Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13370–13391.
DOI:10.3934/math.2021774
Received: 04 April 2021
Accepted: 07 September 2021
Published: 17 September 2021

Research article

Solving singular coupled fractional differential equations with integral boundary constraints by coupled fixed point methodology

Hasanen A. Hammad[1] **and Watcharaporn Chaolamjiak**[2,*]

[1] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
[2] School of Science, University of Phayao, Phayao 56000, Thailand

* **Correspondence:** Email: watcharaporn.ch@up.ac.th.

Abstract: This manuscript was originally built to establish some coupled common fixed point results for rational contractive mapping in the framework of b-metric spaces. Thereafter, the existence and uniqueness of the boundary value problem for a singular coupled fractional differential equation of order ν via coupled fixed point techniques are discussed. At the last, some supportive examples to illustrate the theoretical results are presented.

Keywords: coupled fractional differential equations; coupled fixed point techniques; b-metric spaces; rational contractive mappings; Green's functions
Mathematics Subject Classification: 34B15, 54H10, 54H25

1. Background materials

The science of fractional differentiation is related to many engineering disciplines because its basis is based on differential equations that have a long history in chemistry, polymer rheology electrodynamics, physics and aerodynamics. Derivatives of fractional order are also included in mathematical simulations of structures and processes [1–3]. More broadly, differential equations of fractional order often become means of multiple perspectives on control systems, fluid dynamics, and so on.

Another reason for the importance of studying fractional order differential equations is that the fractional order models are more accurate than the correct order models and they also seem to have a greater degree of freedom. To learn more recent results about this branch, we cite [4–13].

The integral boundary stipulations play a prominent role in many applications such as thermo-elasticity, population dynamics, problems with blood flow and underground water supply. To obtain a full and comprehensive explanation of the terms of integral boundaries, we direct the reader to certain recent publications [14–24].

The fixed point technique is one of the final modeling methods for many fields. In engineering, it is used to achieve solutions or search for more effective results. In general, this method has become one of the best methods used in modern mathematics, especially functional analysis. This method relates to the existence, uniqueness and characteristics of a specified operator's fixed points.

One of the very important discoveries of this technique is the Banach contraction principle [25], as it contributed greatly to spread after exploring generalized metric areas that were greatly enamored by the authors in the field of fractional differential equations. For further clarification, see [7, 10, 26–29].

The notion of coupled fixed points (CFPs) was introduced in 1987 by Guo and Lakshmikantham [30] and applications to it were recounted by Bhaskar and Lakshmikantham [31] who were able to study the monotone property and applied the theoretical results to find a unique solution to periodic boundary value problems (BVPs). In abstract spaces, a lot of authors generalized this concept and obtained pivotal results and more applications. For more details, see [32–37].

Coupled fixed points are not only an abstract definition but have many vital applications in some models of economics such as equilibrium in duopoly markets and variational principle, for instance, see [38, 39].

In the framework of b-metric space (bMS), our main aim of this paper is to establish some common CFP results for two rational contractive mappings under mild conditions. The theoretical results are applied to discuss the existence and uniqueness of the solution for a singular coupled fractional differential equation (CFDE) of order ν in the form of:

$$\begin{cases} {}^{c}\Theta^{\nu}z(\tau) + \Xi(\tau, z(\tau), w(\tau)) = 0, \ \tau \in (0, 1), \\ {}^{c}\Theta^{\nu}w(\tau) + \Xi(\tau, w(\tau), z(\tau)) = 0, \ \tau \in (0, 1), \\ \Lambda'''(0) = \Lambda''(0) = 0, \\ \Lambda' = \Lambda(1) = \alpha \int_0^1 \Lambda(\theta) d\theta, \end{cases} \tag{1.1}$$

where $\Lambda \in C[0, 1] \times C[0, 1]$ and given by $\Lambda(\tau) = (z(\tau), w(\tau))$, $\nu \in (3, 4)$, $\alpha \in (0, 2)$, ${}^{c}\Theta^{\nu}$ is the Caputo fractional derivative and Ξ may be singular at $z = 0$ and $w = 0$.

2. Preliminaries

The concept of bMSs initiated by Czerwik [40] in the year 1993, as a generalization of ordinary metric spaces. Just it's multiplying the constant b at the right-hand side of the triangle inequality.

Definition 2.1. A b-metric on a nonempty set Ω is a function $\mu_b : \Omega \times \Omega \to \mathbb{R}^+$ such that for all $a_1, a_2, a_3 \in \Omega$ and a constant $b \geq 1$, the hypotheses below hold:

i. $\mu_b(a_1, a_2) = 0$ if and only if $a_1 = a_2$;
ii. $\mu_b(a_1, a_2) = \mu_b(a_2, a_1)$;
iii. $\mu_b(a_1, a_3) \leq b\left[\mu_b(a_1, a_2) + \mu_r(a_2, a_3)\right]$.

The pair (Ω, μ_b) is called bMS with parameter b.

Example 2.1. [41] Let $\ell_p(0 < p < 1) = \{\{a_i\} \in \mathbb{R} : \sum\limits_{i=1}^{+\infty} |a_i|^p < \infty\}$ and $\mu_b : \ell_p \times \ell_p \to \mathbb{R}^+$ be a function described as $\mu_b(a_1, a_2) = \left(\sum\limits_{i=1}^{+\infty} \left|a_1^i - a_2^i\right|^p\right)^{\frac{1}{p}}$, where $a_1 = \{a_1^i\}$; $a_2 = \{a_2^i\} \in \ell_p$. Then $\left(\ell_p, \mu_b\right)$ is a bMS with $b = 2^{\frac{1}{p}}$.

Example 2.2. [41] Let $L_p(0 < p < 1)$ be the space of all real continuous functions $a(\tau)$, $\tau \in [0,1]$ so that $\int_0^1 |a(\tau)|^p \, d\tau \le \infty$ and $\mu_b : L_p \times L_p \to \mathbb{R}^+$ be a function described as $\mu_b(a_1, a_2) = \left(\int_0^1 |a_1(\tau) - a_2(\tau)|^p \, d\tau \right)^{\frac{1}{p}}$, for each $a_1, a_2 \in L_p$. Then $\left(L_p, \mu_b\right)$ is a bMS with $b = 2^{\frac{1}{p}}$.

Definition 2.2. [41] Let (Ω, μ_b) be a bMS, the sequence $\{a_i\}$ in Ω is called:

(i) convergent to $a \in \Omega$ if for each $\varepsilon > 0$, there exists $Q(\varepsilon) \in \mathbb{N}$ so that $\mu_r(a_i, a) < \varepsilon$ for all $i \ge Q(\varepsilon)$ and we write $\lim_{i \to +\infty} a_i = a$;
(ii) a Cauchy sequence if for each $\varepsilon > 0$, there exists $Q(\varepsilon) \in \mathbb{N}$ so that $\mu_r(a_i, a_j) < \varepsilon$ for all $i, j \ge Q(\varepsilon)$.

If every Cauchy sequence in Ω converges in Ω, then a bMS is called complete.

It should be noted that in a bMS, a convergent sequence has a unique limit and every convergent sequence is Cauchy.

Definition 2.3. [42] Assume that $\Upsilon, \Xi : \Omega \times \Omega \to \Omega$ are two mappings on a bMS (Ω, μ_b), the pair $(a_1, a_2) \in \Omega \times \Omega$ is called:

i. a CFP of Υ if $a_1 = \Upsilon(a_1, a_2)$ and $a_2 = \Upsilon(a_2, a_1)$;
ii. a coupled coincidence point of Υ and Ξ if $\Upsilon(a_1, a_2) = \Xi(a_1, a_2)$ and $\Xi(a_2, a_1) = \Upsilon(a_2, a_1)$;
iii. a common CFP of Υ and Ξ if $a_1 = \Upsilon(a_1, a_2) = \Xi(a_1, a_2)$ and $a_2 = \Xi(a_2, a_1) = \Upsilon(a_2, a_1)$.

For the convenience of the reader, we present some definitions and necessary lemmas from the theory of fractional analysis.

Definition 2.4. [43] The Caputo derivative of fractional order $\nu > 0$, $n - 1 < \nu < n$, $n \in \mathbb{N}$, for the function $z(\tau) : [0, \infty) \to \mathbb{R}$ is described as

$$^{C}\Theta^{\nu} z(\tau) = \frac{1}{\Gamma(n-\nu)} \int_0^{\tau} \frac{z^n(\theta)}{(\tau - \theta)^{\nu - n + 1}} d\theta, \; n = [\nu] + 1,$$

where $[\nu]$ represents the integer part of the real number ν.

Definition 2.5. [43] For a function $z(\tau) : [0, \infty) \to \mathbb{R}$, the Riemann–Liouville fractional integral of order ν is described as

$$I^{\nu} z(\tau) = \frac{1}{\Gamma(\nu)} \int_0^{\tau} (\nu - \theta)^{\nu - 1} z(\theta) d\theta, \; \nu > 0,$$

provided that an integral exists.

Lemma 2.1. *[44] Consider* $\nu > 0$, *then*

$$I^{\nu C}\Theta^{\nu} z(\tau) = z(\tau) - C_0 - C_1\tau - C_2\tau^2 - ... - C_{n-1}\tau^{n-1},$$

where $C_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, $n = [\nu] + 1$.

Lemma 2.2. *[1] If* $\nu > 0$ *and* $\alpha > 0$, *then*

i. $^{c}\Theta^{\nu} \tau^{\alpha - 1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \nu)} \tau^{\alpha - \nu - 1}$, *for* $\alpha > n$;
ii. $^{c}\Theta^{\nu} \tau^{k} = 0$, *for* $k = 0, 1, ..., n-1$.

3. Common coupled fixed point results

Theorem 3.1. *Assume that* (Ω, μ_r) *is a complete bMS with a coefficient* $r(= b) \geq 1$ *and let the mappings* $\Upsilon, \Xi : \Omega \times \Omega \to \Omega$ *satisfy*

$$\begin{aligned}\mu_r(\Upsilon(\rho,a), \Xi(\sigma,b)) \leq\ & \lambda\frac{\mu_r(\rho,\sigma)+\mu_r(a,b)}{2} + \tau\frac{[1+\mu_r(\rho,\Upsilon(\rho,a))]\,\mu_r(\sigma,\Xi(\sigma,b))}{(1+\mu_r(\rho,\sigma)+\mu_r(a,b))} \\ & +\zeta\frac{\mu_r(\sigma,\Upsilon(\rho,a))\,\mu_r(\rho,\Xi(\sigma,b))}{(1+\mu_r(\rho,\sigma)+\mu_r(a,b))}. \end{aligned} \tag{3.1}$$

for all $\rho, a, \sigma, b \in \Omega$ *and* $\lambda, \tau, \zeta \geq 0$ *with* $r\lambda + \tau < 1$ *and* $\lambda + \zeta < 1$.

