x, u)\partial_x u + g(x, u)$ . Assume that  $\{X^\alpha\}_{\alpha \geq 0}$  is the Hilbert semiscale generated by  $A$  and  $\mathcal{H}^s = \mathcal{H}^s(J)$  are generalized Sobolev  $L^2$ -spaces (spaces of Bessel potentials [1, 16]) of scalar functions on  $J$  with arbitrary  $s \geq 0$ . If  $s > 1/2$ , then  $\mathcal{H}^s \subset C(J)$  and  $\mathcal{H}^s$  is a Banach algebra [16, Section 2.8.3]. The differentiation operator  $\partial_x$  belongs to  $\mathcal{L}(\mathcal{H}^{s+1}, \mathcal{H}^s)$ . As the phase space we choose  $X^\alpha = \mathcal{H}^{2\alpha}(J, \mathbb{R}^m)$  with arbitrary  $\alpha \in (3/4, 1)$  which is fixed below.

We will generalize the conclusions of [9, pages 991 and 992] about the smoothness of the nonlinear function  $F$  and the phase dynamics of (1.1) to the case  $m > 1$ . We let the symbol  $\hookrightarrow$  denote linear continuous embeddings of function spaces and shall use necessary results obtained in [1, 16]. For an arbitrary  $C^\infty$ -function  $z : J \times \mathbb{R}^m \rightarrow \mathbb{R}$ , the mapping  $\psi : u \rightarrow z(x, u)$  is a function of class  $BC^\nu$  from  $C^s(J)$  in  $C^s(J)$  for all  $\nu, s \in \mathbb{N}$ . Since  $\mathcal{H}^{2\alpha} \hookrightarrow C^1(J)$ , we have  $\psi \in BC^\nu(\mathcal{H}^{2\alpha}, C^1(J))$ . Embedding theorems imply that  $\psi \in BC^\nu(\mathcal{H}^s(J, \mathbb{R}^m), \mathcal{H}^s(J))$ . As we see,  $F \in BC^2(X^1, X^{1/2})$  and  $F \in BC^1(X^{3/2}, X^1)$ . Moreover,  $X^\alpha \hookrightarrow C^1(J, \mathbb{R}^m) \hookrightarrow C(J, \mathbb{R}^m) \hookrightarrow X$ , and hence  $F \in BC^3(X^\alpha, X)$ . We also note that  $X^{3/2} \hookrightarrow C^2(J, \mathbb{R}^m)$  and  $X^2 \hookrightarrow C^3(J, \mathbb{R}^m)$ .

In the case of finite functions  $f = f(u)$ ,  $g = g(u)$ , the dissipativity of system (1.1) with phase space  $X^{1/2}$ , and hence also with  $X^\alpha$ ,  $3/4 < \alpha < 1$ , was proved in [11, Theorem 3.1]. This result can easily be transferred to the case of functions  $f(x, u)$  and  $g(x, u)$  that are finite in  $u$  and can also be generalized in other directions. Anyway, we further assume that system (1.1) is dissipative in  $X^\alpha$  and there exists the global attractor  $\mathcal{A} \subset X^\alpha$ . Using the above-listed properties of nonlinearity  $F$  and following the reasoning in [9, page 992], we formulate the following remark.

**Remark 3.1** (see [9, Remark 5.2]). The following assertions hold: (a) The attractor  $\mathcal{A}$  is bounded in  $X^2$ ; (b) If  $Y$  is a Banach space, then each vector field  $\Pi : \mathcal{N} \rightarrow Y$  continuous in the  $(X^\alpha \times X^\alpha)$ -metric can be continued to  $C^1$ -mapping  $X^1 \times X^1 \rightarrow Y$  regularly in the sense of Definition 2.1.

Our goal is to apply Theorem 2.2 to system (1.1) and to prove that the final dynamics is finite-dimensional. Let

$$G(u) = -Au + F(u) = D\partial_{xx}u - u + f(x, u)\partial_x u + g(x, u) \quad (3.1)$$

be the vector field of system (1.1), and let  $\mathcal{N} = \mathcal{A} \times \mathcal{A} \subset X^\alpha \times X^\alpha$ . The main idea, as in [9], is related to the change of variable in the linear differential expression with respect to  $x \in J$  for the difference  $G(u) - G(v)$  for a fixed  $(u, v) \in \mathcal{N}$ , which allows one to eliminate the dependence on  $\partial_x h$ ,  $h(x) = u(x) - v(x)$ . Along with the convection matrix  $f = \{f_{ij}\}$  we consider the  $m \times m$  function matrices

$$g_u = \left\{ \frac{\partial g_i}{\partial u_j} \right\}, \quad f_u \partial_x u = \left\{ \sum_{l=1}^m \frac{\partial f_{il}}{\partial u_j} \partial_x u_l \right\}, \quad i, j \in \overline{1, m}.$$

We put

$$B_0(x; u, v) = -E + \int_0^1 (f_u(x, w(x))w_x(x) + g_u(x, w(x)))d\tau, \quad (3.2.1)$$

