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Research article

Left almost semihyperrings characterized by their hyperideals

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Abstract: The notion of left almost semihyperrings (briefly, *LA*-semihyperrings), as a generalization of left almost semirings (briefly, *LA*-semirings), was introduced by Nawaz, Rehman and Gulistan in 2018. The purpose of this article is to study the classes of weakly regular *LA*-semihyperrings and regular *LA*-semihyperrings. Then, characterizations of weakly regular *LA*-semihyperrings and regular *LA*-semihyperrings in terms of their hyperideals have been obtained.

Keywords: *LA*-semihypergroup; *LA*-semihyperring; regular *LA*-semihyperring; weakly regular *LA*-semihyperring
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1. Introduction

The study of left almost semigroups (briefly, *LA*-semigroups), as a generalization of commutative semigroups, was first introduced in 1972 by Kazim and Naseeruddin [25]. It is also called an Abel-Grassmann's groupoid (briefly, *AG*-groupoid) [32]. An *LA*-semigroup is a non-associative and non-commutative algebraic structure midway between a groupoid and a commutative semigroup. Mushtaq and Yousuf [27] examined some basic results of the structure of *LA*-semigroups, for examples, a commutative monoid is an *LA*-semigroup with right identity, every left cancellative *LA*-semigroup is right cancellative and every right cancellative *LA*-monoid is left cancellative. On *LA*-semigroups, regularities are interesting and essential properties to investigate. Khan and Asif [20] classified intra-regular *LA*-semigroups based on the features of their fuzzy ideals in 2010. Abdullah, Aslam and Amin [2] began discussing regular *LA*-semigroup categorizations in terms of interval (α,β)-fuzzy ideals. Also, Khan, Jun and Yousafzai [22] used their fuzzy left ideals and fuzzy right ideals to characterize right regular *LA*-semigroups. In 2016, Khan, Yousafzai and Khan [24] defined a class of (m,n)-regular *LA*-semigroups based on their (m,n)-ideals. Several characterizations of weakly regular *LA*-semigroups by using the smallest ideals and fuzzy ideals of *LA*-semigroups were investigated by Yousafzai, Iampan and Tang [46]. Furthermore, Sezer [36] has developed soft sets to characterize regular, intra-regular, completely regular, weakly regular and quasi-regular *LA*-semigroups. Recently, various properties of *LA*-semigroups have been studied by many mathematicians (see, e.g., [4, 8, 14, 15, 44, 47]). Additionally, the notion of left almost semirings (briefly, *LA*-semirings), which is a generalization of left almost rings (briefly, *LA*-rings) [37], has been considered different properties by some mathematicians (see, e.g., [12, 13, 33]). Moreover, the concept of left almost was studied in other algebraic structures (for example, in ordered *LA*-(Γ)-semigroups [5, 7, 19, 45], in gamma *LA*-rings and gamma *LA*-semigroups [23], in *LA*-polygroups [3, 40, 42]).

The concept of hyperstructures was introduced by Marty [26] in the 8th Congress of Scandinavian Mathematicians. There are many authors expanded the concept of hyperstructures (see, e.g., ([9–11, 28, 29, 31, 38, 39]). Hila and Dine [18] introduced the notion of left almost semihypergroups (briefly, *LA*-semihypergroups) which is a generalization of *LA*-semigroups and commutative semihypergroups. It is a useful non-associative algebraic hyperstructure, midway between a hypergroupoid and a commutative semihypergroup, with wide applications in the theory of flocks etc. In 2013, Yaqoob, Corsini and Yousafzai [41] used the properties of their left and right hyperideals to characterize intra-regular *LA*-semihypergroups. Then, the class of regular *LA*-semihypergroups was characterized in terms of $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic (resp., left, right, two-sided, bi, generalized bi, interior, quasi)-hyperideals of *LA*-semihypergroups by Gulistan, Khan, Yaqoob and Shahzad [16]. In addition, Khan, Farooq, Izhar and Davvaz [21] studied into some properties of fuzzy left and right hyperideals in regular and intra-regular *LA*-semihypergroups. In terms of soft interior hyperideals, Abbasi, Khan, Talee and Khan [1] gave different essential characterizations of left regular *LA*-semihypergroups. On the other hand, Yaqoob and Gulistan [43] introduced the concept of ordered *LA*-semihypergroups which is a generalization of *LA*-semihypergroups. Next, the results of fuzzy hyperideals and generalized fuzzy hyperideals of ordered *LA*-semihypergroups were then examined by Azhar, Gulistan, Yaqoob and Kadry (see, [6, 17]).

