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Research article

On investigations of graphs preserving the Wiener index upon vertex removal

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Abstract: In this paper, we present solutions of two open problems regarding the Wiener index $W(G)$ of a graph G. More precisely, we prove that for any $r \geq 2$, there exist infinitely many graphs G such that $W(G) = W(G - \{v_1, \ldots, v_r\})$, where $v_1, \ldots, v_r$ are r distinct vertices of G. We also prove that for any $r \geq 1$ there exist infinitely many graphs G such that $W(G) = W(G - \{v_i\})$, $1 \leq i \leq r$, where $v_1, \ldots, v_r$ are r distinct vertices of G.

Keywords: Wiener index; vertex removal; topological index
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1. Introduction

The topological indices (TIs) are found to be one of the effective tools in determining the properties such as: boiling point, melting point and bond energy of each compound. These are certain parameters that represent the physical and biochemical properties of different compounds depending upon their structures. TIs are helpful in describing the qualitative as well as quantitative analysis of a given molecular structure. Several aspects of TIs are still being vastly studied by different researchers, some recent contributions include [13, 16]. Use of TIs in drugs and chemistry are mentioned in [3, 19–22] and few other applications can be seen in [9, 10, 12, 15].

Before proceeding further, we set the notations of this paper. We denote an undirected connected graph with no multiple edge or loop by G, the number of edges and vertices by m and n, respectively, a cycle with n vertices by C_n, complete graph by K_n, degree of a vertex v in G by $d_G(v)$ (or simply $d(v)$) and the distance between the vertices u and v by $d(u, v)$. The Wiener index of G is defined as

$$W(G) = \sum_{\{u,v\}\in V(G)} d(u, v).$$

The main objective of producing this index was to determine the boiling points of alkanes [17]. Wiener index is also implicitly used in network topology as it plays a vital role in the computation of average distance which is an important measure in the network topology. It is also known due to its frequent use in communication theory and facility location, sociometry and crystallography, for details see [2], [4] and references therein. Due to its vast applications, the Wiener index and its different variants have been studied extensively, for example see [1, 6, 7, 18] and surveys [8, 11].

Although, the study of several papers comprises of different aspects of Wiener index, but there are still unsolved problems related to this descriptor. The attempts to solve these problems have resulted in to formulations and solutions of new variants of related problems as well. In [14], Šoltés posed the question of finding all graphs G for which $W(G)$ does not change upon removal of any vertex. Up to now, the only such known graph is C_{11}. Recently, Knor et al. [4] studied some relaxed versions of this problem and produced partial solutions to those problems. In this paper, we solved an open problem which was formulated by Knor et al. during their investigations on relaxed version of Šoltés problem.

Before presenting the main results, we recall the formulations of the related problems: we start with the Šoltés [14] problem:
Problem 1: Find all possible graphs G such that $W(G) = W(G - \{v\})$, for any vertex v in G. The first graph, which is C_{11}, with this property was found by Šoltés himself. Till now, several efforts have been made to solve this problem. However, Problem 1 still remains open until now. Knor et al. [5] investigated a relaxed version of the Problem 1 by considering:
Problem 2: Find all possible graphs G such that $W(G) = W(G - \{v\})$ is true for some $v \in V(G)$. They constructed many infinite classes of graphs G in which $W(G) = W(G - \{v\})$ for some vertex v of degree 2. Recently, in [4], some infinite classes of graphs G with a vertex v such that $d(v) \geq 3$ and $W(G) = W(G - \{v\})$ are constructed. Other than that, the existence of vertices of degree $n - 2$ and $n - 1$ in an n-vertex graph G with $W(G) = W(G - \{v\})$ have also been shown in [4]. But, finding all such graphs is still a quite challenging problem. Knor et al. [4] continued to study this problem and formulated the following three problems as well:
Problem 3: Are there k-regular connected graphs $G \neq C_{11}$ with $W(G) = W(G - \{v\})$ for some $v \in V(G)$?
Problem 4: Find connected graphs G with $W(G) = W(G - S)$ where $S \subseteq V(G)$ is arbitrary and $|S| \geq 2$.
Problem 5: For a given r, find (infinitely many) graphs G with $W(G) = W(G - \{v_1\}) = W(G - \{v_2\}) = \cdots = W(G - \{v_r\})$ for some distinct vertices $v_1, \cdots, v_r \in V(G)$.
An infinite family of graphs G with $W(G) = W(G - \{v_i\})$ for some vertices v_i of $G, i = 1, ..., k$ (k is an arbitrary natural number) is constructed in [23]. Moreover, the symmetry of G leads to the following property: $W(G) = W(G - \{v_1\}) = ... = W(G - \{v_k\})$.

