Research article

A Lie group integrator to solve the hydromagnetic stagnation point flow of a second grade fluid over a stretching sheet

Mir Sajjad Hashemi[1]**, Hadi Rezazadeh**[2]**, Hassan Almusawa**[3] **and Hijaz Ahmad**[4,5,*]

[1] Department of Mathematics, Basic Science Faculty, University of Bonab, P.O. Box 55513-95133, Bonab, Iran

[2] Faculty of Engineering Technology, Amol University of Special Modern Technological, Amol, Iran

[3] Department of Mathematics, College of Sciences, Jazan University, Jazan 45142, Saudi Arabia

[4] Department of Basic Sciences, University of Engineering and Technology, Peshawar, Pakistan

[5] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39,00186 Roma, Italy

* **Correspondence:** Email: hijaz555@gmail.com.

Abstract: In the present paper, a Lie-group integrator, based on $GL(4, \mathbb{R})$ has been newly constructed to consider the flow characteristics in an electrically conducting second grade fluid over a stretching sheet. Present method which have a very fast convergence, permits us to explore some missing initial values at the left-end. Accurate initial values can be achieved when the determined target equation is valid, and then we can apply the group preserving scheme (GPS) as a geometric approach to obtain a rather accurate numerical solution. Finally, effects of magnetic parameter, viscoelastic parameter, stagnation point flow and stretching of the sheet parameters are illustrated.

Keywords: $GL(4, \mathbb{R})$ Lie group method; group preserving scheme; second grade fluid; stagnation point flow

Mathematics Subject Classification: 34Bxx, 34A26, 65Lxx

1. Introduction

Stagnation point flow reveals in practically all flow branches of engineering and science. Free stagnation point or a line exists interior of the fluid domain while in some situations flow is stagnated by a solid wall.

Hiemenz [1] demonstrated that Navier-Stokes equations are able to describe the stagnation point flow. Plane stagnation point flow which is called the Hiemenz flow, firstly discussed by Hiemenz as

follows:

$$f''' + f f'' - (f')^2 + 1 = 0, \tag{1.1}$$

with the boundaries:

$$f(0) = 0, \ f'(0) = 0, \ f'(\infty) = 1. \tag{1.2}$$

Gersten et al. [2] generalized these results to the stagnation point flow in three dimensions. Homotopy analysis method is used by Liao [3] to the 2D laminar viscous flow over a semi-infinite plate. Existence and uniqueness consequences for nonlinear third-order ODEs appeared in the stagnation point flow of a hydromagnetic fluid are discussed by Van Gorder et al. [4]. The flow characteristics in an electrically conducting second grade fluid pervaded by a uniform magnetic field over a stretching sheet is discussed by Vajravelu et al. [5]. Steady 2D stagnation-point flow of an incompressible viscous electrically conducting fluid over a flat deformable sheet is studied by Mahapatra et al. [6]. Van Gorder et al. [7] established the existence and uniqueness consequences for fourth-order ODEs over the semi-infinite interval, arising in the hydromagnetic stagnation point flow of a second grade fluid over a stretching sheet.

The concept of Lie groups plays an significant rule in establishing some intense numerical approaches to integrate ODEs, which conserve the geometrical properties. Preserving the geometry construction under discretization, is essential in the improvement of a qualitatively true performance in the minimization of approximation errors. So, along with the geometric structure and invariance of the ODEs, novel approaches may be invented, which are more stable and effective than the common approximation techniques.

A novel geometric method based on $GL(6, \mathbb{R})$ is introduced for the magneto-hemodynamic flow in a semi-porous channel in [8]. Moreover, in [9], a nonlinear heat transfer equation is considered by $SL(2, \mathbb{R})$. Order of differential equations and different initial or boundary conditions conclude different geometric algorithms for solving differential equations. That is, numerical algorithms based on $e.g.$ $GL(6, \mathbb{R})$ and $GL(4, \mathbb{R})$ is completely different. Combination of radial basis functions with GPS are discussed in [10]. Some other recent papers of geometric numerical approach, Lie group analysis, and exact solution can be found in [11–33].

This paper is organized as follows. In the further section, we will touch only a few aspects of the Mathematical formulation for the hydromagnetic stagnation point flow. In Section 3, we derive an interesting algorithm to integrate the underlying nonlinear equation. Section 4 deals with the numerical results. Final remarks are provided in the conclusion section.

2. Mathematical formulation

Let us investigate the 2D steady stagnation-point flow of an electrically conducting fluid in the existence of a uniform transverse magnetic field towards a flat surface coinciding with the plane $y = 0$, the flow being enclosed to the region $y > 0$. Two equivalent and converse forces are considered in direction of the x−axis such that the wall is stretched keeping the origin fixed, and a uniform magnetic field B_0 is forced along the y−axis (Figure 1).
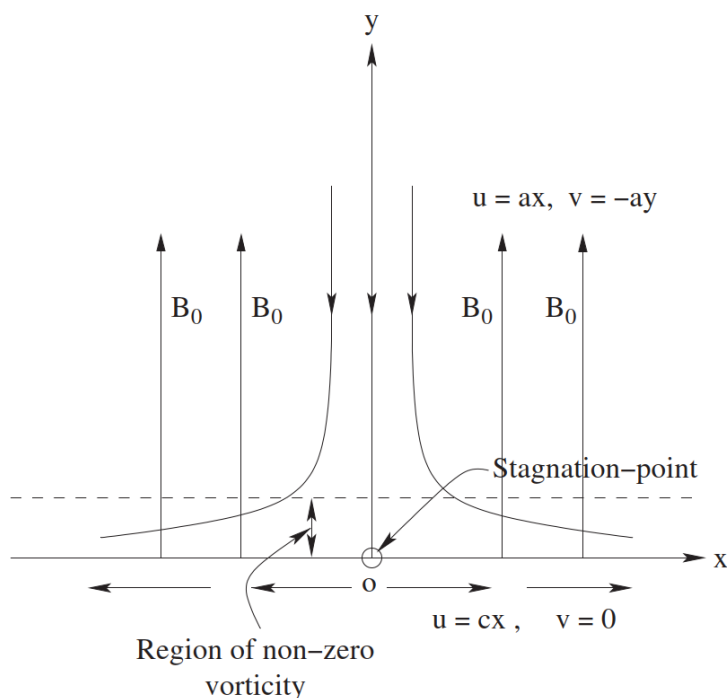
Figure 1. A profile of the physical problem.

