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Research article

On lightlike geometry of indefinite Sasakian statistical manifolds

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Abstract: In the present study, the concept of Sasakian statistical manifold has been generalized to indefinite Sasakian statistical manifolds. We also introduce lightlike hypersurfaces of an indefinite Sasakian statistical manifold and establish relations between induced geometrical objects with respect to dual connections. Finally, invariant lightlike submanifold of indefinite Sasakian statistical manifold is proved to be an indefinite Sasakian statistical manifold.

Keywords: lightlike hypersurface; statistical manifolds; dual connections; indefinite Sasakian; indefinite Sasakian statistical manifold
Mathematics Subject Classification: 53C15, 53C25, 53C40

1. Introduction

Neural networks is useful for solving numerous complex optimization problems in electromagnetic theory. Applied physicist B. Bartlett presented unsupervised machine learning model for computing approximate electromagnetic field solutions [7]. In April 2019, the Event Horizon Telescope (EHT) collaboration released the first image of the shadow of a black hole with the help of deep learning algorithms. This image provides direct evidence for the existence of black holes and general theory of relativity and indirect evidence for the existence of lightlike geometry in the universe [19]. A statistical manifold is emerging branch of mathematics that generalizes the Riemannian manifold and is used to model the information; and also uses tools of differential geometry to study statistical inference, information loss and estimation [9]. Statistical manifolds are applicable to many areas such as neural networks, machine learning and artificial intelligence. On the other hand, the study of lightlike manifolds is one of the most important research areas in differential geometry with many applications in physics and mathematics, such as general relativity, electromagnetism and black hole theory (Please see [7, 8, 10, 11, 19]).

There exist qualified papers dealing with statistical manifolds and their submanifolds admitting various differentiable structures. In 1975, Efron [14] was the first researcher who emphasized the role of differential geometry in statistics and was further studied with the help of differential geometrical tools by Amari [1,2]. In 1989, Vos [30] obtained fundamental equations for submanifolds of statistical manifolds. Takano define a Sasaki-like statistical manifold and he study Sasaki-like statistical submersion with the property that the curvature tensor with respect to the affine connection of the total space satisfies the condition [28]. In [29], the authors define the concept of quaternionic Kahler-like statistical manifold and derive the main properties of quaternionic Kahler-like statistical submersions, extending in a new setting some previous results obtained by Takano concerning statistical manifolds endowed with almost complex and almost contact structures. The geometry of hypersurfaces of statistical manifolds was presented by Furuhata in [15, 16]. Statistical manifolds admitting contact structures or complex structures and submanifolds of these kinds of manifolds were investigated in [18, 24]. In degenerate case, lightlike hypersurfaces of statistical manifolds were introduced by the Bahadir and Tripathi [6]. Also, statistical lightlike hypersurfaces were studied by Jain, Singh and Kumar in [20]. In addition, many studies have been conducted in relation to these concepts [3–5, 21, 22, 25, 26].

Motivated by these developments, we dedicate the present study to introduce the lightlike geometry of an indefinite Sasakian statistical manifold. In Section 2, we present basic definitions and results about statistical manifolds and lightlike hypersurfaces. In Section 3, we show that the induced connections on a lightlike hypersurface of a statistical manifold need not be dual and a lightlike hypersurface need not be a statistical manifold. Moreover, we show that the second fundamental forms are not degenerate. We conclude the section with an example. In Section 4, we defined indefinite Sasakian statistical manifolds and obtain the characterization theorem for indefinite Sasakian statistical manifolds. This section is concluded with two examples. In Section 5, we consider lightlike hypersurfaces of indefinite Sasakian statistical manifolds. We characterize the parallelness, totaly geodeticity and integrability of some distributions. In this section we also give two examples. In Section 6, we prove that an invariant lightlike submanifold of indefinite Sasakian statistical manifold is an indefinite Sasakian statistical manifold.

2. Preliminaries

Let $(\widetilde{M}, \widetilde{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold with index$(\widetilde{g}) = q \geq 1$ and (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g})$ with the induced metric g from $\widetilde{g}$. If the induced metric g on M is degenerate, then M is called a lightlike (null or degenerate) hypersurface.

