



Research article

Some Ostrowski type inequalities via n -polynomial exponentially s -convex functions and their applications

Muhammad Tariq¹, Soubhagya Kumar Sahoo², Jamshed Nasir³, Hassen Aydi^{4,5,6} and Habes Alsamir^{7,*}

¹ Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan

² Department of Mathematics, Institute of Technical Education and Research, Siksha O Anusandhan University, Bhubaneswar 751030, Odisha, India

³ Virtual University Islamabad, Lahore Campus, Pakistan

⁴ Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia

⁵ Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

⁶ China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

⁷ College of Business Administration-Finance Department, Dar Al Uloom University, Saudi Arabia

* **Correspondence:** Email: habes@dau.edu.sa.

Abstract: This paper deals with introducing and investigating a new convex mapping namely, n -polynomial exponentially s -convex. Here, we present some algebraic properties and some logical examples to validate the theory of newly introduced convexity. Some novel adaptations of the well-known Hermite-Hadamard and Ostrowski type inequalities for this convex function have been established. Additionally, some special cases of the newly established results are derived as well. Finally, as applications some new limits for special means of positive real numbers are given. These new outcomes yield a few generalizations of the earlier outcomes already published in the literature.

Keywords: Ostrowski inequality; Hölder's inequality; power-mean integral inequality; n -polynomial exponentially s -convex function

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1. Introduction and preliminaries

These days, the investigation on convexity theory is considered as a unique symbol in the study of the theoretical conduct of mathematical inequalities. As of late, a few articles have been published with a special reference to integral inequalities for convex functions. Specifically, much consideration has been given to the theoretical investigations of inequalities on various kinds of convex functions, for example, s -type convex functions, Harmonic convex functions, $tg s$ -convex functions, Exponential type convex functions, GA -convex functions, (α, m) -convex functions, MT -convex functions, Hyperbolic convex functions, Trigonometrically convex functions, Exponential s -type convex functions, and so on. Many researchers have worked on the above mentioned convexities in different directions with some innovative applications. The most interesting aspect of these variants of convex functions is that each definition is generalization of other one for some specific values. For example, if we choose $s = 1$ in exponentially s -convex function, it simply reduces to exponentially convex function. For the attention of the readers, see the references [1–8].

In 1938, Ostrowski introduced the following useful and interesting integral inequality (see [9], page 468).

Let $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° , the interior of the interval J , such that $\varphi \in \mathcal{L}[\alpha_1, \alpha_2]$, where $\alpha_1, \alpha_2 \in J$ with $\alpha_2 > \alpha_1$. If $|\varphi'(z)| \leq K$, for all $z \in [\alpha_1, \alpha_2]$, then the following inequality:

$$\left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq K(\alpha_2 - \alpha_1) \left[\frac{1}{4} + \frac{\left(z - \frac{\alpha_1 + \alpha_2}{2}\right)^2}{(\alpha_2 - \alpha_1)^2} \right] \quad (1.1)$$

holds. The above inequality (1.1), in literature is known as the well known Ostrowski inequality. For some detailed knowledge about the recent researches on this inequality and related generalizations and extensions, see ([10–16]). This inequality yields an upper bound for the approximation of the integral average $\frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du$ by the value of $\varphi(u)$ at the point $u \in [\alpha_1, \alpha_2]$.

Since the time it is established, broad research history on establishing numerous generalizations of Ostrowski type inequalities have been directed. Most of the earlier and current investigations use different properties of the function and additionally use convexity and bounded variation. The posterity of Ostrowski type inequalities has a great role in the numerical integration theory as utilized by numerous mathematicians. They furnish the numerical integration field with a huge class of quadrature and cubature rules as well.

In assorted and rival research, inequalities have a ton of utilizations in measure theory, mathematical finance, statistical problems, probability, and numerical quadrature formulas. Meng et al. in [17, 18] applied the convex model for optimization via probabilistic approach. Brown in his article [19], explained the relationship between inequalities and measure theory. Numerous broadly known outcomes about inequalities can be acquired utilizing the properties of convex functions. In 1994, for the first-time Hudzik and Maligranda [20] presented the class of s -convex functions in the second sense. Further towards this path, Dragomir and Fitzpatrick [21] put endeavors and set up new fundamental inequalities by means of s -convex functions. In 2019, İşcan [22] developed some new Hermite-Hadamard type inequalities for s -convex functions with the help of notable inequalities like improved power-mean Integral inequality and Hölder-İşcan integral inequality.

In the frame of simple calculus, we explore and attain the novel refinements of Ostrowski type

inequalities. To the best of our knowledge, a comprehensive investigation of newly introduced definition, namely n -polynomial exponentially s -convex function in the present paper is new. Recently, it is seen that many scientists are interested in big data analysis, deep learning and information theory utilizing the concept of exponentially convex functions.

Motivated by the ongoing research and literature, the present paper is structured in the following way, first in Section 2, we will give some necessary known definitions and literature. Second in Section 3, we will explore the concept of n -polynomial exponentially s -convex functions. In addition, some algebraic properties and examples for the newly introduced definition are elaborated. In Section 4, we attain the new sort of Hermite-Hadamard type inequality. Further, in Section 5, we investigate some refinements of the Ostrowski type inequality and some special cases. Finally, in the next section we present some applications to special means and a conclusion.

2. Preliminaries

In this section, we recall some known concepts.

Definition 2.1. [23] Let $\varphi : I \rightarrow \mathbb{R}$ be a real valued function. A function φ is said to be convex, if

$$\varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \leq \chi\varphi(\alpha_1) + (1 - \chi)\varphi(\alpha_2), \quad (2.1)$$

holds for all $\alpha_1, \alpha_2 \in I$ and $\chi \in [0, 1]$.