In the usual notation, the MHD equations for steady 2D stagnation-point flow in the boundary layer along the stretching surface are [6, 34]:

$$u_x + v_y = 0, \tag{2.1}$$

$$uu_x + vu_y = UU_x + vu_{yy} + \frac{\sigma B_0^2}{\rho}(U - u), \tag{2.2}$$

where ρ, σ, B_0 are density, electrical conductivity and magnetic field respectively. Also, (u, v) are the fluid velocity components toward x and y. $U(x)$ stands for the stagnation-point velocity in the inviscid free stream.

For this problem, the boundaries are

$$u = cx, \ v = 0, \ at \ y = 0, \tag{2.3}$$

$$u \rightarrow U(x) = ax, \ as \ y \rightarrow \infty, \tag{2.4}$$

where $c, a \in \mathbb{R}^+$. Eqs (2.1) and (2.2) together with the conditions (2.3) and (2.4) have the similarities of the form

$$u(x, y) = cxf'(\xi), \ v(x, y) = -\sqrt{cv}f(\xi), \ \xi = y\sqrt{\frac{c}{v}}. \tag{2.5}$$

Substituting (2.5) into (2.2), concludes

$$f'''(\xi) + f(\xi)f''(\xi) - (f'(\xi))^2 - Mf'(\xi) + C(C + M) = 0, \tag{2.6}$$

where $M = B_0^2\left(\frac{\sigma}{\rho c}\right)$, $C = \frac{a}{c}$ and $\sqrt{M}$ is the Hartmann number.

With the use of the transformation (2.5), the boundary conditions (2.3) and (2.4) read $f(0) = 0$, $f'(0) = 1$, $f'(\infty) = C$.

The self-similar solutions will normally be made to meet Eq (2.6) with additional nonlinear terms

$$K[2f'(\xi)f'''(\xi) - (f''(\xi))^2 - f(\xi)f^{(iv)}(\xi)], \tag{2.7}$$

in the governing differential equation, where $K \geq 0$ is the viscoelastic parameter. By the additional condition $f''(\infty) = 0$, we investigate the following fourth order differential equation:

$$f'''(\xi) + f(\xi)f''(\xi) - (f'(\xi))^2 - Mf'(\xi)$$
$$+K[2f'(\xi)f'''(\xi) - (f''(\xi))^2 - f(\xi)f^{(iv)}(\xi)] + C(C + M) = 0, \tag{2.8}$$

with

$$f(0) = s \geq 0, \quad f'(0) = \chi \geq 0, \quad f'(\infty) = C \geq 0, \quad f''(\infty) = 0, \tag{2.9}$$

where $M = \sigma_0 B_0^2 / \rho B \geq 0$ is the magnetic parameter, $K = \omega B / \nu \geq 0$ is the viscoelastic parameter, $s \geq 0$, C is the parameter due to the stagnation point flow, and χ is the stretching of the sheet.

3. An $GL(4, \mathbb{R})$ Lie group method

There is no loss of generality in assuming

$$f(\xi) > 0 \ \& \ f'''(\xi) > 0, \ \forall \xi \in [0, \infty). \tag{3.1}$$

This supposition is possible. Because in the other case, we can obtain a constant $\gamma > 0$ such that

$$F'''(\xi) = f'''(\xi) + \gamma > Max\{0, \Upsilon_1, \Upsilon_2\}, \tag{3.2}$$

$$F''(\xi) = f''(\xi) + \gamma\xi + c_1, \tag{3.3}$$

$$F'(\xi) = f'(\xi) + \frac{\gamma\xi^2}{2} + c_1\xi + c_2, \tag{3.4}$$

$$F(\xi) = f(\xi) + \frac{\gamma\xi^3}{6} + \frac{c_1\xi^2}{2} + c_2\xi + c_3, \tag{3.5}$$

where

$$\Upsilon_1 = \min\{|f(\xi)| \ : \xi \in [0, \infty)\}, \quad \Upsilon_2 = \min\{|f'''(\xi)| \ : \xi \in [0, \infty)\}, \tag{3.6}$$

and c_1–c_3 are integration constants which we can assume all of them to be positive. Also, from this assumption we clearly have $|F(\xi)| > 0$. Using these transformations, Eq (2.8) converts into:

$$F''' - \gamma + \left(F - \frac{\gamma\xi^3}{6} - \frac{c_1\xi^2}{2} - c_2\xi - c_3\right)(F'' - \gamma\xi - c_1)$$

$$-M\left(F'(\xi) - \frac{\gamma\xi^2}{2} - c_1\xi - c_2\right) + K\left[2\left(F'(\xi) - \frac{\gamma\xi^2}{2} - c_1\xi - c_2\right)(F''' - \gamma)\right.$$

$$\left. - (F'' - \gamma\xi - c_1)^2 - F^{(iv)}\left(F - \frac{\gamma\xi^3}{6} - \frac{c_1\xi^2}{2} - c_2\xi - c_3\right)\right] + C(C + M) = 0,$$

subject to

$$F(0) = s + c_3 > 0, \quad F'(0) = \chi + c_2 > 0, \quad F'(\infty) \to \infty, \quad F''(\infty) \to \infty.$$

Now, we are willing to erect $GL(4, \mathbb{R})$ Lie group method. Suppose $y_1(\xi) = f(\xi)$, $y_2(\xi) = f'(\xi)$, $y_3(\xi) = f''(\xi)$ and $y_4(\xi) = f'''(\xi)$. Then equivalent system of (2.8) leaves

$$
\begin{cases}
y_1'(\xi) = y_2(\xi), \\
y_2'(\xi) = y_3(\xi), \\
y_3'(\xi) = y_4(\xi), \\
y_4'(\xi) = \frac{2y_2y_4 - y_3^2}{y_1} + \frac{y_4 + y_1y_3 - y_2^2 - My_2 + C(C+M)}{Ky_1},
\end{cases}
\tag{3.7}
$$

with boundary conditions:

$$
y_1(0) = y_1^0 = s, \ y_2(0) = y_2^0 = \chi, \ y_2(\infty) = y_2^f = C, \ y_3(\infty) = y_3^f = 0.
\tag{3.8}
$$