In this paper, we extend the study of the work presented in [4] by solving Problem 5. The following section contains the main results of this paper.

2. Results

We start the section by introducing the following symbols and notations. For an integer $r \geq 1$, a star with $r+1$ vertices is called an $(r+1)$-star and the vertex with the maximum degree is called the center. For $r_1 \geq 1$, $r_2 \geq 2$, $H(r_1, r_2)$ is a graph obtained from the (r_1+1)-star and K_{r_2} by adding an edge uv where u is the center of an (r_1+1)-star and v is a vertex of K_{r_2}, so $H(r_1, r_2)$ has $r_1 + r_2 + 1$ vertices. We declare that $t_G(u) = \sum_{v \in V(G)-\{u\}} d(u, v)$.

For a given graph G, the integers $r \geq 2$ and $|V(G)| + r - 1 \leq q \leq \frac{r(r+2|V(G)|-1)}{2}$, we define a set of graphs $\mathcal{T}(G, r, q)$ as follows:

$$\begin{aligned}\mathcal{T}(G, r, q) \;=\; & \{G' | V(G') = V(G) \cup U, E(G') = E(G) \cup E', \\ & \text{where } U = \{u_1, \ldots, u_r\}, |E'| = q \text{ and } u_1 v \in E' \text{ for every } v \in U \cup V(G) - \{u_1\}, \\ & E' \subset U \times (U \cup V(G))\},\end{aligned}$$

i.e. $\mathcal{T}(G, r, q)$ is a set of the graphs G' obtained from G by adding r vertices $\{u_1, \ldots, u_r\}$ and q edges such that $d_{G'}(u_1) = |V(G)| + r - 1$, $V(G') = \{u_1, \ldots, u_r\} \cup V(G)$, $G'[V(G)] = G$. Clearly, if $|V(G)| + r - 1 \leq q \leq \frac{r(r+2|V(G)|-1)}{2}$, we have $\mathcal{T}(G, r, q) \neq \emptyset$.

Given two trees T_1, T_2 with $v_i \in V(T_i)$ for $i = 1, 2$ and an integer $k \geq 1$, let $T_1^j \cong T_1$ and $T_2^j \cong T_2$ with $f(v_1^j) = v_1$ and $f(v_2^j) = v_2$ under an isomorphic mapping f for $j = 1, 2, \cdots, k$. Now we define a graph, denoted by $G' = G(T_1, T_2, k, v_1, v_2)$ (see Figure 1), with

$$\begin{aligned}V(G') &= \{u\} \cup \left(\cup_{j=1}^{k} V(T_1^j)\right) \cup \left(\cup_{j=1}^{k} V(T_2^j)\right) \cup \{p_1^j, p_2^j, w_j : j = 1, 2, \cdots, k\}, \\ E(G') &= \left(\cup_{j=1}^{k} E(T_1^j)\right) \cup \left(\cup_{j=1}^{k} E(T_1^j)\right) \cup \{uw_j, uv_1^j, v_1^j w_j, w_j v_2^j, v_1^j p_1^j, p_1^j p_2^j, p_2^j v_2^j : j = 1, 2, \cdots, k\}.\end{aligned}$$