The Hermite-Hadamard inequality states that, if a mapping $\varphi : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is convex on J for $\alpha_1, \alpha_2 \in J$ and $\alpha_2 > \alpha_1$, then

$$\varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \leq \frac{\varphi(\alpha_1) + \varphi(\alpha_2)}{2}. \quad (2.2)$$

Interested readers can refer to [21, 24].

Definition 2.2. [25] A function $\varphi : [0, +\infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense for a real number $s \in (0, 1]$ or φ belongs to the class of \mathbb{K}_s^2 , if

$$\varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \leq \chi^s\varphi(\alpha_1) + (1 - \chi)^s\varphi(\alpha_2) \quad (2.3)$$

holds for all $\alpha_1, \alpha_2 \in [0, +\infty)$ and $\chi \in [0, 1]$.

Breckner in his article [26] introduced, s -convex functions. Hudzik in his paper [20] presented a few properties and connections with s -convexity in the first sense. Usually, when we put $s = 1$ for s -convexity, it reduces to the classical convexity. In [21], Dragomir et al. proved a generalized Hadamard's inequality, which holds for s -convex functions in the second sense.

Recently, many researchers investigated about the importance and development of the theory of exponentially convex functions. In 2020, Kadakal et al. [27], investigated a new class of exponential convexity, which is stated as follows:

Definition 2.3. [27] A non-negative real-valued function $\varphi : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is known to be an exponential convex function if the following inequality holds:

$$\varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \leq (e^\chi - 1)\varphi(\alpha_1) + (e^{(1-\chi)} - 1)\varphi(\alpha_2). \quad (2.4)$$

Definition 2.4. [28] A non-negative real-valued function $\varphi : J \rightarrow \mathbb{R}$ is called n -polynomial convex, if

$$\varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \leq \frac{1}{n} \sum_{i=1}^n [1 - (1 - \chi)^i] \varphi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n [1 - \chi^i] \varphi(\alpha_2), \quad (2.5)$$

holds for every $\alpha_1, \alpha_2 \in J, \chi \in [0, 1], s \in [0, 1]$ and $n \in \mathbb{N}$.

Employing the above concepts Iscan et al. in [27, 28] proved the following results respectively:

Theorem 2.5. Let $\varphi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be an exponential type convex function. If $\alpha_1 < \alpha_2$ and $\varphi \in \mathcal{L}[\alpha_1, \alpha_2]$, then the following Hermite-Hadamard type inequality holds:

$$\frac{1}{2(e^{\frac{1}{2}} - 1)} \varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \leq (e - 2) \left[\varphi(\alpha_1) + \varphi(\alpha_2) \right]. \quad (2.6)$$

Theorem 2.6. Let $\varphi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be an n polynomial convex function. If $\alpha_1 < \alpha_2$ and $\varphi \in \mathcal{L}[\alpha_1, \alpha_2]$, then the following Hermite-Hadamard type inequality holds:

$$\frac{1}{2} \left(\frac{n}{n + 2^{-n} - 1} \right) \varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \leq \left(\frac{\varphi(\alpha_1) + \varphi(\alpha_2)}{n} \right) \sum_{s=1}^n \frac{s}{s + 1}. \quad (2.7)$$

In [30], the authors proved some refinements of Hermite-Hadamard inequality for exponential convex function. Qi et al. [31], presented Hermite-Hadamard inequality for exponential $(\alpha, (h - m))$ convex function. Recently, Naz et al. [32], used k -Hilfer Katugampola derivative to establish Ostrowski type inequality for n -polynomial \mathcal{P} -convex function. For some recent developments on n -polynomial convexity interested readers can refer to [33, 34] and the references cited therein.

3. Generalized exponentially s -convex function and its properties

Next, due to afore-mentioned research activities, we are able and capable to introduce the generalized form of exponential type convexity, which is called an n -polynomial exponentially s -convex function. Further, we will try to discuss and explore its properties.

Definition 3.1. Let $n \in \mathbb{N}$ and $s \in [\ln 2.4, 1]$. Then the non-negative real-valued function $\varphi : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is known to be an n -polynomial exponentially s -convex function if the following inequality holds:

$$\varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \leq \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \varphi(\alpha_2). \quad (3.1)$$

We represent the class of all n -polynomial exponentially type convex functions on the interval J as $POLEXPC(J)$ for each $\alpha_1, \alpha_2 \in J$ and $\chi \in [0, 1]$.

Remark 1. In above Definition 3.1, if $n = s = 1$, then we get (2.4) given by İşcan in [27].

Remark 2. The range of the exponentially s -convex functions for some fixed $s \in [\ln 2.4, 1]$ is $[0, +\infty)$.

Lemma 3.2. For all $\chi \in [0, 1]$ and for some fixed $s \in [\ln 2.4, 1]$ the following inequalities $\frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \geq \chi$ and $\frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \geq (1 - \chi)$ hold.

Proof. The proof is simple. \square

Lemma 3.3. For all $\chi \in [0, 1]$ and for some fixed $s \in [\ln 2.4, 1]$ the following inequalities $\frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \geq \chi^s$ and $\frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \geq (1 - \chi)^s$ hold.

Proof. The proof is simple. \square

From the above lemmas, it can be clearly seen that, the new class of n -polynomial exponentially convex function is very larger when compared to the known class of functions like convex, exponentially convex, s -convex and exponentially s -convex. This is an added advantage of the newly proposed Definition 3.1.

Proposition 1. Every nonnegative convex function is an n -polynomial exponentially s -convex function for $s \in [\ln 2.4, 1]$.

Proof. Applying Lemma 3.2 and $s \in [\ln 2.4, 1]$, we have

$$\begin{aligned} \varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) &\leq \chi\varphi(\alpha_1) + (1 - \chi)\varphi(\alpha_2) \\ &\leq \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \varphi(\alpha_2). \end{aligned}$$

\square

Proposition 2. Every nonnegative s -convex function is an n -polynomial exponentially s -convex function for $s \in [\ln 2.4, 1]$.