In 2018, Nawaz, Rehman and Gulistan [30] defined the idea of left almost semihyperrings (briefly, *LA*-semihyperrings), as a generalization of *LA*-semirings, and studied at some of their basic properties. In 2020, Rahman, Hidayat and Alghofari [34] applied the concept of fuzzy sets to define the new algebraic structure, namely, fuzzy left almost semihyperrings, and they have shown that the set of all fuzzy subsets in *LA*-semihyperrings is also *LA*-semihyperrings. In this paper, we are interesting in the classes of weakly regular *LA*-semihyperrings and regular *LA*-semihyperrings. Then, we give some characterizations of weakly regular *LA*-semihyperrings and regular *LA*-semihyperrings in terms of their hyperideals.

2. Preliminaries

Firstly, we recall some of the basic concepts and properties, which are necessary for this paper. Let H be a nonempty set. Then, the map $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation* on H where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ denotes the set of all nonempty subsets of H. A *hypergroupoid* is called the pair $(H, \circ)$, where $\circ$ is a hyperopartion on a nonempty set H. If $x \in H$ and A, B are two nonempty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hypergroupoid $(H, \circ)$ is called an *LA-semihypergroup* [18] if for all $x, y, z \in H$, $(x \circ y) \circ z = (z \circ y) \circ x$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in z \circ y} v \circ x.$$

This law is known as a left invertive law.

For any nonempty subsets A, B and C of an LA-semihypergroup $(H, \circ)$, we have that

$$(A \circ B) \circ C = (C \circ B) \circ A.$$

The following the notion, which appears in [30], will be considering in this study.

A hyperstructure $(S, +, \cdot)$ is called an *LA-semihyperring* if it satisfies the following conditions:

(i) $(S, +)$ is an LA-semihypergroup;
(ii) $(S, \cdot)$ is an LA-semihypergroup;
(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for all $x, y, z \in S$.

Example 2.1. [30] Let $S = \{a, b, c\}$ be a set with the hyperoperations + and $\cdot$ on S defined as follows:

$+$	a	b	c
a	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$
b	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$
c	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$

$\cdot$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, b, c\}$	$\{c\}$
c	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$

Then, $(S, +, \cdot)$ is an LA-semihyperring.

For more convenient, we say an LA-semihyperring S instead of an LA-semihyperring $(S, +, \cdot)$ and we write xy instead of $x \cdot y$ for any $x, y \in S$.

In an LA-semihyperring S, the medial law $(xy)(zw) = (xz)(yw)$ holds for all $x, y, z, w \in S$. An element e of an LA-semihyperring S is called a *left identity* (resp., *pure left identity*) if for all $x \in S$, $x \in ex$ (resp., $x = ex$). If an LA-semihyperring S contains a pure left identity e, then it is unique. In an LA-semihyperring S with a pure left identity e, the paramedial law $(xy)(zw) = (wy)(zx)$ holds for all $x, y, z, w \in S$.

An element a of an LA-semihyperring S with a left identity (resp., pure left identity) e is called a *left invertible* (resp., *pure left invertible*) if there exists $x \in S$ such that $e \in xa$ (resp., $e = xa$). An LA-semihyperring S is called a *left invertible* (resp., *pure left invertible*) if every element of S is a left invertible (resp., pure left invertible).

We observe that if an element e is a pure left identity of an LA-semihyperring S, then e is a left identity. But the converse is not true in general, as the following example.

Example 2.2. Let $S = \{a, b, c\}$ be a set with the hyperoperations + and $\cdot$ on S defined as follows:

$+$	a	b	c
a	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$
b	$\{a, b, c\}$	$\{b, c\}$	$\{b, c\}$
c	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$

$\cdot$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, b, c\}$	$\{c\}$
c	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$

Then, $(S, +, \cdot)$ is an LA-semihyperring [35]. One can see that b is a left identity, but it is not a pure left identity.