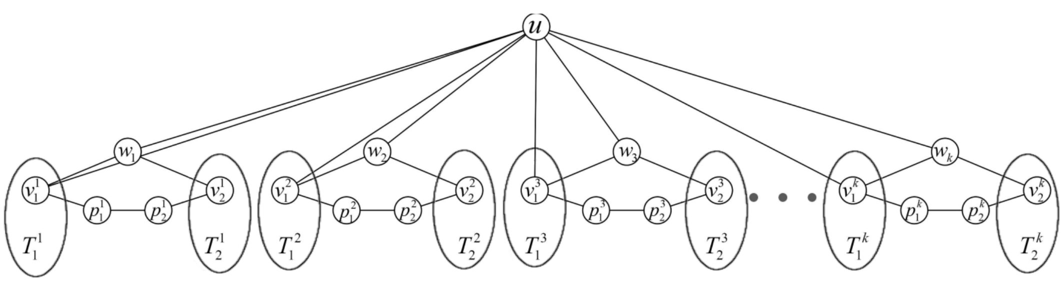

Figure 1. The graph $G(T_1, T_2, k, v_1, v_2)$ constructed from an integer k and trees T_1, T_2 with $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$.

Lemma 1. *For a graph G, if there is a vertex $v \in V(G)$ with $d(v) = n - 1$, then we have $W(G) = n(n-1) - m$.*

Proof. Since there is a vertex $v \in V(G)$ with $d(v) = n-1$, then for any vertex $u \in V(G)$, we have $d(u,v) \leq 2$. Since there are m pairs of vertices u, v with $d(u,v) = 1$, there are $\frac{n(n-1)}{2} - m$ pairs of vertices u, v with $d(u,v) = 2$. Therefore, we have $W(G) = m + 2(\frac{n(n-1)}{2} - m) = n(n-1) - m$. □

Lemma 2. *For integers $r_1 \geq 1$ and $r_2 \geq 2$, let $H = H(r_1, r_2)$. Then we have $W(H) = n^2 - \frac{3}{2}r_2^2 + nr_2 - 3n + \frac{3}{2}r_2 + 1$ and $m = n - r_2 + \frac{r_2(r_2-1)}{2}$.*

Proof. By the definition of $H(r_1, r_2)$, we assume that G_1 is the (r_1+1)-star, G_2 is a K_{r_2}, and $uv \in E(H)$ with $u \in V(G_1), v \in V(G_2)$. Then $V(H(r_1, r_2)) = V(G_1) \cup V(G_2)$. For any vertex $w_1 \in V(G_1) - \{u\}$, we have $t_H(w_1) = 2r_1 + 1 + 3(r_2 - 1)$. For any vertex $w_2 \in V(G_2) - \{v\}$, we have $t_H(w_2) = (r_2 - 1) + 2 + 3r_1$. Since $t_H(u) = r_1 + 1 + 2(r_2 - 1), t_H(v) = r_2 + 2r_1$, therefore

$$W(H) = \frac{1}{2}(r_1(2r_1 + 1 + 3(r_2 - 1)) + (r_2 - 1)((r_2 - 1) + 2 + 3r_1) + r_1 + 1 + 2(r_2 - 1) + r_2 + 2r_1). \tag{2.1}$$

Also, we have

$$r_1 + r_2 + 1 = n. \tag{2.2}$$

By using Eq (2.2) in Eq (2.1), we get

$$W(H) = n^2 - \frac{3}{2}r_2^2 + nr_2 - 3n + \frac{3}{2}r_2 + 1.$$

By counting the edges of H, we have $m = r_1 + 1 + \frac{r_2(r_2-1)}{2} = n - r_2 + \frac{r_2(r_2-1)}{2}$. □

Lemma 3. *For any integer $r \geq 2$, let y, r_2, q be three integers with $r_2 = 2r, y \geq r + \frac{4r^2-1}{r-1}, q = -1 - r + r^2 + r_2^2 + y + 2yr - yr_2$. Let $H = H(y - 1 - r_2, r_2)$. Then, for any graph $G \in \mathcal{T}(H, r, q)$, we have $W(G) = W(H)$.*

Proof. We first show that $r + y - 1 \leq q \leq \frac{r(r+2y-1)}{2}$. Since $q = -1 - r + r^2 + r_2^2 + y + 2yr - yr_2$, we have $q-(r+y-1) = -2r+r^2+r_2^2+2yr-yr_2$. Moreover, $r \geq 2, r_2 = 2r \geq 4$ yield $r^2-2r \geq 0, r_2^2 > 0, 2yr-yr_2 = 0$, thus we have $q - (r + y - 1) \geq 0$. Now, we will show that $q \leq \frac{r(r+2y-1)}{2}$:

$$\begin{aligned} q - \frac{r(r+2y-1)}{2} &= -1 + \frac{(-1+r)r}{2} + r_2^2 + y + (r - r_2)y \\ &= (1-r)y + \frac{r^2 - r}{2} + 4r^2 - 1. \end{aligned}$$