where  $E$  is the unit  $m \times m$  matrix, and

$$B(x; u, v) = \int_0^1 f(x, w(x))d\tau \quad (3.2.2)$$

for  $u, v \in X^\alpha$ ,  $w(x) = \tau u(x) + (1 - \tau)v(x)$ ,  $x \in J$ . The elements of the matrices  $B_0$  and  $B$  are continuous functions, and for  $u, v \in \mathcal{A}$ , function of class  $C^2$  on  $J$ . If necessary, it is convenient to treat expressions (3.2) as Bochner integrals ranging in some function spaces. By  $\mathbb{M}^m$  we denote the algebra of numerical  $m \times m$  matrices with Euclidean norm, and by  $Y(J, \mathbb{M}^m)$  we denote the linear spaces of such matrices with elements from some Banach space  $Y$  of scalar functions on  $J$ . Using the  $C^1$ -smoothness of the mappings  $(u, v) \rightarrow f_u(x, w)w_x + g_u(x, w)$  and  $(u, v) \rightarrow f(x, w)$ ,  $X^\alpha \times X^\alpha \rightarrow C(J, \mathbb{M}^m)$  for a fixed  $\tau \in [0, 1]$  and differentiating the expression under the integral sign in (3.2) with respect to the parameter  $(u, v)$ , we conclude that the mappings  $(u, v) \rightarrow B_0(\cdot; u, v)$  and  $(u, v) \rightarrow B(\cdot; u, v)$  are of class  $C^1(X^\alpha \times X^\alpha, C(J, \mathbb{M}^m))$ . By the integral mean-value theorem for nonlinear operators, we have

$$\begin{aligned} G(u) - G(v) &= -Ah + \left( \int_0^1 DF(\tau u + (1 - \tau)v) d\tau \right) h \\ &= Dh_{xx} + B_0(x; u, v)h + B(x; u, v)h_x \doteq Rh, \end{aligned}$$

where  $h = u - v$ ,  $u, v \in \mathcal{A}$ , and  $\tau u + (1 - \tau)v \in \text{co } \mathcal{A}$ . Here  $D$  is the Fréchet differentiation. Proceeding as in [17], we apply the transformation  $h = U\eta$  to the differential expression  $Rh$ , where the  $m \times m$  matrix function  $U(x) = U(x; u, v)$ ,  $x \in [0, 1]$ , is a solution of the linear Cauchy problem

$$U_x = -\frac{1}{2}D^{-1}B(x)U, \quad U(0) = E. \quad (3.3)$$

Similar problems are considered in [18, Chapters 3 and 5]. We often write  $B_0$ ,  $B$ , and  $U$  omitting the dependence on  $u$  and  $v$  and sometimes on  $x$ . Taking into account the fact that

$$U_{xx} = -\frac{1}{2}D^{-1}(B_x(x)U + B(x)U_x) = -\frac{1}{2}D^{-1}B_x(x)U + \frac{1}{4}D^{-1}B(x)D^{-1}B(x)U,$$

we have

$$\begin{aligned} Rh &= RU\eta = D(U\eta_{xx} + 2U_x\eta_x + U_{xx}\eta) + B_0(x)U\eta + B(x)(U_x\eta + U\eta_x) \\ &= DU\eta_{xx} - B(x)U\eta_x - \frac{1}{2}B_x(x)U\eta + \frac{1}{4}B(x)D^{-1}B(x)U\eta + B_0(x)U\eta + B(x)U\eta_x \\ &\quad + B(x)\left(-\frac{1}{2}D^{-1}B(x)U\eta\right) = DU\eta_{xx} + (B_0(x) - \frac{1}{2}B_x(x) - \frac{1}{4}B(x)D^{-1}B(x))U\eta. \end{aligned}$$

Now we write a decomposition of the form (2.2) for the vector field (3.1) of evolution Eq (1.1) on the attractor  $\mathcal{A}$  with linear components

$$T_0(u, v)h = \omega h + (B_0(x) - \frac{1}{2}B_x(x) - \frac{1}{4}B(x)D^{-1}B(x))h, \quad (3.4.1)$$

$$T(u, v)h = \omega h - DU\partial_{xx}U^{-1}h, \quad (3.4.2)$$

where the numerical parameter  $\omega > 0$  will be chosen later.

We slightly generalize the fact that linear problem (3.3) can be solved explicitly under the condition that the operators  $D^{-1}B(x)$  are commutative in  $x \in J$ .

**Lemma 3.2.** *Let  $D^{-1}B(x) = CW(x)C^{-1}$  with constant nondegenerate matrix  $C$  and matrix function  $W \in C(J, \mathbb{M}^m)$ , and let  $W(x_1)W(x_2) = W(x_2)W(x_1)$  for  $x_1, x_2 \in J$ . Then  $U(x) = C\bar{U}(x)C^{-1}$  with*

$$\bar{U}(x) = \exp\left(-\frac{1}{2} \int_0^x W(\xi) d\xi\right) \quad (3.5)$$

is the solution of the Cauchy problem (3.3) on  $[0, 1]$ .

**Proof.** Under the conditions of the lemma, we have  $\overline{U}(x)W(x) = W(x)\overline{U}(x)$ , and hence  $\overline{U}_x = -\frac{1}{2}W(x)\overline{U}$ . Further,  $\overline{U}(0) = U(0) = E$  and

$$\begin{aligned} U_x &= C\overline{U}_xC^{-1} = C\left(-\frac{1}{2}W\overline{U}\right)C^{-1} \\ &= C\left(-\frac{1}{2}C^{-1}D^{-1}BC \cdot C^{-1}UC\right)C^{-1} = -\frac{1}{2}D^{-1}BU. \quad \square \end{aligned}$$

Now we prove regularity in the sense of Definition 2.1 of some vector fields on the compact set  $\mathcal{N} \subset X^\alpha \times X^\alpha$ . If  $Y \hookrightarrow Y_1$  for the function spaces  $Y$  and  $Y_1$ , then the regularity of the field  $\Pi : \mathcal{N} \rightarrow Y$  implies the regularity of  $\Pi : \mathcal{N} \rightarrow Y_1$ .