Proof. Applying Lemma 3.3 and $s \in [\ln 2.4, 1]$, we have

$$\begin{aligned} \varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) &\leq \chi^s \varphi(\alpha_1) + (1 - \chi)^s \varphi(\alpha_2) \\ &\leq \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \varphi(\alpha_2). \end{aligned}$$

\square

Now, we will make some examples in the support of the newly introduced function.

Example 1. If $\varphi(x) = e^x$, for all $x > 0$ is a nonnegative convex function, from Proposition 1, it is also an n -polynomial exponentially s -convex function for $s \in [\ln 2.4, 1]$.

Example 2. If $\varphi(x) = c$ is a nonnegative convex function on R for any $c \geq 0$, from Proposition 1, it is also an n -polynomial exponentially s -convex function for $s \in [\ln 2.4, 1]$.

Example 3. If $\varphi(x) = \frac{1}{x}$, for all $x > 0$ is a nonnegative convex function, from Proposition 1, it is also an n -polynomial exponentially s -convex function for $s \in [\ln 2.4, 1]$.

Example 4. $\varphi(x) = \frac{q}{s+q} x^{\frac{s}{q}+1}$, for $s > 1$ is a nonnegative convex function. By using Proposition 1, it is also an n -polynomial exponentially s -convex function for $s \in [\ln 2.4, 1]$.

Example 5. The Great mathematician Dragomir clearly investigated and proved that in published article [21], the function $\varphi(x) = x^{ls}$, $x > 0$ is s -convex function, for the all mentioned conditions $s \in (0, 1)$ and $1 \leq l \leq \frac{1}{s}$. But, using Proposition 2, it is also an n -polynomial exponentially s -convex function for $s \in [\ln 2.4, 1]$.

Theorem 3.4. Let $\psi, \varphi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$. If ψ and φ are n -polynomial exponentially s -convex functions and $s \in [\ln 2.4, 1]$ then

(i) $\psi + \varphi$ is an n -polynomial exponentially s -convex function.

(ii) For nonnegative real number c , $c\psi$ is an n -polynomial exponentially s -convex function.

Proof. (i) Let ψ and φ be n -polynomial exponentially s -convex functions, then

$$\begin{aligned} & (\psi + \varphi) \left[(\chi\alpha_1 + (1 - \chi)\alpha_2) \right] \\ &= \psi(\chi\alpha_1 + (1 - \chi)\alpha_2) + \varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \\ &\leq \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \psi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i \psi(\alpha_2) \\ &+ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i \varphi(\alpha_2) \\ &= \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i [\psi(\alpha_1) + \varphi(\alpha_1)] + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i [\psi(\alpha_2) + \varphi(\alpha_2)] \\ &= \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i (\psi + \varphi)(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i (\psi + \varphi)(\alpha_2). \end{aligned}$$

(ii) Let ψ be an n -polynomial exponentially s -convex function, then

$$\begin{aligned} (c\psi) \left[(\chi\alpha_1 + (1 - \chi)\alpha_2) \right] &= c \left[\psi(\chi\alpha_1 + (1 - \chi)\alpha_2) \right] \\ &\leq c \left[\frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \psi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i \psi(\alpha_2) \right] \\ &= \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i c\psi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i c\psi(\alpha_2) \\ &= \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i (c\psi)(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i (c\psi)(\alpha_2). \end{aligned}$$

□

Remark 3. If we choose $s = 1$ in Theorem 3.4, then we get Theorem 2.1 in [27].

Theorem 3.5. Let $\varphi_i : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ be an arbitrary family of n -polynomial exponentially s -convex functions for the same fixed $s \in [\ln 2.4, 1]$ and let $\varphi(\alpha) = \sup_i \varphi_i(\alpha)$. If $P = \{\alpha \in [\alpha_1, \alpha_2] : \varphi(\alpha) < +\infty\} \neq \emptyset$, then P is an interval and φ is an n -polynomial exponentially s -convex function on P .

Proof. For all $\alpha_1, \alpha_2 \in P$ and $\chi \in [0, 1]$, and for the same fixed numbers $s \in [\ln 2.4, 1]$, we have

$$\begin{aligned} \varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) &= \sup_i \varphi_i(\chi\alpha_1 + (1 - \chi)\alpha_2) \\ &\leq \sup_i \left[\frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi_i(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i \varphi_i(\alpha_2) \right] \\ &\leq \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \sup_i \varphi_i(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i \sup_i \varphi_i(\alpha_2) \\ &= \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i \varphi(\alpha_2) < +\infty. \end{aligned}$$

This shows that P is an interval. \square

Theorem 3.6. *If the function $\varphi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ is n -polynomial exponentially s -convex for some fixed $s \in [\ln 2.4, 1]$, then φ is bounded on $[\alpha_1, \alpha_2]$.*

Proof. Let $\mathbb{L} = \max\{\varphi(\alpha_1), \varphi(\alpha_2)\}$ and $x \in [\alpha_1, \alpha_2]$ be an arbitrary point. Then there exists $\chi \in [0, 1]$ such that $x = \chi\alpha_1 + (1 - \chi)\alpha_2$. Thus, since $e^{s\chi} \leq e^s$ and $e^{(1-\chi)s} \leq e^s$ for some fixed $s \in [\ln 2.4, 1]$, we have

$$\begin{aligned} \varphi(x) &= \varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \\ &\leq \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{(1-\chi)s} - 1)^i \varphi(\alpha_2) \\ &\leq \frac{1}{n} \sum_{i=1}^n (e^s - 1)^i \mathbb{L} + \frac{1}{n} \sum_{i=1}^n (e^s - 1)^i \mathbb{L} = \mathbb{L} \frac{1}{n} \sum_{i=1}^n (e^s - 1)^i = \mathbb{M}. \end{aligned}$$

We have proved that φ is bounded above by \mathbb{M} . \square

Remark 4. If we choose $n = s = 1$ in Theorem 3.6, then we get Theorem 2.4 in [27].