For a lightlike hypersurface (M, g) of $(\widetilde{M}, \widetilde{g})$, there exists a non-zero vector field ξ on M such that

$$g(\xi, X) = 0, \qquad X \in \Gamma(TM), \tag{2.1}$$

Here the vector field ξ is called a null vector ([11–13]). The radical or the null space Rad T_xM at each point $x \in M$ is defined as

$$\text{Rad } T_xM = \{\xi \in T_xM : g_x(\xi, X) = 0,\ \forall X \in \Gamma(TM)\}. \tag{2.2}$$

The dimension of Rad T_xM is called the nullity degree of g. We recall that the nullity degree of g for a lightlike hypersurface is equal to 1. Since g is degenerate and any null vector being orthogonal to itself, the normal space $T_xM^{\perp}$ is a null subspace. Also, we have

$$\text{Rad } T_xM = T_xM^{\perp}. \tag{2.3}$$

The complementary vector bundle $S(TM)$ of Rad TM in TM is called the screen bundle of M. We note that any screen bundle is non-degenerate. Therefore we can write the following decomposition:

$$TM = \text{Rad}\, TM \perp S(TM). \tag{2.4}$$

Here $\perp$ denotes the orthogonal-direct sum. The complementary vector bundle $S(TM)^{\perp}$ of $S(TM)$ in $T\widetilde{M}$ is called the screen transversal bundle. Since Rad TM is a lightlike subbundle of $S(TM)^{\perp}$, there exists a unique local section N of $S(TM)^{\perp}$ such that we have

$$\widetilde{g}(N, N) = 0, \quad \widetilde{g}(\xi, N) = 1. \tag{2.5}$$

Note that N is transversal to M and $\{\xi, N\}$ is a local frame field of $S(TM)^{\perp}$ and there exists a line subbundle ltr(TM) of $T\widetilde{M}$. This set is called the lightlike transversal bundle, locally spanned by N. Hence we have the following decomposition:

$$T\widetilde{M} = TM \oplus ltr(TM) = S(TM) \perp Rad\, TM \oplus ltr(TM), \tag{2.6}$$

where $\oplus$ is the direct sum but not orthogonal ([11, 12]).

For details of lightlike submanifolds, we may refer to [8, 11–13].

Definition 1. [28] A statistical manifold is a triple $(\widetilde{M}, \widetilde{g}, \widetilde{D})$ formed of a semi-Riemannian manifold and a torsion free connection $\widetilde{D}$ subject to the following identity
$(\widetilde{D}_X\widetilde{g})(Y, Z) = (\widetilde{D}_Y\widetilde{g})(X, Z)$, for all $X, Y, Z \in \Gamma(T\widetilde{M})$.

Given statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D})$ the $\widetilde{g}-$ dual of $\widetilde{D}$, $\widetilde{D}^*$ is defined by the following identity

$$\widetilde{g}(X, \widetilde{D}^*_Z Y) = Z\widetilde{g}(X, Y) - \widetilde{g}(\widetilde{D}_Z X, Y). \tag{2.7}$$

It is easy to check that $\widetilde{D}^*$ is torsion free and $(\widetilde{M}, \widetilde{g}, \widetilde{D}^*)$ is a statistical manifold. Also, an equivalent of Definition 1 according to definition D^* is as follows:

Definition 2. A statistical manifold is a quadruple $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$ formed of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and a pair of torsion free connection $(\widetilde{D}, \widetilde{D}^*)$ subject the following identity
$Z\widetilde{g}(X, Y) = \widetilde{g}(X, \widetilde{D}^*_Z Y) + \widetilde{g}(\widetilde{D}_Z X, Y)$, for all $X, Y, Z \in \Gamma(T\widetilde{M})$.
A statistical manifold will be represented by $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$.

If $\widetilde{D}^0$ is Levi-Civita connection of $\widetilde{g}$, then we have

$$\widetilde{D}^0 = \frac{1}{2}(\widetilde{D} + \widetilde{D}^*). \tag{2.8}$$

If we choose $\widetilde{D}^* = \widetilde{D}$ in the Eq (2.8), then Levi-Civita connection is obtained.