**Lemma 3.3.** *The field of operators  $T_0$  on  $\mathcal{N}$  is bounded ranging in  $\mathcal{L}(X^\alpha)$  and regularly ranging in  $\mathcal{L}(X^\alpha, X)$ .*

**Proof.** Let  $\|\cdot\|_{\alpha,\alpha}$  and  $\|\cdot\|_{\alpha,0}$  be norms of operators in  $\mathcal{L}(X^\alpha)$  and  $\mathcal{L}(X^\alpha, X)$ . We assume that  $T_0h = Q(x; u, v)h$  in (3.4.1) with  $h \in \text{co}\mathcal{A} \subset X^\alpha$ . By Remark 3.1.(a), the convex hull of the attractor is bounded in the norm of  $X^2$  which is equivalent to the norm  $\mathcal{H}^4(J, \mathbb{R}^m)$ , and hence the matrix functions  $B, B_0$ , and  $BD^{-1}B$  are uniformly bounded with respect to  $(u, v) \in \mathcal{N}$  in  $\mathcal{H}^3(J, \mathbb{M}^m)$ . Thus, the matrix functions  $B_x$  and  $Q$  are bounded on  $\mathcal{N}$  in the norm  $\mathcal{H}^2(J, \mathbb{M}^m)$  and  $T_0$  is the operator of multiplication of vector functions in  $X^\alpha = \mathcal{H}^{2\alpha}(J, \mathbb{R}^m)$  by the matrix  $Q \in \mathcal{H}^{2\alpha}(J, \mathbb{M}^m)$  with  $2\alpha \in (3/2, 2)$ . Since  $\mathcal{H}^{2\alpha}(J)$  is a Banach algebra, we see that  $T_0(u, v) \in \mathcal{L}(X^\alpha)$  and  $\|T_0(u, v)\|_{\alpha,\alpha} \leq \text{const}$  on  $\mathcal{N}$ .

Since  $\mathcal{H}^{2\alpha}(J) \hookrightarrow C(J) \hookrightarrow L^2(J)$ , we have  $\mathcal{H}^{2\alpha}(J, \mathbb{M}^m) \hookrightarrow C(J, \mathbb{M}^m) \hookrightarrow L^2(J, \mathbb{M}^m)$  and  $\|T_0\|_{\alpha,0} \leq c\|Q\|_{0,0}$ , where  $\|Q\|_{0,0}$  is the norm of  $Q$  as an operator in  $\mathcal{L}(X)$  and  $c = c(\mathcal{A})$ . Therefore, the field of operators  $T_0 : \mathcal{N} \rightarrow \mathcal{L}(X^\alpha, X)$  is regular if the field of matrix functions  $Q : \mathcal{N} \rightarrow L^2(J, \mathbb{M}^m)$  is regular. The function  $u \rightarrow f(x, u)$ ,  $X^\alpha \rightarrow C(J, \mathbb{M}^m)$ , is of class  $C^1$ . Since the mappings  $(u, v) \rightarrow B_0(\cdot; u, v)$  and  $(u, v) \rightarrow B(\cdot; u, v)$  are of class  $C^1(X^\alpha \times X^\alpha, C(J, \mathbb{M}^m))$ , it follows that their restrictions to  $\mathcal{N}$  are regular. The regularity of the field  $BD^{-1}B : \mathcal{N} \rightarrow C(J, \mathbb{M}^m)$  follows from the regularity of the fields  $B$  and  $D^{-1} = \text{const}$  with the multiplicative structure of  $C(J, \mathbb{M}^m)$  taken into account. Moreover, the fields of matrix functions  $B, B_0, BD^{-1}B$  on  $\mathcal{N}$  are regular with values in  $L^2(J, \mathbb{M}^m)$ .

Now we prove the regularity of the field  $\Pi \doteq B_x : \mathcal{N} \rightarrow L^2(J, \mathbb{M}^m)$ . Let  $\Pi_\tau(u, v) = (f(x, w))_x$  with  $w = \tau u(x) + (1 - \tau)v(x)$  for a fixed  $\tau \in [0, 1]$  and arbitrary  $u = u(x), v = v(x) \in X^1$ . Then  $\Pi(u, v) = (B(x; u(x), v(x)))_x$  is the result of integration of  $\Pi_\tau(u, v)$  over  $\tau$ . The mapping  $u \rightarrow f(x, u)$  belongs at least to the class  $BC^1(X^1, X^{1/2})$ , and hence  $\Pi_\tau \in C^1(X^1 \times X^1, L^2(J, \mathbb{M}^m))$ . Differentiating the integral expression for  $\Pi(u, v)$  with respect to the parameter  $(u, v) \in X^1 \times X^1$ , we obtain  $\Pi \in C^1(X^1 \times X^1, L^2(J, \mathbb{M}^m))$ . It remains to verify that the fields  $\Pi : \mathcal{N} \rightarrow L^2(J, \mathbb{M}^m)$  are continuous and to use Remark 3.1.(b). By [9, Lemma 1.1], the function  $u \rightarrow Au$ ,  $\mathcal{A} \rightarrow X$ , with  $Au = u - Du_{xx}$  is continuous in the  $X^\alpha$ -metric; the same holds for the mappings  $u \rightarrow u_{xx}$  and  $u \rightarrow u_x$  of the set  $\text{co}\mathcal{A} \subset X^\alpha$  into  $X$  for  $u \in \text{co}\mathcal{A} \subset X^1$ . In the relation  $(f(x, u))_x = f_x + f_u u_x$ , the operators  $u \rightarrow f_x(x, u)$ ,  $u \rightarrow f_u(x, u)u_x$  continuously act from  $X^\alpha$  to  $C(J, \mathbb{M}^m)$ , and hence  $\Pi_\tau, \Pi \in C(\mathcal{N}, L^2(J, \mathbb{M}^m))$ . The proof of the lemma is complete.  $\square$

Everywhere below,  $I \doteq \text{Id}$  in a Banach space. The matrix functions  $B(x)$  and  $U(x)$  in the Cauchy problem (3.3) can be treated as bounded linear operators in  $X$ . The following assertion is related to the smooth dependence of solutions of differential equations on a parameter.