Since $y \geq \frac{4r^2-1}{r-1}, 1 - r < 0$, we have $y(1-r) \leq (1-r)\frac{4r^2-1}{r-1} \leq -4r^2 + 1$. Then, we have

$$\begin{aligned} q - \frac{r(r+2y-1)}{2} &= (1-r)y + \frac{r^2 - r}{2} + 4r^2 - 1 \\ &\leq -4r^2 + 1 + 4r^2 - 1 \\ &= 0. \end{aligned}$$

Consequently, we have $r+y-1 \leq q \leq \frac{r(r+2y-1)}{2}$. Then $\mathcal{T}(H, r, q) \neq \emptyset$. By Lemma 2, $W(H) = y^2 - \frac{3}{2}r_2^2 + yr_2 - 3y + \frac{3}{2}r_2 + 1$. Furthermore, for a graph $G \in \mathcal{T}(H, r, q)$, $|V(G)| = n = y + r$, $|E(G)| = m = |E(H)| + q$.

By Lemma 1, we have $W(G) = n(n-1) - m = (y+r)(y+r-1) - |E(H)| - q$, and by Lemma 2, we have $|E(H)| = (y - r_2 + \frac{r_2(r_2-1)}{2})$, then we have $W(G) = y^2 + 2yr + r^2 - y - r - (y - r_2 + \frac{r_2(r_2-1)}{2}) - (-1 - r + r^2 + r_2^2 + y + 2yr - yr_2) = y^2 + r_2 y - 3y - \frac{3r_2^2}{2} + \frac{3r_2}{2} + 1 = W(H)$. Thus for any graph $G \in \mathcal{T}(H, r, q)$ $W(G) = W(H)$ which completes the proof. □

Lemma 4. *For a graph G and $r \geq 1$, let there be r vertices $u_1, \ldots, u_r$ such that $W(G) = W(G - \{u_1, \ldots, u_r\})$. Denote by $G' = G - \{u_1, \ldots, u_r\}$, then there is a vertex $v \in V(G')$ with $t_{G'}(v) - t_G(v) \geq r$.*

Proof. Suppose to the contrary, we have $t_{G'}(v) - t_G(v) < r$ for any $v \in V(G')$. Clearly, we have $t_G(v) \geq n-1$ for any $v \in V(G)$. Since $W(G) = \frac{1}{2}\sum_{i=1}^{n} t_G(u_i)$, $W(G') = \frac{1}{2}\sum_{i=1}^{n-r} t_{G'}(u_{i+r})$, $W(G) = W(G')$. Then, we have $2(W(G) - W(G')) = \sum_{i=1}^{r}(t_G(u_i)) + \sum_{i=r+1}^{n}(t_G(u_i) - t_{G'}(u_i)) > r(n-1) - r(n-r) = r^2 - r \geq 0$, a contradiction with $W(G) = W(G')$. □

Lemma 5. *For three integers r, m, k with $m \geq r \geq 2$, $\frac{(1+rk)rk}{2} \geq m$, there is a set of trees $\mathcal{F} \neq \emptyset$ such that $|V(T)| = rk + 1$ and there is a vertex $v \in V(T)$ with $t_T(v) = mk$ for any $T \in \mathcal{F}$.*

Proof. First we consider following equations:

$$\begin{aligned} x_1 + x_2 + x_3 + \ldots + x_j &= z_1 \\ x_2 + x_3 + \ldots + x_j &= z_2 \\ &\vdots \\ x_j &= z_j \end{aligned}$$

Let $V_i(T, v_0) = \{v | d_T(v, v_0) = i\}$. Clearly, for $x_1, x_2, \ldots, x_j$, if all $x_i's$ are in N^+, then there is a tree T and $v_0 \in V(T)$ with $|V_i(T, v_0)| = x_i, |V(T)| = z_1 + 1$. And for this tree T, we have $|V(T)| = z_1 + 1, t_T(v_0) = z_1 + \ldots + z_j$. Clearly, if $z_{i+1} < z_i$ for any $1 \leq i \leq j-1$, there is a solution $x_1, x_2, \ldots, x_j$ in which all x_i's are in N^+.