Then there is a unique common CFP of Υ *and* Ξ.

Proof. Assume that $\rho_0, a_0 \in \Omega$ is arbitrary points. Describe the sequences $\{\rho_{2i+1}\}_{i=0}^{+\infty}$, $\{a_{2i+1}\}_{i=0}^{+\infty}$, $\{\rho_{2i+2}\}_{i=0}^{+\infty}$ and $\{a_{2i+2}\}_{i=0}^{+\infty}$ as

$$\rho_{2i+1} = \Upsilon(\rho_{2i}, a_{2i}),\ a_{2i+1} = \Xi(a_{2i}, \rho_{2i}),\ \rho_{2i+2} = \Xi(\rho_{2i+1}, a_{2i+1}) \text{ and } a_{2i+2} = \Xi(a_{2i+1}, \rho_{2i+1}),$$

for all $i = 0, 1, 2, ...$, then by (3.1), we have

$$\begin{aligned}\mu_r(\rho_{2i+1},\rho_{2i+2}) &= \mu_r(\Upsilon(\rho_{2i},a_{2i}), \Xi(\rho_{2i+1},a_{2i+1})) \\ &\leq \lambda\frac{\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})}{2} \\ &\quad +\tau\frac{[1+\mu_r(\rho_{2i},\Upsilon(\rho_{2i},a_{2i}))]\,\mu_r(\rho_{2i+1},\Xi(\rho_{2i+1},a_{2i+1}))}{1+\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})} \\ &\quad +\zeta\frac{\mu_r(\rho_{2i+1},\Upsilon(\rho_{2i},a_{2i}))\,\mu_r(\rho_{2i},\Xi(\rho_{2i+1},a_{2i+1}))}{1+\mu_r(\rho,\sigma)+\mu_r(a,b)} \\ &= \lambda\frac{\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})}{2} \\ &\quad +\tau\frac{[1+\mu_r(\rho_{2i},\rho_{2i+1})]\,\mu_r(\rho_{2i+1},\rho_{2i+2})}{1+\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})} \\ &\quad +\zeta\frac{\mu_r(\rho_{2i+1},\rho_{2i+1})\,\mu_r(\rho_{2i},\rho_{2i+2})}{1+\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})} \\ &= \lambda\frac{\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})}{2} + \tau\frac{[1+\mu_r(\rho_{2i},\rho_{2i+1})]\,\mu_r(\rho_{2i+1},\rho_{2i+2})}{1+\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})} \\ &\leq \lambda\frac{\mu_r(\rho_{2i},\rho_{2i+1})}{2} + \lambda\frac{\mu_r(a_{2i},a_{2i+1})}{2} + \tau\mu_r(\rho_{2i+1},\rho_{2i+2}),\end{aligned}$$

this implies that

$$\mu_r(\rho_{2i+1},\rho_{2i+2}) \leq \frac{\lambda}{2(1-\tau)}\mu_r(\rho_{2i},\rho_{2i+1}) + \frac{\lambda}{2(1-\tau)}\mu_r(a_{2i},a_{2i+1}). \tag{3.2}$$

By following the same approach, we can write

$$\mu_r(a_{2i+1},a_{2i+2}) \leq \frac{\lambda}{2(1-\tau)}\mu_r(a_{2i},a_{2i+1}) + \frac{\lambda}{2(1-\tau)}\mu_r(\rho_{2i},\rho_{2i+1}). \tag{3.3}$$

Adding (3.2) and (3.3), we get

$$\begin{aligned}\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2}) &\leq \frac{\lambda}{(1-\tau)}\left(\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})\right)\\ &= \rho\left(\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})\right),\end{aligned}$$

where $0<\rho=\frac{\lambda}{(1-\tau)}<1$.

Also, we can write

$$\mu_r(\rho_{2i+2},\rho_{2i+3}) \leq \frac{\lambda}{2(1-\tau)}\mu_r(\rho_{2i+1},\rho_{2i+2})+\frac{\lambda}{2(1-\tau)}\mu_r(a_{2i+1},a_{2i+2}). \tag{3.4}$$

Similarly,

$$\mu_r(a_{2i+2},a_{2i+3}) \leq \frac{\lambda}{2(1-\tau)}\mu_r(a_{2i+1},a_{2i+2})+\frac{\lambda}{2(1-\tau)}\mu_r(\rho_{2i+1},\rho_{2i+2}). \tag{3.5}$$

Adding (3.4) and (3.5), we have

$$\begin{aligned}\mu_r(\rho_{2i+2},\rho_{2i+3})+\mu_r(a_{2i+2},a_{2i+3}) &\leq \frac{\lambda}{(1-\tau)}\left(\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2})\right)\\ &= \rho\left(\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2})\right),\end{aligned}$$

Continuing with the same previous approach, we find repeatedly that, for all $i\geq 0$,

$$\begin{aligned}\mu_r(\rho_i,\rho_{i+1})+\mu_r(a_i,a_{i+1}) &\leq \rho\left(\mu_r(\rho_{i-1},\rho_i)+\mu_r(a_{i-1},a_i)\right)\\ &\leq \rho^2\left(\mu_r(\rho_{i-2},\rho_{i-1})+\mu_r(a_{i-2},a_{i-1})\right)\\ &\leq \cdots\\ &\leq \rho^i\left(\mu_r(\rho_0,\rho_1)+\mu_r(a_0,a_1)\right).\end{aligned} \tag{3.6}$$

Now, if $\mu_r(\rho_i,\rho_{i+1})+\mu_r(a_i,a_{i+1})=\Lambda^i$, then (3.6) is reduces to

$$\Lambda_i \leq \rho\Lambda_{i-1} \leq \rho^2\Lambda_{i-2} \leq \cdots \leq \rho^i\Lambda_0.$$

For $j>i$, we get

$$\begin{aligned}\mu_r(\rho_i,\rho_j)+\mu_r(a_i,a_j) &\leq r\left(\mu_r(\rho_i,\rho_{i+1})+\mu_r(a_i,a_{i+1})\right)+\cdots+r^{j-i}\left(\mu_r(\rho_j,\rho_{j+1})+\mu_r(a_j,a_{j+1})\right)\\ &\leq r\rho^i\Lambda_0+r^2\rho^{i+1}\Lambda_0+\cdots+r^{j-i}\rho^{j-1}\Lambda_0\\ &< r\rho^i\left(1+(r\rho)+(r\rho)^2+...\right)\Lambda_0\\ &= \left(\frac{r\rho^i}{1-r\rho}\right)\Lambda_0 \to 0 \text{ as } i\to+\infty.\end{aligned}$$

This proves that $\{\rho_i\}$ and $\{a_i\}$ are Cauchy sequences in Ω. The completeness of Ω leads to there are $\rho,a\in\Omega$ so that $\lim_{i\to+\infty}\rho_i=\rho$ and $\lim_{i\to+\infty}a_i=a$.

Now, we claim that $\rho=\Upsilon(\rho,a)$ and $a=\Upsilon(a,\rho)$. Suppose that the contradiction, that is $\rho\neq\Upsilon(\rho,a)$ and $a\neq\Upsilon(a,\rho)$ so that $\mu_r(\rho,\Upsilon(\rho,a))=\ell_1>0$ and $\mu_r(a,\Upsilon(a,\rho))=\ell_2>0$.

Consider

$$\ell_1 = \mu_r(\rho,\Upsilon(\rho,a))$$

$$
\begin{aligned}
&\le r\left(\mu_r\left(\rho,\rho_{2i+2}\right)+\mu_r\left(\rho_{2i+2},\Upsilon\left(\rho,a\right)\right)\right)\\
&= r\mu_r\left(\rho,\rho_{2i+2}\right)+r\mu_r\left(\Xi\left(\rho_{2i+1},a_{2i+1}\right),\Upsilon\left(\rho,a\right)\right)\\
&= r\mu_r\left(\rho,\rho_{2i+2}\right)+r\lambda\frac{\mu_r\left(\rho,\rho_{2i+1}\right)+\mu_r\left(a,a_{2i+1}\right)}{2}\\
&\quad +r\tau\frac{\left[1+\mu_r\left(\rho,\Upsilon\left(\rho,a\right)\right)\right]\mu_r\left(\rho_{2i+1},\Xi\left(\rho_{2i+1},a_{2i+1}\right)\right)}{\left(1+\mu_r\left(\rho,\rho_{2i+1}\right)+\mu_r\left(a,a_{2i+1}\right)\right)}\\
&\quad +r\zeta\frac{\mu_r\left(\rho_{2i+1},\Upsilon\left(\rho,a\right)\right)\mu_r\left(\rho,\Xi\left(\rho_{2i+1},a_{2i+1}\right)\right)}{\left(1+\mu_r\left(\rho,\rho_{2i+1}\right)+\mu_r\left(a,a_{2i+1}\right)\right)}\\
&= r\mu_r\left(\rho,\rho_{2i+2}\right)+r\lambda\frac{\mu_r\left(\rho,\rho_{2i+1}\right)+\mu_r\left(a,a_{2i+1}\right)}{2}\\
&\quad +r\tau\frac{\left[1+\mu_r\left(\rho,\Upsilon\left(\rho,a\right)\right)\right]\mu_r\left(\rho_{2i+1},\rho_{2i+2}\right)}{\left(1+\mu_r\left(\rho,\sigma\right)+\mu_r\left(a,b\right)\right)}\\
&\quad +r\zeta\frac{\mu_r\left(\rho_{2i+1},\Upsilon\left(\rho,a\right)\right)\mu_r\left(\rho,\rho_{2i+2}\right)}{\left(1+\mu_r\left(\rho,\rho_{2i+1}\right)+\mu_r\left(a,a_{2i+1}\right)\right)}.
\end{aligned}
\tag{3.7}
$$

Passing $i \to +\infty$ in (3.7), we have $\ell_1 \le 0$, which is a contradiction. This conclude that $\mu_r\left(\rho,\Upsilon\left(\rho,a\right)\right)=0$, i.e., $\rho=\Upsilon\left(\rho,a\right)$, similarly, one can obtain that $a=\Upsilon\left(a,\rho\right)$. It follows similarly that $\rho=\Xi\left(\rho,a\right)$ and $a=\Xi\left(a,\rho\right)$.