Equivalent system of (3.7) is:

$$
\frac{d}{d\xi}
\begin{pmatrix}
y_1(\xi) \\
y_2(\xi) \\
y_3(\xi) \\
y_4(\xi)
\end{pmatrix}
=
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & \Omega(\xi, y_1, y_2, y_3, y_4)
\end{pmatrix}
\begin{pmatrix}
y_1(\xi) \\
y_2(\xi) \\
y_3(\xi) \\
y_4(\xi)
\end{pmatrix},
\tag{3.9}
$$

where

$$
\Omega(\xi, y_1, y_2, y_3, y_4) := \frac{2y_2y_4 - y_3^2}{y_1y_4} + \frac{y_4 + y_1y_3 - y_2^2 - My_2 + C(C+M)}{Ky_1y_4},
\tag{3.10}
$$

which is well-defined because of supposition (3.1). System (3.9), in spite of its nonlinearity, admits a Lie-group of $GL(4, \mathbb{R})$ and we have:

$$
\frac{d}{d\xi}\mathbb{G} = \mathcal{A}\mathbb{G}, \ \ \mathbb{G}(0) = I_{4\times 4},
\tag{3.11}
$$

where

$$
\mathcal{A} =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & \Omega
\end{pmatrix}.
\tag{3.12}
$$

Now, we are able to erect a GPS to consider Eq (3.9) as follows:

$$
\mathbf{Y}_{n+1} = \mathbb{G}(n)\mathbf{Y}_n, \ \ \mathbb{G}(n) \in GL(4, \mathbb{R}),
\tag{3.13}
$$

where $\mathbf{Y}_n = \mathbf{Y}|_{\xi=\xi_n}$. Integrating (3.9) by (3.13) along with initial condition $\mathbf{Y}(0) = \mathbf{Y}_0$, results the demanded value $\mathbf{Y}(\xi)$. Suppose the GPS method:

$$
\mathbf{Y}_{n+1} = \mathbf{Y}_n + \frac{(\alpha_n - 1)\mathcal{F}_n.\mathbf{Y}_n + \beta_n\|\mathbf{Y}_n\|\|\mathcal{F}_n\|}{\|\mathcal{F}_n\|^2},
\tag{3.14}
$$

where

$$
\alpha_n = \cosh\left(\frac{\Delta\xi\|\mathcal{F}_n\|}{\|\mathbf{Y}_n\|}\right), \ \beta_n = \sinh\left(\frac{\Delta\xi\|\mathcal{F}_n\|}{\|\mathbf{Y}_n\|}\right),
\tag{3.15}
$$

and $\Delta\xi = 1/N$ is the GPS iteration stepsize. In fact, this approach approximates the solution of

$$\begin{cases} \mathbf{Y}' = \mathcal{F}(\xi, \mathbf{Y}), \\ \quad \mathbf{Y}(0) = \mathbf{Y}_0. \end{cases} \tag{3.16}$$

Then

$$\mathbf{Y}_f = \mathbb{G}_N(\Delta\xi) \cdots \mathbb{G}_1(\Delta\xi)\mathbf{Y}_0, \tag{3.17}$$

computes $\mathbf{Y}(\xi)|_{\xi=\eta_\infty},^*$ at $\xi = \eta_\infty$. From the closure property of the Lie groups, $\mathbb{G} := \mathbb{G}_N(\Delta\xi) \cdots \mathbb{G}_1(\Delta\xi) \in GL(4, \mathbb{R})$. Therefore, we can develop a one-step transformation from $\mathbf{Y}_0$ to $\mathbf{Y}_f$ as:

$$\mathbf{Y}_f = \mathbb{G}\mathbf{Y}_0, \quad \mathbb{G} \in GL(4, \mathbb{R}). \tag{3.18}$$

From Eqs (3.11) and (3.18) we obtain

$$\mathbb{G}(\xi) = exp\left(\int_0^\xi \mathcal{A}(\eta)d\eta\right). \tag{3.19}$$

Along with (3.8) and generalized mid-point rule, let us assume

$$\begin{aligned} \hat{\xi} &= r\xi_0 + (1-r)\xi_f = (1-r)\eta_\infty, \\ \hat{y}_1 &= ry_1^0 + (1-r)y_1^f = rs + (1-r)y_1^f, \\ \hat{y}_2 &= ry_2^0 + (1-r)y_2^f = r\chi + (1-r)C, \\ \hat{y}_3 &= ry_3^0 + (1-r)y_3^f = ry_3^0, \\ \hat{y}_4 &= ry_4^0 + (1-r)y_4^f = ry_4^0 + (1-r)y_4^f, \end{aligned}$$

where $r \in [0, 1]$ has to be determined. Therefore Eq (3.19) may be rewritten as:

$$\mathbb{G}(r) = exp\left(\mathcal{A}(\hat{\xi}, \hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4)\right), \tag{3.20}$$

where

$$\mathcal{A}(\hat{\xi}, \hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4) =: \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \quad \omega = \Omega(\hat{\xi}, \hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4). \tag{3.21}$$

In this letter, $y_1^0 = s$, $y_2^0 = \chi$, $y_2^f = C$, $y_3^f = 0$ are known and y_1^f, y_3^0, y_4^0, y_4^f are unknown values of the current problem, which after determining y_3^0 and y_4^0, the B.V.P. (2.8) converts to an I.V.P.

For the fixed $\mathcal{A} \in gl(4, \mathbb{R}),^\dagger$ we have

$$\mathbb{G}(r) = \begin{pmatrix} 1 & \eta_\infty & \frac{\eta_\infty^2}{2} & \frac{2e^{\eta_\infty\omega} - 2\eta_\infty\omega - (\eta_\infty\omega)^2 - 2}{2\omega^3} \\ 0 & 1 & \eta_\infty & \frac{e^{\eta_\infty\omega} - \eta_\infty\omega - 1}{\omega^2} \\ 0 & 0 & 1 & \frac{e^{\eta_\infty\omega} - 1}{\omega} \\ 0 & 0 & 0 & e^{\eta_\infty\omega} \end{pmatrix}, \tag{3.22}$$

---

where

$$\omega = \frac{2\hat{y}_2\hat{y}_4 - \hat{y}_3^2}{\hat{y}_1\hat{y}_4} + \frac{\hat{y}_4 + \hat{y}_1\hat{y}_3 - \hat{y}_2^2 - M\hat{y}_2 + C(C+M)}{K\hat{y}_1\hat{y}_4}. \tag{3.23}$$

Therefore, each of this Lie group elements may be erected by a single parameter $\mathbb{G}(r)$, with $r \in [0, 1]$.