4. Hermite-Hadamard type inequality

In this section, we present one Hermite-Hadamard type inequality for the n -polynomial exponentially s -convex function.

Theorem 4.1. *Suppose $s \in [\ln 2.4, 1]$, $\alpha \in (0, 1]$, $\alpha_2 > \alpha_1$ and $\varphi : J = [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ is an n -polynomial exponentially s -convex function such that $\varphi' \in \mathcal{L}[\alpha_1, \alpha_2]$. Then one has*

$$\frac{1}{2^{\frac{1}{n}} \sum_{i=1}^n (e^{\frac{s}{2}} - 1)^i} \varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \leq \frac{1}{n} \sum_{i=1}^n \left(\frac{e^s - s - 1}{s}\right)^i \left[\varphi(\alpha_1) + \varphi(\alpha_2)\right]. \quad (4.1)$$

Proof. Let $z_1, z_2 \in J$. Then it follows from the n -polynomial exponentially s -convex function for φ on J that

$$\varphi\left(\frac{z_1 + z_2}{2}\right) \leq \frac{1}{n} \sum_{i=1}^n (e^{\frac{s}{2}} - 1)^i [\varphi(z_1) + \varphi(z_2)] \quad (4.2)$$

Suppose $z_1 = \chi\alpha_2 + (1 - \chi)\alpha_1$ and $z_2 = \chi\alpha_1 + (1 - \chi)\alpha_2$.

Then (4.2) leads to

$$\varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{n} \sum_{i=1}^n \left(e^{\frac{s}{2}} - 1\right)^i [\varphi(\chi\alpha_2 + (1 - \chi)\alpha_1) + \varphi(\chi\alpha_1 + (1 - \chi)\alpha_2)]. \quad (4.3)$$

Now, integrating both sides of the inequality (4.3) with respect to χ from 0 to 1, one has

$$\varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{n} \sum_{i=1}^n \left(e^{\frac{s}{2}} - 1\right)^i \left[\int_0^1 \varphi(\chi\alpha_2 + (1 - \chi)\alpha_1) d\chi + \int_0^1 \varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) d\chi \right].$$

$$\frac{1}{2^{\frac{1}{n}} \sum_{i=1}^n \left(e^{\frac{s}{2}} - 1\right)^i} \varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du.$$

This completes the proof of first part of inequality (4.1). Next, we prove the second part of inequality (4.1). Let $\chi \in [0, 1]$. Using the fact that φ is an n -polynomial exponentially s -convex function, we obtain

$$\varphi(\chi\alpha_2 + (1 - \chi)\alpha_1) \leq \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi(\alpha_2) + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \varphi(\alpha_1) \quad (4.4)$$

and

$$\varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \leq \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i \varphi(\alpha_1) + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \varphi(\alpha_2). \quad (4.5)$$

By adding the above inequalities, we obtain

$$\begin{aligned} & \varphi(\chi\alpha_2 + (1 - \chi)\alpha_1) + \varphi(\chi\alpha_1 + (1 - \chi)\alpha_2) \\ & \leq [\varphi(\alpha_1) + \varphi(\alpha_2)] \left\{ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \right\}. \end{aligned} \quad (4.6)$$

Now, integrating both sides of the above inequality with respect to χ from 0 to 1, then making the change of variable, we obtain

$$\frac{2}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \leq [\varphi(\alpha_1) + \varphi(\alpha_2)] \int_0^1 \left\{ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i \right\} d\chi,$$

which leads to the conclusion that

$$\frac{2}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \leq \frac{2}{n} \sum_{i=1}^n \left(\frac{e^s - s - 1}{s} \right)^i [\varphi(\alpha_1) + \varphi(\alpha_2)].$$

The proof is completed. \square

Remark 5. If we choose $n = s = 1$, then Theorem 4.1 becomes to [Theorem 3.1, [27]].

5. Refinements of Ostrowski type inequality involving n -polynomial exponentially s -convex functions

In this section, we present some enhancements of the Ostrowski type inequality for differentiable n -polynomial exponentially s -convex function. Here, we need the following lemma as given in [29].

Lemma 5.1. *Suppose $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable mapping on J^o , where $\alpha_1, \alpha_2 \in J$ with $\alpha_1 < \alpha_2$. If $\varphi' \in \mathcal{L}[\alpha_1, \alpha_2]$, then the following equality holds:*

$$\begin{aligned} & \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \\ &= \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi \varphi'(\chi z + (1 - \chi)\alpha_1) d\chi - \frac{(\alpha_2 - z)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi \varphi'(\chi z + (1 - \chi)\alpha_2) d\chi, \end{aligned} \quad (5.1)$$

for each $z \in [\alpha_1, \alpha_2]$.

Theorem 5.2. *Suppose $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable mapping on J^o , where $\alpha_1, \alpha_2 \in J$ with $\alpha_1 < \alpha_2$. If $|\varphi'|$ is an n -polynomial exponentially s -convex function on $[\alpha_1, \alpha_2]$ for some $s \in [\ln 2.4, 1]$, $\varphi' \in \mathcal{L}[\alpha_1, \alpha_2]$ and $|\varphi'(z)| \leq K$, for all $z \in [\alpha_1, \alpha_2]$, then the following inequality holds:*

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \right| \\ & \leq \frac{K}{(\alpha_2 - \alpha_1)n} \left[(z - \alpha_1)^2 \left\{ \sum_{i=1}^n \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\} \right. \\ & \quad \left. + (\alpha_2 - z)^2 \left\{ \sum_{i=1}^n \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\} \right], \end{aligned} \quad (5.2)$$

for each $z \in [\alpha_1, \alpha_2]$.