So if we can find $z_1, \ldots, z_j$ with $z_1 = rk, z_1 + \ldots + z_j = mk$, and $z_{i+1} < z_i$ for any $1 \leq i \leq j-1$, then we can get the tree as required. If $m \geq r \geq 2$, $\frac{(1+rk)rk}{2} \geq m$, we do following procedure:

(1) Let $z_1 = rk, i = 1$.

(2) If $mk - \sum_{k=1}^{i} z_k > z_i - 1$, let $z_{i+1} = z_i - 1, i = i + 1$ and go to step 2, else let $z_{i+1} = mk - \sum_{k=1}^{i} z_k$.

Since $rk + (rk-1) + \ldots + 1 = \frac{(1+rk)rk}{2} \geq m$, we can get $z_1, \ldots, z_j$ with $z_1 = rk, z_1 + \ldots + z_j = mk$, and $z_{i+1} < z_i$ for any $1 \leq i \leq j-1$.

□

Theorem 1. *For any $r \geq 2$, there are infinitely many graphs G for which $W(G) = W(G - \{v_1, \ldots, v_r\})$ for some distinct vertices $v_1, \ldots, v_r \in V(G)$.*

Proof. By Lemma 3, there exist graphs G, H, with $H = G - \{u_1, \ldots, u_r\}, W(G) = W(H)$. By Lemma 4, there is a vertex $u \in V(G)$ with $u \notin \{u_1, \ldots, u_r\}, t_H(u) - t_G(u) \geq r$. Clearly, we can find integer k with $\frac{(1+rk)r}{2} \geq t_H(u) - t_G(u)$, then by lemma 5, there is a tree T_1 with $|V(T_1)| = rk + 1$ and a vertex

$v \in V(T_1)$ such that $t_{T_1}(v) = (t_H(u) - t_G(u))k$. By identifying u with v, we obtain G_1. Then, we have $W(G_1) = W(G) + W(T_1) + (|V(T_1)| - 1)t_G(u) + (|V(G)| - 1)t_{T_1}(v)$, and we have:

$$\begin{aligned} W(G_1 - \{u_1, \ldots, u_r\}) =& W(H) + W(T_1) + (|V(T_1)| - 1)t_H(u) + (|V(H)| - 1)t_{T_1}(v) \\ =& W(H) + W(T_1) + (kr + 1 - 1)t_H(u) + (|V(G)| - 1 - r)t_{T_1}(v) \\ =& W(H) + W(T_1) + (kr + 1 - 1)t_H(u) + (|V(G)| - 1)t_{T_1}(v) - rt_{T_1}(v) \\ =& W(H) + W(T_1) + krt_H(u) + (|V(G)| - 1)t_{T_1}(v) - r(t_H(u) - t_G(u))k \\ =& W(G) + W(T_1) + (|V(G)| - 1)t_{T_1}(v) + rkt_G(u) \\ =& W(G) + W(T_1) + (|V(G)| - 1)t_{T_1}(v) + (|V(T_1)| - 1)t_G(u) \\ =& W(G_1). \end{aligned} \tag{2.3}$$

Then, for G_1, take $G'_1 = G_1 - \{u_1, \ldots, u_r\}$, by Lemma 4, there is a vertex $u \in V(G_1)$ with $u \notin \{u_1, \ldots, u_r\}, (t'_{G_1}(u) - t_{G_1}(u)) \geq r$. By Lemma 5, there is a tree T_2 with $V(T_2) = rk_2 + 1$ such that there is a vertex $v \in V(T)$ with $t_{T_2}(v) = (t_{G'_1}(u) - t_{G_1}(u))k_2$. By identifying u with v, we obtain G_2. Similarly, we have $W(G_2 - \{u_1, \ldots, u_r\}) = W(G_2)$. Similarly, we can obtain $G_3, G_4, \ldots$, which completes the proof. □

Remark 1. *For an integer $k > 0$, there exist a rational number $u = \frac{a}{b}$ and integer $n_2 > 0$ such that*

(*i*) $0 < \frac{8uk-4u+8k-8-ku^2}{4k} < \frac{1}{3}$.