**Lemma 3.4.** *The field of operators  $U : \mathcal{N} \rightarrow \mathcal{L}(X)$  is regular.*

**Proof.** We consider (3.3) for arbitrary  $u, v \in X^\alpha$  as the non-autonomous evolution problem

$$\partial_x U = -\frac{1}{2}D^{-1}B(x; (u, v))U, \quad U(0) = I$$

in the Banach algebra  $\mathcal{L}(X)$  with identically zero sectorial linear part and the parameter  $(u, v) \in X^\alpha \times X^\alpha$ . The function

$$(x, U, (u, v)) \rightarrow -\frac{1}{2}D^{-1}B(x; (u, v))U$$

ranging in  $\mathcal{L}(X)$  is Lipschitz in  $x$ , linear in  $U \in \mathcal{L}(X)$  and of class  $C^1$  with respect to the parameter  $(u, v)$ . Under these conditions, by [1, Theorem 3.4.4], the mapping  $(u, v) \rightarrow U(x; (u, v))$ ,  $X^\alpha \times X^\alpha \rightarrow \mathcal{L}(X)$  is continuously differentiable, and hence the operator field  $U : \mathcal{N} \rightarrow \mathcal{L}(X)$  is regular.  $\square$

Now we formulate an important condition on the diffusion matrix  $D$  and the convection matrix  $f$  of system (1.1).

**Assumption 3.5.**  $Df(x, u) = f(x, u)D$  for  $x \in J$ ,  $u \in \text{co } \mathcal{A}$ .

For the scalar diffusion matrix  $D = dE$ , this assumption is satisfied automatically. In the case of  $m$  distinct diffusion coefficients  $d_j$ , Assumption 3.5 holds under the condition that the matrix  $f$  is diagonal on  $\text{co } \mathcal{A}$ , and in the case of  $s$  distinct diffusion coefficients,  $1 < s < m$ , it holds under the condition that the matrix  $f$  on  $\text{co } \mathcal{A}$  inherits the block structure (with respect to the same  $d_j$ ) of the matrix  $D = \text{diag}\{d_j\}$ .

**Lemma 3.6.** *If Assumption 3.5 holds, then*

$$T(u, v) = U(u, v)(\omega I - D\partial_{xx})U^{-1}(u, v)$$

for  $u, v \in \mathcal{A}$ .

**Proof.** Assumption 3.5 implies (for any  $x \in J$  and  $u, v \in \mathcal{A}$ ) that  $DB(x) = B(x)D$  for the matrices  $B(x) = B(x; u, v)$  in (3.2.2). Thus, the matrices  $B(x)$  and  $D^{-1}B(x)$  inherit the block structure (with respect to the same  $d_j$ ) of the diffusion matrix  $D = \text{diag}\{d_1, \dots, d_m\}$ . Therefore, the same also holds for the solutions  $U(x)$  of problem (3.3), and hence  $DU(x) = U(x)D$ ,  $x \in [0, 1]$ , and the assertion of the lemma follows from (3.4.2).  $\square$

#### 4. Main results

The conditions for the dynamics to be finite-dimensional on the attractor will depend on the structure of the diffusion matrix  $D$  and the nonlinear function  $f$  in (1.1). By Theorem 2.2, we need to prove that the operators  $T(u, v)$  in (3.4.2) are “uniformly and regularly” similar, like (2.4), to the normal operators in  $X$  and to establish the required sparseness (2.5) of the total spectrum  $\Sigma_T$ .

We note that  $B(0) = B(1)$ ,  $B_x(0) = B_x(1)$  for the matrix function  $B(x) = B(x; u, v)$  in (3.2.2) defined on  $J \times \mathcal{N}$ . The matrix function  $V(x) = U^{-1}(x)$ ,  $x \in [0, 1]$ , is a solution (see [18, Section 3.1.3]) of the Cauchy problem adjoint to (3.3):

$$V_x = \frac{1}{2}VD^{-1}B(x), \quad V(0) = E, \quad (4.1)$$

and we have

$$\begin{aligned}\eta &= Vh, \quad \eta_x = V_x h + Vh_x, \quad V_x = \frac{1}{2}VD^{-1}B, \\ V_x(0) &= \frac{1}{2}V(0)D^{-1}B(0), \quad V_x(1) = \frac{1}{2}V(1)D^{-1}B(1), \\ \eta(0) &= h(0), \quad \eta(1) = V(1)h(1), \\ \eta_x(0) &= \frac{1}{2}D^{-1}B(0)h(0) + h_x(0), \quad \eta_x(1) = \frac{1}{2}V(1)D^{-1}B(1)h(1) + V(1)h_x(1).\end{aligned}$$

So the periodic boundary conditions  $h(1) = h(0)$ ,  $h_x(1) = h_x(0)$  become

$$\eta(1) = V(1)\eta(0), \quad \eta_x(1) = V(1)\eta_x(0), \quad (4.2)$$

where  $V(1) \neq E$  in general. Further, we use the notation  $U_1 = U_1(u, v)$ ,  $V_1 = V_1(u, v)$  with  $(u, v) \in \mathcal{N}$  for the operators  $U(1)$ ,  $V(1) \in \mathcal{L}(\mathbb{R}^m) = \mathbb{M}^m$ . Operators  $U_1$  and  $V_1$  are monodromy operators [18, Section 5.1] Cauchy problems (3.3) and (4.1) respectively.

The following assertion plays the key role.