Proof. From Lemma 5.1, the fact that $|\varphi'|$ is n -polynomial exponentially s -convex and $|\varphi'| \leq K$, we have

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \right| \\ & \leq \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_1)| d\chi + \frac{(\alpha_2 - z)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_2)| d\chi \\ & \leq \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi \left\{ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i |\varphi'(z)| + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i |\varphi'(\alpha_1)| \right\} d\chi \\ & \quad + \frac{(\alpha_2 - z)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi \left\{ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i |\varphi'(z)| + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i |\varphi'(\alpha_2)| \right\} d\chi \\ & \leq \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \left[|\varphi'(z)| \int_0^1 \chi \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i d\chi + |\varphi'(\alpha_1)| \int_0^1 \chi \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i d\chi \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{(\alpha_2 - z)^2}{\alpha_2 - \alpha_1} \left[|\varphi'(z)| \int_0^1 \chi \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i d\chi + |\varphi'(\alpha_2)| \int_0^1 \chi \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i d\chi \right] \\
& \leq \frac{K(z - \alpha_1)^2}{(\alpha_2 - \alpha_1)n} \left\{ \sum_{i=1}^n \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\} \\
& + \frac{K(\alpha_2 - z)^2}{(\alpha_2 - \alpha_1)n} \left\{ \sum_{i=1}^n \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\} \\
& = \frac{K}{(\alpha_2 - \alpha_1)n} \left[(z - \alpha_1)^2 \left\{ \sum_{i=1}^n \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\} \right. \\
& \left. + (\alpha_2 - z)^2 \left\{ \sum_{i=1}^n \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\} \right].
\end{aligned}$$

The proof is completed. □

Corollary 1. *If we choose $n = 1$ in Theorem 5.2, we obtain*

$$\begin{aligned}
& \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \\
& \leq \frac{K}{(\alpha_2 - \alpha_1)} \left[(z - \alpha_1)^2 \left\{ \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\} \right. \\
& \left. + (\alpha_2 - z)^2 \left\{ \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right) + \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\} \right].
\end{aligned}$$

Corollary 2. *Under the similar consideration in Theorem 5.2, by choosing $s = 1$, we obtain*

$$\begin{aligned}
& \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \\
& \leq \frac{K}{n(\alpha_2 - \alpha_1)} \left[(z - \alpha_1)^2 \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\} \right. \\
& \left. + (\alpha_2 - z)^2 \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\} \right].
\end{aligned}$$

Also, we have

1. If we choose $z = \frac{\alpha_1 + \alpha_2}{2}$ in Corollary 2, then we have the following mid-point inequality:

$$\left| \varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq \frac{K}{n} \frac{(\alpha_2 - \alpha_1)}{2} \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\}.$$

2. If we choose $z = \alpha_1$ in Corollary 2, then we have the following inequality:

$$\left| \varphi(\alpha_1) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq \frac{K}{n} (\alpha_2 - \alpha_1) \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\}.$$

3. If we choose $z = \alpha_2$ in Corollary 2, then we have the following inequality:

$$\left| \varphi(\alpha_2) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq \frac{K}{n} (\alpha_2 - \alpha_1) \left\{ \sum_{i=1}^n \left(\frac{1}{2}\right)^i + \sum_{i=1}^n \left(\frac{2e-5}{2}\right)^i \right\}.$$

Theorem 5.3. Suppose $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable mapping on J° , where $\alpha_1, \alpha_2 \in J$ with $\alpha_1 < \alpha_2$. If $|\varphi'|^q$ is n -polynomial exponentially s -convex on $[\alpha_1, \alpha_2]$ for some $s \in [\ln 2.4, 1]$, $q > 1$, $q^{-1} = 1 - p^{-1}$, $\varphi' \in \mathcal{L}[\alpha_1, \alpha_2]$ and $|\varphi'(z)| \leq K$, for all $z \in [\alpha_1, \alpha_2]$, then the following inequality holds:

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \right| \\ & \leq \frac{2^{\frac{1}{q}} K}{(\alpha_2 - \alpha_1) \sqrt[q]{n}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[(z - \alpha_1)^2 \left\{ \sum_{i=1}^n \left(\frac{e^s - s - 1}{s}\right)^i \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + (\alpha_2 - z)^2 \left\{ \sum_{i=1}^n \left(\frac{e^s - s - 1}{s}\right)^i \right\}^{\frac{1}{q}} \right], \end{aligned} \quad (5.3)$$

for each $z \in [\alpha_1, \alpha_2]$.

Proof. From Lemma 5.1 and the famous Hölder's inequality, we have

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \right| \\ & \leq \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_1)| d\chi \\ & \quad + \frac{(\alpha_2 - z)^2}{\alpha_1 - \alpha_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_2)| d\chi \\ & \leq \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \left(\int_0^1 \chi^p d\chi \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(\chi z + (1 - \chi)\alpha_1)|^q d\chi \right)^{\frac{1}{q}} \\ & \quad + \frac{(\alpha_2 - z)^2}{\alpha_2 - \alpha_1} \left(\int_0^1 \chi^p d\chi \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(\chi z + (1 - \chi)\alpha_2)|^q d\chi \right)^{\frac{1}{q}}. \end{aligned} \quad (5.4)$$