From (3.18) and (3.22) we have:

$$\begin{pmatrix} y_1^f \\ y_2^f \\ y_3^f \\ y_4^f \end{pmatrix} = \begin{pmatrix} 1 & \eta_\infty & \frac{\eta_\infty^2}{2} & \frac{2e^{\eta_\infty\omega} - 2\eta_\infty\omega - (\eta_\infty\omega)^2 - 2}{2\omega^3} \\ 0 & 1 & \eta_\infty & \frac{e^{\eta_\infty\omega} - \eta_\infty\omega - 1}{\omega^2} \\ 0 & 0 & 1 & \frac{e^{\eta_\infty\omega} - 1}{\omega} \\ 0 & 0 & 0 & e^{\eta_\infty\omega} \end{pmatrix} \begin{pmatrix} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{pmatrix}. \tag{3.24}$$

Thus, for a given r, we may get the values y_1^f, y_3^0, y_4^0 and y_4^f by the further explained process.

Imposing the initial guesses of y_3^0, y_4^0, y_1^f and y_4^f, and the given boundary values of $y_1^0 = s$, $y_2^0 = \chi$, $y_2^f = C$ and $y_3^0 = 0$, with a specified $r \in [0, 1]$, and from

$$\begin{cases} y_3^0 = \dfrac{\omega(C - \chi)(e^{\eta_\infty\omega} - 1)}{(\eta_\infty\omega - 1)e^{\eta_\infty\omega} + 1}, \\[3mm] y_4^0 = \dfrac{\omega^2(\chi - C)}{(\eta_\infty\omega - 1)e^{\eta_\infty\omega} + 1}, \\[3mm] y_1^f = \dfrac{e^{\eta_\infty\omega}\left(2s\eta_\infty\omega^2 - 2s\omega + \chi\eta_\infty^2\omega^2 - 2\chi\eta_\infty\omega + C\eta_\infty^2\omega^2 - 2C + 2\chi\right) + 2s\omega + 2C\eta_\infty\omega + 2C - 2\chi}{2\omega((\eta_\infty\omega - 1)e^{\eta_\infty\omega} + 1)}, \\[3mm] y_4^f = \dfrac{\omega^2(\chi - C)e^{\eta_\infty\omega}}{(\eta_\infty\omega - 1)e^{\eta_\infty\omega} + 1}, \end{cases} \tag{3.25}$$

we may produce new values of y_1^f, y_3^0, y_4^0 and y_4^f. If these values satisfy the convergence criterion:

$$\sqrt{\sum_{i \in \{3,4\}} \left(y_{i\,New}^0 - y_{i\,Old}^0\right)^2 + \sum_{i \in \{1,4\}} \left(y_{i\,New}^f - y_{i\,Old}^f\right)^2} \le \epsilon, \tag{3.26}$$

then iterations stop.

For a fixed value of r, we are able to compute y_3^0 and y_4^0 from Eq (3.25) and then approximately integrate Eq (3.7) by GPS from 0 to η_∞, and compare the end value of y_2^f and y_3^f with the exact ones $y_2(\eta_\infty) = C$ and $y_3(\eta_\infty) = 0$ respectively. That is, we have to find the minimum value of

$$\min_{r \in [0,1]} \sqrt{\left(y_2^f - C\right)^2 + (y_3^f)^2}. \tag{3.27}$$

In the appendix, we give a brief algorithm of our proposed method to find the initial values of problem.

In the non-geometric numerical methods, investigation of the method stability is an important task which guarantees the validity of techniques. Without considering the convergence and error analysis, results may lead to solutions with no physical meaning or high-level of errors. This issue may arise

from that, we do not know information about the geometric structure of original problem. However, in this paper we introduce a $GL(4, \mathbb{R})$ method which conserve the geometrical properties of problem. Preserving the geometry construction under discretization, is essential in the improvement of a qualitatively true performance in the minimization of approximation errors.

4. Numerical results

We emphasize that the stepping approach developed for integrating the IVP needs the initial values $f_1(0) = f(0)$, $f_2(0) = f'(0)$, $f_3(0) = f''(0)$ and $f_4(0) = f'''(0)$ for the fourth-order ODEs. Both of the last initial values in Eq (2.8) are unknown. Therefore, approximating $f''(0)$ and $f'''(0)$ converts Eq (2.8) into an IVP. It is worth pointing out that every IVP can be integrated by GPS.

Indeed, our proposed method in this paper is split into two parts. In the first part, initial conditions of the problem are obtain by the $GL(4, \mathbb{R})$ method. Our method is implemented by the convergence criterion of (3.26) with $\epsilon = 10^{-8}$. For this value of convergence criterion, only four or five steps of algorithm, mentioned in the Appendix, are needed. So, very fast convergence occur in this part of the proposed method Second part of this technique starts when we obtain the initial values of BVP from the first part. Second part which is nominated as the GPS, is an explicit iteration method which is well-known as the weighted Euler method. Convergence of this part can be guaranteed from the geometric structure of method.

The results of this paper were announced with $\eta_\infty = 7$ and $\Delta\xi = 0.01$ for the GPS procedure. There is no loss of generality in assuming $y_3^0 = y_4^0 = y_1^f = y_4^f = 1$ as initial guesses.