Lemma 2.1. *For statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$, if we set*

$$\overline{\mathbb{K}} = \widetilde{D} - \widetilde{D}^0. \tag{2.9}$$

Then we have

$$\overline{\mathbb{K}}(X, Y) = \overline{\mathbb{K}}(Y, X), \; \widetilde{g}(\overline{\mathbb{K}}(X, Y), Z) = \widetilde{g}(\overline{\mathbb{K}}(X, Z), Y), \tag{2.10}$$

for any $X, Y, Z \in \Gamma(TM)$.

Conversely, for a Riemannian metric g, if $\overline{\mathbb{K}}$ satisfies (2.10), the pair $(\widetilde{D} = \widetilde{\nabla} + \overline{\mathbb{K}}, \widetilde{g})$ is a statistical structure on $\widetilde{M}$ [18].

3. Lightlike hypersurface of a statistical manifold

Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then, Gauss and Weingarten formulas with respect to dual connections are given as follows:

$$\widetilde{D}_X Y = D_X Y + B(X, Y)N, \tag{3.1}$$

$$\widetilde{D}_X N = -A_N X + \tau(X)N \tag{3.2}$$

$$\widetilde{D}^*_X Y = D^*_X Y + B^*(X, Y)N, \tag{3.3}$$

$$\widetilde{D}^*_X N = -A^*_N X + \tau^*(X)N, \tag{3.4}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltrTM)$, where $D_X Y$, $D^*_X Y$, $A_N X$, $A^*_N X \in \Gamma(TM)$ and

$$B(X, Y) = \widetilde{g}(\widetilde{D}_X Y, \xi), \quad \tau(X) = \widetilde{g}(\widetilde{D}_X N, \xi),$$

$$B^*(X, Y) = \widetilde{g}(\widetilde{D}^*_X Y, \xi), \quad \tau^*(X) = \widetilde{g}(\widetilde{D}^*_X N, \xi).$$

Here, D, D^*, B, B^*, A_N and A^*_N are called the induced connections on M, the second fundamental forms and the Weingarten mappings with respect to $\widetilde{D}$ and $\widetilde{D}^*$, respectively ([6, 15]). Using Gauss formulas and the Eq (2.8), we obtain

$$\begin{aligned} Xg(Y, Z) &= g(\widetilde{D}_X Y, Z) + g(Y, \widetilde{D}^*_X Z), \\ &= g(D_X Y, Z) + g(Y, D^*_X Z) + B(X, Y)\eta(Z) + B^*(X, Z)\eta(Y). \end{aligned} \tag{3.5}$$

From the Eq (3.5), we have the following result.

Theorem 3.1. *[6] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then we have the following expressions*:

- **(i)** *Induced connections D and D^* need not be dual.*
- **(ii)** *A lightlike hypersurface of a statistical manifold need not be a statistical manifold with respect to the dual connections.*

Using Gauss and Weingarten formulas in (3.5), we get

$$\begin{aligned} (D_X g)(Y, Z) + (D^*_X g)(Y, Z) &= B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \\ &\quad + B^*(X, Y)\eta(Z) + B^*(X, Z)\eta(Y). \end{aligned} \tag{3.6}$$

Proposition 3.2. *[6] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the following assertions are true*:

- **(i)** *Induced connections D and D^* are symmetric connection.*
- **(ii)** *The second fundamental forms B and B^* are symmetric.*

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.4). For all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we have

$$D_X PY = \nabla_X PY + \overline{h}(X, PY), \tag{3.7}$$

$$D_X \xi = -\overline{A}_\xi X + \overline{\nabla}^t_X \xi, \tag{3.8}$$

where $\nabla_X PY$ and $\overline{A}_\xi X$ belong to $\Gamma(S(TM))$. Also we can say that ∇ and $\overline{\nabla}^t$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. Here, $\overline{h}$ and $\overline{A}$ are called screen second fundamental form and screen shape operator of $S(TM)$, respectively. If we define

$$C(X, PY) = g(\overline{h}(X, PY), N), \tag{3.9}$$

$$\varepsilon(X) = g(\overline{\nabla}^t_X \xi, N), \ \forall X, Y \in \Gamma(TM). \tag{3.10}$$