For uniqueness: Assume that $(\widetilde{\rho},\widetilde{a})\in\Omega\times\Omega$ is a different common CFP of Υ and Ξ. Then

$$
\begin{aligned}
\mu_r\left(\rho,\widetilde{\rho}\right) &= \mu_r\left(\Upsilon\left(\rho,a\right),\Xi\left(\widetilde{\rho},\widetilde{a}\right)\right)\\
&\le \lambda\frac{\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)}{2}+\tau\frac{\left[1+\mu_r\left(\rho,\Upsilon\left(\rho,a\right)\right)\right]\mu_r\left(\widetilde{\rho},\Xi\left(\widetilde{\rho},\widetilde{a}\right)\right)}{\left(1+\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)\right)}\\
&\quad +\zeta\frac{\mu_r\left(\widetilde{\rho},\Upsilon\left(\rho,a\right)\right)\mu_r\left(\rho,\Xi\left(\widetilde{\rho},\widetilde{a}\right)\right)}{\left(1+\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)\right)}\\
&= \lambda\frac{\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)}{2}+\tau\frac{\left[1+\mu_r\left(\rho,\rho\right)\right]\mu_r\left(\widetilde{\rho},\widetilde{\rho}\right)}{\left(1+\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)\right)}+\zeta\frac{\mu_r\left(\widetilde{\rho},\rho\right)\mu_r\left(\rho,\widetilde{\rho}\right)}{\left(1+\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)\right)}\\
&\le \lambda\frac{\mu_r\left(\rho,\widetilde{\rho}\right)}{2}+\lambda\frac{\mu_r\left(a,\widetilde{a}\right)}{2}+\zeta\mu_r\left(\rho,\widetilde{\rho}\right),
\end{aligned}
$$

this yields

$$
\mu_r\left(\rho,\widetilde{\rho}\right)\le\left(\frac{\lambda}{2-\lambda-2\zeta}\right)\mu_r\left(a,\widetilde{a}\right). \tag{3.8}
$$

By the same manner, one can write

$$
\mu_r\left(a,\widetilde{a}\right)\le\left(\frac{\lambda}{2-\lambda-2\zeta}\right)\mu_r\left(\rho,\widetilde{\rho}\right). \tag{3.9}
$$

Adding (3.8) and (3.9), one sees that

$$
\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)\le\left(\frac{\lambda}{2-\lambda-2\zeta}\right)\left(\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)\right),
$$

this leads to, $\left(2-2\lambda-2\zeta\right)\left(\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)\right)\le 0$, since $\lambda+\zeta<1$, then we have $\left(\mu_r\left(\rho,\widetilde{\rho}\right)+\mu_r\left(a,\widetilde{a}\right)\right)=0$. This is only achieved when $\rho=\widetilde{\rho}$ and $a=\widetilde{a}$. Therefore, (ρ,a) is a unique common CFP of Υ and Ξ. □

If we put $\Upsilon = \Xi$ in the above theorem, we get the result below.

Corollary 3.1. *Assume that* (Ω, μ_r) *is a complete bMS with a coefficient* $r \geq 1$ *and let the mapping* $\Upsilon : \Omega \times \Omega \to \Omega$ *verifies*

$$\begin{aligned}\mu_r(\Upsilon(\rho, a), \Upsilon(\sigma, b)) \quad \leq \quad & \lambda \frac{\mu_r(\rho, \sigma) + \mu_r(a, b)}{2} + \tau \frac{[1 + \mu_r(\rho, \Upsilon(\rho, a))]\, \mu_r(\sigma, \Upsilon(\sigma, b))}{(1 + \mu_r(\rho, \sigma) + \mu_r(a, b))} \\ & + \zeta \frac{\mu_r(\sigma, \Upsilon(\rho, a))\, \mu_r(\rho, \Xi(\sigma, b))}{(1 + \mu_r(\rho, \sigma) + \mu_r(a, b))}.\end{aligned}$$

for all $\rho, a, \sigma, b \in \Omega$ *and* $\lambda, \tau, \zeta \geq 0$ *with* $r\lambda + \tau < 1$ *and* $\lambda + \zeta < 1$.

Then there is a unique CFP of Υ.

The following Corollary is very important in the next section (applications).

Corollary 3.2. *Let* (Ω, μ_r) *be a complete bMS with a coefficient* $r \geq 1$ *and let the mapping* $\Upsilon : \Omega \times \Omega \to \Omega$ *verifies*

$$\mu_r(\Upsilon(\rho, a), \Upsilon(\sigma, b)) \leq \lambda \frac{\mu_r(\rho, \sigma) + \mu_r(a, b)}{2}, \tag{3.10}$$

for all $\rho, a, \sigma, b \in \Omega$ *and* $\lambda \geq 0$ *with* $r\lambda < 1$.*Then* Υ *has a unique CFP.*

Proof. Just put $\Upsilon = \Xi$ and $\tau = \zeta = 0$ in Theorem 3.1, we get the proof. □

Theorem 3.2. *Suppose that* (Ω, μ_r) *is a complete bMS with a coefficient* $r \geq 1$ *and let the mappings* $\Upsilon, \Xi : \Omega \times \Omega \to \Omega$ *verify*

$$\begin{aligned}&\mu_r(\Upsilon(\rho, a), \Xi(\sigma, b)) \\ \leq \quad & \begin{cases} \lambda \frac{\mu_r(\rho,\sigma)+\mu_r(a,b)}{2} + \tau \frac{\mu_r(\rho,\Upsilon(\rho,a))\mu_r(\sigma,\Xi(\sigma,b))}{Q} + \zeta \frac{\mu_r(\sigma,\Upsilon(\rho,a))[1+\mu_r(\rho,\Xi(\sigma,b))]}{1+\mu_r(\rho,\sigma)+\mu_r(a,b)}, & \text{if } Q \neq 0, \\ 0 & \text{if } Q = 0,\end{cases}\end{aligned}$$

for all $\rho, a, \sigma, b \in \Omega$, *where*

$$Q = Q(\rho, a, \sigma, b) = r(\mu_r(\sigma, \Upsilon(\rho, a)) + \mu_r(\rho, \Xi(\sigma, b)) + \mu_r(\rho, \sigma) + \mu_r(a, b)),$$

and $\lambda, \tau, \zeta \geq 0$ *with* $r(\lambda + \tau + \zeta) < 1$. *Then* Υ *and* Ξ *have a unique common coupled fixed point.*

Proof. Let $\rho_0, a_0 \in \Omega$ be an arbitrary points. Define the sequences $\{\rho_{2i+1}\}_{i=0}^{+\infty}$, $\{a_{2i+1}\}_{i=0}^{+\infty}$, $\{\rho_{2i+2}\}_{i=0}^{+\infty}$ and $\{a_{2i+2}\}_{i=0}^{+\infty}$ by

$$\rho_{2i+1} = \Upsilon(\rho_{2i}, a_{2i}), \ a_{2i+1} = \Xi(a_{2i}, \rho_{2i}), \rho_{2i+2} = \Xi(\rho_{2i+1}, a_{2i+1}) \text{ and } a_{2i+2} = \Xi(a_{2i+1}, \rho_{2i+1}),$$

for all $i = 0, 1, 2, ...$ Consider

$$Q_1 = Q(\rho_{2i}, a_{2i}, \rho_{2i+1}, a_{2i+1}) \neq 0 \text{ and } Q_2 = Q(a_{2i}, \rho_{2i}, a_{2i+1}, \rho_{2i+1}) \neq 0.$$

Then

$$\begin{aligned}\mu_r(\rho_{2i+1}, \rho_{2i+2}) \quad &= \quad \mu_r(\Upsilon(\rho_{2i}, a_{2i}), \Xi(\rho_{2i+1}, a_{2i+1})) \\ &\leq \quad \lambda \frac{\mu_r(\rho_{2i}, \rho_{2i+1}) + \mu_r(a_{2i}, a_{2i+1})}{2}\end{aligned}$$

$$
\begin{aligned}
&+\tau\frac{\mu_r(\rho_{2i},\Upsilon(\rho_{2i},a_{2i}))\,\mu_r(\rho_{2i+1},\Xi(\rho_{2i+1},a_{2i+1}))}{Q_1}\\
&+\zeta\frac{\mu_r(\rho_{2i+1},\Upsilon(\rho_{2i},a_{2i}))\,[1+\mu_r(\rho_{2i},\Xi(\rho_{2i+1},a_{2i+1}))]}{1+\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})}\\
=\;&\lambda\frac{\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})}{2}\\
&+\tau\frac{\mu_r(\rho_{2i},\rho_{2i+1})\,\mu_r(\rho_{2i+1},\rho_{2i+2})}{r(\mu_r(\rho_{2i},\rho_{2i+2})+\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1}))}\\
&+\zeta\frac{\mu_r(\rho_{2i+1},\rho_{2i+1})\,[1+\mu_r(\rho_{2i},\rho_{2i+2})]}{1+\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})}\\
\le\;&\lambda\frac{\mu_r(\rho_{2i},\rho_{2i+1})}{2}+\lambda\frac{\mu_r(a_{2i},a_{2i+1})}{2}+\tau\mu_r(\rho_{2i},\rho_{2i+1}),
\end{aligned}
$$

Hence, one can write

$$
\mu_r(\rho_{2i+1},\rho_{2i+2})\le\left(\frac{2\tau+\lambda}{2}\right)\mu_r(\rho_{2i},\rho_{2i+1})+\frac{\lambda}{2}\mu_r(a_{2i},a_{2i+1}). \tag{3.11}
$$

Similarly, one can easily prove via assumption $Q_2\neq 0$ that

$$
\mu_r(a_{2i+1},a_{2i+2})\le\left(\frac{2\tau+\lambda}{2}\right)\mu_r(a_{2i},a_{2i+1})+\frac{\lambda}{2}\mu_r(\rho_{2i},\rho_{2i+1}). \tag{3.12}
$$

By adding (3.11) to (3.12), we find that

$$
\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2})\le(\tau+\lambda)\left[\mu_r(\rho_{2i},\rho_{2i+1})+\mu_r(a_{2i},a_{2i+1})\right].
$$

