AIMS *Mathematics*

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13311–13326.
DOI:10.3934/math.2021770
Received: 07 June 2021
Accepted: 31 August 2021
Published: 16 September 2021

Research article

Two meromorphic functions on annuli sharing some pairs of small functions or values

Hongzhe Cao*

Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, China

* **Correspondence:** Email: hongzhecao@ncu.edu.cn.

Abstract: In this paper, we prove that two admissible meromorphic functions on an annulus must be linked by a quasi-Möbius transformation if they share some pairs of small function with multiplicities truncated by 4. We also give the representation of Möbius transformation between two admissible meromorphic functions on an annulus if they share four pairs of values with multiplicities truncated by 4. In our results, the zeros with multiplicities more than a certain number are not needed to be counted if their multiplicities are bigger than a certain number.

Keywords: meromorphic function; small function; quasi-Möbius transformation; Nevanlinna theory
Mathematics Subject Classification: 32H30, 32A22

1. Introduction

In 1925, R. Nevanlinna [8] extended the classical theorems of Picard and Borel by developing the value distribution theory of meromorphic functions on the complex plane $\mathbb{C}$, which is now called Nevanlinna theory. As its application, Nevanlinna derived the well known five values theorem and four values theorem: two nonconstant distinct meromorphic functions on $\mathbb{C}$ cannot have the same inverse images of five distinct values; they must be linked by a Möbius transformation if they share four values counting multiplicities. Since then, the uniqueness problem related to sharing values or functions has been widely studied.

In 1997, T. Czubiak and G. Gundersen [3] proved the following result.

Theorem A. *Let f and g be two non constant meromorphic functions on $\mathbb{C}$ that share six pairs of values (a_i, b_i), $1 \le i \le 6$, IM (ignoring multiplicities), where $a_i, b_i \in \mathbb{C}$ and $a_i \neq a_j$, $b_i \neq b_j$ whenever $i \neq j$, i.e.,*

$$\min\{\nu^0_{f-a_i}, 1\} = \min\{\nu^0_{g-b_i}, 1\}\ (1 \le i \le 6).$$

Then f is a Möbius transformation of g.

Li and Yang [6] gave an example which was found by G. G. Gundersen in 1979 and showed that the number of pairs of values in Theorem A cannot be replaced by a smaller one. After that, many authors studied the sharing pairs of values problem of meromorphic functions on $\mathbb{C}$. There are some extensions of the above result, where the pairs of values are replaced by pairs of small functions (e.g., see [7, 13, 15]).

By the Doubly Connected Mapping Theorem, each doubly connected domain is conformally equivalent to the annulus $\mathbb{A}(r; R) = \{z : 0 \le r < |z| < R \le +\infty\}$. However $\mathbb{A}(r; R)$ is biholomorphic to $\mathbb{A}(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$ for some $R_0 \in (1, +\infty]$. In fact, if $r > 0$ and $R = +\infty$, set $z \to \frac{1}{z-r}$, $R_0 = +\infty$; if $r = 0$ and $R < +\infty$, set $z \to \frac{z}{R-z}$, $R_0 = +\infty$; if $0 < r < R < +\infty$, set $z \to \frac{z}{\sqrt{rR}}$, $R_0 = \sqrt{\frac{R}{r}}$.

Recently, Khrystiyanyn and Kondratyuk (see [4, 5]) proposed the Nevanlinna theory for meromorphic functions on annuli. Using the second main theorem for meromorphic functions on annuli, Cao, Yi and Xu in [2] proved a uniqueness theory of meromorphic functions on annuli sharing values. Quang and Tran [13] studied the case where the meromorphic functions on annuli share some pairs of values with truncated multiplicities and obtained the following result.

Theorem B. *Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \le +\infty)$. Let $\{(a_i, b_i)\}_{i=1}^q$ $(q \ge 6)$ be q pairs of values, where $a_i \ne a_j$, $b_i \ne b_j$ whenever $i \ne j$. Let k_i $(1 \le i \le q)$ be q positive integers or $+\infty$ with $k_1 \ge k_2 \ge \cdots \ge k_q$ such that*

$$\sum_{i=m+1}^{q} \frac{1}{k_i + 1} + \left(m - \frac{3q}{5}\right)\frac{1}{k_m + 1} < \frac{2q - 10}{5}$$

for an integer number $m \in \{1, 2, \ldots, q\}$. Assume that

$$\min\{\nu^0_{f-a_i,\le k_i}, 1\} = \min\{\nu^0_{g-b_i,\le k_i}, 1\}\ (1 \le i \le q).$$

Then f is a Möbius transformation of g.

Here, the meromorphc function f on $\mathbb{A}(R_0)$ is said to be *admissible* if it satisfies

$$\limsup_{r\to+\infty} \frac{T_0(r, f)}{\log r} = +\infty \quad \text{in the case} \quad R_0 = +\infty$$

or

$$\limsup_{r\to R^-} \frac{T_0(r, f)}{-\log(R_0 - r)} = +\infty \quad \text{in the case} \quad R_0 < +\infty,$$

where $T_0(r, f)$ is the characteristic function of f (see Section 2 for details).

For the case of meromorphic functions on the complex plane $\mathbb{C}$, there is a sharp second main theorem for small functions given by Yamanoi [14]. Therefore, in that case, all authors used the Cartans auxialiary functions and got nice results. However, in the case of functions on annuli, there is no sharp second main theorem for small function. In the light of [9], we study the case that the functions on annuli share q $(q \ge 5)$ small functions and get the following result.

Theorem 1.1. *Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \le +\infty)$. Let $\{(a_i(z), b_i(z))\}_{i=1}^q$ $(q \ge 5)$ be q pairs of small (with respect to f and g) functions on $\mathbb{A}(R_0)$, where $a_i(z) \not\equiv a_j(z)$, $b_i(z) \not\equiv b_j(z)$ for every $z \in \mathbb{A}(R_0)$ and $i \ne j$. Let $k_i \ge 4$ $(1 \le i \le q)$ be q positive integers or $+\infty$ such that*

$$\sum_{i=1}^{q} \frac{289}{k_i + 289} < \frac{2q}{5}.$$

Assume that

$$\min\{\nu^0_{f-a_i,\le k_i},4\} = \min\{\nu^0_{g-b_i,\le k_i},4\}\ (1 \le i \le q).$$

Then f is a quasi-Möbius transformation of g.

Here, we say that f is a *quasi-Möbius* transformation of g if there exist four small functions (with respect to f and g) a, b, c, d with $ad - bc \not\equiv 0$ such that $f = \frac{ag+b}{cg+d}$.

For the case of meromorphic functions sharing four pairs of small functions, we will prove the following.

Theorem 1.2. *Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \le +\infty)$. Let $\{(a_i(z), b_i(z))\}_{i=1}^4$ be four pairs of small (with respect to f and g) functions on $\mathbb{A}(R_0)$, where $a_i(z) \not\equiv a_j(z)$, $b_i(z) \not\equiv b_j(z)$ whenever $i \ne j$. Let $k_i \ge 4$ $(1 \le i \le 4)$ be positive integers or $+\infty$ such that*

$$\sum_{i=1}^{4} \frac{289}{k_i + 289} < \frac{4}{3}.$$

Assume that

$$\min\{\nu^0_{f-a_i,\le k_i},4\} = \min\{\nu^0_{g-b_i,\le k_i},4\}\ (1 \le i \le 4).$$

Then f is a Möbius transformation of g.