**Lemma 4.1.** *If system (1.1) is dissipative in  $X^\alpha$  with  $\alpha \in (3/4, 1)$ , then the phase dynamics on the attractor is finite-dimensional in each of the following two cases.*

(i) *The diffusion matrix  $D$  is scalar and, for all  $u, v \in \mathcal{A}$ , the monodromy operators  $V_1(u, v)$  are similar to self-adjoint positive definite ones with a fixed similarity matrix  $C = C(\mathcal{A})$ .*

(ii) *Assumption 3.5 holds and, for all  $u, v \in \mathcal{A}$ , the monodromy operators  $V_1(u, v)$  are similar to diagonal positive definite ones with a fixed similarity matrix  $C = C(\mathcal{A})$ .*

**Proof.** By the conditions of the lemma, we have  $V_1 = C^{-1}\mathcal{V}C$  for self-adjoint positive definite operators  $\mathcal{V} = \mathcal{V}(u, v)$  in  $\mathbb{R}^m$ . For fixed  $u, v \in \mathcal{A}$ , we let  $\varphi_j \in \mathbb{R}^m$  and  $\mu_j > 0$  denote orthonormal eigenvectors and eigenvalues of the operator  $\mathcal{V}$  with  $j \in \overline{1, m}$ . We assume that  $H_0 = H_0(u, v) = \omega I - D\partial_{xx}$ ,  $D = \text{diag}\{d_j\}$ , with boundary conditions (4.2) on  $(0, 1)$  for some  $\omega > 0$ . We also assume that

$$\chi_{k,j}(x) = e^{2\pi kix} \cdot \varphi_j, \quad x \in J, \quad k \in \mathbb{Z}, \quad j \in \overline{1, m}.$$

Since  $\mathcal{V}\varphi_j = \mu_j\varphi_j$ , we have  $V_1C^{-1}\varphi_j = \mu_jC^{-1}\varphi_j$  and, for the functions  $\psi_{k,j}(x) = \mu_j^x \cdot C^{-1}\chi_{k,j}$ ,  $\psi_{k,j}(0) = C^{-1}\varphi_j$ ,  $\psi_{k,j}(1) = V_1C^{-1}\varphi_j$ ,

$$(\psi_{k,j})_x(0) = (\ln \mu_j + 2\pi ki)C^{-1}\varphi_j, \quad (\psi_{k,j})_x(1) = (\ln \mu_j + 2\pi ki)V_1C^{-1}\varphi_j.$$

As we see,  $\psi_{k,j}$  are eigenfunctions of the operator  $H_0$  with eigenvalues

$$\lambda_{k,j} = \omega - d_j(\ln \mu_j + 2\pi ki)^2 = \omega + d_j(2\pi k - i \ln \mu_j)^2, \quad (4.3)$$

where  $d_j \equiv d > 0$  in case (i). The operators  $V_1(u, v)$  continuously depend on  $(u, v) \in \mathcal{N}$ , and hence this also holds for their spectrum. By the compactness of  $\mathcal{N} \subset X^\alpha \times X^\alpha$ , we have  $0 < c_1 \leq \mu_j \leq c_2$ ,  $j \in \overline{1, m}$ , for some  $c_1(\mathcal{A})$ ,  $c_2(\mathcal{A})$ . Thus, the values  $|\ln \mu_j|$  are uniformly bounded in  $j \in \overline{1, m}$  and  $u, v \in \mathcal{A}$ . We put

$$S_0(x) = CV_1^{-x} = \mathcal{V}^{-x}C$$

for  $x \in J$  and  $H = S_0 H_0 S_0^{-1}$ . Then  $S_0 \psi_{k,j} = \chi_{k,j}$  and

$$H\chi_{k,j} = S_0 H_0 \psi_{k,j} = \lambda_{k,j} S_0 \psi_{k,j} = \lambda_{k,j} \chi_{k,j}.$$

Since the system of functions  $\{\chi_{k,j}\}$  is complete and orthonormal in  $X = L^2(J, \mathbb{R}^m)$ , it follows that the operators  $H = H(u, v)$  are normal in  $X$  for  $u, v \in \mathcal{A}$ . Let  $S = S_0 U^{-1}(x) = S_0 V(x)$ . We use Lemma 3.6 to write decomposition (2.4) of the vector field (3.1) on the attractor  $\mathcal{A}$  with

$$T(u, v) = UH_0U^{-1} = US_0^{-1}HS_0U^{-1} = S^{-1}(u, v)H(u, v)S(u, v)$$

and operators  $T_0(u, v)$  of the form (3.4.1). We see that  $S^{-1} = U(x)S_0^{-1}$ .

By Lemma 3.4, the operator field  $U$  on  $\mathcal{N}$  ranging in the Banach algebra  $\mathcal{L}(X)$  is regular, and hence, the field of inverse operators  $V : \mathcal{N} \rightarrow \mathcal{L}(X)$  is regular. Since  $V_1 = V(1)$  and  $\mathcal{V} = CV_1C^{-1}$ , it follows that the operator field  $\mathcal{V} : \mathcal{N} \rightarrow \mathbb{M}^m$  is regular and  $\|\partial_t \mathcal{V}(u, v)\| \leq c_3$  for the derivative  $\partial_t \mathcal{V}$  at zero for all  $(u, v) \in \mathcal{N}$  (see Definition 2.1). Here  $\|\cdot\|$  is the Euclidean norm of matrices. Let  $b = 2 \max(c_2, c_3)$  and  $\delta = 1 - c_1/b$ , then  $\delta \in (0, 1)$ . Since the spectrum  $\sigma(b^{-1}\mathcal{V} - E) \subset (-\delta, 0)$  and  $\|b^{-1}\mathcal{V} - E\| < \delta$ , it follows that the matrix representation