Put $Q_3=Q(\rho_{2i+2},a_{2i+2},\rho_{2i+1},a_{2i+1})\neq 0$, then we get

$$
\begin{aligned}
\mu_r(\rho_{2i+2},\rho_{2i+3})\;=\;&\mu_r(\Upsilon(\rho_{2i+1},a_{2i+1}),\Xi(\rho_{2i+2},a_{2i+2}))\\
\le\;&\lambda\frac{\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2})}{2}\\
&+\tau\frac{\mu_r(\rho_{2i+1},\Upsilon(\rho_{2i+1},a_{2i+1}))\,\mu_r(\rho_{2i+2},\Xi(\rho_{2i+2},a_{2i+2}))}{Q_3}\\
&+\zeta\frac{\mu_r(\rho_{2i+2},\Upsilon(\rho_{2i+1},a_{2i+1}))\,[1+\mu_r(\rho_{2i+1},\Xi(\rho_{2i+2},a_{2i+2}))]}{1+\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2})}\\
=\;&\lambda\frac{\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2})}{2}\\
&+\tau\frac{\mu_r(\rho_{2i+1},\rho_{2i+2})\,\mu_r(\rho_{2i+2},\rho_{2i+3})}{r(\mu_r(\rho_{2i+1},\rho_{2i+3})+\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2}))}\\
&+\zeta\frac{\mu_r(\rho_{2i+2},\rho_{2i+2})\,[1+\mu_r(\rho_{2i+1},\rho_{2i+3})]}{1+\mu_r(\rho_{2i+1},\rho_{2i+2})+\mu_r(a_{2i+1},a_{2i+2})}\\
\le\;&\lambda\frac{\mu_r(\rho_{2i+1},\rho_{2i+2})}{2}+\lambda\frac{\mu_r(a_{2i+1},a_{2i+2})}{2}+\tau\mu_r(\rho_{2i+1},\rho_{2i+2}),
\end{aligned}
$$

this leads to

$$
\mu_r(\rho_{2i+2},\rho_{2i+3})\le\left(\frac{2\tau+\lambda}{2}\right)\mu_r(\rho_{2i+1},\rho_{2i+2})+\frac{\lambda}{2}\mu_r(a_{2i+1},a_{2i+2}). \tag{3.13}
$$

Similarly, one can easily prove via assumption $Q_4 = Q(a_{2i+2}, \rho_{2i+2}, a_{2i+1}, \rho_{2i+1}) \neq 0$ that

$$\mu_r(a_{2i+2}, a_{2i+3}) \leq \left(\frac{2\tau+\lambda}{2}\right)\mu_r(a_{2i+1}, a_{2i+2}) + \frac{\lambda}{2}\mu_r(\rho_{2i+1}, \rho_{2i+2}). \tag{3.14}$$

By adding (3.13) to (3.14), we can write

$$\mu_r(\rho_{2i+2}, \rho_{2i+3}) + \mu_r(a_{2i+2}, a_{2i+3}) \leq (\tau+\lambda)\left[\mu_r(\rho_{2i+1}, \rho_{2i+2}) + \mu_r(a_{2i+1}, a_{2i+2})\right].$$

Continuing with the same scenario for all $i \geq 0$, we get

$$\begin{aligned}\mu_r(\rho_i, \rho_{i+1}) + \mu_r(a_i, a_{i+1}) &\leq (\tau+\lambda)\left[\mu_r(\rho_{i-1}, \rho_i) + \mu_r(a_{i-1}, a_i)\right]\\ &= \sigma\left[\mu_r(\rho_{i-1}, \rho_i) + \mu_r(a_{i-1}, a_i)\right],\end{aligned}$$

where $\sigma = \tau + \lambda < 1$. Now if, $\mu_r(\rho_i, \rho_{i+1}) + \mu_r(a_i, a_{i+1}) = \varpi_i$, then

$$\varpi_i \leq \sigma\varpi_{i-1} \leq \sigma^2\varpi_{i-2} \leq ... \leq \sigma^i\varpi_0,$$

Suppose that for $j.i \in \mathbb{N} \cup \{0\}$ so that $j > i$, then

$$\begin{aligned}\mu_r(\rho_i, \rho_j) + \mu_r(a_i, a_j) &\leq r(\mu_r(\rho_i, \rho_{i+1}) + \mu_r(a_i, a_{i+1})) + \cdots + r^{j-i}\left(\mu_r(\rho_j, \rho_{j+1}) + \mu_r(a_j, a_{j+1})\right)\\ &\leq r\sigma^i\varpi_0 + r^2\sigma^{i+1}\varpi_0 + \cdots + r^{j-i}\sigma^{j-1}\varpi_0\\ &< r\sigma^i\left(1 + (r\sigma) + (r\sigma)^2 + ...\right)\varpi_0\\ &= \left(\frac{r\sigma^i}{1-r\sigma}\right)\varpi_0 \to 0 \text{ as } i \to +\infty.\end{aligned}$$

this proves that the two sequences $\{\rho_i\}$ and $\{a_i\}$ are Cauchy. Because Ω is complete, then there exist $\rho, a \in \Omega$ so that $\lim_{i\to+\infty}\rho_i = \rho$ and $\lim_{i\to+\infty} a_i = a$.

Now, we show that $\rho = \Upsilon(\rho, a)$ and $a = \Upsilon(a, \rho)$. Suppose the opposite, that is $\rho \neq \Upsilon(\rho, a)$ and $a \neq \Upsilon(a, \rho)$ so that $\mu_r(\rho, \Upsilon(\rho, a)) = \ell_1 > 0$ and $\mu_r(a, \Upsilon(a, \rho)) = \ell_2 > 0$.

Consider

$$\begin{aligned}\ell_1 &= \mu_r(\rho, \Upsilon(\rho, a))\\ &\leq r(\mu_r(\rho, \rho_{2i+2}) + \mu_r(\rho_{2i+2}, \Upsilon(\rho, a)))\\ &= r\mu_r(\rho, \rho_{2i+2}) + r\mu_r(\Xi(\rho_{2i+1}, a_{2i+1}), \Upsilon(\rho, a))\\ &= r\mu_r(\rho, \rho_{2i+2}) + r\lambda\frac{\mu_r(\rho, \rho_{2i+1}) + \mu_r(a, a_{2i+1})}{2}\\ &\quad + r\tau\frac{\mu_r(\rho, \Upsilon(\rho, a))\mu_r(\rho_{2i+1}, \Xi(\rho_{2i+1}, a_{2i+1}))}{r(\mu_r(\rho_{2i+1}, \Upsilon(\rho, a)) + \mu_r(\rho, \Xi(\rho_{2i+1}, a_{2i+1})) + \mu_r(\rho, \rho_{2i+1}) + \mu_r(a_{2i+1}, a))}\\ &\quad + r\zeta\frac{\mu_r(\rho_{2i+1}, \Upsilon(\rho, a))\left[1 + \mu_r(\rho, \Xi(\rho_{2i+1}, a_{2i+1}))\right]}{1 + \mu_r(\rho, \rho_{2i+1}) + \mu_r(a, a_{2i+1})}\\ &= r\mu_r(\rho, \rho_{2i+2}) + r\lambda\frac{\mu_r(\rho, \rho_{2i+1}) + \mu_r(a, a_{2i+1})}{2}\\ &\quad + \tau\frac{\mu_r(\rho, \Upsilon(\rho, a))\mu_r(\rho_{2i+1}, \rho_{2i+2})}{(\mu_r(\rho_{2i+1}, \Upsilon(\rho, a)) + \mu_r(\rho, \Xi(\rho_{2i+1}, a_{2i+1})) + \mu_r(\rho, \rho_{2i+1}) + \mu_r(a_{2i+1}, a))}\end{aligned}$$

$$+r\zeta\frac{\mu_r(\rho_{2i+1},\Upsilon(\rho,a))[1+\mu_r(\rho,\rho_{2i+2})]}{(1+\mu_r(\rho,\rho_{2i+1})+\mu_r(a,a_{2i+1}))}. \tag{3.15}$$

Letting $i\to+\infty$ in (3.15), we get $\ell_1\le r\zeta\mu_r(\rho,\Upsilon(\rho,a))=r\zeta\ell_1$, this leads to $(1-r\zeta)\ell_1\le 0$, so either $r\zeta\ge 1$, this contradicts the condition $r(\lambda+\tau+\zeta)<1$, or $\ell_1\le 0$ and this contradicts the condition $\ell_1>0$. Thus, in both cases we have a contradiction. This implies that $\mu_r(\rho,\Upsilon(\rho,a))=0$, i.e., $\rho=\Upsilon(\rho,a)$, similarly, one can obtain that $a=\Upsilon(a,\rho)$. It follows similarly that $\rho=\Xi(\rho,a)$ and $a=\Xi(a,\rho)$.

For uniqueness: Assume that $(\widetilde{\rho},\widetilde{a})\in\Omega\times\Omega$ is a different common CFP of Υ and Ξ. Then

$$\begin{aligned}\mu_r(\rho,\widetilde{\rho}) &= \mu_r(\Upsilon(\rho,a),\Xi(\widetilde{\rho},\widetilde{a}))\\ &\le \lambda\frac{\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a})}{2}+\tau\frac{\mu_r(\rho,\Upsilon(\rho,a))\mu_r(\widetilde{\rho},,\Xi(\widetilde{\rho},\widetilde{a}))}{r(\mu_r(\widetilde{\rho},\Upsilon(\rho,a))+\mu_r(\rho,\Xi(\widetilde{\rho},\widetilde{a}))+\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a}))}\\ &\quad+\zeta\frac{\mu_r(\widetilde{\rho},\Upsilon(\rho,a))[1+\mu_r(\rho,\Xi(\widetilde{\rho},\widetilde{a}))]}{1+\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a})}\\ &= \lambda\frac{\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a})}{2}+\tau\frac{\mu_r(\rho,\rho)\mu_r(\widetilde{\rho},\widetilde{\rho})}{r(\mu_r(\widetilde{\rho},\rho)+\mu_r(\rho,\widetilde{\rho})+\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a}))}\\ &\quad+\zeta\frac{\mu_r(\widetilde{\rho},\rho)[1+\mu_r(\rho,\widetilde{\rho})]}{(1+\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a}))}\\ &\le \lambda\frac{\mu_r(\rho,\widetilde{\rho})}{2}+\lambda\frac{\mu_r(a,\widetilde{a})}{2}+\zeta\mu_r(\rho,\widetilde{\rho}),\end{aligned}$$

this implies that

$$\mu_r(\rho,\widetilde{\rho})\le\left(\frac{\lambda}{2-\lambda-2\zeta}\right)\mu_r(a,\widetilde{a}). \tag{3.16}$$

By the same method, one can obtain

$$\mu_r(a,\widetilde{a})\le\left(\frac{\lambda}{2-\lambda-2\zeta}\right)\mu_r(\rho,\widetilde{\rho}). \tag{3.17}$$