Let $k_1 = \cdots = k_q = k$, we have the following corollary.

Corollary 1.3. *Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \le +\infty)$. Let $\{(a_i(z), b_i(z))\}_{i=1}^q$ be q pairs of small (with respect to f and g) functions on $\mathbb{A}(R_0)$, where $a_i(z) \not\equiv a_j(z)$, $b_i(z) \not\equiv b_j(z)$ whenever $i \ne j$. Let k be a positive integer or $+\infty$ such that $k > 433$ if $q \ge 5$ and $k > 578$ if $q = 4$. Assume that*

$$\min\{\nu^0_{f-a_i,\le k},4\} = \min\{\nu^0_{g-b_i,\le k},4\}\ (1 \le i \le q).$$

Then f is a quasi-Möbius transformation of g.

Furthermore, we consider the two admissible meromorphic functions on $\mathbb{A}(R_0)$ sharing 4 pairs of values and obtain:

Theorem 1.4. *Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \le +\infty)$. Let $\{(a_i, b_i)\}_{i=1}^4$ be four pairs of values, where $a_i \ne a_j$, $b_i \ne b_j$ whenever $i \ne j$. Let $k_i \ge 4$ $(1 \le i \le 4)$ be positive integers or $+\infty$ such that $\sum_{i=1}^4 \frac{289}{k_i+289} < 2$. Assume that*

$$\min\{\nu^0_{f-a_i,\le k_i},4\} = \min\{\nu^0_{g-b_i,\le k_i},4\}\ (1 \le i \le 4).$$

Then f is a Möbius transformation of g. Moreover there is a permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$ such that

$$\frac{f-a_{i_1}}{f-a_{i_2}} \cdot \frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}} \cdot \frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}}, \quad \frac{f-a_{i_1}}{f-a_{i_2}} \cdot \frac{a_{i_4}-a_{i_2}}{a_{i_4}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}} \cdot \frac{b_{i_4}-b_{i_2}}{b_{i_4}-b_{i_1}},$$

$$\text{or} \quad \frac{f-a_{i_1}}{f-a_{i_2}} \cdot \frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}} \cdot \frac{b_{i_4}-a_{i_2}}{b_{i_4}-a_{i_1}}.$$

When $k_1 = \cdots = k_4 = k$, Theorem 1.4. implies the following corollary.

Corollary 1.5. *Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \le +\infty)$. Let $\{(a_i, b_i)\}_{i=1}^4$ be four pairs of values, where $a_i \ne a_j$, $b_i \ne b_j$ whenever $i \ne j$. Let $k > 289$ be a positive integer or $+\infty$ such that*

$$\min\{\nu^0_{f-a_i,\le k}, 4\} = \min\{\nu^0_{g-b_i,\le k}, 4\}\ (1 \le i \le 4).$$

Then f is a Möbius transformation of g.Moreover there is a permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$ such that

$$\frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}},\quad \frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_4}-a_{i_2}}{a_{i_4}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_4}-b_{i_2}}{b_{i_4}-b_{i_1}},$$

$$\text{or}\quad \frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_4}-a_{i_2}}{b_{i_4}-a_{i_1}}.$$

2. Some definitions and results from Nevanlinna theory on annuli

First of all, we will recall some basic notions of Nevanlinna theory for meromorphic functions on annuli from [6] (see also [1, 4, 5]).

For a divisor ν on $\mathbb{A}(R_0)$ $(1 < R_0 \le +\infty)$, which we may regard as a function on $\mathbb{A}(R_0)$ with values in $\mathbb{Z}$ whose support is discrete subset of $\mathbb{A}(R_0)$, and for a positive integer M (maybe $M = \infty$), we define the counting function of ν as follows

$$n_0^{[M]}(t) = \begin{cases} \sum_{1\le|z|\le t} \min\{M, \nu(z)\} & \text{if} \quad 1 \le t < R_0, \\ \sum_{1\le|z|\le t} \min\{M, \nu(z)\} & \text{if} \quad \frac{1}{R_0} < t < 1. \end{cases}$$

and

$$N_0^{[M]}(r, \nu) = \int_{\frac{1}{r}}^1 \frac{n_0^{[M]}(t)}{t}dt + \int_1^r \frac{n_0^{[M]}(t)}{t}dt\ (1 < r < R_0).$$

For brevity, we will omit the character $[M]$ if $M = \infty$. For a divisor ν and a positive integer k or $+\infty$, we define:

$$\nu_{\le k}(z) = \begin{cases} \nu(z) & \text{if} \quad \nu(z) \le k \\ 0 & \text{otherwise} \end{cases}, \qquad \nu_{>k}(z) = \begin{cases} \nu(z) & \text{if} \quad \nu(z) > k, \\ 0 & \text{otherwise.} \end{cases}$$

For a meromorphic function φ, we define ν^0_φ (resp. ν^∞_φ) the divisor of zeros (resp. divisor of poles) of φ; $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$; $\nu^0_{\varphi,\le k} = (\nu^0_\varphi)_{\le k}$, $\nu^0_{\varphi,>k} = (\nu^0_\varphi)_{>k}$. Similarly, we define $\nu^\infty_{\varphi,\le k}$, $\nu^\infty_{\varphi,>k}$, $\nu_{\varphi,\le k}$, $\nu_{\varphi,>k}$ and their counting functions. For a discrete subset $S \subset \mathbb{A}(R_0)$, we consider it as a reduced divisor (denoted again by S) whose support is S, and denote by $N_0(r, S)$ its counting function. We also set $\chi_S(z) = 0$ if $z \notin S$ and $\chi_S(z) = 1$ if $z \in S$.

Let f be a nonconstant meromorphic function on $\mathbb{A}(R_0)$. The proximity function of f is defined by

$$m_0(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(\frac{e^{i\theta}}{r})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta - \frac{1}{\pi}\int_0^{2\pi} \log^+ |f(e^{i\theta})|d\theta$$

and the characteristic function of f is defined by

$$T_0(r, f) = N_0(r, \nu_f^\infty) + m_0(r, f).$$

Throughout this paper, we denote by $S_f(r)$ quantities satisfying:
(i) in the case $R_0 = +\infty$,

$$S_f(r) = O(\log(rT_0(r, f))).$$

for $r \in (1, R_0)$ except for a set Δ_R such that $\int_{\Delta_R} r^{\lambda-1}dr < +\infty$ for some $\lambda \geq 0$.
(ii) in the case $R_0 < +\infty$,

$$S_f(r) = O(\log(\frac{T_0(r, f)}{R_0 - r})),$$

for $r \in (1, R_0)$ except for a set Δ'_R such that $\int_{\Delta'_R} \frac{1}{(R_0-r)^{\lambda+1}}dr < +\infty$ for some $\lambda \geq 0$.

Thus for an admissible meromorphic function f on the annulus $\mathbb{A}(R_0)$, we have $S_f(r) = o(T_0(r, f))$ as $r \to R_0$ for all $1 \leq r < R_0$ except for the set Δ_R or the set Δ'_R mentioned above, respectively [1]. A meromorphic function a on $\mathbb{A}(R_0)$ is said to be small with respect to f if $T_0(r, a) = S_f(r)$.