Lemma 2.3. [30] *Let S be an LA-semihyperring with a pure left identity e. Then $x(yz) = y(xz)$ for all $x, y, z \in S$.*

For any LA-semihyperring S, the following law holds $(AB)(CD) = (AC)(BD)$ for all nonempty subsets A, B, C, D of S. If an LA-semihyperring S contains the pure left identity e, then $(AB)(CD) = (DB)(CA)$ and $A(BC) = B(AC)$ for every nonempty subsets A, B, C, D of S.

Now, we recall the concepts of different types of hyperideals of LA-semihyperrings which occurred in [30] as follows. Let S be an LA-semihyperring and a nonempty subset A of S such that $A + A \subseteq A$. Then:

(i) A is called a *left hyperideal* of S if $SA \subseteq A$;
(ii) A is called a *right hyperideal* of S if $AS \subseteq A$;
(iii) A is called a *hyperideal* of S if it is both a left and a right hyperideal of S;
(iv) A is called a *quasi-hyperideal* of S if $SA \cap AS \subseteq A$;
(v) A is called a *bi-hyperideal* of S if $AA \subseteq A$ and $(AS)A \subseteq A$.

Example 2.4. Let $S = \{a, b, c, d, e\}$ be a set with the hyperoperations $+$ and $\cdot$ on S defined as follows:

$+$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
e	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$

$\cdot$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, e\}$	$\{a, e\}$	$\{a, c\}$	$\{a, e\}$
c	$\{a\}$	$\{a, e\}$	$\{a, e\}$	$\{a, b\}$	$\{a, e\}$
d	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{d\}$	$\{a, e\}$
e	$\{a\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$	$\{a, e\}$

Then, $(S, +, \cdot)$ is an LA-semihyperring. Now, we can see that $A = \{a, b, e\}$ is a left hyperideal of S, but it is not a right hyperideal, because $b \cdot d = \{a, c\} \nsubseteq \{a, b, e\}$.

Proposition 2.5. *Let S be an LA-semihyperring such that $S = S^2$. Then every right hyperideal of S is a hyperideal.*

Proof. Let R be a right hyperideal of S. Let $a \in SR$. Then $a \in sr$ for some $r \in R$ and $s \in S$. Since $S = S^2$, $s \in xy$ for some $x, y \in S$. By using the left invertive law, we have

$$a \in sr \subseteq (xy)r = (ry)x \subseteq (RS)S \subseteq RS \subseteq R.$$

Thus, $SR \subseteq R$. This shows that R is a left hyperideal of S. Therefore, R is a hyperideal of S. □

For any LA-semihyperring S with a pure left identity e, we have that $S = S^2$. Then, we have the following lemma.

Lemma 2.6. *Let S be an LA-semihyperring with a pure left identity e. Then every right hyperideal of S is a hyperideal of S.*

Lemma 2.7. *Every left (resp., right) hyperideal of an LA-semihyperring S is a quasi-hyperideal of S.*

Proof. Let Q be a left hyperideal of an LA-semihyperring S. Then, $Q+Q \subseteq Q$ and $SQ \cap QS \subseteq SQ \subseteq Q$. Hence, Q is a quasi-hyperideal of S. For the case of the right hyperideal, we can prove similarly. □

Lemma 2.8. *The intersection of a left hyperideal L and a right hyperideal R of an LA-semihyperring S is a quasi-hyperideal of S.*

Proof. It is easy to show that $L \cap R + L \cap R \subseteq L \cap R$. Next, consider

$$S(L \cap R) \cap (L \cap R)S \subseteq SL \cap RS \subseteq L \cap R.$$

Hence, $L \cap R$ is a quasi-hyperideal of S. □

Lemma 2.9. *Let S be an LA-semihyperring with a left identity e such that $(xe)S \subseteq xS$ for all $x \in S$. Then every quasi-hyperideal of S is a bi-hyperideal of S.*

Proof. Let B be a quasi-hyperideal of S. Then, $B + B \subseteq B$ and $SB \cap BS \subseteq B$. Clearly, $BB \subseteq B$. Now, we have that $(BS)B \subseteq SB$. Next, let $x \in (BS)B$. So, $x \in (as)b$ for some $a, b \in B$ and $s \in S$. By assumption and using the medial law, we have

$$x \in (as)b \subseteq (as)(eb) = (ae)(sb) \subseteq (ae)S \subseteq aS \subseteq BS.$$