One can show that

$$\varepsilon(X) = -\tau(X).$$

Therefore, we have

$$D_X PY = \nabla_X PY + C(X, PY)\xi, \tag{3.11}$$

$$D_X \xi = -\overline{A}_\xi X - \tau(X)\xi, \ \forall X, Y \in \Gamma(TM). \tag{3.12}$$

Here $C(X, PY)$ is called the local screen fundamental form of $S(TM)$.

Similarly, the relations of induced dual objects on $S(TM)$ are given by

$$D^*_X PY = \nabla^*_X PY + C^*(X, PY)\xi, \tag{3.13}$$

$$D^*_X \xi = -\overline{A}^*_\xi X - \tau^*(X)\xi, \ \forall X, Y \in \Gamma(TM). \tag{3.14}$$

Using (3.5), (3.11), (3.13) and Gauss-Weingarten formulas, the relationship between induced geometric objects are given by

$$B(X, \xi) + B^*(X, \xi) = 0, \ g(A_N X + A^*_N X, N) = 0, \tag{3.15}$$

$$C(X, PY) = g(A^*_N X, PY), \ C^*(X, PY) = g(A_N X, PY). \tag{3.16}$$

Now, using the Eq (3.15), we can state the following result.

Proposition 3.3. *[6] Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then second fundamental forms B and B^* are not degenerate.*

Additionally, due to $\widetilde{D}$ and $\widetilde{D}^*$ are dual connections, we obtain

$$B(X, Y) = g(\overline{A}^*_\xi X, Y) + B^*(X, \xi)\eta(Y), \tag{3.17}$$

$$B^*(X, Y) = g(\overline{A}_\xi X, Y) + B(X, \xi)\eta(Y). \tag{3.18}$$

Using (3.17) and (3.18) we get

$$\overline{A}^*_\xi \xi + \overline{A}_\xi \xi = 0.$$

Example 1. Let $(R_2^4, \widetilde{g})$ be a 4-dimensional semi-Euclidean space with signature $(-,-,+,+)$ of the canonical basis $(\partial_0, \dots, \partial_3)$. Consider a hypersurface M of R_2^4 given by

$$x_0 = x_1 + \sqrt{2}\sqrt{x_2^2 + x_3^2}.$$

For simplicity, we set $f = \sqrt{x_2^2 + x_3^2}$. It is easy to check that M is a lightlike hypersurface whose radical distribution $RadTM$ is spanned by

$$\xi = f(\partial_0 - \partial_1) + \sqrt{2}(x_2\partial_2 + x_3\partial_3).$$

Then the lightlike transversal vector bundle is given by

$$ltr(TM) = Span\left\{N = \frac{1}{4f^2}\left\{f(-\partial_0 + \partial_1) + \sqrt{2}(x_2\partial_2 + x_3\partial_3)\right\}\right\}.$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = \partial_0 + \partial_1,\ W_2 = -x_3\partial_2 + x_2\partial_3\}.$$

Then, by direct calculations we obtain

$$\widetilde{\nabla}_X W_1 = \widetilde{\nabla}_{W_1} X = 0,$$

$$\widetilde{\nabla}_{W_2} W_2 = -x_2\partial_2 - x_3\partial_3,$$

$$\widetilde{\nabla}_\xi \xi = \sqrt{2}\xi,\ \widetilde{\nabla}_{W_2}\xi = \widetilde{\nabla}_\xi W_2 = \sqrt{2}W_2,$$

for any $X \in \Gamma(TM)$ (see [13], Example 2, pp. 48–49).