By adding (3.16) to (3.17), one can write

$$\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a})\le\left(\frac{\lambda}{2-\lambda-2\zeta}\right)(\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a})),$$

this implies that, $(2-2\lambda-2\zeta)(\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a}))\le 0$, since $\lambda+\zeta<1$, then we have $(\mu_r(\rho,\widetilde{\rho})+\mu_r(a,\widetilde{a}))=0$. This is only holds when $\rho=\widetilde{\rho}$ and $a=\widetilde{a}$. Therefore, (ρ,a) is a unique common CFP of Υ and Ξ. □

If we set $\Upsilon=\Xi$ in Theorem 3.2, we get the following result:

Corollary 3.3. *Suppose that (Ω,μ_r) is a complete bMS with a coefficient $r\ge 1$ and let the mappings $\Upsilon:\Omega\times\Omega\to\Omega$ verifies*

$$\begin{aligned}&\mu_r(\Upsilon(\rho,a),\Upsilon(\sigma,b))\\ &\le\begin{cases}\lambda\frac{\mu_r(\rho,\sigma)+\mu_r(a,b)}{2}+\tau\frac{\mu_r(\rho,\Upsilon(\rho,a))\mu_r(\sigma,\Upsilon(\sigma,b))}{Q}+\zeta\frac{\mu_r(\sigma,\Upsilon(\rho,a))[1+\mu_r(\rho,\Upsilon(\sigma,b))]}{1+\mu_r(\rho,\sigma)+\mu_r(a,b)}, & \text{if } Q\ne 0,\\ 0 & \text{if } Q=0,\end{cases}\end{aligned}$$

for all $\rho, a, \sigma, b \in \Omega$, *where*

$$Q = Q(\rho, a, \sigma, b) = r(\mu_r(\sigma, \Upsilon(\rho, a)) + \mu_r(\rho, \Upsilon(\sigma, b)) + \mu_r(\rho, \sigma) + \mu_r(a, b)),$$

and $\lambda, \tau, \zeta \geq 0$ *with* $r(\lambda + \tau + \zeta) < 1$. *Then* Υ *has a unique common CFP.*

4. The existence solution of singular CFDEs

In this part, we discuss the existence and uniqueness of the solution of the nonlinear singular CFDE in the setting of bMSs.

Here, we begin with the proof of the following lemma which demonstrate that Green's function of a FDE with integral boundary conditions.

Lemma 4.1. *Given the pair* $(a, q) \in (C(0,1) \cap L(0,1)) \times (C(0,1) \cap L(0,1))$, $\nu \in (3,4)$, $\alpha \in (0,2)$, *such that* $\nu \neq \alpha$, *the unique solution of*

$$\begin{cases} {}^{c}\Theta^{\nu}(\Lambda(\tau)) + (a(\tau), q(\tau)) = 0, \ \tau \in (0,1), \\ \Lambda'''(0) = \Lambda''(0) = 0, \\ \Lambda' = \Lambda(1) = \alpha \int_0^1 \Lambda(\theta) d\theta, \end{cases} \tag{4.1}$$

is

$$\Lambda(\tau) = \int_0^1 \Im(\tau, \theta)(a(\theta), q(\theta)) d\theta,$$

where $\Lambda(\tau) = (z(\tau), w(\tau))$ *and*

$$\Im(\tau, \theta) = \frac{1}{\nu(2-\alpha)\Gamma(\nu)} \begin{cases} \begin{array}{l} (\nu(2-\alpha) + 2\alpha\tau(\nu - 1 + \theta))(1-\theta)^{\nu-1} \\ \quad -\nu(2-\alpha)(\tau-\theta)^{\nu-1}, \end{array} & if\, 0 \leq \theta \leq \tau \leq 1, \\ (\nu(2-\alpha) + 2\alpha\tau(\nu - 1 + \theta))(1-\theta)^{\nu-1}, & if\, 0 \leq \tau \leq \theta \leq 1, \end{cases} \tag{4.2}$$

Proof. Based on Lemma 2.1, problem (4.1) can be reduced to the equivalent integral equation

$$\begin{aligned} \Lambda(\tau) &= -I^{\nu}(a(\tau), q(\tau)) + C_0 + C_1\tau + C_2\tau^2 + C_3\tau^3 \\ &= -\frac{1}{\Gamma(\nu)} \int_0^{\tau} (\tau-\theta)^{\nu-1}(a(\theta), q(\theta)) d\theta + C_0 + C_1\tau + C_2\tau^2 + C_3\tau^3. \end{aligned} \tag{4.3}$$

Since $(z''(0), w''(0)) = (z'''(0), w'''(0)) = (0,0)$, so we get $\Lambda''(0) = \Lambda'''(0)$, thus, $C_2 = C_3 = 0$, and we can write

$$\begin{aligned} \Lambda(\tau) &= -\frac{1}{\Gamma(\nu)} \int_0^{\tau} (\tau-\theta)^{\nu-1}(a(\theta), q(\theta)) d\theta + C_0 + C_1\tau, \\ \Lambda'(\tau) &= -\frac{\nu-1}{\Gamma(\nu)} \int_0^{\tau} (\tau-\theta)^{\nu-2}(a(\theta), q(\theta)) d\theta + C_1. \end{aligned}$$

Because $(z'(0), w'(0)) = (z(1), w(1)) = \left(\alpha \int_0^1 z(\theta)d\theta, \alpha \int_0^1 w(\theta)d\theta\right)$, hence $\Lambda'(0) = \Lambda(1) = \alpha \int_0^1 \Lambda(\theta)d\theta$, then one sees that

$$\begin{aligned}
\Lambda'(0) &= C_1 = \alpha \int_0^1 \Lambda(\theta)d\theta, \\
\Lambda(1) &= -\frac{1}{\Gamma(\nu)} \int_0^1 (1-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta + C_0 + C_1 = \alpha \int_0^1 \Lambda(\theta)d\theta,
\end{aligned} \tag{4.4}$$

So, by (4.1), we conclude that

$$C_0 = \frac{1}{\Gamma(\nu)} \int_0^1 (1-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta,$$

From the previous equality, we obtain that

$$\Lambda(\tau) = -\frac{1}{\Gamma(\nu)} \int_0^\tau (\tau-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta + \frac{1}{\Gamma(\nu)} \int_0^1 (1-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta + C_1\tau. \tag{4.5}$$

By integrating both sides of Eq (4.5) from 0 to 1, one can get

$$\begin{aligned}
\int_0^1 \Lambda(\tau)d\tau &= -\frac{1}{\Gamma(\nu)} \int_0^1 \int_0^\tau (\tau-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta d\tau \\
&\quad + \frac{1}{\Gamma(\nu)} \int_0^1 \int_0^1 (1-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta d\tau + \int_0^1 C_1\tau d\tau. \\
&= -\frac{1}{\Gamma(\nu)} \int_0^1 \frac{(1-\theta)^\nu}{\nu}(a(\theta), q(\theta))d\theta + \frac{1}{\Gamma(\nu)} \int_0^1 (1-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta + \frac{C_1}{2}.
\end{aligned} \tag{4.6}$$

From (4.4) and (4.6), one can write

$$\begin{aligned}
C_1 &= \alpha \int_0^1 \Lambda(\theta)d\theta \\
&= -\frac{\alpha}{\nu\Gamma(\nu)} \int_0^1 (1-\theta)^\nu (a(\theta), q(\theta))d\theta + \frac{\alpha}{\Gamma(\nu)} \int_0^1 (1-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta + \frac{\alpha C_1}{2},
\end{aligned}$$

Thus, we get

$$C_1 = -\frac{2\alpha}{\nu\Gamma(\nu)(2-\alpha)} \int_0^1 (1-\theta)^\nu (a(\theta), q(\theta))d\theta + \frac{2\alpha}{\Gamma(\nu)(2-\alpha)} \int_0^1 (1-\theta)^{\nu-1} (a(\theta), q(\theta))d\theta.$$

Substituting the values of the constants in (4.5) we obtain

$$\begin{aligned}
\Lambda(\tau) &= -\frac{1}{\Gamma(\nu)}\int_0^\tau (\tau-\theta)^{\nu-1}(a(\theta),q(\theta))d\theta + \frac{1}{\Gamma(\nu)}\int_0^1 (1-\theta)^{\nu-1}(a(\theta),q(\theta))d\theta + \\
&\quad -\frac{2\alpha\tau}{\nu\Gamma(\nu)(2-\alpha)}\int_0^1 (1-\theta)^{\nu}(a(\theta),q(\theta))d\theta + \frac{2\alpha\tau}{\Gamma(\nu)(2-\alpha)}\int_0^1 (1-\theta)^{\nu-1}(a(\theta),q(\theta))d\theta \\
&= -\frac{1}{\Gamma(\nu)}\int_0^\tau (\tau-\theta)^{\nu-1}(a(\theta),q(\theta))d\theta \\
&\quad +\frac{1}{\nu\Gamma(\nu)(2-\alpha)}\int_0^1 [\alpha(2-\alpha)+2\alpha\tau(\nu-1+\theta)](1-\theta)^{\nu}(a(\theta),q(\theta))d\theta \\
&= \frac{1}{\nu\Gamma(\nu)(2-\alpha)}\int_0^1 \left\{[\alpha(2-\alpha)+2\alpha\tau(\nu-1+\theta)](1-\theta)^{\nu} - \alpha(2-\alpha)(\nu-\theta)^{\nu-1}\right\}(a(\theta),q(\theta))d\theta \\
&\quad +\frac{1}{\nu\Gamma(\nu)(2-\alpha)}\int_\tau^1 [\alpha(2-\alpha)+2\alpha\tau(\nu-1+\theta)](1-\theta)^{\nu}(a(\theta),q(\theta))d\theta \\
&= \int_0^1 \Im(\tau,\theta)(a(\theta),q(\theta))d\theta,
\end{aligned}$$

and this completes the proof. □

The following lemma estimates the green's function $\Im(\tau,\theta)$ of FDE with integral boundary stipulations described in (4.1) on $L_2(0,1)$.