Since $|\varphi'|^q$ is n -polynomial exponentially s -convex and $|\varphi'(z)| \leq K$, we obtain

$$\begin{aligned} & \int_0^1 |\varphi'(\chi z + (1 - \chi)\alpha_1)|^q d\chi \\ & = \int_0^1 \left\{ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i |\varphi'(z)|^q + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i |\varphi'(\alpha_1)|^q \right\} d\chi \\ & \leq K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{e^s - s - 1}{s}\right)^i + K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{e^s - s - 1}{s}\right)^i \\ & \leq 2K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{e^s - s - 1}{s}\right)^i \end{aligned} \quad (5.5)$$

and

$$\begin{aligned}
 & \int_0^1 |\varphi'(\chi z + (1-\chi)\alpha_2)|^q d\chi \\
 &= \int_0^1 \left\{ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i |\varphi'(z)|^q + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i |\varphi'(\alpha_2)|^q \right\} d\chi \\
 &\leq K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{e^s - s - 1}{s} \right)^i + K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{e^s - s - 1}{s} \right)^i \\
 &\leq 2K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{e^s - s - 1}{s} \right)^i.
 \end{aligned} \tag{5.6}$$

By connecting (5.5) and (5.6) with (5.4), we get (5.3). The proof is completed. \square

Corollary 3. *If we choose $n = 1$ in Theorem 5.3, we obtain*

$$\begin{aligned}
 & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \\
 &\leq \frac{2^{\frac{1}{q}} K}{(\alpha_2 - \alpha_1) \left(\frac{1}{p+1} \right)^{\frac{1}{p}}} \left[(z - \alpha_1)^2 \left(\frac{e^s - s - 1}{s} \right)^{\frac{1}{q}} + (\alpha_2 - z)^2 \left(\frac{e^s - s - 1}{s} \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

Corollary 4. *Under the similar consideration in Theorem 5.3, by choosing $s = 1$, we obtain*

$$\begin{aligned}
 & \left| \varphi(x) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \\
 &\leq \frac{2^{\frac{1}{q}} K}{\sqrt[q]{n} (\alpha_2 - \alpha_1) \left(\frac{1}{p+1} \right)^{\frac{1}{p}}} \left[(z - \alpha_1)^2 \left\{ \sum_{i=1}^n (e - 2)^i \right\}^{\frac{1}{q}} + (\alpha_2 - z)^2 \left\{ \sum_{i=1}^n (e - 2)^i \right\}^{\frac{1}{q}} \right].
 \end{aligned}$$

Also we have

1. If we choose $z = \frac{\alpha_1 + \alpha_2}{2}$ in Corollary 4, then we have the following mid-point inequality:

$$\left| \varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq \frac{2^{\frac{1}{q}-1} K (\alpha_2 - \alpha_1) \left(\frac{1}{p+1} \right)^{\frac{1}{p}}}{\sqrt[q]{n}} \left\{ \sum_{i=1}^n (e - 2)^i \right\}^{\frac{1}{q}}.$$

2. If we choose $z = \alpha_1$ in Corollary 4, then we have the following inequality:

$$\left| \varphi(\alpha_1) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq 2^{\frac{1}{q}} \frac{K}{\sqrt[q]{n}} (\alpha_2 - \alpha_1) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \sum_{i=1}^n (e - 2)^i \right\}^{\frac{1}{q}}.$$

3. If we choose $z = \alpha_2$ in Corollary 4, then we have the following inequality:

$$\left| \varphi(\alpha_2) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq 2^{\frac{1}{q}} \frac{K}{\sqrt[q]{n}} (\alpha_2 - \alpha_1) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \sum_{i=1}^n (e - 2)^i \right\}^{\frac{1}{q}}.$$

Theorem 5.4. Suppose $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable mapping on J° , where $\alpha_1, \alpha_2 \in J$ with $\alpha_1 < \alpha_2$. If $|\varphi'|^q$ is n -polynomial exponentially s -convex on $[\alpha_1, \alpha_2]$ for some $s \in [\ln 2.4, 1]$, $q \geq 1$, $q^{-1} = 1 - p^{-1}$, $\varphi' \in \mathcal{L}[\alpha_1, \alpha_2]$ and $|\varphi'(z)| \leq K$, for all $z \in [\alpha_1, \alpha_2]$, then the following inequality holds:

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \right| \\ & \leq \frac{K}{\sqrt[q]{n}(\alpha_2 - \alpha_1) 2^{1-\frac{1}{q}}} \left[(z - \alpha_1)^2 \left\{ \sum_{i=1}^n \left(\frac{2 + (2s-2)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + (\alpha_2 - z)^2 \left\{ \sum_{i=1}^n \left(\frac{2 + (2s-2)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\}^{\frac{1}{q}} \right], \end{aligned} \quad (5.7)$$

for each $z \in [\alpha_1, \alpha_2]$.

Proof. From Lemma 5.1 and power mean inequality, we have

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \right| \\ & \leq \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_1)| d\chi + \frac{(\alpha_2 - z)^2}{\alpha_2 - \alpha_1} \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_2)| d\chi \\ & \leq \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \left(\int_0^1 \chi d\chi \right)^{1-\frac{1}{q}} \left(\int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_1)|^q d\chi \right)^{\frac{1}{q}} \\ & \quad + \frac{(\alpha_2 - z)^2}{\alpha_2 - \alpha_1} \left(\int_0^1 \chi d\chi \right)^{1-\frac{1}{q}} \left(\int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_2)|^q d\chi \right)^{\frac{1}{q}}. \end{aligned} \quad (5.8)$$

Since $|\varphi'|^q$ is n -polynomial exponentially s -convex and $|\varphi'(z)| \leq K$, we obtain

$$\begin{aligned} & \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_1)|^q d\chi \\ & = \int_0^1 \chi \left\{ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i |\varphi'(z)|^q + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i |\varphi'(\alpha_1)|^q \right\} d\chi \\ & \leq K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{(2s-2)e^s - s^2 + 2}{2s^2} \right)^i + K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} & \int_0^1 \chi |\varphi'(\chi z + (1 - \chi)\alpha_2)|^q d\chi \\ & = \int_0^1 \chi \left\{ \frac{1}{n} \sum_{i=1}^n (e^{s\chi} - 1)^i |\varphi'(z)|^q + \frac{1}{n} \sum_{i=1}^n (e^{s(1-\chi)} - 1)^i |\varphi'(\alpha_2)|^q \right\} d\chi \\ & \leq K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{(2s-2)e^s - s^2 + 2}{2s^2} \right)^i + K^q \frac{1}{n} \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i. \end{aligned} \quad (5.10)$$