Lemma 2.1. (Lemma on logarithmic derivatives [1,4–6]) *Let f be a nonzero meromorphic function on $\mathbb{A}(R_0)$. Then for each $k \in \mathbb{N}$ we have*

$$m_0(r, \frac{f^{(k)}}{f}) = S_f(r) \quad (1 < r < R).$$

Theorem 2.2. (First main theorem for meromorphic functions and small functions [11]) *Let f be a meromorphic function on $\mathbb{A}(R_0)$ and a be a small function with respect to f. Then we have*

$$T_0(r, f) = T_0(r, \frac{1}{f - a}) + S_f(r) \quad (1 < r < R).$$

Theorem 2.3. (Second main theorem [1]) *Let f be a nonconstant meromorphic function on $\mathbb{A}(R_0)$. Let $a_1, \ldots, a_q$ be q distinct values in $\mathbb{C} \cup \{\infty\}$. We have*

$$(q - 2)T_0(r, f) \leq \sum_{i=1}^{q} N_0^{[1]}(r, \nu^0_{f-a_i}) + S_f(r) \quad (1 < r < R_0).$$

Theorem 2.4. (Second main theorem with small function (see [15], Lemmas 3.1, 3.2)) *Let f be a nonconstant meromorphic function on $\mathbb{A}(R_0)$. Let $a_1, \ldots, a_5$ be 5 distinct small functions with respect to f. We have*

$$2T_0(r, f) \leq \sum_{i=1}^{5} N_0^{[1]}(r, \nu^0_{f-a_i}) + S_f(r) \quad (1 < r < R_0).$$

From Theorem 2.4, we easily get the following theorem.

Theorem 2.5. (Second main theorem with small function) *Let f be a nonconstant meromorphic function on $\mathbb{A}(R_0)$. Let $a_1, \ldots, a_q$ be q ($q \geq 5$) distinct small functions with respect to f. We have*

$$\frac{2q}{5}T_0(r, f) \leq \sum_{i=1}^{q} N_0^{[1]}(r, \nu^0_{f-a_i}) + S_f(r) \quad (1 < r < R_0).$$

3. Nevanlinna theory for holomorphic mappings from an annulus into a projective space

Let f be a holomorphic mapping from an annulus $\mathbb{A}(R_0)$ into $P^N(\mathbb{C})$ with a reduced representation $f = (f_0 : \cdots : f_N)$. For $1 < r < R_0$, the Nevanlinna-Cartan's characteristic function $T_0(r, f)$ of f is defined by

$$\begin{aligned} T_0(r, f) =& \frac{1}{2\pi}\int_0^{2\pi} \log \| f(r^{-1}e^{i\theta}) \| \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log \| f(re^{i\theta}) \| \, d\theta \\ & - \frac{1}{\pi}\int_0^{2\pi} \log \| f(e^{i\theta}) \| \, d\theta \end{aligned}$$

where $\| f \|= (|f_0|^2 + \cdots + |f^N|^2)^{\frac{1}{2}}$.

Let H be a hyperplane in $P^N(\mathbb{C})$ given by $H = \{(\omega_0 : \cdots : \omega_N)|a_0\omega_0 + \cdots + a_N\omega_N = 0\}$. We set $(f, H) = a_0 f_0 + \cdots + a_N f_N$. The proximity function of f with respect to H is defined by

$$\begin{aligned} m_0(r, f, H) =& \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\| f(r^{-1}e^{i\theta}) \| \| H \|}{| (f(r^{-1}e^{i\theta})) |} d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\| f(re^{i\theta}) \| \| H \|}{| (f(re^{i\theta})) |} d\theta \\ & - \frac{1}{\pi}\int_0^{2\pi} log \frac{\| f(re^{i\theta}) \| \| H \|}{| (f(e^{i\theta})) |} d\theta \end{aligned}$$

where $\| H \|= (|a_0|^2 + \cdots + |a^N|^2)^{\frac{1}{2}}$. By Jensens formula, we have the First Main Theorem for a holomorphic mapping from an annulus $\mathbb{A}(R_0)$ into $P^N(\mathbb{C})$ as follows.

$$T_0(r, f) = m_0(r, f, H) + N_0(r, f^* H),$$

where $f^* H$ denotes the pull back divisor of H by f.

Remark *Let f be a meromorphic function on $\mathbb{A}(R_0)$. We may regard f as a holomorphic curve from $A(R_0)$ into $P^1(\mathbb{C})$. Similarly to the case of meromorphic functions on $\mathbb{C}$, we see that the above two definitions of characteristic function $T_0(r, f)$ coincide to each other up to a constant.*

Proposition 3.1. [11] *Let f be a holomorphic mapping from an annulus $\mathbb{A}(R_0)$ into $P^N(\mathbb{C})$. Let H and G be two distinct hyperplanes of $P^N(\mathbb{C})$, then we have*

$$T_0(r, \frac{(f, H)}{(f, G)}) \leq T_0(r, f) + O(1).$$

Proof. By the definition of the characteristic function and the property of the function $\log^+$ (for positive numbers a and b, $\log^+(ab) + \log b \leq \log(a^2 + b^2)^{\frac{1}{2}}, a, b > 0$), we have

$$\begin{aligned} T_0(r, \frac{(f, H)}{(f, G)}) =& m_0(r, \frac{(f, H)}{(f, G)}) + N_0(r, \nu^{\infty}_{\frac{(f,H)}{(f,G)}}) \\ \leq & \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| \frac{(f, H)(re^{i\theta})}{(f, G)(re^{i\theta})} \right| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| \frac{(f, H)(r^{-1}e^{i\theta})}{(f, G)(r^{-1}e^{i\theta}} \right| d\theta \\ & + \frac{1}{2\pi}\int_0^{2\pi} \log |(f, G)(re^{i\theta}) | \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log | (f, G)(r^{-1}e^{i\theta}) | \, d\theta + O(1) \end{aligned}$$

$$
\begin{aligned}
\leq&\frac{1}{2\pi}\int_0^{2\pi}\log\left(|\,(f,H)(re^{i\theta})\,|^2+|\,(f,G)(re^{i\theta})\,|^2\right)^{\frac{1}{2}}\\
&+\frac{1}{2\pi}\int_0^{2\pi}\log\left(|\,(f,H)(r^{-1}e^{i\theta})\,|^2+|\,(f,G)(r^{-1}e^{i\theta})\,|^2\right)^{\frac{1}{2}}+O(1)\\
\leq&\frac{1}{2\pi}\int_0^{2\pi}\log\|\,f(r^{-1}e^{i\theta})\,\|\,d\theta+\frac{1}{2\pi}\int_0^{2\pi}\log\|\,f(re^{i\theta})\,\|\,d\theta+O(1)\\
=&T_0(r,f)+O(1).
\end{aligned}
$$

□

Let $\{H_i\}_{i=1}^q$ $(q\geq N+2)$ be a set of q hyperplanes in $P^N(\mathbb{C})$. We say that the family$\{H_i\}_{i=1}^q$ is in general position if $\cap_{j=1}^{N+1}H_{i_j}=\emptyset$ for any $1\leq i_1<\cdots<i_{N+1}\leq q$.

In 2015, H. T. Phuong and N. V. Thin [10] proved the following Second Main Theorem for holomorphic curves from an annulus into $P^N(\mathbb{C})$.