(*ii*) $n_2 \equiv 0 \ (\mathrm{mod}\ 4ka^2)$, $0 < \omega_2 = \lambda_2 n_2^2 + \lambda_1 n_2 + \lambda_0 < \frac{n_2(n_2-1)}{2}, 0 < \omega_1 = \frac{n_1^2}{4} < \frac{n_1(n_1-1)}{2}$, *where* $n_1 = un_2 - 4$, $\lambda_2 = \frac{8uk-4u+8k-8-ku^2}{4k} = \frac{8bka-4ba+8ka^2-8a^2+kb^2}{4ka^2}, \lambda_1 = \frac{-5k+2+u}{k} = \frac{-5ka+2a+b}{ak}$, *and* $\lambda_0 = -5$

Proof. (i) Let $f_k(u) = \frac{8uk-4u+8k-8-ku^2}{4k}$. If $k = 1$, we have $f_k(4) = 0, f_k(0) = 0$. Since $f_k(u)$ is a quadratic function, there is a rational number u with $0 < f_k(u) < \frac{1}{3}$. If $k > 1$, we have $f_k(0) = 2 - \frac{2}{k} > 0$. Then there is a real number $x > 0$ with $f_k(x) = 0$. Then there is a rational number u with $0 < f_k(u) < \frac{1}{3}$.

(ii)Since $0 < \lambda_2 < \frac{1}{3} < \frac{1}{2}$, we have $\lambda_2 n_2^2 + \lambda_1 n_2 + \lambda_0 < \frac{n_2(n_2-1)}{2}$ when n_2 is large enough. And clearly, we have $0 < \omega_1 = \frac{n_1^2}{4} < \frac{n_1(n_1-1)}{2}$ when n_1 is large enough. Now, let $n_2 \equiv 0 \ (\mathrm{mod}\ 4ka^2)$ and large enough. Let $n_1 = un_2 - 4$. (ii) holds.

□

Clearly, $n_1, n_2, \omega_1, \omega_2$ all are positive integers.

Lemma 6. *For an integer $k \geq 1$, let T_1, T_2 be two trees with $|V(T_1)| = n_1, t_{T_1}(v_1) = \omega_1, |V(T_2)| = n_2, t_{T_2}(v_2) = \omega_2$ and $G = G(T_1, T_2, k, v_1, v_2)$. Then we have $W(G) = W(G - \{w_1\}) = \ldots = W(G - \{w_k\})$.*

Proof. We will only show $W(G) = W(G - \{w_1\})$. Let $G - \{w_1\} = G'$, then we have:

$$\begin{aligned}
W(G) - W(G') &= \sum_{u,v \in V(G-\{w_1\}), u \neq v} (d(u,v|G) - d(u,v|G')) + t_G(w_1) \\
&= \sum_{u,v \in V(T_2^1), u \neq v} (d(u,v|G) - d(u,v|G')) + \sum_{u,v \in V(G')-V(T_2^1), u \neq v} (d(u,v|G) - d(u,v|G')) \\
&\quad + \sum_{u \in V(T_2^1), v \in V(G')-V(T_2^1)} (d(u,v|G) - d(u,v|G')) + t_G(w_1) \\
&= \sum_{u \in V(T_2^1), v \in V(G')-V(T_2^1)} (d(u,v|G) - d(u,v|G')) + t_G(w_1) \\
&= -(n_2 n_1 + 2(1 + (k-1)(n_1 + n_2 + 3))n_2) + n_2 + t_{T_2}(v_2) + n_1 + t_{T_1}(v_1) + 4 + 1 \\
&\quad + (k-1)(2n_1 + t_{T_1}(v_1)) + 2(k-1) + (k-1)(3n_2 + t_{T_2}(v_2)) \\
&\quad + 3(k-1) + 4(k-1).
\end{aligned} \tag{2.4}$$