By connecting (5.9) and (5.10) with (5.8), we get (5.7). The proof is completed. \square

Corollary 5. *If we choose $n = 1$ in Theorem 5.4, we obtain*

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \\ & \leq \frac{K}{(\alpha_2 - \alpha_1) 2^{1-\frac{1}{q}}} \left[(z - \alpha_1)^2 \left\{ \left(\frac{2 + (2s - 2)e^s - s^2}{2s^2} \right) + \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + (\alpha_2 - z)^2 \left\{ \left(\frac{2 + (2s - 2)e^s - s^2}{2s^2} \right) + \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right) \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

Corollary 6. *Under the similar consideration in Theorem 5.4, by choosing $s = 1$, we obtain*

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \\ & \leq \frac{K}{\sqrt[q]{n} (\alpha_2 - \alpha_1) 2^{1-\frac{1}{q}}} \left[(z - \alpha_1)^2 \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + (\alpha_2 - z)^2 \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

We also have

1. If we choose $z = \frac{\alpha_1 + \alpha_2}{2}$ in Corollary 6, then we have the following mid-point inequality:

$$\left| \varphi\left(\frac{\alpha_1 + \alpha_2}{2}\right) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq \frac{K}{2\sqrt[q]{n}} \frac{(\alpha_2 - \alpha_1)}{2^{1-\frac{1}{q}}} \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\}^{\frac{1}{q}}.$$

2. If we choose $z = \alpha_1$ in Corollary 6, then we have the following inequality:

$$\left| \varphi(\alpha_1) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq \frac{K}{\sqrt[q]{n}} \frac{(\alpha_2 - \alpha_1)}{2^{1-\frac{1}{q}}} \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\}^{\frac{1}{q}}.$$

3. If we choose $z = \alpha_2$ in Corollary 6, then we have the following inequality:

$$\left| \varphi(\alpha_2) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \leq \frac{K}{\sqrt[q]{n}} \frac{(\alpha_2 - \alpha_1)}{2^{1-\frac{1}{q}}} \left\{ \sum_{i=1}^n \left(\frac{1}{2} \right)^i + \sum_{i=1}^n \left(\frac{2e - 5}{2} \right)^i \right\}^{\frac{1}{q}}.$$

Theorem 5.5. *Suppose $\varphi : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable mapping on J° , where $\alpha_1, \alpha_2 \in J$ with $\alpha_2 > \alpha_1$. If $|\varphi'|^q$ is n -polynomial exponentially s -concave on $[\alpha_1, \alpha_2]$, for some $s \in [\ln 2.4, 1]$, $\varphi' \in \mathcal{L}[\alpha_1, \alpha_2]$, $p, q > 1$, $q^{-1} + p^{-1} = 1$, then the following inequality holds:*

$$\begin{aligned} & \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \right| \\ & \leq \frac{1}{\alpha_2 - \alpha_1} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{2^{\frac{1}{n}} \sum_{i=1}^n (e^{\frac{s}{2}} - 1)^i} \right)^{\frac{1}{q}} \\ & \quad \times \left\{ (z - \alpha_1)^2 \left| \varphi' \left(\frac{z + \alpha_1}{2} \right) \right| + (\alpha_2 - z)^2 \left| \varphi' \left(\frac{z + \alpha_2}{2} \right) \right| \right\}. \end{aligned} \tag{5.11}$$

Proof. Suppose $p > 1$, by Lemma 5.1 and using the Hölder inequality, we obtain

$$\begin{aligned} \left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\chi) d\chi \right| &\leq \frac{(z - \alpha_1)^2}{\alpha_2 - \alpha_1} \left(\int_0^1 \chi^p d\chi \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(\chi z + (1 - \chi)\alpha_1)|^q d\chi \right)^{\frac{1}{q}} \\ &\quad + \frac{(\alpha_2 - z)^2}{\alpha_2 - \alpha_1} \left(\int_0^1 \chi^p d\chi \right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(\chi z + (1 - \chi)\alpha_2)|^q d\chi \right)^{\frac{1}{q}} \end{aligned} \quad (5.12)$$

$|\varphi'|^q$ is n -polynomial exponentially s -concave, so using the left hand side of (4.1) inequality, we obtain

$$\int_0^1 |\varphi'(\chi z + (1 - \chi)\alpha_1)|^q d\chi \geq \left(\frac{1}{2^{\frac{1}{n}} \sum_{i=1}^n (e^{\frac{s}{2}} - 1)^i} \right) \left| \varphi' \left(\frac{z + \alpha_1}{2} \right) \right|^q \quad (5.13)$$

and

$$\int_0^1 |\varphi'(\chi z + (1 - \chi)\alpha_2)|^q d\chi \geq \left(\frac{1}{2^{\frac{1}{n}} \sum_{i=1}^n (e^{\frac{s}{2}} - 1)^i} \right) \left| \varphi' \left(\frac{z + \alpha_2}{2} \right) \right|^q. \quad (5.14)$$

By connecting the (5.13), (5.14) with (5.12), we get (5.11). The proof is completed. \square

Corollary 7. *If we choose $n = 1$ in Theorem 5.5, we obtain*

$$\begin{aligned} &\left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \\ &\leq \frac{1}{\alpha_2 - \alpha_1} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{2(e^{\frac{s}{2}} - 1)} \right)^{\frac{1}{q}} \left\{ (z - \alpha_1)^2 \left| \varphi' \left(\frac{z + \alpha_1}{2} \right) \right| + (\alpha_2 - z)^2 \left| \varphi' \left(\frac{z + \alpha_2}{2} \right) \right| \right\}. \end{aligned}$$