Theorem 3.2. *Let $f:\mathbf{A}(R_0)\to P^N(\mathbb{C})$ be a linearly nondegenerate holomorphic mapping. Let $\{H_i\}_{i=i}^q$ $(q\geq N+2)$ be a set of q hyperplanes in $P^N(\mathbb{C})$ in general position. Then*

$$(q-N-1)T_0(r,f)\leq\sum_{i=1}^q N_0^{[N]}(r,f^*H_i)+S_f(r),$$

where f^*H_i denotes the pull back divisor of H_i by f.

4. Proofs of main theorems

In order to prove the main theorems, we need some lemmas.

Lemma 4.1. *Let f be a nonconstant meromorphic function on $\mathbb{A}(R_0)$ and $a(z)$ be a small function (with respect to f) on $\mathbb{A}(R_0)$. Then for each positive integer k (may be ∞) we have*

$$N_0^{[1]}(r,\nu_{f-a}^0)\leq\frac{k}{k+1}N_0^{[1]}(r,\nu_{f-a,\leq k}^0)+\frac{1}{k+1}T_0(r,f)+S_f(r).$$

Proof. Since $N_0^{[1]}(r,\nu_{f-a,\leq k})\leq N_0(r,\nu_{f-a,\leq k})$ and $N_0(r,\nu_{f-a})\leq T_0(r,f)+S_f(r)$, we have

$$
\begin{aligned}
N_0^{[1]}(r,\nu_{f-a}^0)=&N_0^{[1]}(r,\nu_{f-a,\leq k}^0)+N_0^{[1]}(r,\nu_{f-a,>k}^0)\\
\leq&N_0^{[1]}(r,\nu_{f-a,\leq k}^0)+\frac{1}{k+1}N_0(r,\nu_{f-a,>k}^0)\\
=&N_0^{[1]}(r,\nu_{f-a,\leq k}^0)+\frac{1}{k+1}\left(N_0(r,\nu_{f-a}^0)-N_0(r,\nu_{f-a,\leq k}^0)\right)\\
\leq&\frac{k}{k+1}N_0^{[1]}(r,\nu_{f-a,\leq k}^0)+\frac{1}{k+1}T_0(r,f)+S_f(r).
\end{aligned}
$$

□

Lemma 4.2. *Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \leq +\infty)$. Let $\{(a_i(z), b_i(z))\}_{i=1}^3$ be three pairs small (with respect to f and g) functions on $\mathbb{A}(R_0)$, where $a_i(z) \not\equiv a_j(z)$, $b_i(z) \not\equiv b_j(z)$ whenever $i \neq j$. Assume that f is not a quasi-Möbius transformation of g. We have the following inequality*

$$N_0(r, \nu) \leq N_0^{[1]}\left(r, |\ \nu^0_{f-a_1} - \nu^0_{g-b_1}\ |\right) + N_0^{[1]}\left(r, |\ \nu^0_{f-a_2} - \nu^0_{g-b_2}\ |\right) + S(r),$$

where ν is the divisor defined by $\nu(z) = \max\{0, \min\{\nu^0_{f-a_3}, \nu^0_{g-b_3}\} - 1\}$ and $S(r) = S_f(r) + S_g(r)$.

Proof. By replacing f and g by $\frac{(f-a_1)(a_3-a_2)}{(f-a_2)(a_3-a_1)}$ and $\frac{(g-b_1)(b_3-b_2)}{(g-b_2)(b_3-b_1)}$ if necessary, we may assume that $a_1 = b_1 = 0, a_2 = b_2 = \infty$ and $a_3 = b_3 = 1$. Since f is not quasi-Möbius transformation of g,

$$h := \frac{f'}{f} - \frac{g'}{g} = \frac{(f/g)'}{f/g} \not\equiv 0.$$

By the lemma on logarithmic derivatives, it follows that

$$m_0(r, h) \leq m_0(r, \frac{f'}{f}) + m_0(r, \frac{g'}{g}) = S(r).$$

We also see that h has only simple poles, and it must be either $\nu^0_f(z) \neq \nu^0_g(z)$, or $\nu^\infty_f(z) \neq \nu^\infty_g(z)$. Then

$$N_0(r, \nu^\infty_h) \leq N_0^{[1]}(r, |\ \nu^0_f - \nu^0_g)\ | + N_0^{[1]}(r, |\ \nu^\infty_f - \nu^\infty_g).$$

On the other hand,

$$h = \frac{((f-g)/g)'}{f/g}.$$

This yields

$$N_0(r, \nu^0_h) \geq N_0(r, \nu).$$

By the first main theorem, we easily see that

$$\begin{aligned} N_0(r, \nu) \leq & N_0(r, \nu^0_h) \leq T_0(r, h) = m_0(r, h) + N_0(r, \nu^\infty_h) \\ \leq & N_0^{[1]}(r, |\ \nu^0_f - \nu^0_g\ |) + N_0^{[1]}(r, |\ \nu^\infty_f - \nu^\infty_g\ |) + S(r). \end{aligned}$$

The Lemma is proved. □

Lemma 4.3. *Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \leq +\infty)$. Let $\{(a_i, b_i)\}_{i=1}^4$ be four pairs of small (with respect to f and g) functions on $\mathbb{A}(R_0)$, where $a_i \not\equiv a_j$, $b_i(z) \not\equiv b_j(z)$ whenever $i \neq j$. Let k_i $(1 \leq i \leq 4)$ be positive integers or $+\infty$ such that*

$$\min\{\nu^0_{f-a_i, \leq k_i}, 4\} = \min\{\nu^0_{g-b_i, \leq k_i}, 4\}\ (1 \leq i \leq 4).$$

Assume that f is not a quasi-Möbius transformation of g. Then

$$\sum_{i=1}^4 \left(N_0^{[1]}(r, \nu^0_{f-a_i, \leq k_i}) + N_0^{[1]}(r, \nu^0_{g-b_i, \leq k_i})\right)$$

$$\leq 288 \sum_{i=1}^{4} \left(N_0^{[1]}(r, \nu^0_{f-a_i,>k_i}) + N_0^{[1]}(r, \nu^0_{g-b_i,>k_i}) \right) + S(r)$$

where $S(r) = S_f(r) + S_g(r)$.

Proof. For each $1 \leq i \leq 4$, we define the divisors ν_i and μ_i as follows

$$\nu_i(z) = \max\{0, \min\{\nu^0_{f-a_i}, \nu^0_{g-b_i}\} - 1\}$$
$$\mu_i(z) = \min\{1, | \nu^0_{f-a_i} - \nu^0_{g-b_i} |\}.$$

Take three indices $i, j, t \in \{1, 2, 3, 4\}$. By Lemma 4.2, we have

$$\begin{aligned} 3N_0^{[1]}(r, \mu_i(z)) \leq & 3\left(N_0^{[1]}(r, 3 < \nu^0_{f-a_i} \leq k_i) + N_0^{[1]}(r, \nu^0_{f-a_i,>k_i}) + N_0^{[1]}(r, \nu^0_{g-b_i,>k_i}) \right) \\ \leq & N_0(r, \nu_i) + 3\left(N_0^{[1]}(r, \nu^0_{f-a_i,>k_i}) + N_0^{[1]}(r, \nu^0_{g-b_i,>k_i}) \right) \\ \leq & N_0^{[1]}(r, \mu_j) + N_0^{[1]}(r, \mu_t) + 3\left(N_0^{[1]}(r, \nu^0_{f-a_i,>k_i}) + N_0^{[1]}(r, \nu^0_{g-b_i,>k_i}) \right). \end{aligned}$$