Lemma 4.2. *Suppose that* $\nu\in(3,4)$, $\alpha\in(0,2)$, *such that* $\nu\neq\alpha$, *then for all* $\tau,\theta\in(0,1)$, *the Green's functions* $\Im(\tau,.)\in L_2$ *verifies*

$$\int_0^1 \left|\Im(\tau,\theta)\right|^2 d\theta < \frac{1}{\Gamma^2(\nu)}\left(\frac{4}{5}+\frac{8\alpha}{3|\alpha-2|}+\frac{4\alpha^2}{9(\alpha-2)^2}\right).$$

Proof. When $\nu\in(3,4)$, $\alpha\in(0,2)$, such that $\nu\neq\alpha$, there are two clarifications:

(i) For $0\le\theta\le\tau\le1$,

$$\begin{aligned}
\left|\Im(\tau,\theta)\right| &\le \frac{2\nu|\alpha-2|+2\alpha\tau(\nu-1+\theta)}{\nu|\alpha-2|\Gamma(\nu)}(1-\theta)^{\nu-1} \\
&= \frac{1}{\Gamma(\nu)}\left(2+\frac{2\alpha}{\nu|\alpha-2|}(\nu-1+\theta)\right)(1-\theta)^{\nu-1}. \qquad (4.7)
\end{aligned}$$

(ii) For $0\le\tau\le\theta\le1$,

$$\left|\Im(\tau,\theta)\right| \le \frac{\nu|\alpha-2|+2\alpha\tau(\nu-1+\theta)}{\nu|\alpha-2|\Gamma(\nu)}(1-\theta)^{\nu-1}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\nu)}\left(1+\frac{2\alpha}{\nu\,|\alpha-2|}\,(\nu-1+\theta)\right)(1-\theta)^{\nu-1} \\
&\le \frac{1}{\Gamma(\nu)}\left(2+\frac{2\alpha}{\nu\,|\alpha-2|}\,(\nu-1+\theta)\right)(1-\theta)^{\nu-1}. \qquad (4.8)
\end{aligned}
$$

It follow from (4.7) and (4.8) that

$$
\left|\Im(\tau,\theta)\right|^2 \le \frac{1}{\Gamma^2(\nu)}\left(4+\frac{8\alpha}{\nu\,|\alpha-2|}\,(\nu-1+\theta)+\frac{4\alpha^2\,(\nu-1+\theta)^2}{\nu^2\,(\alpha-2)^2}\right)(1-\theta)^{2\nu-2},
$$

which yields

$$
\begin{aligned}
&\int_0^1 \left|\Im(\tau,\theta)\right|^2 d\theta \\
\le\ & \frac{1}{\Gamma^2(\nu)}\left\{\int_0^1 4\,(1-\theta)^{2\nu-2}\,d\theta+\frac{8\alpha}{\nu\,|\alpha-2|}\int_0^1 (\nu-1+\theta)\,(1-\theta)^{2\nu-2}\,d\theta\right. \\
&\left.+\frac{4\alpha^2}{\nu^2\,(\alpha-2)^2}\int_0^1 (\nu-1+\theta)^2\,(1-\theta)^{2\nu-2}\,d\theta\right\} \\
=\ & \frac{1}{\Gamma^2(\nu)}\left(\frac{4}{2\nu-1}+\frac{8\alpha}{\nu\,|\alpha-2|}\left(\frac{2\nu^2-2\nu+1}{2\nu(2\nu-1)}\right)+\frac{4\alpha^2}{\nu^2\,(\alpha-2)^2}\left(\frac{2\nu^2-3\nu+1}{2\nu(2\nu-1)(2\nu+1)}\right)\right) \\
<\ & \frac{1}{\Gamma^2(\nu)}\left(\frac{4}{5}+\frac{8\alpha}{3\,|\alpha-2|}+\frac{4\alpha^2}{9\,(\alpha-2)^2}\right).
\end{aligned}
$$

□

Assume that $\Xi(.,z(.),w(.)) \in L_2$ for any $z,w \in C[0,1]$ and describe a mapping $\Upsilon : C[0,1] \times C[0,1] \to C[0,1]$ as follows:

$$
\Upsilon(z(\tau),w(\tau)) = \int_0^1 \Im(\tau,\theta)\Xi(\theta,z(\theta),w(\theta))d\theta, \qquad (4.9)
$$

where $\theta \to \Im(\tau,\theta)$ is continuous from $[0,1]$ to L_2. Suppose that $\tau_n \in [0,1]$ with $\tau_n \to \tau$. Because $\Im(\tau,.), \Xi(.,z(.),w(.)) \in L_2$ for any $z,w \in C[0,1]$ and $\tau \in [0,1]$, therefore $\Im(\tau,.)\Xi(.,z(.),w(.))$ is integrable function. Hence, using the Lebesgue dominated convergence theorem, one can write

$$
\begin{aligned}
\lim_{n\to+\infty} \Upsilon\,(z(\tau_n),w(\tau_n)) &= \lim_{n\to+\infty}\int_0^1 \Im(\tau_n,\theta)\Xi(\theta,z(\theta),w(\theta))d\theta \\
&= \int_0^1 \lim_{n\to+\infty} \Im(\tau_n,\theta)\Xi(\theta,z(\theta),w(\theta))d\theta \\
&= \int_0^1 \Im(\tau,\theta)\Xi(\theta,z(\theta),w(\theta))d\theta = \Upsilon(z(\tau),w(\tau)).
\end{aligned}
$$

Similarly

$$
\lim_{n\to+\infty} \Upsilon\,(w(\tau_n),z(\tau_n)) = \lim_{n\to+\infty}\int_0^1 \Im(\tau_n,\theta)\Xi(\theta,w(\theta),z(\theta))d\theta
$$

$$
\begin{aligned}
&= \int_0^1 \lim_{n\to+\infty} \Im(\tau_n,\theta)\Xi(\theta,w(\theta),z(\theta))d\theta \\
&= \int_0^1 \Im(\tau,\theta)\Xi(\theta,w(\theta),z(\theta))d\theta = \Upsilon(w(\tau),z(\tau)),
\end{aligned}
$$

this shows that $\Upsilon \in C[0,1]$. Thus, the mapping $\Upsilon : C[0,1]\times C[0,1] \to C[0,1]$ is well-defined.

Suppose that $\mu_r : C[0,1]\times C[0,1] \to \mathbb{R}^+$ is described as

$$
\mu_r(z,w) = \sup_{\tau\in[0,1]} |z(\tau)-w(\tau)|^2, \tag{4.10}
$$

Then the pair (Ω,μ_r) is complete bMS with coefficient $r = 2$.

Lemma 4.3. *Assume that Υ is a mapping described as (4.8) and $z,w \in C[0,1]$. The pair $(z(\tau),w(\tau))$ is a solution of BVP (1.1) iff it is a CFP of Υ.*

Proof. Assume that the form of a solution of BVP (1.1) is $(z(\tau),w(\tau))$. Then from Lemma 4.1, the unique solution can be described as

$$
z(\tau) = \int_0^1 \Im(\tau,\theta)\Xi(\theta,z(\theta),w(\theta))d\theta,
$$

and

$$
w(\tau) = \int_0^1 \Im(\tau,\theta)\Xi(\theta,w(\theta),z(\theta))d\theta,
$$

where $\Im(\tau,\theta)$ defined in (4.2). Thus, $(z(\tau),w(\tau))$ is a CFP of Υ. □

Conversely, suppose that $(z(\tau),w(\tau))$ is a CFP of Υ. Since $\alpha < n-1$, by Lemma 2.2, we get

$$
\begin{aligned}
{}^c\Theta^\nu(z(\tau)) &= {}^c\Theta^\nu\left(\int_0^1 \Im(\tau,\theta)\Xi(\theta,z(\theta),w(\theta))d\theta\right) \\
&= \frac{1}{\nu(2-\alpha)\Gamma(\nu)}\left\{{}^c\Theta^\nu\left[\int_0^t \left(\left(-\nu(2-\alpha)(\tau-\theta)^{\nu-1}\right)\right.\right.\right. \\
&\quad \left. + (\nu(2-\alpha)+2\alpha\tau(\nu-1+\theta))(1-\theta)^{\nu-1}\right)\Xi(\theta,z(\theta),w(\theta))d\theta \\
&\quad \left.\left. + \int_\tau^1 (\nu(2-\alpha)+2\alpha\tau(\nu-1+\theta))(1-\theta)^{\nu-1}\,\Xi(\theta,z(\theta),w(\theta))d\theta\right]\right\} \\
&= -\Xi(\tau,z(\tau),w(\tau)) +{}^c \Theta^\nu\left(\frac{\tau^0}{\Gamma(\nu)}\int_0^1 (1-\theta)^{\nu-1}\,\Xi(\theta,z(\theta),w(\theta))d\theta\right. \\
&\quad \left. + \frac{2\alpha\tau}{\nu(2-\alpha)\Gamma(\nu)}\int_0^1 (\nu-1+\theta)(1-\theta)^{\nu-1}\,\Xi(\theta,z(\theta),w(\theta))d\theta\right. \\
&= -\Xi(\tau,z(\tau),w(\tau))
\end{aligned}
$$

Similarly, one can prove that

$$
{}^c\Theta^\nu(w(\tau)) = -\Xi(\tau,w(\tau),z(\tau)).
$$

Hence $(z(\tau), w(\tau))$ verifies problem (1.1), further, it is easy to verify that $\Lambda''(0) = \Lambda''(0) = 0$, $\Lambda' = \Lambda(1) = \alpha \int_0^1 \Lambda(\theta) d\theta$, which implies that $(z(\tau), w(\tau))$ is a solution to problem (1.1). Because Υ is continuous and has a CFP $(z(\tau), w(\tau))$, thus $(z(\tau), w(\tau))$ is a continuous solution for the given BVP.

In order to study the existence and uniqueness solution for the BVP (1.1), we propose the following theorem.