$$\ln \mathcal{V} = \ln(bE) + \ln(E + b^{-1}\mathcal{V} - E) = \ln(bE) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (b^{-1}\mathcal{V} - E)^n \quad (4.4)$$

converges uniformly on  $\mathcal{N}$ . By [19, Section 5.8, Exercise 3], we have

$$\partial_t (b^{-1}\mathcal{V} - E)^n = \sum_{i=1}^n (b^{-1}\mathcal{V} - E)^{i-1} \partial_t (b^{-1}\mathcal{V}) (b^{-1}\mathcal{V} - E)^{n-i},$$

and therefore,  $\|\partial_t (b^{-1}\mathcal{V} - E)^n\| < n\delta^{n-1}$ . If we differentiate (4.4) with respect to  $t$ , we obtain a uniformly converging series with the estimate

$$\|\partial_t \ln \mathcal{V}(u, v)\| < \sum_{n=1}^{\infty} \delta^{n-1} = \frac{1}{1 - \delta}, \quad (u, v) \in \mathcal{N}.$$

So the operator field  $\ln \mathcal{V} : \mathcal{N} \rightarrow \mathbb{M}^m$  is regular. We have  $\mathcal{V}^{-x} = \exp(-x \ln \mathcal{V})$  and the standard decomposition of the matrix exponent guarantees that the field  $\mathcal{V}^{-x} : \mathcal{N} \rightarrow \mathcal{L}(X)$  is regular, which implies the regularity of fields of the operators  $S_0, S, S^{-1} : \mathcal{N} \rightarrow \mathcal{L}(X)$ .

Finally, by Lemma 3.3, the field of operators  $T_0$  in (3.4.1) is regular on  $\mathcal{N}$  with values in  $\mathcal{L}(X^\alpha, X)$  and bounded with values in  $\mathcal{L}(X^\alpha)$ .

Let  $\Sigma_H = \Sigma_T$  be the total spectrum of the field of operators  $H(u, v)$  on  $\mathcal{N}$ . Using (4.3), we choose a parameter  $\omega > 0$  that ensures the inclusion  $\Sigma_H \subset \Omega(c, \theta)$  of the form (2.3) with  $\theta = 1/2$  and an appropriate  $c > 0$ . As we can see, operators  $H(u, v)$  are sectorial. Since  $3/4 < \alpha < 1$ , we have  $\beta = (\alpha + \theta)/3 < 1/2$ . Moreover, from (4.3) we derive that the set  $\mathbb{C} \setminus \Sigma_H$  contains vertical strips  $\Gamma(a_k, \xi_k)$  with

$$a_k \sim 4\pi^2 k^2, \quad \xi_k \sim 4\pi^2 k$$

and hence  $a_k^\beta = o(\xi_k)$  as  $k \rightarrow \infty$ . Thus, all conditions of Theorem 2.2 are satisfied and the dynamics of system (1.1) is finite-dimensional on the attractor.  $\square$

**Remark 4.2.** For all  $u, v \in \mathcal{A}$ , the monodromy operators  $U_1 = U_1(u, v)$  and  $V_1 = V_1(u, v) = U_1^{-1}$  are self-adjoint and positive definite if any of the following two conditions is satisfied:

- (a) The matrices  $D^{-1}B(x_1)$  and  $D^{-1}B(x_2)$  are symmetric and commutative for  $x_1, x_2 \in J$ ;
- (b)  $(D^{-1}B(x))^t = D^{-1}B(1-x)$  for all  $x \in J$ , where  $(\cdot)^t$  is the operation of transposition.

Under conditions (a), Lemma 3.2 holds with  $C = E$  and the matrix

$$U_1 = \exp\left(-\frac{1}{2} \int_0^1 D^{-1}B(x)dx\right)$$

is symmetric and positive definite. The sufficiency of condition (b) was proved in [20, Proposition 2.3].

**Theorem 4.3.** Assume that system (1.1) is dissipative in  $X^\alpha$  with  $\alpha \in (3/4, 1)$  and the convection matrix  $f$  is diagonal on  $\text{co } \mathcal{A}$ . Then the phase dynamics is finite-dimensional on the attractor.

**Proof.** Under the conditions of the theorem, the matrices  $B(\cdot; u, v)$  from (3.2.2), and hence (see the proof of Lemma 3.6), also the matrices  $U(\cdot; u, v), V(\cdot; u, v)$ , are diagonal on  $\mathcal{A} \times \mathcal{A}$ . According to Remark 4.2.(a) matrices  $U(\cdot; u, v), V(\cdot; u, v)$  are positive definite. The monodromy operators  $V_1(u, v)$  are also positive definite and are diagonal, so we can refer to Lemma 4.1.(ii) with  $C = E$ .  $\square$

**Lemma 4.4.** Assume that system (1.1) is dissipative in  $X^\alpha$  with  $\alpha \in (3/4, 1)$  and  $D = dE$ . Then the phase dynamics is finite-dimensional on the attractor if, for  $(x, u) \in J \times \text{co } \mathcal{A}$ ,

$$D^{-1}f(x, u(x)) = CH(x, u(x))C^{-1}, \quad (4.5)$$

where the symmetric matrix functions  $H(x; u)$  commute with each other for any  $(x, u) \in J \times \text{co } \mathcal{A}$  and  $C$  is a constant nondegenerate matrix.

**Proof.** From (3.2.2) we derive that  $D^{-1}B(x) = CW(x)C^{-1}$ , where

$$W(x) = W(x; u, v) = \int_0^1 H(x; w(x))d\tau$$

for  $u, v \in \mathcal{A}$ ,  $w(x) = \tau u(x) + (1-\tau)v(x)$ ,  $x \in J$ . By Lemma 3.2, the monodromy operator  $V_1(u, v) = U_1^{-1}(u, v)$ ,  $u, v \in \mathcal{A}$ , satisfies the relation  $V_1 = C(\bar{U}(1))^{-1}C^{-1}$  with operator  $\bar{U}$  given in formula (3.5).