Some mis-matching errors with various M, C, s, K and χ are plotted in Figure 2(a)–(d). Desired values of $r = 0.5314$, $r = 0.4993$, $r = 0.814$ and $r = 0.9546$ are noticeable from these figures.
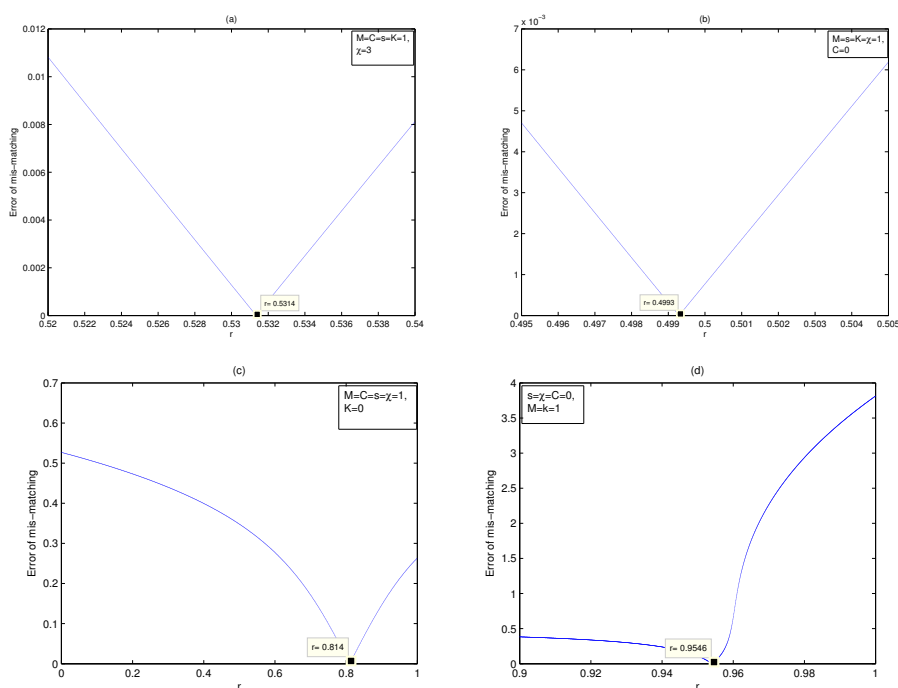

Figure 2. Plots of mis-matching errors for some values of M, C, s, K and χ.

In Figure 3(a)–(d), we plot the profiles of $f(\xi)$, $f'(\xi)$, $f''(\xi)$ and $f'''(\xi)$ for various values of magnetic parameter (the Hartmann number) M ($\sqrt{M}$), and fixed values of the other parameters. It is observable that the profiles of $f(\xi)$ and $f'(\xi)$ decrease, when the magnetic parameter increases, while the reverse situation occur for $f''(\xi)$ and it decrease more rapidly toward zero as ξ tends to η_∞.
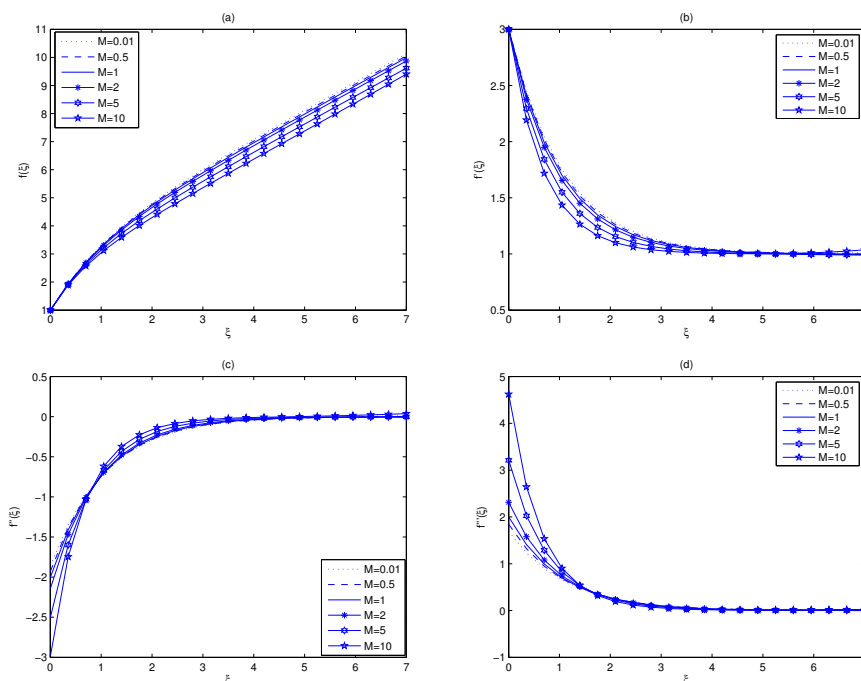

Figure 3. Profiles of (a) : $f(\xi)$, (b) : $f'(\xi)$, (c) : $f''(\xi)$ and (d) : $f'''(\xi)$, for various M when $C = s = K = 1$ and $\chi = 3$.

Figure 4(a)–(d) show the absorbing dependence of the solutions on the various viscoelastic parameter K. Uniform decrease in the magnitudes of $f(\xi)$ and $f'(\xi)$ are observable for increasing values in K. Inverse results occur for the profiles of $f(\xi)$ and $f'(\xi)$ when χ is greater than the boundary value C; see Figure 5(a)–(d). Notice that the plots are in a good agreement with the boundary conditions $f'(\eta_\infty) = C$ and $f''(\eta_\infty) = 0$.

It may be seen from Figure 6(a)–(d) that for fixed values of K, s, M and χ, the transverse velocity increases with increase in the stagnation point flow C. Notice the behavior of solutions in the profiles of $f(\xi)$, $f'(\xi)$ and $f''(\xi)$ for different cases $\chi < C$ and $\chi > C$ which are against to the profile of $f'''(\xi)$.