Theorem 4.1. *Suppose that* $\nu \in (3, 4)$ *and*

$$\lambda = \frac{1}{\Gamma^2(\nu)} \left(\frac{4}{5} + \frac{8\alpha}{3|\alpha - 2|} + \frac{4\alpha^2}{9(\alpha - 2)^2} \right) < 1, \tag{4.11}$$

holds foe any $\alpha \in (0, 2)$, $\alpha \neq \nu$. *Let* $\Xi(., z(.), w(.))$ *be a function in* L_2 *for any* $z, w \in C[0, 1]$ *and for any* $z^*, w^* \in C[0, 1]$ *the inequality below holds*

$$|\Xi(\theta, z(\theta), w(\theta)) - \Xi(\theta, z^*(\theta), w^*(\theta))|^2 \leq \frac{|z(\theta) - z^*(\theta)|^2 + |w(\theta) - w^*(\theta)|^2}{2}, \ \theta \in [0, 1], \tag{4.12}$$

Then the mapping Υ *has a unique CFP, which is a unique solution to the BVP (1.1).*

Proof. Based on the Cauchy–Schwarz inequality, the mapping Υ given in (4.9) and Lemma 4.2, one can write

$$\begin{aligned}
& |\Upsilon(z(\tau), w(\tau)) - \Upsilon(z^*(\tau), w^*(\tau))|^2 \\
= & \left| \int_0^1 \mathfrak{I}(\tau, \theta) \left[\Xi(\theta, z(\theta), w(\theta)) - \Xi(\theta, z^*(\theta), w^*(\theta)) \right] d\theta \right|^2 \\
\leq & \left(\int_0^1 \left| \mathfrak{I}(\tau, \theta) \right|^2 d\theta \right) \left(\int_0^1 |\Xi(\theta, z(\theta), w(\theta)) - \Xi(\theta, z^*(\theta), w^*(\theta))|^2 d\theta \right) \\
\leq & \frac{1}{\Gamma^2(\nu)} \left(\frac{4}{5} + \frac{8\alpha}{3|\alpha - 2|} + \frac{4\alpha^2}{9(\alpha - 2)^2} \right) \left(\int_0^1 \frac{|z(\theta) - z^*(\theta)|^2 + |w(\theta) - w^*(\theta)|^2}{2} d\theta \right) \\
= & \lambda \left(\int_0^1 \frac{|z(\theta) - z^*(\theta)|^2 + |w(\theta) - w^*(\theta)|^2}{2} d\theta \right).
\end{aligned}$$

Taking the supremum over [0, 1], we have

$$\mu_r(\Upsilon(z, w), \Upsilon(z^*, w^*)) \leq \lambda \frac{\mu_r(z, z^*) + \mu_r(w, w^*)}{2}.$$

Hence the contractive stipulation (3.10) of Corollary 3.2 is fulfilled, then the mapping Υ have a unique CFP. Thus, by Lemma 4.3 the BVP (1.1) has a unique solution in $C[0, 1]$. □

5. Supportive examples

This part is devoted to support the theoretical results, where some illustrative examples are presented.

Example 5.1. Assume that $\Omega = \mathbb{R}$ and $\Upsilon, \Xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two mappings defined by

$$\Upsilon(a_1, a_2) = (a_1)^2 (a_2)^2 \text{ and } \Xi(a_1, a_2) = \frac{4}{3}(a_1 + a_2),$$

for all $a_1, a_2 \in \mathbb{R}$, then (0, 0), (1, 2) and (2, 1) are coupled coincidence points of Υ and Ξ.

Example 5.2. Let $\Omega = \mathbb{R}$ and $\Upsilon, \Xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two mappings described as

$$\Upsilon(a_1, a_2) = a_1 + a_2 - a_1 a_2 - \sin(a_1 - a_2) \text{ and } \Xi(a_1, a_2) = a_1 + a_2 + \cos(a_1 + a_2)\,,$$

for all $a_1, a_2 \in \mathbb{R}$, then $(0, \frac{\pi}{4})$ and $(\frac{\pi}{4}, 0)$ are coupled coincidence points of Υ and Ξ.

Example 5.3. Let $\Omega = \mathbb{R}$ and $\Upsilon, \Xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two mappings described as

$$\Upsilon(a_1, a_2) = a_1 a_2 \text{ and } \Xi(a_1, a_2) = a_1 + (a_2 - a_1)^2,$$

for all $a_1, a_2 \in \mathbb{R}$, then $(0, 0)$ and $(1, 1)$ are common CFP of Υ and Ξ.

Example 5.4. Let $\Omega = [0, \infty)$. Define $\mu_r : [0, \infty) \times [0, \infty) \to \mathbb{R}^+$ by

$$\mu_r(\rho, \sigma) = (\rho - \sigma)^2\,, \text{ for all } \rho, \sigma \in \Omega.$$

Then (Ω, μ_r) is bMS with $r = 2$. Define the mappings $\Upsilon, \Xi : \Omega \times \Omega \to \Omega$ as follows:

$$\Upsilon(\rho, \sigma) = \begin{cases} \frac{\rho-\sigma}{4}, & \text{if } \rho > \sigma, \\ 0, & \text{if } \rho \leq \sigma, \end{cases} \text{ and } \Xi(\rho, \sigma) = \begin{cases} \frac{\rho-\sigma}{5}, & \text{if } \rho > \sigma, \\ 0, & \text{if } \rho \leq \sigma, \end{cases}$$

To fulfill the rational contractive condition (3.1) of Theorem 3.1, we consider the cases below:

(i) If $\rho > a > \sigma > b$, then $\Upsilon(\rho, \sigma) = \frac{\rho-\sigma}{4}$ and $\Xi(\rho, \sigma) = \frac{\rho-\sigma}{5}$. Consider

$$\begin{aligned}
\mu_r(\Upsilon(\rho, a), \Xi(\sigma, b)) &= (\Upsilon(\rho, a) - \Xi(\sigma, b))^2 = \left(\frac{\rho - a}{4} - \frac{\sigma - b}{5}\right)^2 \\
&\leq \frac{(\rho - a)^2}{16} + \frac{(\sigma - b)^2}{25} \\
&\leq \frac{((\rho - \sigma) + (\sigma - a))^2}{16} + \frac{((\sigma - a) + (a - b))^2}{16} \\
&\leq \frac{(\rho - \sigma)^2}{16} + \frac{(a - b)^2}{16}, \text{ since } \sigma < a \\
&= \frac{1}{8}\left(\frac{(\rho - \sigma)^2 + (a - b)^2}{2}\right) \\
&= \lambda\frac{(\rho - \sigma)^2 + (a - b)^2}{2} \\
&= \lambda\frac{\mu_r(\rho, \sigma) + \mu_r(a, b)}{2} \\
&\leq \lambda\frac{\mu_r(\rho, \sigma) + \mu_r(a, b)}{2} + \tau\frac{[1 + \mu_r(\rho, \Upsilon(\rho, a))]\,\mu_r(\sigma, \Xi(\sigma, b))}{(1 + \mu_r(\rho, \sigma) + \mu_r(a, b))} \\
&\quad + \zeta\frac{\mu_r(\sigma, \Upsilon(\rho, a))\,\mu_r(\rho, \Xi(\sigma, b))}{(1 + \mu_r(\rho, \sigma) + \mu_r(a, b))}.
\end{aligned}$$

Hence for any value of τ and ζ with $\lambda = \frac{1}{8}$ so that $\lambda + \tau < 1$ and $\lambda + \zeta < 1$, we find that the condition (3.1) holds.

(ii) If $\rho \leq a \leq \sigma \leq b$, then $\Upsilon(\rho, \sigma) = 0$ and $\Xi(\rho, \sigma) = 0$. It is a trivial case. Thus, all requirements of Theorem 3.1 are fulfilled and $(0, 0)$ is a unique common CFP of Υ and Ξ.

Example 5.5. Let $\Omega = \{0, 1\}$. Consider a $b-$metric $\mu_r : \Omega \times \Omega \to \mathbb{R}^+$ by

$$\mu_r(\rho, \sigma) = \frac{2}{3}(\rho - \sigma)^2, \text{ for all } \rho, \sigma \in \Omega.$$

Then (Ω, μ_r) is bMS with parameter $r = 2$. Define the mappings $\Upsilon, \Xi : \Omega \times \Omega \to \Omega$ by $\Upsilon(\rho, \sigma) = \frac{\rho\sigma}{4}$ and $\Xi(\rho, \sigma) = \frac{\rho\sigma}{3}$, for all $\rho, \sigma \in \Omega$. It is easy to conclude that the stipulation (3.1) of Theorem 3.1 is fulfilled with $\lambda = \frac{3}{8}$, $\tau = \frac{1}{5}$ and $\zeta = \frac{2}{5}$. Hence $(0, 0)$ is a unique common CFP of Υ and Ξ.

Example 5.6. Consider the BVP of fractional order below:

$$^{c}\Theta^{\frac{7}{2}} z(\tau) + \Xi(\tau, z(\tau), w(\tau)) = 0, \ \tau \in (0, 1), \tag{5.1}$$

where Ξ described as

$$\Xi(\tau, z(\tau), w(\tau)) = \begin{cases} \frac{1}{2\sqrt{z(\tau)-w(\tau)}}, & \text{if } -1 \le z, w < 1, \\ \frac{1}{2\sqrt{z(\tau)+w(\tau)}}, & \text{otherwise}, \end{cases}$$

which is a singular at $z = 0 = w$, with the stipulations

$$\Lambda'''(0) = \Lambda''(0) = 0, \ \Lambda' = \Lambda(1) = \frac{1}{3}\int_0^1 \Lambda(\theta) d\theta,$$

for each $\Lambda \in [0, 1] \times [0, 1]$. It is clear that the solution of the differential equation of fractional order (5.1) can be satisfied to fulfill the integral equations below:

$$z(\tau) = \int_0^1 \Im(\tau, \theta)\Xi(\theta, z(\theta), w(\theta)) d\theta,$$

and

$$w(\tau) = \int_0^1 \Im(\tau, \theta)\Xi(\theta, w(\theta), z(\theta)) d\theta,$$

where $\Im(\tau, \theta)$ is given by

$$\Im(\tau, \theta) = \frac{6}{35\Gamma(\frac{7}{2})} \begin{cases} \left(\frac{35}{6} + \frac{2}{3}\tau\left(\frac{5}{2} + \theta\right)\right)(1 - \theta)^{\frac{5}{2}} - \frac{35}{6}(\tau - \theta)^{\frac{5}{2}}, & \text{if } 0 \le \theta \le \tau \le 1, \\ \left(\frac{35}{6} + \frac{2}{3}\tau\left(\frac{5}{2} + \theta\right)\right)(1 - \theta)^{\frac{5}{2}}, & \text{if } 0 \le \tau \le \theta \le 1, \end{cases} \tag{5.2}$$

here $\alpha = \frac{1}{3}$ and $\nu = \frac{7}{2}$, which verify the assumption (4.11). It follows from Lemma 4.2 that

$$\lambda = \int_0^1 \left|\Im(\tau, \theta)\right|^2 d\theta < \frac{64}{225\pi} \times \frac{304}{225} \approx 0.1223 < 1. \tag{5.3}$$