In this case,  $\bar{U}_x = -\frac{1}{2}W(x)\bar{U}$  and the matrices  $W(x; u, v)$  are symmetric and commutative on  $J$  for all  $u, v \in \mathcal{A}$ . By Remark 4.2.(a), the operator  $\bar{U}(1)$  is self-adjoint and positive definite and the assertion of the theorem follows from Lemma 4.1.(i).  $\square$

We shall give two more arguments ensuring that the final dynamics is finite-dimensional.

**Theorem 4.5.** Assume that system (1.1) is dissipative in  $X^\alpha$  with  $\alpha \in (3/4, 1)$  and  $D = dE$ . Then the phase dynamics is finite-dimensional on the attractor if the following two conditions are satisfied:

- (i) The numerical matrices  $f(x, u(x))$  have  $m$  distinct real eigenvalues for each  $(x, u) \in J \times \text{co } \mathcal{A}$ ;
- (ii) The matrices  $f(x, u)$  commute with each other for any  $(x, u) \in J \times \text{co } \mathcal{A}$ .

**Proof.** Condition (ii) and assumption  $D = dE$  imply that the matrices  $D^{-1}f(x, u(x))$  commute with each other on  $J \times \text{co } \mathcal{A}$ . It is known [19, Theorem 8.6.1] that two simple (similar to diagonal) commutative  $m \times m$  matrices have a common set of  $m$  of linearly independent eigenvectors. By condition (i), all

eigenvalues of each numerical matrix  $f(x, u(x))$  with  $(x, u) \in J \times \text{co } \mathcal{A}$  are real and distinct, and hence there exists a unique (up to permutations and multiplications by  $-1$ ) common (for all these matrices) normalized basis  $\mathcal{E} = (e_1, \dots, e_m)$  of their eigenvectors in  $\mathbb{R}^m$ . By  $C$  we denote the constant matrix of transition from the canonical basis in  $\mathbb{R}^m$  to the basis  $\mathcal{E}$ , and by  $H(x)$  we denote diagonal (symmetric) matrices of linear operators  $D^{-1}f(x, u(x)) \in \mathcal{L}(\mathbb{R}^m)$  in this basis. We see that relation (4.5) is satisfied and it remains to apply Lemma 4.4.  $\square$

**Theorem 4.6.** *Assume that system (1.1) is dissipative in  $X^\alpha$  with  $\alpha \in (3/4, 1)$  and  $D = dE$ . Then the phase dynamics is finite-dimensional on the attractor if the matrices  $f(x, u)$  are symmetric and commute with each other for any  $(x, u) \in J \times \text{co } \mathcal{A}$ .*

**Proof.** The conditions of the theorem guarantee that the numerical matrices  $D^{-1}f(x, u(x))$  commute with each other on  $J \times \text{co } \mathcal{A}$ . As in the proof of Lemma 4.4, from formula (3.2.2) for  $B(x)$ , we derive that the matrices  $D^{-1}B(x_1)$  and  $D^{-1}B(x_2)$  are symmetric and commutative for arbitrary  $x_1, x_2 \in J$ . By Remark 4.2.(a), the monodromy operators  $V_1(u, v)$  are positive definite for any  $u, v \in \mathcal{A}$  and the assertion of the theorem follows from Lemma 4.1.(i) with  $C = E$ .  $\square$

In contrast to Theorem 4.5, we here admit the multiplicity of eigenvalues of the numerical matrices  $f(x, u(x))$ , but we assume that these matrices are symmetric.

## 5. Some examples

We consider several examples illustrating the above-described theory in terms of properties of the convection matrix  $f$ . Here we restrict ourselves to the case of scalar diffusion and assume that system (1.1) is dissipative in the phase space  $X^\alpha$  with  $\alpha \in (3/4, 1)$ . We assume that all the conditions assumed below on  $f = f(x, u)$  are valid for  $x \in J$  and  $u = u(x)$ ,  $u \in \text{co } \mathcal{A}$ .

**Proposition 5.1.** *Assume that  $D = dE$  and  $f(x, u) = f_1(x, u)Q$  with a scalar  $C^\infty$  function  $f_1$  and numerical  $m \times m$  matrix  $Q$ . Then, the dynamics on the attractor of system (1.1) is finite-dimensional if any of the following two conditions is satisfied:*

- (i) *The matrix  $Q$  has  $m$  distinct real eigenvalues and  $f_1(x, u(x)) \neq 0$  for  $x \in J$  and  $u \in \text{co } \mathcal{A}$ ;*
- (ii) *The matrix  $Q$  is symmetric.*

**Proof.** The numerical matrices  $f = f_1(x, u(x))Q$  are commutative. In the case of (i), each of these matrices has distinct real eigenvalues  $\lambda_j f_1(x, u(x))$ , where  $\lambda_1, \dots, \lambda_m$  are eigenvalues of  $Q$ , and Theorem 4.5 can be applied. In the case of (ii), the fact that the dynamics is finite-dimensional on the attractor is a direct consequence of Theorem 4.6.  $\square$

**Remark 5.2.** Condition (i) in Proposition 5.1 is satisfied on  $Q$  for upper-triangular and lower-triangular matrices with distinct elements on the diagonal. For  $m = 2$  and  $Q = \{q_{jl}\}$ , this condition precisely means that  $(q_{11} - q_{22})^2 + 4q_{12}q_{21} > 0$ .