From the Figure 7(a)–(d), an increase in the stretching sheet parameter χ results in a increase in the profiles of $f(\xi)$, $f'(\xi)$ and $f'''(\xi)$. Also, an increase in χ consequences in a decrease in the magnitude of the shear stress at the wall, $f''(0)$.
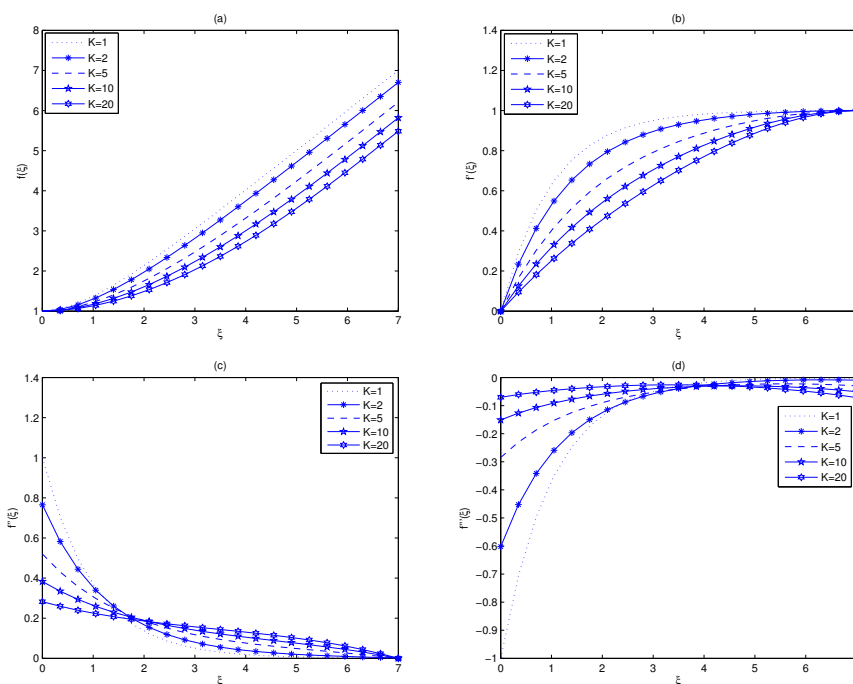
Figure 4. Variation of $(a) : f(\xi)$, $(b) : f'(\xi)$, $(c) : f''(\xi)$ and $(d) : f'''(\xi)$, for various viscoelastic parameter K when $C = s = M = 1$, and $\chi = 0$.
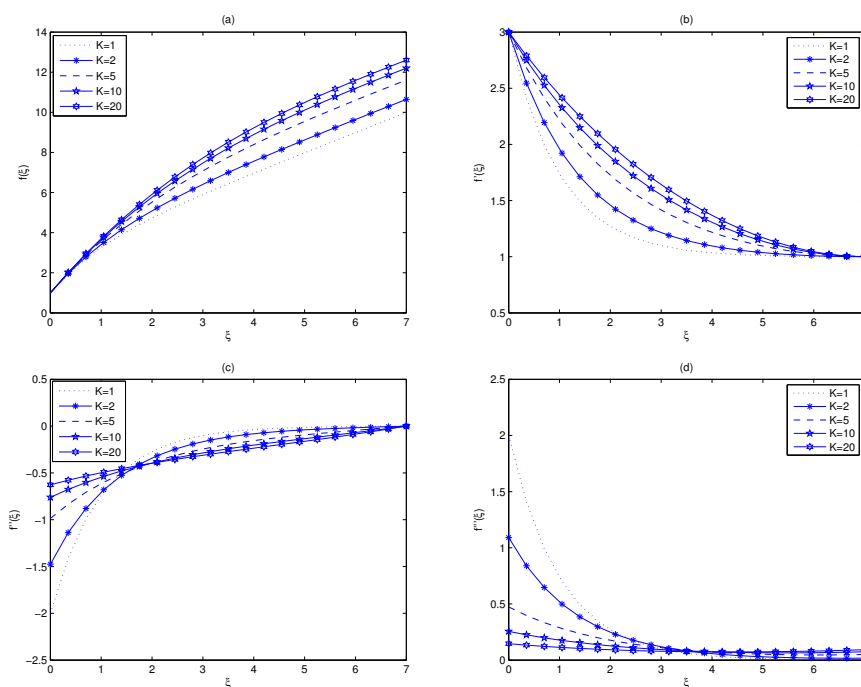

Figure 5. Profiles of $(a) : f(\xi)$, $(b) : f'(\xi)$, $(c) : f''(\xi)$ and $(d) : f'''(\xi)$, for various viscoelastic parameter K when $C = s = M = 1$, and $\chi = 3$.
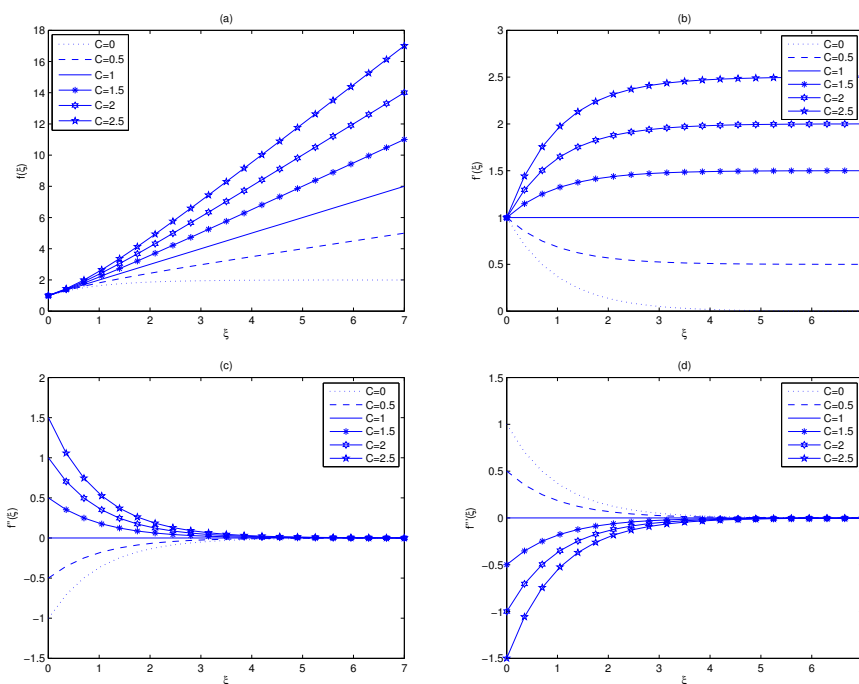
Figure 6. Variation of (a) : $f(\xi)$, (b) : $f'(\xi)$, (c) : $f''(\xi)$ and (d) : $f'''(\xi)$, for various stagnation point flow C when $K = s = M = 1$, and $\chi = 1$.
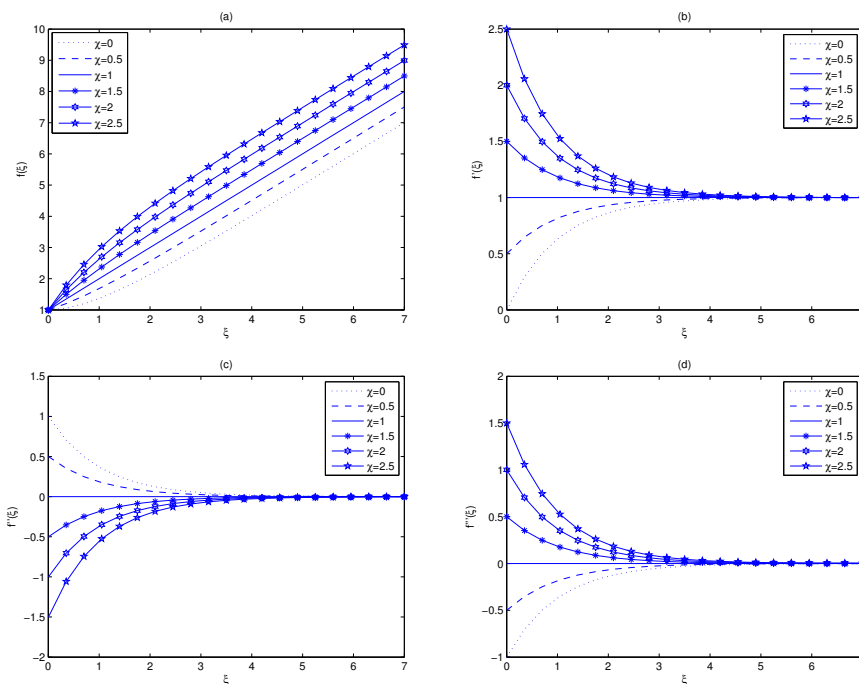