Since $n_1 = un_2 - 4, t_{T_2}(v_2) = \omega_2 = \lambda_2 n_2^2 + \lambda_1 n_2 + \lambda_0, t_{T_1}(v_1) = \omega_1 = \frac{n_1^2}{4}$, then we have:

$$\begin{aligned}
W(G) - W(G') &= -(n_2 n_1 + 2(1 + (k-1)(n_1 + n_2 + 3))n_2) + n_2 + t_{T_2}(v_2) + n_1 + t_{T_1}(v_1) + 4 + 1 \\
&\quad + (k-1)(2n_1 + t_{T_1}(v_1)) + 2(k-1) + (k-1)(3n_2 + t_{T_2}(v_2)) \\
&\quad + 3(k-1) + 4(k-1) \\
&= (\lambda_2 k + \frac{u^2 k}{4} - 2ku + u - 2k + 2)n_2^2 + (\lambda_1 k + 5k - 2 - u)n_2 + \lambda_0 k + 5k.
\end{aligned} \tag{2.5}$$

Since $\lambda_2 = \frac{8uk-4u+8k-8-ku^2}{4k}, \lambda_1 = \frac{-5k+2+u}{k}, \lambda_0 = -5$, therefore, we have:

$$\begin{aligned}
W(G) - W(G') &= (\lambda_2 k + \frac{u^2 k}{4} - 2ku + u - 2k + 2)n_2^2 + (\lambda_1 k + 5k - 2 - u)n_2 + \lambda_0 k + 5k \\
&= 0.
\end{aligned} \tag{2.6}$$

□

For example, let $b = 13, a = 2, u = \frac{13}{2}, n_1 = 2076, n_2 = 320, \omega_1 = 1077444, \omega_2 = 18955$. Let $P_{607} = v_1 v_2 \ldots v_{607}$ be a path with 607 vertices, $P_{65} = u_1 u_2 \ldots u_{65}$ be a path with 66 vertices. Let S_{1234} be a (1234)-star centered at v_{608} and S_{236} be a (236)-star centered at u_{66}. Now we obtain T_1 by adding v_{608} to v_{607} and we obtain T_2 by adding u_{66} to u_{65}.

Let $G = G(T_1, T_2, 2, v_1, u_1)$, $G' = G - \{w_1\}$.

$$\begin{aligned}W(G)-W(G') =& \sum_{u,v\in V(G-\{w_1\}),u\neq v}(d(u,v|G)-d(u,v|G'))+t_G(w_1)\\
=& \sum_{u,v\in V(T_2^1),u\neq v}(d(u,v|G)-d(u,v|G'))+\sum_{u,v\in V(G')-V(T_2^1),u\neq v}(d(u,v|G)-d(u,v|G'))\\
&+\sum_{u\in V(T_2^1),v\in V(G')-V(T_2^1)}(d(u,v|G)-d(u,v|G'))+t_G(w_1)\\
=& \sum_{u\in V(T_2^1),v\in V(G')-V(T_2^1)}(d(u,v|G)-d(u,v|G'))+t_G(w_1)\\
=& -(n_2n_1+2(1+(k-1)(n_1+n_2+3))n_2)+n_2+t_{T_2}(v_2)+n_1+t_{T_1}(v_1)+4+1\\
&+(k-1)(2n_1+t_{T_1}(v_1))+2(k-1)+(k-1)(3n_2+t_{T_2}(v_2))\\
&+3(k-1)+4(k-1)\\
=&0.\end{aligned} \tag{2.7}$$

Theorem 2. *For any $k>0$, there are infinitely many graphs G for which $W(G)=W(G-\{w_1\})=W(G-\{w_2\})=\ldots=W(G-\{w_k\})$ for some distinct vertices $w_1,\ldots,w_k\in V(G)$.*