**Example 5.3.** The dynamics on the attractor of system (1.1) is finite-dimensional in the case of  $m = 2$ ,  $D = dE$  and  $f(x, u) = \{f_{ji}(x, u)\}$  with  $f_{11} = f_{22}$  and  $f_{12} = f_{21}$ . This is a consequence of Theorem 4.6 and the commutativity of numerical matrices of the form  $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ .

**Example 5.4.** Assume that  $D = dE$ , the matrix  $f = P_n(Q)$ , where  $P_n$  is a polynomial of degree  $n \geq 0$  with coefficients  $a_i = a_i(x, u)$ ,  $0 \leq i \leq n$ ,  $a_i \in C^\infty(J \times \mathbb{R}^m, \mathbb{R})$ , and the numerical matrix  $Q$  is symmetric.

Then the dynamics of the attractor of system (1.1) is finite-dimensional. This is also a consequence of Theorem 4.6.

**Proposition 5.5.** *Assume that  $D = dE$  and  $f = Q(x)$ , where  $Q$  is a  $C^\infty$  function matrix. Then the dynamics of system (1.1) is finite-dimensional on the attractor if  $Q^t(x) = Q(1 - x)$  for  $x \in J$ .*

**Proof.** Since  $f = Q(x)$ , the matrix  $B(x)$  in (3.2.2) satisfies the condition  $(D^{-1}B(x))^t = D^{-1}B(1 - x)$  for all  $x \in J$  and  $u, v \in \mathcal{A}$ . By Remark 4.2.(b), the monodromy operators  $V_1(u, v)$  are positive definite for all  $u, v \in \mathcal{A}$ , and we can apply Lemma 4.1.(i) with  $C = E$ .  $\square$

## 6. Other possible approaches

The above presentation is based on Theorem 2.2, which means the verification of regularity (in the sense of Definition 2.1) of operator vector fields on the attractor. Alternatively, one can obtain a finite-dimensional reduction of one-dimensional parabolic systems by using the technique [5, Sections 2.3, 2.4 and 3.3] closely related to the results obtained in [21] about the dichotomies of *non-autonomous* parabolic equations. Here we will discuss the  $X^{1/2}$ -dissipative systems of general form (1.2). In our short description (on the sketch level), we omit technical details and refer to the criteria for the final dynamics to be finite-dimensional, i.e., criteria (FI) and (GrF) in Section 2. The results of [4, Section 3.6] about inverse uniqueness of solutions of SPE (2.1) allow one to conclude that the phase semiflow on the attractor  $\mathcal{A}$  expands to the continuous flow  $\{\Phi_t\}_{t \in \mathbb{R}}$ . If  $u_1, u_2 \in \mathcal{A}$  and  $h(t) = \Phi_t u_1 - \Phi_t u_2$  for  $t \in \mathbb{R}$ , then

$$h_t = Dh_{xx} + B_0(t, x)h + B(t, x)h_x, \quad (6.1)$$

$$B_0(t, x) = \int_0^1 f_u(x, w, w_x) d\tau, \quad B(t, x) = \int_0^1 f_{u_x}(x, w, w_x) d\tau,$$

$w = \tau \Phi_t u_1 + (1 - \tau) \Phi_t u_2$ , with matrix functions  $B_0$  and  $B$  sufficiently smooth in  $(t, x) \in \mathbb{R} \times J$  and bounded in  $(u_1, u_2)$ . We assume that the invertible change  $v(t, x) = S(t, x)h(t, x)$  with operators  $S, S^{-1} \in \mathcal{L}(X)$  depending on  $u_1, u_2$  allows one to reduce (6.1) to the equation

$$v_t = -Hv + R(t)v,$$

where  $H = H(u_1, u_2) \in \mathcal{L}(X^1, X)$  are normal (or uniformly with respect to  $(u_1, u_2)$  similar to normal) sectorial operators and  $R = R(\cdot; u_1, u_2) : \mathbb{R} \rightarrow \mathcal{L}(X)$  is a continuous operator function. Assume that the norms of the operators  $S, S^{-1}$ , and  $R$  are uniformly bounded in the parameter  $(u_1, u_2)$ . In this case, if the spectrum  $\Sigma_H$  combined over  $u_1, u_2 \in \mathcal{A}$  is “sufficiently rare”, then using the technique give in [5], one can verify that the phase flow is Lipschitzian on the attractor and then apply criterion (FI). In contrast to the preceding presentation, we here have to deal with second-order linear differential expression in  $t \in \mathbb{R}$  and not in  $x \in J$ . The assertions of Section 4 can be obtained in this way after the change  $v(t, x) = V^{-x}(t, 1)V(t, x)h(t, x)$ , where  $V(\cdot, x)$  is a solution of the Cauchy problem (4.1).

Another possible approach to the problem of finite-dimension of the final dynamics is related to the verification of criterion (GrF). In [22], a scalar parabolic equation of the form (1.2) is considered in a rectangle with Dirichlet boundary condition. The authors present conditions under which the attractor is a Lipschitz graph over finitely many first modes of the Laplace operator. And they use the *cone condition* well-known in the literature [3–5]. In this connection, it seems to be very perspective to study the problem of finite-dimensional reduction of systems of Eq (1.2) on the two-dimensional torus  $\mathbb{T}^2$ .

## 7. Conclusions

We show that for new wide classes of 1D periodic systems of reaction-diffusion-convection Eq (1.1) the final (for large values of time) phase dynamics is finite-dimensional. For this, we develop in a nontrivial way the general methodology of [9]. The difficulties associated with the transition from a scalar equation to a system of equations have been successfully overcome. Alternative approaches to the problem of finite-dimensional reduction for more general systems (1.2) are also discussed.

## Conflict of interest

The authors declare no conflict of interest.

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