We define an affine connection $\widetilde{D}$ as follows:

$$\begin{aligned} &\widetilde{D}_X W_1 = \widetilde{D}_{W_1} X = 0,\ \widetilde{D}_{W_2} W_2 = -2x_2\partial_2 \\ &\widetilde{D}_\xi \xi = \sqrt{2}\xi, \\ &\widetilde{D}_{W_2}\xi = \widetilde{D}_\xi W_2 = \sqrt{2}W_2. \end{aligned} \tag{3.19}$$

Then we obtain

$$\begin{aligned} &\widetilde{D}^*_X W_1 = \widetilde{D}^*_{W_1} X = 0,\ \widetilde{D}^*_{W_2} W_2 = -2x_3\partial_3, \\ &\widetilde{D}^*_\xi \xi = \sqrt{2}\xi, \\ &\widetilde{D}^*_{W_2}\xi = \widetilde{D}^*_\xi W_2 = \sqrt{2}W_2. \end{aligned} \tag{3.20}$$

Then $\widetilde{D}$ and $\widetilde{D}^*$ are dual connections. Here, one can easily see that $T^{\widetilde{D}} = 0$ and $\widetilde{D}\widetilde{g} = 0$. Thus, we can easily see that $(R_2^4, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$ is a statistical manifold.

4. Indefinite Sasakian statistical manifolds

In order to call a differentiable semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension $n = 2m + 1$ as practically contact metric one, a $(1, 1)$ tensor field $\widetilde{\varphi}$, a contravariant vector field ν, a $1-$ form η and a Riemannian metric $\widetilde{g}$ should be admitted, which satisfy

$$\widetilde{\varphi}\nu = 0, \ \eta(\widetilde{\varphi}X) = 0, \ \eta(\nu) = \epsilon, \tag{4.1}$$
$$\widetilde{\varphi}^2(X) = -X + \eta(X)\nu, \ \widetilde{g}(X, \nu) = \epsilon\eta(X), \tag{4.2}$$
$$\widetilde{g}(\widetilde{\varphi}X, \widetilde{\varphi}Y) = \widetilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \ \epsilon = \mp 1, \tag{4.3}$$

for all the vector fields X, Y on $\widetilde{M}$. When a practically contact metric manifold performs

$$(\widetilde{\nabla}_X\widetilde{\varphi})Y \ = \ \widetilde{g}(X, Y)\nu - \epsilon\eta(Y)X, \tag{4.4}$$
$$\widetilde{\nabla}_X\nu \ = \ -\widetilde{\varphi}X, \tag{4.5}$$

$\widetilde{M}$ is regarded as an indefinite Sasakian manifold. In this study, we assume that the vector field ν is spacelike.

Definition 3. Let $(\widetilde{g}, \widetilde{\varphi}, \nu)$ be an indefinite Sasakian structure on $\widetilde{M}$. A quadruplet $(\widetilde{D} = \widetilde{\nabla} + \overline{\mathbb{K}}, \widetilde{g}, \widetilde{\varphi}, \nu)$ is called a indefinite Sasakian statistical structure on $\widetilde{M}$ if $(\widetilde{D}, \widetilde{g})$ is a statistical structure on $\widetilde{M}$ and the formula

$$\overline{\mathbb{K}}(X, \widetilde{\varphi}Y) = -\widetilde{\varphi}\overline{\mathbb{K}}(X, Y) \tag{4.6}$$

holds for any $X, Y \in \Gamma(T\widetilde{M})$. Then $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$ is said to an indefinite Sasakian statistical manifold.

An indefinite Sasakian statistical manifold will be represented by $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. We remark that if $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$ is an indefinite Sasakian statistical manifold, so is $(\widetilde{M}, \widetilde{D}^*, \widetilde{g}, \varphi, \nu)$ [17, 18].

Theorem 4.1. *Let $(\widetilde{M}, \widetilde{D}, \widetilde{g})$ be a statistical manifold and $(\widetilde{g}, \widetilde{\varphi}, \nu)$ an almost contact metric structure on $\widetilde{M}$. $(\widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$ is an indefinite Sasakian statistical struture if and only if the following conditions hold:*

$$\widetilde{D}_X\varphi Y - \widetilde{\varphi}\widetilde{D}^*_X Y = \widetilde{g}(Y, X)\nu - \widetilde{g}(Y, \nu)X, \tag{4.7}$$
$$\widetilde{D}_X\nu = -\widetilde{\varphi}X + g(\widetilde{D}_X\nu, \nu)\nu, \tag{4.8}$$