Summing-up both sides of the above inequality over all subsets $\{i, j, t\}$ of $\{1, 2, 3, 4\}$, we obtain

$$\sum_{i=1}^{4} N_0(r, \mu_i) \leq 3 \sum_{i=1}^{4} \left(N_0^{[1]}(r, \nu^0_{f-a_i,>k_i}) + N_0^{[1]}(r, \nu^0_{g-b_i,>k_i}) \right) + S(r). \tag{4.1}$$

Put

- $c_1 = \dfrac{a_3 - a_2}{a_2 - a_1},\ c_2 = \dfrac{a_3 - a_1}{a_2 - a_1},\ c'_1 = \dfrac{b_3 - b_2}{b_2 - b_1},\ c'_2 = \dfrac{b_3 - b_1}{b_2 - b_1},$
- $F_1 = c_1(f - a_1),\ F_2 = c_2(f - a_2),\ G_1 = c'_1(g - b_1),\ G_2 = C'_2(g - b_2),$
- $h_1 = \dfrac{F_1}{G_1},\ h_2 = \dfrac{F_2}{G_2},\ h_3 = \dfrac{f_2 - F_1}{G_2 - G_1} = \dfrac{f - a_3}{g - b_3} \cdot \dfrac{b_2 - b_1}{a_2 - a_1},$
- $\alpha = \dfrac{c_1(a_4 - a_1)}{c_2(a_4 - a_2)},\quad \beta = \dfrac{c'_1(b_4 - b_1)}{c'_2(b_4 - b_2)},$
- $h_4 = \dfrac{F_1 - \alpha F_2}{G_1 - \beta G_2} = \dfrac{(a_3 - a_2)(b_4 - b_2)}{(a_4 - a_2)(b_3 - b_2)} \cdot \dfrac{f - a_4}{g - b_4}.$

We easily see that $c_1 \not\equiv c_2, c'_1 \not\equiv c'_2, \alpha \not\equiv 1, \beta \not\equiv 1$, all c_i, c'_i $(i = 1, 2), \alpha,\ \beta$ are small with respect to f and g, and

$$N_0^{[1]}(r, \nu^0_{h_i}) + N_0^{[1]}(r, \nu^\infty_{h_i}) = N_0^{[1]}(r, \mu_i) + S(r) \quad (1 \leq i \leq 4). \tag{4.2}$$

Moreover,we have the following equations system:

$$\begin{cases} F_1 & - h_1 G_1 & = 0 \\ F_2 & - h_2 G_2 & = 0 \\ F_1 - F_2 - h_3 G_1 + h_3 G_2 & & = 0 \\ F_1 - \alpha F_2 - h_4 G_1 + h_4 \beta G_2 & & = 0 \end{cases}$$

Thus

$$\begin{vmatrix} 1 & 0 & -h_1 & 0 \\ 0 & 1 & 0 & -h_2 \\ 1 & -1 & -h_3 & h_3 \\ 1 & -\alpha & -h_4 & \beta h_4 \end{vmatrix} = 0.$$

This implies

$$(1-\alpha)h_1h_2 - h_1h_3 + \beta h_1h_4 + \alpha h_2h_3 - h_2h_4 + (1-\beta)h_3h_4 = 0.$$

Denote by $\mathcal{I}$ the set of all subsets $I = \{i, j\}$ of the set $\{1, 2, 3, 4\}$. For $I \in \mathcal{I}$, we define the function h_I as follows:

$$h_{\{1,2\}} = (1-\alpha)h_1h_2,\ h_{\{1,3\}} = -h_1h_3,\ h_{\{1,4\}} = \beta h_1h_4,$$
$$h_{\{2,3\}} = \alpha h_2h_3,\ h_{\{2,4\}} = -h_2h_4,\ h_{\{3,4\}} = (1-\beta)h_3h_4.$$

Therefore $\sum_{I\in\mathcal{I}} h_I = 0$. Choose a meromorphic function d on $\mathbb{A}(R_0)$ such that dh_I $(I \in \mathcal{I})$ are all holomorphic functions on $\mathbb{A}(R_0)$ without common zero. By Eq (4.2), we have

$$\begin{aligned} \sum_{I\in\mathcal{I}} N_0^{[1]}(r, \nu^0_{dh_I}) &\le 3\sum_{i=1}^{4}\left(N_0^{[1]}(r, \nu^0_{h_i}) + N_0^{[1]}(r, \nu^\infty_{h_i})\right) + S(r) \\ &= 3\sum_{i=1}^{4} N_0^{[1]}(r, \mu_i) + S(r). \end{aligned} \tag{4.3}$$

Take each $I_0 \in \mathcal{I}$, then

$$dh_{I_0} = -\sum_{I\neq I_0, I\in\mathcal{I}} dh_I.$$

Denote by t_{I_0} $(1 \le t_{I_0} \le 5)$ the minimum number satisfying the following: There exist t_{I_0} elements $I_1, \cdots, I_t \in \mathcal{I}$ and t_{I_0} nonzero constants $\beta_\nu \in \mathbb{C}$ $(1 \le \nu \le t_{I_0})$ such that $dh_{I_0} = \sum_{\nu=1}^{t_{I_0}} \beta_\nu dh_{I_\nu}$. Set $t := \max\{t_{I_0} : I_0 \in \mathcal{I}\}$.

By the minimality of t, the family $\{dh_{I_1}, \cdots, dh_{I_t}\}$ is linearly independent over $\mathbb{C}$.

Case 1. $t = 1$, then for each $I \in \mathcal{I}$, there exists $J \in \mathcal{I} \setminus \{I\}$ such that $\frac{h_I}{h_J} \in \mathbb{C} \setminus \{0\}$. We consider the following two cases:

(a). There exist $I = \{i, j\}, J = \{i, l\}, j \neq l$ such that $\frac{h_I}{h_J} = a$, where $a \in \mathbb{C} \setminus \{0\}$. Then $h_j = ah_l$. Therefore, f is a quasi-Möbius transformation of g. This is a contradiction.

(b). Otherwise, there exist nonzero constants $b, c \in \mathbb{C}\setminus\{0\}$ such that $h_{\{1,2\}} = bh_{\{3,4\}}$ and $h_{\{1,3\}} = ch_{\{2,4\}}$. This implies that

$$(1-\alpha)h_1h_2 = b(1-\beta)h_3h_4, \quad h_1h_3 = ch_2h_4.$$

Then $\left(\frac{h_1}{h_4}\right)^2 = \frac{bc(1-\beta)}{1-\alpha}$. Hence f is a quasi-Möbius transformation of g. This is a contradiction.