Thus, $(BS)B \subseteq BS$. It follows that $(BS)B \subseteq SB \cap BS \subseteq B$. Therefore, B is a bi-hyperideal of S. □

Lemma 2.10. *If S is an LA-semihyperring with a pure left identity e, then for every $a \in S$, a^2S is a hyperideal of S such that $a^2 \subseteq a^2S$.*

Proof. Assume that S is an LA-semihyperring with a pure left identity e. Let $a \in S$. By using the left invertive law, we have

$$\begin{aligned} a^2S + a^2S &= (aa)S + (aa)S = (Sa)a + (Sa)a \\ &= ((S + S)a)a \subseteq (Sa)a \\ &= (aa)S = a^2S. \end{aligned}$$

Then, using Lemma 2.3, the left invertive law and the paramedial law, we have

$$S(a^2S) = a^2(SS) \subseteq a^2S$$

and

$$(a^2S)S = ((aa)S)S = ((Sa)a)S = (Sa)(Sa) = (aa)(SS) \subseteq a^2S.$$

Hence, a^2S is a hyperideal of S. Now, using the left invertive law, we have

$$a^2 = aa = (ea)a = (aa)e \subseteq a^2S.$$

This completes the proof. □

3. Weakly regular *LA*-semihyperrings

In this section, the class of weakly regular *LA*-semihyperrings has been studied, we give some characterizations of weakly regular *LA*-semihyperrings by using the concepts of left hyperideals and right hyperideals of *LA*-semihyperrings.

Definition 3.1. An element a of an *LA*-semihyperrnig S is said to be *weakly regular* if there exist $x, y \in S$ such that $a \in (ax)(ay)$. The *LA*-semihyperring S is called *weakly regular* if every element of S is weakly regular.

Example 3.2. In Example 2.1, we can show that there exist $x, y \in S$ such that $a \in (ax)(ay)$ for all $a \in S$. Therefore, S is weakly regular.

Theorem 3.3. *Let S be a pure left invertible LA-semihyperring with a pure left identity e. Then S is weakly regular if and only if $R_1 \cap R_2 \subseteq R_1R_2$, where both R_1 and R_2 are right hyperideals of S.*

Proof. Assume that S is weakly regular. Let R_1 and R_2 be right hyperideals of S and $a \in R_1 \cap R_2$. Then, there exist $x, y \in S$ such that $a \in (ax)(ay) \subseteq (R_1S)(R_2S) \subseteq R_1R_2$. Hence, $R_1 \cap R_2 \subseteq R_1R_2$.

Conversely, let $a \in S$. Since S is a pure left invertible *LA*-semihyperring, there exists $x \in S$ such that $e = xa$. By Lemma 2.10, we have that a^2S is a right hyperideal of S and $a^2 \subseteq a^2S$. Then, by using assumption, the left invertive law and Lemma 2.3, we have

$$\begin{aligned} a^2 &\subseteq (a^2S) \cap (a^2S) \\ &\subseteq (a^2S)(a^2S) \\ &= a^2((a^2S)S) \\ &= a^2((SS)a^2) \\ &\subseteq (aa)(S(aa)) \\ &= (aa)(a(Sa)) \\ &\subseteq (aS)(aS). \end{aligned}$$

Next, using the left invertive law and Lemma 2.3, we have

$$\begin{aligned} a = ea = (xa)a = (aa)x &\subseteq ((aS)(aS))x \\ &= (x(aS))(aS) = (a(xS))(aS) \\ &\subseteq (aS)(aS). \end{aligned}$$

This implies that $a \in (ax)(ay)$ for some $x, y \in S$. Therefore, S is weakly regular. □

The proof of the following theorem is similar to Theorem 3.3.