Figure 7. Profiles of (a) : $f(\xi)$, (b) : $f'(\xi)$, (c) : $f''(\xi)$ and (d) : $f'''(\xi)$, for various stretching sheet parameter χ when $K = s = M = 1$, and $C = 1$.

The double mesh principle is applied to estimate the error

$$e^N(\xi) = |f^N(\xi) - f^{2N}(\xi)|, \tag{4.1}$$

where N is the number of GPS iterations. Residual errors of the present method applied for Eq (2.8) with $N = 10000$ are plotted in Figure 8 for various parameters of equation. This figure shows that our method is reliable for current types of equations. Besides, obtained solutions and results are in good agreement with the results of [7].
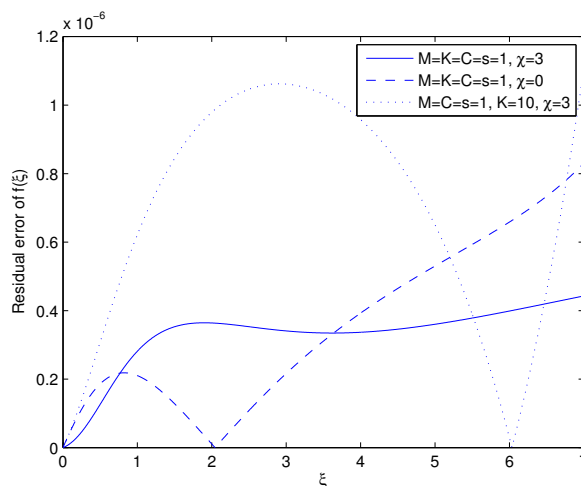

Figure 8. Plot of estimate errors of $f(\xi)$ for some values of M, C, K, s and χ.

5. Conclusions

In this paper, a novel iterative technique based on the Lie group of $GL(4, \mathbb{R})$, as the first time, is proposed to solve the hydromagnetic stagnation point flow of a second grade fluid over a stretching sheet. Structures of the Lie group elements in $GL(4, \mathbb{R})$ and corresponding Lie algebra elements in $gl(4, \mathbb{R})$ are completely innovative for the underlying problem in this paper. Against the non-geometric numerical techniques, the proposed method uses the geometric structure of original problem and this guarantees that obtained results are reliable and the proposed method avoid the ghost solutions. Effects of equation parameters are graphically considered. Point-wise errors for various parameters show the power and effectiveness of proposed method.

Conflict of interest

No potential competing interest was reported by the authors.

References

1. K. Hiemenz, Boundary Layer for a homogeneous flow around a dropping cylinder, *Dinglers Polytech. J.*, **326** (1911), 215–220.

2. K. Gersten, H. Papenfuss, J. Gross, Influence of the Prandtl number on second-order heat transfer due to surface curvature at a three dimensional stagnation point, *Int. J. Heat. Mass Transfer*, **21** (1978), 275–284.

3. S. Liao, A uniformly valid analytic solution of two-dimensional viscous flow over a semi-infinite flat plate, *J. Fluid Mech.*, **385** (1999), 101–128.

4. R. Van Gorder, K. Vajravelu, I. Pop, Hydromagnetic stagnation point flow of a viscous fluid over a stretching or shrinking sheet, *Meccanica*, **47** (2012), 31–50.

5. K. Vajravelu, D. Rollins, Hydromagnetic flow of a second grade fluid over a stretching sheet, *Appl. Math. Comput.*, **148** (2004), 783–791.

6. T. Mahapatra, A. Gupta, Magnetohydrodynamic stagnation-point flow towards a stretching sheet, *Acta Mech.*, **152** (2001), 191–196.

7. R. Van Gorder, K. Vajravelu, Hydromagnetic stagnation point flow of a second grade fluid over a stretching sheet, *Mech. Res. Commun.*, **37** (2010), 113–118.

8. M. Hashemi, A novel simple algorithm for solving the magneto-hemodynamic flow in a semi-porous channel, *Eur. J. Mech.-B/Fluids*, **65** (2017), 359–367.

9. M. Hashemi, Constructing a new geometric numerical integration method to the nonlinear heat transfer equations, *Commun. Nonlinear. Sci. Numer. Simul.*, **22** (2015), 990–1001.

10. M. Hajiketabi, S. Abbasbandy, F. Casas, The lie-group method based on radial basis functions for solving nonlinear high dimensional generalized benjamin-bona-mahony-burgers equation in arbitrary domains, *Appl. Math. Comput.*, **321** (2018), 223–243.