Case 2. $2 \le t \le 5$, consider the linearly non-degenerate holomorphic mapping $h : \mathbb{C} \to P^{t-1}(\mathbb{C})$ with the representation $h = (dh_{I_1} : \cdots : dh_{I_t})$. Applying Theorem 3.2 and the inequality (4.3), we have

$$
\begin{aligned}
T_0(r,h) \leq& \sum_{\vartheta=1}^{t} N_0^{[t-1]}(r,\nu^0_{dh_{I_\vartheta}}) + N_0^{[t-1]}(r,\nu^0_{dh_{I_0}}) + S(r)\\
\leq&(t-1)\sum_{\vartheta=1}^{t} N_0^{[1]}(r,\nu^0_{dh_{I_\vartheta}}) + (t-1)N_0^{[1]}(r,\nu^0_{dh_{I_0}}) + S(r)\\
\leq&3(t-1)\sum_{I\in\mathcal{I}} N_0^{[1]}(r,\nu^0_{dh_I}) + S(r)\\
\leq&12\sum_{i=1}^{4} N_0^{[1]}(r,\mu_i) + S(r).
\end{aligned}
\tag{4.4}
$$

We define the following rational functions:

$$
\begin{aligned}
H_1(X,Y) =& \frac{c_1(X-a_1)}{c_1'(Y-b_1)},\ H_2(X,Y) = \frac{c_2(X-a_2)}{c_2'(Y-b_2)},\\
H_3(X,Y) =& \frac{b_2-b_1}{a_2-a_1}\cdot\frac{X-a_3}{Y-b_3},\\
H_4(X,Y) =& \frac{(a_3-a_2)(b_4-b_2)}{(a_4-a_2)(b_3-b_2)}\cdot\frac{X-a_4}{Y-b_4}.
\end{aligned}
$$

For each $I \subset \{1,\cdots,4\}$, put $I^c = \{1,\cdots,4\}\setminus I$. For $0 \leq u, v \leq t,\ u \neq v$, then $((I_u\cup I_v)\setminus(I_u\cap I_v))^c = \emptyset$, or there exist $i, j \in ((I_u \cup I_v) \setminus (I_u \cap I_v))^c$ and $i \neq j$. Hence

$$
\begin{aligned}
T_0(r,\frac{h_{I_u}}{h_{I_v}}) =&T_0(r,\frac{\prod_{j\in I_u} h_j}{\prod_{j\in I_v} h_j}) + S(r)\\
\geq&N_0(r,\nu^0_{\frac{\prod_{j\in I_u} h_j}{\prod_{j\in I_v} h_j}-\frac{\prod_{j\in I_u} H_j(a_i,b_i)}{\prod_{j\in I_v} H_j(a_i,b_i)}}) + S(r)\\
\geq&N_0^{[1]}(r,\nu_{f-a_i,\leq k_i}) + S(r)
\end{aligned}
$$

Similarly, we have

$$
T_0(r,\frac{h_{I_u}}{h_{I_v}}) \geq N_0^{[1]}(r,\nu_{f-a_j,\leq k_j}) + S(r).
$$

We denote that $((I_0\cup I_1)\setminus(I_0\cap I_1))^c \cup ((I_1\cup I_2)\setminus(I_1\cap I_2))^c \cup ((I_2\cup I_0)\setminus(I_2\cap I_0))^c = \{1,\cdots,4\}$. If $((I_0\cup I_1)\setminus(I_0\cap I_1))^c = \emptyset$, then $((I_1\cup I_2)\setminus(I_1\cap I_2))^c \cup ((I_2\cup I_0)\setminus(I_2\cap I_0))^c = \{1,\cdots,4\}$, by Proposition 3.1, we have

$$
\begin{aligned}
4T_0(r,h) \geq&2T_0(r,\frac{h_{I_1}}{h_{I_2}}) + 2T_0(r,\frac{h_{I_2}}{h_{I_0}}) + S(r)\\
\geq&\sum_{i=1}^{4} N_0^{[1]}(r,\nu_{f-a_i,\leq k_i}) + S(r).
\end{aligned}
$$

If $((I_0 \cup I_1) \setminus (I_0 \cap I_1))^c \neq \emptyset$, by Proposition 3.1, we have

$$\begin{aligned} 4T_0(r,h) \geq & 2T_0(r,\frac{h_{I_0}}{h_{I_1}}) + T_0(r,\frac{h_{I_1}}{h_{I_2}}) + T_0(r,\frac{h_{I_2}}{h_{I_0}}) + S(r) \\ \geq & \sum_{i=1}^{4} N_0^{[1]}(r,\nu_{f-a_i,\leq k_i}) + S(r). \end{aligned}$$

Therefore, we obtain

$$4T_0(r,h) \geq \sum_{i=1}^{4} N_0^{[1]}(r,\nu_{f-a_i,\leq k_i}) + S(r). \tag{4.5}$$

Using the inequalities (4.1), (4.4) and (4.5), we get

$$\sum_{i=1}^{4} N_0^{[1]}(r,\nu_{f-a_i,\leq k_i}) \leq 144 \sum_{i=1}^{4} \left(N_0^{[1]}(r,\nu^0_{f-a_i,>k_i}) + N_0^{[1]}(r,\nu^0_{g-b_i,>k_i}) \right) + S(r).$$

The lemma is proved. □

Proof of Theorem 1.1. Suppose that f is not a quasi-Möbius transformation of g. By Theorem 2.5 and Lemma 4.1, we have

$$\begin{aligned} \frac{2q}{5}T_0(r,f) \leq & \sum_{i=1}^{q} N_0^{[1]}(r,\nu^0_{f-a_i}) + S_f(r) \\ \leq & \sum_{i=1}^{q} \frac{k_i}{k_i+1} N_0^{[1]}(r,\nu^0_{f-a_i,\leq k_i}) + \sum_{i=1}^{q} \frac{1}{k_i+1} T_0(r,f) + S_f(r) \\ \leq & qT_0(r,g) + \sum_{i=1}^{q} \frac{1}{k_i+1} T_0(r,f) + S_f(r). \end{aligned}$$

From the assumption of the theorem, $\sum_{i=1}^{q} \frac{1}{k_i+1} < \sum_{i=1}^{q} \frac{289}{k_i+289} < \frac{2q}{5}$. This implies that $T_0(r,f) = O(T_0(r,g))$. Similarly, $T_0(r,g) = O(T_0(r,f))$. Thus $S_f(r) = S_g(r)$.

Denote $T_0(r) = T_0(r,f) + T_0(r,g), S(r) := S_f(r) = S_g(r)$, applying Theorem 2.5 and Lemma 4.3:

$$\begin{aligned} \frac{2q}{5}T_0(r) \leq & \sum_{i=1}^{q} \sum_{u=f-a_i,g-b_i} N_0^{[1]}(r,\nu^0_u) + S(r) \\ \leq & \sum_{i=1}^{q} \sum_{u=f-a_i,g-b_i} \frac{289}{k_i+289} N_0^{[1]}(r,\nu^0_{u,\leq k_i}) \\ & + \sum_{i=1}^{q} \sum_{u=f-a_i,g-b_i} \left(\frac{288k_i}{k_i+289} + 1 \right) N_0^{[1]}(r,\nu^0_{u,>k_i}) + S_f(r) \\ \leq & \sum_{i=1}^{q} \sum_{u=f-a_i,g-b_i} \frac{289}{k_i+289} \left(N_0^{[1]}(r,\nu^0_{u,\leq k_i}) + N_0(r,\nu^0_{u,>k_i}) \right) + S_f(r) \end{aligned}$$

$$\leq \sum_{i=1}^{q} \frac{289}{k_i + 289} T_0(r) + S(r).$$

Letting $r \to R_0$, we get $\sum_{i=1}^{q} \frac{289}{k_i+289} \geq \frac{2q}{5}$. This is a contradiction. Hence f is a quasi-Möbius transformation of g.