Theorem 3.4. *Let S be a pure left invertible LA-semihyperring with a pure left identity e. Then S is weakly regular if and only if $L_1 \cap L_2 \subseteq L_1L_2$, where both L_1 and L_2 are left hyperideals of S.*

Theorem 3.5. *Let S be a pure left invertible LA-semihyperring with a pure left identity e. Then S is weakly regular if and only if $R \cap L \subseteq L^2R^2$, for every right hyperideal R and left hyperideal L of S.*

Proof. Assume that S is weakly regular. Let R be a right hyperideal and L be a left hyperideal of S and $a \in R \cap L$. Then, there exist $x, y \in S$ such that $a \in (ax)(ay)$. By using the left invertive law, the medial law, the paramedial law and Lemma 2.3, we have

$$\begin{aligned} a &\in (ax)(ay) \\ &\subseteq (((ax)(ay))x)(((ax)(ay))y) \\ &= ((ax)(ay))((((ax)(ay))x)y) \\ &= ((ax)(ay))((yx)((ax)(ay))) \\ &= ((ax)(ay))((ax)((yx)(ay))) \\ &= ((ax)(ay))(((ay)(yx))(xa)) \\ &= ((ax)(ay))((((yx)y)a)(xa)) \\ &\subseteq ((aS)(aS))((Sa)(Sa)) \\ &= ((Sa)(Sa))((aS)(aS)) \\ &\subseteq ((SL)(SL))((RS)(RS)) \\ &\subseteq L^2R^2. \end{aligned}$$

Therefore, $R \cap L \subseteq L^2R^2$.

Conversely, let R_1 and R_2 be right hyperideals of S. By Lemma 2.6, we have that R_1 is also a left hyperideal of S. By assumption, $R_1 \cap R_2 \subseteq R_1^2R_2^2 \subseteq R_1R_2$. Consequently, S is weakly regular by Theorem 3.3. □

Theorem 3.6. *Let S be a pure left invertible LA-semihyperring with a pure left identity e. Then S is weakly regular if and only if $R \cap L \subseteq L^3R$, for every right hyperideal R and left hyperideal L of S.*

Proof. Let R be a right hyperideal and L be a left hyperideal of S and $a \in R \cap L$. By assumption, there exist $x, y \in S$ such that $a \in (ax)(ay)$. Then, by using the left invertive law, the medial law, the paramedial law and Lemma 2.3, we have

$$\begin{aligned} a &\in (ax)(ay) \\ &\subseteq (((ax)(ay))x)(((ax)(ay))y) \\ &= (y((ax)(ay)))(x((ax)(ay))) \\ &= ((ax)(y(ay)))((ax)(x(ay))) \\ &= ((ax)(ay^2))((ax)(a(xy))) \\ &= ((y^2a)(xa))((ax)(a(xy))) \\ &= (((ax)(a(xy)))(xa))(y^2a) \\ &= ((((xy)a)(xa))(xa))((yy)a) \\ &= ((((xy)a)(xa))(xa))((ay)y) \\ &\subseteq (((SL)(SL))(SL))((RS)S) \\ &\subseteq ((LL)L)R \\ &= L^3R. \end{aligned}$$

Hence, $R \cap L \subseteq L^3R$.

Conversely, let R_1 and R_2 be right hyperideals of S. By Lemma 2.6, we have that R_1 also a left hyperideal of S. By assumption, $R_1 \cap R_2 \subseteq R_1^3 R_2 = ((R_1 R_1) R_1) R_2 \subseteq R_1 R_2$. By Theorem 3.3, S is weakly regular. □

4. Regular LA-semihyperrings

In this section, we characterize the class of regular LA-semihyperrings in terms of (resp., left, right) hyperideals, quasi-hyperideals and bi-hyperideals of LA-semihyperrings.

Definition 4.1. An element a of an LA-semihyperrnig S is said to be *regular* if there exists an element $x \in S$ such that $a \in (ax)a$. The LA-semihyperring S is called *regular* if every element of S is regular.

Example 4.2. In Example 2.2, we have that there exists $x \in S$ such that $a \in (ax)a$ for all $a \in S$. Hence, S is regular.

Lemma 4.3. *Let S be an LA-semihyperring. Then the following conditions are equivalent:*

(i) *S is regular;*
(ii) *$a \in (aS)a$, for every $a \in S$;*
(iii) *$A \subseteq (AS)A$, for all $\emptyset \neq A \subseteq S$.*

Theorem 4.4. *Let S be a pure left invertible LA-semihyperring with a pure left identity e. Then S is regular if and only if $R \cap L = RL$, for every right hyperideal R and left hyperideal L of S.*

Proof. Assume that S is regular. Let R be a right hyperideal and L be a left hyperideal of S and let $a \in R \cap L$. Then, $a \in (aS)a \subseteq (RS)L \subseteq RL$. It follows that $R \cap L \subseteq RL$. Since $RL \subseteq R$ and $RL \subseteq L$, we have $RL \subseteq R \cap L$. Thus, $R \cap L = RL$.