Corollary 8. *Under the similar consideration in Theorem 5.5, by choosing $s = 1$, we obtain*

$$\begin{aligned} &\left| \varphi(z) - \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(u) du \right| \\ &\leq \frac{1}{\alpha_2 - \alpha_1} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{2^{\frac{1}{n}} \sum_{i=1}^n (e^{\frac{s}{2}} - 1)^i} \right)^{\frac{1}{q}} \left\{ (z - \alpha_1)^2 \left| \varphi' \left(\frac{z + \alpha_1}{2} \right) \right| + (\alpha_2 - z)^2 \left| \varphi' \left(\frac{z + \alpha_2}{2} \right) \right| \right\}. \end{aligned}$$

6. Applications

Consider the following special means for different positive real numbers α_1, α_2 and $\alpha_1 < \alpha_2$ as follows:

1. The Arithmetic mean:

$$A(\alpha_1, \alpha_2) = \frac{\alpha_1 + \alpha_2}{2}.$$

2. The Harmonic mean:

$$H(\alpha_1, \alpha_2) = \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2}, \quad \alpha_1, \alpha_2 > 0.$$

3. The log-mean:

$$L = L(\alpha_1, \alpha_2) = \frac{\alpha_2 - \alpha_1}{\ln \alpha_2 - \ln \alpha_1}, \quad \alpha_1 \neq \alpha_2.$$

4. The Generalized log-mean:

$$L_r(\alpha_1, \alpha_2) = \left[\frac{\alpha_2^{r+1} - \alpha_1^{r+1}}{(r+1)(\alpha_2 - \alpha_1)} \right]^{\frac{1}{r}}; \quad r \in \mathbb{R} \setminus \{-1, 0\}.$$

5. The Identric mean:

$$I(\alpha_1, \alpha_2) = \begin{cases} \alpha_1 & \alpha_1 = \alpha_2 \\ \frac{1}{e} \left(\frac{\alpha_2^{\alpha_2}}{\alpha_1^{\alpha_1}} \right)^{\frac{1}{\alpha_2 - \alpha_1}} & \alpha_1 \neq \alpha_2 \end{cases}.$$

Dragomir et al. [21], have proved that for $s \in (0, 1)$, where $1 \leq r \leq \frac{1}{s}$, the function $\varphi(z) = z^{rs}$, $z > 0$ is an s -convex function.

Now, we present some applications to special means for positive real numbers using results of Section 5, as:

Proposition 3. Let $0 < \alpha_1 < \alpha_2$. Then for some fixed $s \in [\ln 2.4, 1)$, we obtain

$$\begin{aligned} & |A^{rs}(\alpha_1, \alpha_2) - L_{rs}(\alpha_1, \alpha_2)| \\ & \leq (\alpha_2 - \alpha_1) \frac{K}{2n} \left\{ \sum_{i=1}^n \left(\frac{2 + 2(s-1)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\}. \end{aligned} \quad (6.1)$$

Proof. The result holds by letting $z = \frac{\alpha_1 + \alpha_2}{2}$ in Theorem 5.2 with n -polynomial exponentially s -convexity for $\varphi : (0, \infty) \rightarrow \mathbb{R}$, $\varphi(z) = z^{rs}$, we obtain the required result. \square

Proposition 4. Let $0 < \alpha_1 < \alpha_2$ and $q > 1$. Then for some fixed $s \in [\ln 2.4, 1)$, we obtain

$$|\ln I(\alpha_1, \alpha_2) - \ln A(\alpha_1, \alpha_2)| \leq 2^{\frac{1}{q}-1} \frac{K}{\sqrt[q]{n}} (\alpha_2 - \alpha_1) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \sum_{i=1}^n \left(\frac{e^s - s - 1}{s} \right)^i \right\}^{\frac{1}{q}}. \quad (6.2)$$

Proof. The result holds by letting $z = \frac{\alpha_1 + \alpha_2}{2}$ in Theorem 5.3 with n -polynomial exponentially s -convexity for $\varphi : (0, \infty) \rightarrow \mathbb{R}$, $\varphi(z) = -\ln z$, we obtain the required result. \square

Proposition 5. Let $0 < \alpha_1 < \alpha_2$ and $q \geq 1$. Then for some fixed $s \in [\ln 2.4, 1)$, we obtain

$$\begin{aligned} & |H(\alpha_1, \alpha_2) - L^{-1}(\alpha_1, \alpha_2)| \\ & \leq \frac{(\alpha_2 - \alpha_1) K}{\sqrt[q]{n} 2^{2-\frac{1}{q}}} \left\{ \sum_{i=1}^n \left(\frac{2 + (2s-2)e^s - s^2}{2s^2} \right)^i + \sum_{i=1}^n \left(\frac{2e^s - s^2 - 2s - 2}{2s^2} \right)^i \right\}^{\frac{1}{q}}. \end{aligned} \quad (6.3)$$

Proof. The result holds by letting $z = \frac{\alpha_1 + \alpha_2}{2}$ in Theorem 5.4 with n -polynomial exponentially s -convexity for $\varphi : (0, \infty) \rightarrow \mathbb{R}$, $\varphi(z) = \frac{1}{z}$, we obtain the required result. \square

7. Conclusions

In this paper, we have taken into consideration a critical extension of convexity, that is referred as n -polynomial exponentially s -convex functions and acquired new variants of Hermite-Hadamard inequality employing this new definition. We have also obtained refinements of the Ostrowski inequality for functions whose first derivatives in absolute value at certain power are n -polynomial exponential-type s -convex. Moreover, for different values of parameters, i.e., s, n and z , we have deduced some special cases of our main results. We presented some applications of our established results to special means of two positive real numbers. In the future, new inequalities for the other n -polynomial convex functions can be obtained by utilising the techniques used in this paper.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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