Proof of Theorem 1.2. Suppose that f is not a quasi-Möbius transformation of g. Take three $i, j, t \in \{1,2,3,4\}$, replacing f and g by $\frac{(f-a_i)(a_t-a_j)}{(f-a_j)(a_t-a_i)}$ and $\frac{(g-b_i)(b_t-b_j)}{(g-b_j)(b_t-b_i)}$, if necessary, we may assume that $a_i = b_i = 0, a_j = b_j = \infty$ and $a_t = b_t = 1$. Apply Theorem 2.3, we have

$$T_0(r, f) \leq N_0^{[1]}(r, \nu^0_{f-a_i}) + N_0^{[1]}(r, \nu^0_{f-a_j}) + N_0^{[1]}(r, \nu^0_{f-a_t}) + S_f(r).$$

Summing-up both sides of the above inequality over all subsets $\{i, j, t\}$ of $\{1,2,3,4\}$, we have

$$\frac{4}{3} T_0(r, f) \leq \sum_{i=1}^{4} N_0^{[1]}(r, \nu^0_{f-a_i}) + S_f(r). \tag{4.6}$$

Using Lemma 4.1, it yields that

$$\begin{aligned}\frac{4}{3} T_0(r, f) \leq & \sum_{i=1}^{4} \frac{k_i}{k_i+1} N_0^{[1]}(r, \nu^0_{f-a_i, \leq k_i}) + \sum_{i=1}^{4} \frac{1}{k_i+1} T_0(r, f) + S_f(r) \\ \leq & 4T_0(r, g) + \sum_{i=1}^{4} \frac{1}{k_i+1} T_0(r, f) + S_f(r).\end{aligned}$$

From the assumption of the theorem $\sum_{i=1}^{4} \frac{1}{k_i+1} < \sum_{i=1}^{4} \frac{289}{k_i+289} < \frac{4}{3}$. This implies that $T_0(r, f) = O(T_0(r, g))$. Similarly, $T_0(r, g) = O(T_0(r, f))$. Thus $S_f(r) = S_g(r)$.

Combinning the above inequality (4.6) and Lemma 4.3, similarly to the proof of Theorem 1.1, we obtain $\sum_{i=1}^{q} \frac{289}{k_i+289} \geq \frac{4}{3}$. This is a contradiction. Hence f is a quasi-Möbius transformation of g.

Lemma 4.4. Let f and g be two admissible meromorphic functions on $\mathbb{A}(R_0)$ $(1 < R_0 \leq +\infty)$. Let $\{(a_i, b_i)\}_{i=1}^{4}$ be four pairs of values, where $a_i \neq a_j, b_i \neq b_j$ whenever $i \neq j$. Let k_i $(1 \leq i \leq 4)$ be positive integers or $+\infty$ with $1 \leq k_1 \leq k_2 \leq \cdots \leq k_4$ such that $\frac{2}{k_1+1} + \frac{1}{k_2+1} < 1$. Assume that

$$\min\{\nu^0_{f-a_i, \leq k_i}, 1\} = \min\{\nu^0_{g-b_i, \leq k_i}, 1\} \ (1 \leq i \leq 4)$$

for all z outside a discrete subset S of counting function equal to $S_f(r) + S_g(r)$. If f is a Möbius transformation of g, then there is a permutation (i_1, i_2, i_3, i_4) of $(1,2,3,4)$ such that

$$\frac{f-a_{i_1}}{f-a_{i_2}} \cdot \frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}} \cdot \frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}}, \quad \frac{f-a_{i_1}}{f-a_{i_2}} \cdot \frac{a_{i_4}-a_{i_2}}{a_{i_4}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}} \cdot \frac{b_{i_4}-b_{i_2}}{b_{i_4}-b_{i_1}}$$

$$or \quad \frac{f-a_{i_1}}{f-a_{i_2}} \cdot \frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}} \cdot \frac{b_{i_4}-a_{i_2}}{b_{i_4}-a_{i_1}}$$

Proof. By Theorem 2.3 and Lemma 4.1, we have

$$2T_0(r, f) \leq \sum_{i=1}^{4} N_0^{[1]}(r, \nu^0_{f-a_i}) + S_f(r)$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{4} \frac{k_i}{k_i+1} N_0^{[1]}(r, \nu^0_{f-a_i,\leq k_i}) + \sum_{i=1}^{4} \frac{1}{k_i+1} T_0(r,f) + S_f(r) \\
&\leq \sum_{i=1}^{4} \frac{k_i}{k_i+1} N_0^{[1]}(r, \nu^0_{g-a_i,\leq k_i}) + \sum_{i=1}^{4} \frac{1}{k_i+1} T_0(r,f) + S_f(r) + S_g(r) \\
&\leq 4T_0(r,g) + \frac{3}{2} T_0(r,f) + S_f(r) + S_g(r).
\end{aligned}
$$

This implies that $T_0(r, f) = O(T_0(r, g))$. Similarly, $T_0(r, g) = O(T_0(r, f))$. Thus $S_f(r) = S_g(r)$.

Suppose that there is only one index $i_0 \in \{1, 2, 3, 4\}$ such that $N_0^{[1]}(r, \nu^0_{f-a_i,\leq k_i}) \neq S_f(r)$. By Theorem 2.3 and Lemma 4.1, we obtain

$$
\begin{aligned}
T_0(r,f) &\leq \sum_{i=1, i\neq i_0}^{4} N_0^{[1]}(r, \nu^0_{f-a_i}) + S_f(r) \\
&\leq \sum_{i=1, i\neq i_0}^{4} \frac{k_i}{k_i+1} N_0^{[1]}(r, \nu^0_{f-a_i,\leq k_i}) + \sum_{i=1, i\neq i_0}^{4} \frac{1}{k_i+1} T_0(r,f) + S_f(r) \\
&= \sum_{i=1, i\neq i_0}^{4} \frac{1}{k_i+1} T_0(r,f) + S_f(r).
\end{aligned}
$$

Letting $r \to R_0$, we have $1 \leq \sum_{i=1,i\neq i_0}^{4} \frac{1}{k_i+1}$. It is contradict with $k_i \geq 4$.

Therefore, there are at least two indices $i_1, i_2 \in \{1, 2, 3, 4\}$ such that

$$
N_0^{[1]}(r, \nu^0_{f-a_j,\leq k_i}) = N_0^{[1]}(r, \nu^0_{g-b_j,\leq k_i}) + S_f(r) \neq S_f(r)(j = 1, 2). \tag{4.7}
$$

Denote by i_3, i_4 the remaining indices. Put

$$
\begin{aligned}
F &= \frac{f - a_{i_1}}{f - a_{i_2}} \cdot \frac{a_{i_3} - a_{i_2}}{a_{i_3} - a_{i_1}}, \quad G = \frac{g - b_{i_1}}{g - b_{i_2}} \cdot \frac{b_{i_3} - b_{i_2}}{b_{i_3} - b_{i_1}}, \\
A &= \frac{a_{i_4} - a_{i_2}}{a_{i_4} - a_{i_1}} \cdot \frac{a_{i_3} - a_{i_2}}{a_{i_3} - a_{i_1}}, \quad B = \frac{b_{i_4} - b_{i_2}}{b_{i_4} - b_{i_1}} \cdot \frac{b_{i_3} - b_{i_2}}{b_{i_3} - b_{i_1}}.
\end{aligned}
$$

Since f and g are Möbius transformations of each other, then so are F and G. Hence, there exist complex values $\alpha, \beta, \gamma, \delta$ with $\alpha\gamma - \beta\delta \neq 0$ such that

$$
G = \frac{\alpha F + \beta}{\gamma F + \delta}.
$$

Since $\min\{\nu^0_{F,\leq k_{i_1}}, 1\} = \min\{\nu^0_{G,\leq k_{i_1}}, 1\}$ and (1) , $\beta = 0$. Similarly, $\{\nu^0_{1/F,\leq k_{i_2}}, 1\} = \min\{\nu^0_{1/G,\leq k_{i_2}}, 1\}$ and the above inequality (4.7) imply $\gamma = 0$. Therefore $G = \frac{\alpha}{\delta} F$.