Conversely, let $a \in S$. Since S is a pure left invertible LA-semihyperring, there exists $x \in S$ such that $e = xa$. By Lemma 2.10, $a^2 S$ is both a right hyperideal and a left hyperideal of S. Moreover, $a^2 \subseteq a^2 S$. Then, by using the given assumption, Lemma 2.3 and the left invertive law, we have

$$\begin{aligned} a^2 &\subseteq (a^2S) \cap (a^2S) \\ &= (a^2S)(a^2S) \\ &= a^2((a^2S)S) \\ &= a^2((SS)a^2) \\ &\subseteq (aa)(S(aa)) \\ &= (aa)(a(Sa)) \\ &= ((a(Sa))a)a \\ &\subseteq ((aS)a)a. \end{aligned}$$

Hence, using the invertive law, we have

$$\begin{aligned} a = ea = (xa)a = (aa)x &\subseteq (((aS)a)a)x \\ &= (xa)((aS)a) = e((aS)a) \\ &= (aS)a. \end{aligned}$$

Therefore, S is regular. □

Theorem 4.5. *Let S be a pure left invertible LA-semihyperring with a pure left identity e such that $(xe)S \subseteq xS$ for all $x \in S$. Then the following statements are equivalent:*

(*i*) *S is regular;*
(*ii*) *$(BS)B = B$, for every bi-hyperideal B of S;*
(*iii*) *$(QS)Q = Q$, for every quasi-hyperideal Q of S.*

Proof. $(i) \Rightarrow (ii)$ Assume that S is regular. Let B be a bi-hyperideal of S and $a \in B$. Then, $a \in (aS)a \subseteq (BS)B$. Thus, $B \subseteq (BS)B$. On the other hand $(BS)B \subseteq B$. Hence, $(BS)B = B$.

$(ii) \Rightarrow (iii)$ It follows from Lemma 2.9.

$(iii) \Rightarrow (i)$ Let R be a right hyperideal and L be a left hyperideal of S. By Lemma 2.8, $R \cap L$ is a quasi-hyperideal of S. By assumption, we have that $R \cap L = ((R \cap L)S)(R \cap L) \subseteq (RS)L \subseteq RL$. Any other way, $RL \subseteq R \cap L$. Thus, $R \cap L = RL$. Therefore, S is regular by Theorem 4.4. □

Theorem 4.6. *Let S be a pure left invertible LA-semihyperring with a pure left identity e such that $(xe)S \subseteq xS$ for all $x \in S$. Then the following statements are equivalent:*

(*i*) *S is regular;*
(*ii*) *$B \cap I \subseteq (BI)B$, for every bi-hyperideal B and hyperideal I of S;*
(*iii*) *$Q \cap I \subseteq (QI)Q$, for every quasi-hyperideal Q and hyperideal I of S.*

Proof. $(i) \Rightarrow (ii)$ Assume that S is regular. Let B be a bi-hyperideal and I be a hyperideal of S. Now, let $a \in B \cap I$. It turns out that $a \in (aS)a$. Thus, by left invertive law and Lemma 2.3, we have

$$\begin{aligned} a \in (aS)a &\subseteq (((aS)a)S)a \\ &= ((Sa)(aS))a \\ &= (a((Sa)S))a \\ &\subseteq (B((SI)S))B \\ &\subseteq (BI)B. \end{aligned}$$

Hence, $B \cap I \subseteq (BI)B$.

$(ii) \Rightarrow (iii)$ By Lemma 2.9, we have that every quasi-hyperideal of S is a bi-hyperideal. Hence, (iii) holds.