Proof. By Remark 1, there exist rational number $u=\frac{a}{b}$ with $0<\frac{8uk-4u+8k-8-ku^2}{4k}<\frac{1}{3}$ and an integer n_2 with $n_2 \pmod{4ka^2}\equiv 0$ such that $0<\omega_2=\lambda_2n_2^2+\lambda_1n_2+\lambda_0<\frac{n_2(n_2-1)}{2}, 0<\omega_1=\frac{n_1^2}{4}<\frac{n_1(n_1-1)}{2}$ where $n_1=un_2-4$, and $\lambda_2=\frac{8uk-4u+8k-8-ku^2}{4k}=\frac{8bka-4ba+8ka^2-8a^2+kb^2}{4ka^2}, \lambda_1=\frac{-5k+2+u}{k}=\frac{-5ka+2a+b}{ak}, \lambda_0=-5$. Since $n_1,n_2,\omega_1,\omega_2$ all are integers such that $0<\omega_2<\frac{n_2(n_2-1)}{2}$ and $0<\omega_1<\frac{n_1(n_1-1)}{2}$, therefore there exist two trees T_1,T_2 with $|V(T_1)|=n_1, t_{T_1}(v_1)=\omega_1, |V(T_2)|=n_2, t_{T_2}(v_2)=\omega_2$. By Lemma 6, there is a graph $G=G(T_1,T_2,k,v_1,v_2)$ with $W(G)=W(G-\{w_1\})=\ldots=W(G-\{w_k\})$. Let $G_i=G-\{w_i\}$. Clearly, we have $t_{G_1}(u)=t_{G_2}(u)=\ldots=t_{G_k}(u)$ and $t_{G_i}(u)-t_G(u)\geq 1$.

Clearly, there is an integer k such that $\frac{(1+rk)r}{2}\geq t_{G_1}(u)-t_G(u)$ for any r. Now by Lemma 5, there is a tree T with $|V(T)|=rk+1, t_T(v)=(t_{G_1}(u)-t_G(u))k$, where $v\in V(T)$ and $r=1$. Now we obtain a graph G^2 by identifying u with v from G,T. Let $n_G=|V(G)|, n_T=|V(T)|$. Since $W(G^2)=W(G)+W(T)+(n_G-1)t_T(v)+(n_T-1)t_G(u), W(G^2-\{w_i\})=W(G_i)+W(T)+(n_G-2)t_T(v)+(n_T-1)t_{G_i}(u)$, then we have:

$$\begin{aligned}W(G^2)-W(G^2-\{w_i\}) =&W(G)+W(T)+(n_G-1)t_T(v)+(n_T-1)t_G(u)\\
&-(W(G_i)+W(T)+(n_G-2)t_T(v)+(n_T-1)t_{G_i}(u))\\
=&t_T(v)+(n_T-1)(t_G(u)-t_{G_i}(u))\\
=&(t_{G_1}(v)-t_G(u))k+k(t_G(u)-t_{G_i}(u))\\
=&(t_{G_i}(v)-t_G(u))k+k(t_G(u)-t_{G_i}(u))\\
=&0.\end{aligned} \tag{2.8}$$

Therefore, $W(G^2)=W(G^2-\{w_1\})=\ldots=W(G^2-\{w_k\})$. Let $G_i^2=G^2-\{w_i\}$, then we have $t_{G_1^2}(u)=t_{G_2^2}(u)=\ldots=t_{G_k^2}(u)$. Let T_4 be a tree with $|V(T_4)|=rk_2+1, t_T(v)=(t_{G_1^2}(u)-t_{G^2}(u))k_2$ where $v\in V(T_4)$ and $r=1$. Now we obtain a graph G^3 by identifying u with v from G^2,T_4. Similarly, we have $W(G^3)=W(G^3-\{w_1\})=\ldots=W(G^3-\{w_k\})$. Similarly, we can obtain $G^4,G^5,\ldots$. Then the proof is completed. □

3. Conclusions

Topological index is a mathematical quantity which is assigned to a graph in order to develop relationships between a graph (or structure of a molecule) and some properties including biological activity, physical properties or chemical reactivity. Due to vast applications in several branches of science, the Wiener index has remained one of the most frequently studied topological index both in pure and applied mathematics. In this paper, we are able to contribute to this topic by means of a study related to the Šoltés problem. In particular, we have solved the problem of finding infinite family of graphs G such that for each G there exist distinct vertices $w_1, \ldots, w_k \in V(G)$ satisfying $W(G) = W(G - \{w_1\}) = W(G - \{w_2\}) = \ldots = W(G - \{w_k\})$. The problem was posed by Knor et al. [4] in 2018 during the study related to Šoltés problem. The solution presented in this paper may be a step forward toward the solution of Šoltés problem and may be used by other mathematicians working in this area.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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