If $\frac{\alpha}{\delta} \in \{1, B, \frac{B}{A}\}$, the Lemma is proved. Otherwise, we get

$$
\begin{aligned}
&N_0^{[1]}(r, \nu^0_{G-1,\leq k_{i_3}}) = S_f(r), \quad N_0^{[1]}(r, \nu^0_{G-B,\leq k_{i_4}}) = S_f(r), \\
&N_0^{[1]}(r, \nu^0_{G-\frac{\alpha}{\delta},\leq k_{i_3}}) = S_f(r) = N_0^{[1]}(r, \nu^0_{F-1,\leq k_{i_3}}) = S_f(r).
\end{aligned}
$$

By Theorem 2.3 and Lemma 4.1, we have

$$\begin{aligned}T_0(r,G) \leq & \sum_{a\in\{1,B,\frac{\alpha}{\delta}\}} N_0^{[1]}(r,\nu^0_{G-a}) + S_G(r)\\ \leq & (\frac{2}{1+k_{i_3}} + \frac{1}{1+k_{i_4}})T_0(r,G) + S_G(r)\end{aligned}$$

Letting $r \to R_0$, we get $1 \leq \frac{2}{1+k_{i_3}} + \frac{1}{1+k_{i_4}}$. On the other hand, $\frac{2}{1+k_{i_3}} + \frac{1}{1+k_{i_4}} \leq \frac{2}{1+k_1} + \frac{1}{1+k_2} < 1$. This is a contradiction. We complete the proof of the lemma. □

Proof of Theorem 1.4. Suppose that f is not a Möbius transformation of g. By Theorem 2.3 and Lemma 4.1, we have

$$\begin{aligned}2T_0(r,f) \leq & \sum_{i=1}^{4} N_0^{[1]}(r,\nu^0_{f-a_i}) + S_f(r)\\ \leq & \sum_{i=1}^{4} \frac{k_i}{k_i+1} N_0^{[1]}(r,\nu^0_{f-a_i,\leq k_i}) + \sum_{i=1}^{4}\frac{1}{k_i+1}T_0(r,f) + S_f(r)\\ \leq & 4T_0(r,g) + \sum_{i=1}^{4}\frac{1}{k_i+1}T_0(r,f) + S_f(r).\end{aligned}$$

From the assumption of the theorem $\sum_{i=1}^4 \frac{1}{k_i+1} < \sum_{i=1}^4 \frac{289}{k_i+289} < 2$. This implies that $T_0(r,f) = O(T_0(r,g))$. Similarly, $T_0(r,g) = O(T_0(r,f))$. Thus $S_f(r) = S_g(r)$.

Denote $T_0(r) = T_0(r,f) + T_0(r,g), S(r) := S_f(r) = S_g(r)$. Applying Theorem 2.3 and Lemma 4.3, we have

$$\begin{aligned}2T_0(r) \leq & \sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i} N_0^{[1]}(r,\nu^0_u) + S(r)\\ = & \sum_{i=1}^{4}\left(N_0^{[1]}(r,\nu^0_{f-a_i,\leq k_i}) + N_0^{[1]}(r,\nu^0_{f-a_i,>k_i})\right) + S_f(r)\\ \leq & \sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i}\frac{289}{k_i+289}N_0^{[1]}(r,\nu^0_{u,\leq k_i})\\ & + \sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i}\left(\frac{288k_i}{k_i+289}+1\right)N_0^{[1]}(r,\nu^0_{u,>k_i}) + S_f(r)\\ \leq & \sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i}\frac{289}{k_i+289}\left(N_0^{[1]}(r,\nu^0_{u,\leq k_i}) + N_0(r,\nu^0_{u,>k_i})\right) + S_f(r)\\ \leq & \sum_{i=1}^{4}\frac{289}{k_i+289}T_0(r) + S(r).\end{aligned}$$

Letting $r \to R_0$, we get $\sum_{i=1}^4 \frac{289}{k_i+289} \geq 2$. This is a contradiction. Hence f is a Möbius transformation of g.

On the other hand, $k_i \geq 4$, thus

$$\frac{2}{k_i+1} + \frac{1}{k_j+1} \leq \sum_{i=1}^{4} \frac{289}{k_i+289} < 2.$$

By Lemma 4.4, The proof of the theorem is completed.

References

1. T. B. Cao, Z. S. Deng, On the uniqueness of meromorphic functions that share three or two finite sets on annuli, *Proc. Indian Acad. Sci.,* **122** (2012), 203–220.
2. T. B. Cao, H. X. Yi, H. Y. Xu, On the multiple values and uniqueness of meromorphic functions on annuli, *Comput. Math. Appl.,* **58** (2009), 1457–1465.
3. T. Czubiak, G. Gundersen, Meromorphic functions that share pairs of values, *Complex Var. Elliptic Equ.,* **34** (1997), 35–46.
4. A. Y. Khrystiyanyn, A. A. Kondratyuk, On the Nevanlinna theory for meromorphic functions on annuli I, *Mat. Stud.,* **23** (2005), 19–30.
5. A. Y. Khrystiyanyn, A. A. Kondratyuk, On the Nevanlinna theory for meromorphicfunctions on annuli II, *Mat. Stud.,* **24** (2005), 57–68.
6. P. Li, C. C. Yang, On two meromorphic functions that share pairs of small functions, *Complex Var. Elliptic Equ.,* **32** (1997), 177–190.
7. M. Lund, Z. Ye, Nevanlinna theory of meromorphic functions on annuli, *Sci. China Math.,* **53** (2010), 547–554.
8. R. Nevanlinna, Zur theorie der meromorphen funktionen, *Acta Math.,* **46** (1925), 1–99.
9. V. A. Nguyen, S. D. Quang, Two meromorphic functions sharing four pairs of small functions, *Bull. Korean Math. Soc.,* **54** (2017), 1159–1171.
10. H. T. Phuong, N. V. Thin, On fundamental theorems for holomorphic curves on the annuli, *Ukra. Math. J.,* **67** (2015), 1111–1115.
11. S. D. Quang, A. H. Tran, H. H. Giang, Two meromorphic functions on annuli sharing few small functions with truncated Complex, *Anal. Oper. Theory,* **13** (2019), 1693–1711.
12. S. D. Quang, L. N. Quynh, Two meromorphic functions sharing some pairs of small functions regardless of multiplicities, *Int. J. Math.,* **25** (2014), 1450014.
13. S. D. Quang, A. H. Tran, Two meromorphic functions on annuli sharing some pairs of values, *Indagationes Math.,* **29** (2018), 561–579.
14. K. Yamanoi, The second main theorem for small functions and related problems, *Acta Math.,* **192** (2004), 225–294.
15. H. X. Yi, C. C. Yang, Uniqueness Theory of Meromorphic Functions, Pure and Applied Mathematics Monographs, Science Press, Beijing, 1995.