Based on Green's funtion (5.2) and the the related mapping Υ described in (4.9), one can write

$$\begin{aligned} & |\Upsilon(z(\tau), w(\tau)) - \Upsilon(z^*(\tau), w^*(\tau))|^2 \\ = & \left|\int_0^1 \Im(\tau, \theta) \left[\Xi(\theta, z(\theta), w(\theta)) - \Xi(\theta, z^*(\theta), w^*(\theta))\right] d\theta\right|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_0^1 \left|\Im(\tau,\theta)\right|^2 d\theta\right)\left(\int_0^1 |\Xi(\theta,z(\theta),w(\theta)) - \Xi(\theta,z^*(\theta),w^*(\theta))|^2 d\theta\right) \text{ (by Cauchy-Schwarz inequality)}\\
&= \lambda \int_0^1 |\Xi(\theta,z(\theta),w(\theta)) - \Xi(\theta,z^*(\theta),w^*(\theta))|^2 d\theta,
\end{aligned}
$$

where $\lambda < 1$ and

$$
\begin{aligned}
&|\Xi(\theta,z(\theta),w(\theta)) - \Xi(\theta,z^*(\theta),w^*(\theta))|\\
&= \begin{cases}
\left|\sqrt{\frac{1}{4(z(\theta)-w(\theta))} - \frac{1}{4(z^*(\theta)-w^*(\theta))}}\right|, \text{ if } z,w,z^*,w^* \in [-1,1),\\
\left|\sqrt{\frac{1}{4(z(\theta)+w(\theta))} - \frac{1}{4(z^*(\theta)+w^*(\theta))}}\right|, \text{ if } z,w,z^*,w^* \in (-\infty,-1)\cup[1,+\infty),\\
\left|\sqrt{\frac{1}{4(z(\theta)-w(\theta))} - \frac{1}{4(z^*(\theta)+w^*(\theta))}}\right|, \text{ if } z,w \in [-1,1),\ z^*,w^* \in (-\infty,-1)\cup[1,+\infty),\\
\left|\sqrt{\frac{1}{4(z(\theta)+w(\theta))} - \frac{1}{4(z^*(\theta)-w^*(\theta))}}\right|, \text{ if } z,w \in (-\infty,-1)\cup[1,+\infty),\ z^*,w^* \in [-1,1).
\end{cases}
\end{aligned}
$$

Now, for $z,w,z^*,w^* \in [-1,1)$, one sees that

$$
\begin{aligned}
&|\Upsilon(z(\tau),w(\tau)) - \Upsilon(z^*(\tau),w^*(\tau))|^2\\
&\leq \lambda \int_0^1 \left|\sqrt{\frac{1}{4\,(z(\theta)-w(\theta))} - \frac{1}{4\,(z^*(\theta)-w^*(\theta))}}\right|^2 d\theta,\\
&= \frac{\lambda}{4}\int_0^1 \left|\frac{(z^*(\theta)-w^*(\theta)) - (z(\theta)-w(\theta))}{(z(\theta)-w(\theta))\,(z^*(\theta)-w^*(\theta))}\right| d\theta\\
&\leq \frac{\lambda}{2}\int_0^1 \left|\frac{(z^*(\theta)-z(\theta)) + (w(\theta)-w^*(\theta))}{(z(\theta)-w(\theta))\,(z^*(\theta)-w^*(\theta))}\right| d\theta\\
&\leq \lambda \int_0^1 \left(\frac{|z^*(\theta)-z(\theta)|^2 + |w(\theta)-w^*(\theta)|^2}{2}\right) d\theta.
\end{aligned}
$$

By taking the supremum over $\tau \in [0,1]$ and put into account the metric distance (4.10), one can write

$$
\mu_r\left(\Upsilon(z,w),\Upsilon(z^*,w^*)\right) \leq \lambda \frac{\mu_r\,(z,z^*) + \mu_r\,(w,w^*)}{2}.
$$

By the same manner, one can show the other selections. Thus, by Corollary 3.2, we conclude that the mapping Υ described in (4.9) has a unique CFP. So we expect a unique solution to the BVP (1.1) in $C[0,1]$.

Conflict of interest

The authors declare that they have no conflict of interest.

Acknowledgments

This research project was supported by the Thailand Science Research and Innovation Fund and the University of Phayao (Grant No. FF64-UoE002).

H. A. Hammad would like to thank Sohag University, Egypt. W. Cholamjiak would like to thank University of Phayao, Thailand.

References

1. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
2. I. Podlubny, *Fractional differential equations*, Academic Press, 1999.
3. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach, 1993.
4. R. P. Agarwal, B. Ahmad, Existence of solutions for impulsive anti-periodic boundary value problems of fractional semilinear evolution equations, *Dynam. Cont. Dis. Ser. A*, **18** (2011), 457–470.
5. B. Ahmad, R. P. Agarwal, On nonlocal fractional boundary value problems, *Dynam. Cont. Dis. Ser. A*, **18** (2011), 535–544.
6. C. Bai, Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative, *J. Math. Anal. Appl.*, **384** (2011), 211–231.
7. H. A. Hammad, H. Aydi, M. De la Sen, Solutions of fractional differential type equations by fixed point techniques for multi-valued contractions, *Complixty*, **2021** (2021), 5730853.
8. M. Cichoń, H. A. H. Salem, On the lack of equivalence between differential and integral forms of the Caputo-type fractional problems, *J. Pseudo-Differ. Oper.*, **11** (2020), 1869–1895.
9. E. Girejko, D. Mozyrska, M. Wyrwas, A sufficient condition of viability for fractional differential equations with the Caputo derivative, *J. Math. Anal. Appl.*, **381** (2011), 146–154.
10. H. A. Hammad, M. De la Sen, Tripled fixed point techniques for solving system of tripled fractional differential equations, *AIMS Mathematics*, **6** (2020), 2330–2343.
11. S. K. Ntouyas, G. Wang, L. Zhang, Positive solutions of arbitrary order nonlinear fractional differential equations with advanced arguments, *Opusc. Math.*, **31** (2011), 433–442.
12. H. A. H. Salem, On functions without pseudo derivatives having fractional pseudo derivatives, *Quaest. Math.*, **42** (2019), 1237–1252.
13. H. A. Hammad, H. Aydi, N. Mlaiki, Contributions of the fixed point technique to solve the 2D Volterra integral equations, Riemann-Liouville fractional integrals, and Atangana-Baleanu integral operators, *Adv. Differ. Equ.*, **2021** (2021), 97.
14. M. Benchohra, J. J. Nieto, Ouahab, Second-order boundary value problem with integral boundary conditions, *Bound. Value Probl.*, **2011** (2011), 260309.
15. M. Feng, X. Zhang, W. Ge, New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions, *Bound. Value Probl.*, **2011** (2011), 720702.
16. T. Jankowski, Positive solutions for fourth-order differential equations with deviating arguments and integral boundary conditions, *Nonlinear Anal. Theor.*, **73** (2020), 1289–1299.
17. H. A. H. Salem, Fractional order boundary value problem with integral boundary conditions involving Pettis integral, *Acta Math. Sci.*, **31** (2011), 661–672.

18. V. Todorčević, Subharmonic behavior and quasiconformal mappings, *Anal. Math. Phys.*, **9** (2019), 1211–1225.

19. V. Todorčević, *Harmonic quasiconformal mappings and hyperbolic type metrics*, Springer International Publishing, 2019.

20. G. Wang, Boundary value problems for systems of nonlinear integro-differential equations with deviating arguments, *J. Comput. Appl. Math.*, **234** (2010), 1356–1363.

21. H. A. Hammad, M. De la Sen, A Solution of Fredholm integral equation by using cyclic η_s^q−rational contractive mappings technique in b-metric-like spaces, *Symmetry*, **11** (2019), 1184.

22. H. A. Hammad, M. De la Sen, Solution of nonlinear integral equation via fixed point of cyclic α_L^ψ-rational contraction mappings in metric-like spaces, *B. Braz. Math. Soc.*, **51** (2020), 81–105.

23. G. Wang, G. Song, L. Zhang, Integral boundary value problems for first order integro-differential equations with deviating arguments, *J. Comput. Appl. Math.*, **225** (2009), 602–611.

24. X. Zhang, M. Feng, W. Ge, Existence result of second-order differential equations with integral boundary conditions at resonance, *J. Math. Anal. Appl.*, **353** (2009), 311–319.

25. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.

26. T. Abdeljawad, R. P. Agrawal, E. Karapınar, P. S. Kumari, Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space, *Symmetry*, **11** (2019), 686.

27. B. Alqahtani, H. Aydi, E. Karapınar, V. Rakočević, A solution for Volterra fractional integral equations by hybrid contractions, *Mathematics*, **7** (2019), 694.

28. N. Fabiano, N. Nikolič, S. Thenmozhi, S. Radenović, N. Cĭtaković, Tenth order boundary value problem solution existence by fixed point theorem, *J. Inequal. Appl.*, **2020** (2020), 166.

29. E. Karapinar, T. Abdeljawad, F. Jarad, Applying new fixed point theorems on fractional and ordinary differential equations, *Adv. Differ. Equ.*, **2019** (2019), 421.

30. D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal. Theor.*, **11** (1987), 623–632.

31. T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal. Theor.*, **65** (2006), 1379–1393.

32. E. Karapinar, Coupled fixed point theorems for nonlinear contractions in cone metric spaces, *Comput. Math. Appl.*, **59** (2010), 3656–3668.

33. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, *Nonlinear Anal. Theor.*, **74** (2011), 983–992.

34. H. A. Hammad, M. De la Sen, A coupled fixed point technique for solving coupled systems of functional and nonlinear integral equations, *Mathematics*, **7** (2019), 634.

35. H. A. Hammad, D. M. Albaqeri, R. A. Rashwan, Coupled coincidence point technique and its application for solving nonlinear integral equations in RPOCbML spaces, *J. Egypt. Math. Soc.*, **28** (2020), 8.

36. B. S. Choudhury, K. Kundu, Two coupled weak contraction theorems in partially ordered metric spaces, *RACSAM Rev. R. Acad. A*, **108** (2014), 335–351.

37. Y. J. Cho, Z. Kadelburg, R. Saadati, W. Shatanawi, Coupled fixed point theorems under weak contractions, *Discrete Dyn. Nat. Soc.*, **2012** (2012), 184534.

38. Y. Dzhabarova, S. Kabaivanov, M. Ruseva, B. Zlatanov, Existence, uniqueness and stability of market equilibrium on oligopoly markets, *Adm. Sci.*, **10** (2020), 70.

39. S. Kabaivanov, B. Zlatanov, A variational principle, coupled fixed points and market equilibrium, *Nonlinear Anal. Model.*, **26** (2021), 169–185.

40. S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Sem. Mat. Fis. Univ. Modena*, **46** (1993), 263–276.

41. M. Boriceanu, Fixed point theory for multivalued generalized contraction on a set with two b-metrics, *Stud. U. Babes-Bol. Mat.*, **3** (2009), 3–14.

42. T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal. Theor.* **65** (2006), 1379–1393.

43. Y. He, Existence and multiplicity of positive solutions for singular fractional differential equations with integral boundary value conditions, *Adv. Differ. Equ.*, **2016** (2016), 31.

44. Z. Bai, T. Qiu, Existence of positive solutions for singular fractional differential equation, *Appl. Math. Comput.*, **215** (2009), 2761–2767.

AIMS Press