$(iii) \Rightarrow (i)$ Let R be a right hyperideal and L be a left hyperideal of S. Then, $R \cap L$ is a quasi-hyperideal of S by Lemma 2.8. Since (iii) holds, we get that $R \cap L = (R \cap L) \cap S \subseteq ((R \cap L)S)(R \cap L) \subseteq (RS)L \subseteq RL$. Also, $R \cap L = RL$. By Theorem 4.4, S is regular. □

Theorem 4.7. *Let S be a pure left invertible LA-semihyperring with a pure left identity e such that $(xe)S \subseteq xS$ for all $x \in S$. Then the following conditions are equivalent:*

(*i*) *S is regular;*
(*ii*) *$B \cap L \subseteq (BS)L$, for every bi-hyperideal B and left hyperideal L of S;*
(*iii*) *$Q \cap L \subseteq (QS)L$, for every quasi-hyperideal Q and left hyperideal L of S.*

Proof. $(i) \Rightarrow (ii)$ Assume that S is regular. Let B be a bi-hyperideal and L be a left hyperideal of S and $a \in B \cap L$. Then, $a \in (aS)a$. By using the left invertive law, we have

$$
\begin{aligned}
a \in (aS)a \subseteq (aS)((aS)a) \\
&= (((aS)a)S)a \\
&\subseteq (((BS)B)S)L \\
&\subseteq (BS)L.
\end{aligned}
$$

Hence, $B \cap L \subseteq (BS)L$.

$(ii) \Rightarrow (iii)$ Since every quasi-hyperideal is a bi-hyperideal of S, (iii) holds.

$(iii) \Rightarrow (i)$ Let R be a right hyperideal and L be a left hyperideal of S. By Lemma 2.7, R is also a quasi-hyperideal of S. By assumption, $R \cap L \subseteq (RS)L \subseteq RL$. So, $R \cap L = RL$. Therefore, S is regular by Theorem 4.4. □

The proof of the following theorem is similar to Theorem 4.7.

Theorem 4.8. *Let S be a pure left invertible LA-semihyperring with a pure left identity e such that $(xe)S \subseteq xS$ for all $x \in S$. Then the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$B \cap R \subseteq (RS)B$, for every bi-hyperideal B and right hyperideal R of S;*

(iii) *$Q \cap R \subseteq (RS)Q$, for every quasi-hyperideal Q and right hyperideal R of S.*

Theorem 4.9. *Let S be a pure left invertible LA-semihyperring with a pure left identity e such that $(xe)S \subseteq xS$ for all $x \in S$. Then the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$B \cap R \cap L \subseteq (BR)L$, for every bi-hyperideal B, right hyperideal R and left hyperideal L of S;*

(iii) *$Q \cap R \cap L \subseteq (QR)L$, for every quasi-hyperideal Q, right hyperideal R and left hyperideal L of S.*

Proof. $(i) \Rightarrow (ii)$ Assume that S is regular. Let B be a bi-hyperideal, R be a right hyperideal and L be a left hyperideal of S and $a \in B \cap R \cap L$. Then, $a \in (aS)a$. By using the medial law, we have

$$
\begin{aligned}
a \in (aS)a &\subseteq (((aS)a)S)((aS)a) \\
&= (((aS)a)(aS))(Sa) \\
&\subseteq (((BS)B)(RS))(SL) \\
&\subseteq (BR)L.
\end{aligned}
$$

This implies that $B \cap R \cap L \subseteq (BR)L$.

$(ii) \Rightarrow (iii)$ The implication follows by Lemma 2.9.

$(iii) \Rightarrow (i)$ Let R be a right hyperideal and L be a left hyperideal of S. By Lemma 2.7, R is also a quasi-hyperideal of S. By the hypothesis, we have that $R \cap L = R \cap R \cap L \subseteq (RR)L \subseteq RL$. Since $RL \subseteq R \cap L$, it follows that $R \cap L = RL$. By Theorem 4.4, S is regular. □

5. Conclusions

In this paper, the classes of weakly regular *LA*-semihyperrings and regular *LA*-semihyperrings have been considered. In Section 3, the characterizations of weakly regular *LA*-semihyperrings by the properties of their left hyperideals and right hyperideals were shown in Theorem 3.3–Theorem 3.6. In Section 4, the fundamental characterization of regular *LA*-semihyperrings by using their left hyperideals and right hyperideals has been given in Theorem 4.4. Finally, we characterized regular *LA*-semihyperrings in terms of (resp., left, right) hyperideals, quasi-hyperideals and bi-hyperideals of *LA*-semihyperrings were shown in Theorem 4.5–Theorem 4.9. In our future work, we will characterize the class of intra-regular *LA*-semihyperrings by using the concept of their hyperideals.

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Conflict of interest

The author declares no conflict of interest.

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