11. P. Bader, S. Blanes, F. Casas, N. Kopylov, E. Ponsoda, Symplectic integrators for second-order linear non-autonomous equations, *J. Comput. Appl. Math.*, **330** (2018), 909–919.

12. H. Ahmad, N. Alam, M. Omri, New computational results for a prototype of an excitable system, *Results Phys.*, **28** (2021), 104666.

13. H. Almusawa, R. Ghanam, G. Thompson, Classification of symmetry lie algebras of the canonical geodesic equations of five-dimensional solvable lie algebras, *Symmetry*, **11** (2019), 1354.

14. H. Almusawa, R. Ghanam, G. Thompson, Lie symmetries of the canonical connection: Codimension one abelian nilradical case, *J. Nonlinear Math. Phys.*, **28** (2021), 242–253.

15. A. Akbulut, H. Almusawa, M. Kaplan, M. S. Osman, On the conservation laws and exact solutions to the (3+ 1)-dimensional modified kdv-zakharov-kuznetsov equation, *Symmetry*, **13** (2021), 765.

16. M. Hashemi, S. Abbasbandy, A geometric approach for solving troesch's problem, *Bull. Malays. Math. Sci. Soc.*, **40** (2017), 97–116.

17. S. Boulaaras, M. Haiour, The theta time scheme combined with a finite-element spatial approximation in the evolutionary hamilton-jacobi-bellman equation with linear source terms, *Comput. Math. model.*, **25** (2014), 423–438.

18. M. S. Hashemi, E. Darvishi, D. Baleanu, A geometric approach for solving the density-dependent diffusion nagumo equation, *Adv. Differ. Equ.*, **2016** (2016), 1–13.

19. S. Boulaaras, A well-posedness and exponential decay of solutions for a coupled lamé system with viscoelastic term and logarithmic source terms, *Appl. Anal.*, **100** (2021), 1514–1532.

20. I. Ahmad, H. Ahmad, M. Inc, H. Rezazadeh, M. A. Akbar, M. M. Khater, et al., Solution of fractional-order korteweg-de vries and burgers' equations utilizing local meshless method, *J. Ocean Eng. Sci.*, in press.

21. A. Yokus, H. Durur, H. Ahmad, P. Thounthong, Y. F. Zhang, Construction of exact traveling wave solutions of the bogoyavlenskii equation by $(g'/g, 1/g)$-expansion and $(1/g')$-expansion techniques, *Results Phys.*, **19** (2020), 103409.

22. A. Akgül, M. Hashemi, M. Inc, S. Raheem, Constructing two powerful methods to solve the Thomas-Fermi equation, *Nonlinear Dyn.*, **87** (2017), 1435–1444.

23. M. Inc, H. Rezazadeh, J. Vahidi, M. Eslami, M. A. Akinlar, M. N. Ali, et al., New solitary wave solutions for the conformable klein-gordon equation with quantic nonlinearity, *AIMS Math.*, **5** (2020), 6972–6984.

24. S. Boulaaras, M. Haiour, The finite element approximation of evolutionary hamilton-jacobi-bellman equations with nonlinear source terms, *Indagat. Math.*, **24** (2013), 161–173.

25. A. Yokus, H. Durur, H. Ahmad, Hyperbolic type solutions for the couple boiti-leon-pempinelli system, *Facta Univ., Ser.: Math. Inform.*, **35** (2020), 523–531.

26. S. Kumar, H. Almusawa, S. K. Dhiman, M. Osman, A. Kumar, A study of Bogoyavlenskii's (2+1)-dimensional breaking soliton equation: Lie symmetry, dynamical behaviors and closed-form solutions, *Results Phys.*, **29** (2021), 104793.

27. S. Kumar, H. Almusawa, A. Kumar, Some more closed-form invariant solutions and dynamical behavior of multiple solitons for the (2+1)-dimensional rddym equation using the lie symmetry approach, *Results Phys.*, **24** (2021), 104201.

28. L. Akinyemi, M. Şenol, M. Mirzazadeh, M. Eslami, Optical solitons for weakly nonlocal schrödinger equation with parabolic law nonlinearity and external potential, *Optik*, **230** (2021), 166281.

29. A. Yokus, H. Durur, H. Ahmad, S. W. Yao, Construction of different types analytic solutions for the zhiber-shabat equation, *Mathematics*, **8** (2020), 908.

30. M. Senol, L. Akinyemi, A. Ata, O. S. Iyiola, Approximate and generalized solutions of conformable type Coudrey-Dodd-Gibbon-Sawada-Kotera equation, *Int. J. Modern Phys. B*, **35** (2021), 2150021.

31. L. Akinyemi, H. Rezazadeh, S. W. Yao, M. A. Akbar, M. M. Khater, A. Jhangeer, et al., Nonlinear dispersion in parabolic law medium and its optical solitons, *Results Phys.*, **26** (2021), 104411.

32. M. A. Akbar, L. Akinyemi, S. W. Yao, A. Jhangeer, H. Rezazadeh, M. M. Khater, et al., Soliton solutions to the boussinesq equation through sine-gordon method and kudryashov method, *Results Phys.*, **25** (2021), 104228.

33. M. S. Hashemi, D. Baleanu, *Lie symmetry analysis of fractional differential equations*, CRC Press, 2020.

34. T. R. Mahapatra, S. Nandy, A. Gupta, Magnetohydrodynamic stagnation-point flow of a power-law fluid towards a stretching surface, *Int. J. Nonlinear Mech.*, **44** (2009), 124–129.

Appendix

Algorithm of finding initial values of the problem:

(i) Take arbitrary initial guesses for y_3^0, y_4^0, y_1^f and y_4^f.
(ii) Specify an $r \in [0, 1]$ and a small $\epsilon \in \mathbb{R}^+$.
(iii) Calculate $\mathbb{G}(r)$ and ω from (3.22) and (3.23), respectively.
(iv) Compute updated values of y_3^0, y_4^0, y_1^f and y_4^f from (3.25).
(v) If convergence criterion (3.26) is valid for given ϵ, then y_3^0 and y_4^0 are desired initial values of the problem. Otherwise, go to Step (ii).