for all the vector fields X, Y on $\widetilde{M}$.
Proof. *Using (2.9) we get*

$$\widetilde{D}_X\widetilde{\varphi}Y - \widetilde{\varphi}\widetilde{D}^*_X Y = (\widetilde{\nabla}_X\widetilde{\varphi})Y + \overline{\mathbb{K}}(X, \widetilde{\varphi}Y) + \widetilde{\varphi}\overline{\mathbb{K}}(X, Y), \tag{4.9}$$

for all the vector fields X, Y on $\widetilde{M}$. If we consider Definition 3 and the Eq (4.4), we have the formula (4.7). If we write $\widetilde{D}^*$ instead of $\widetilde{D}$ in (4.7), we have

$$\widetilde{D}^*_X\widetilde{\varphi}Y - \widetilde{\varphi}\widetilde{D}_X Y = \widetilde{g}(Y, X)\nu - \widetilde{g}(Y, \nu)X, \tag{4.10}$$

Substituting ν for Y in (4.10), we have the Eq (4.8).

Conversely using (4.7), we obtain

$$\widetilde{\varphi}\{\widetilde{D}_X\widetilde{\varphi}^2Y - \widetilde{\varphi}\widetilde{D}^*_X\widetilde{\varphi}Y\} = 0.$$

Assume (4.2) and (4.8) as well, we get

$$0 = -\widetilde{\varphi}\widetilde{D}_XY + \widetilde{g}(Y,\nu)X - \widetilde{g}(X,\nu)\widetilde{g}(Y,\nu)\nu + \widetilde{D}^*_X\widetilde{\varphi}Y - \widetilde{g}(\widetilde{\varphi}X,\widetilde{\varphi}Y)\nu,$$

From (4.3), we have the Eq (4.10).

Now, using (4.7) and (4.10), respectively, we have the following equations:

$$(\widetilde{\nabla}_X\widetilde{\varphi})Y - \widetilde{g}(Y,X)\nu + \widetilde{g}(Y,\nu)X = \overline{\mathbb{K}}(X,\widetilde{\varphi}Y) + \widetilde{\varphi}\overline{\mathbb{K}}(X,Y),$$

and

$$(\widetilde{\nabla}_X\widetilde{\varphi})Y - \widetilde{g}(Y,X)\nu + \widetilde{g}(Y,\nu)X = -\overline{\mathbb{K}}(X,\widetilde{\varphi}Y) - \widetilde{\varphi}\overline{\mathbb{K}}(X,Y).$$

This last two equations verifies (4.4) and (4.6).

Example 2. Let $\widetilde{M} = (R^5_2, \widetilde{g})$ be a semi-Euclidean space, where $\widetilde{g}$ is of the signature $(-,+,-,+,+)$ with respect to canonical basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z}\}$. Defining

$$\eta = dz,\ \nu = \frac{\partial}{\partial z},$$
$$\widetilde{\varphi}\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i},\ \widetilde{\varphi}\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i},\ \widetilde{\varphi}\left(\frac{\partial}{\partial z}\right) = 0,$$

where $i = 1, 2$. It can easily see that $(\widetilde{\varphi}, \nu, \eta, \widetilde{g})$ is an indefinite Sasakian structure on R^5_2. If we choose $\overline{\mathbb{K}}(X,Y) = \widetilde{g}(Y,\nu)\widetilde{g}(X,\nu)\nu$, then $(\widetilde{D} = \widetilde{\nabla} + \overline{\mathbb{K}}, \widetilde{g}, \widetilde{\varphi}, \nu)$ is an indefinite Sasakian statistical structure on $\widetilde{M}$.

Example 3. In a $5-$ dimensional real number space $\widetilde{M} = R^5$, let $\{x_i, y_i, z\}_{1\le i\le 2}$ be cartesian coordinates on $\widetilde{M}$ and $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z}\}_{1\le i\le 2}$ be the natural field of frames. If we define $1-$ form η, a vector field ν and a tensor field $\widetilde{\varphi}$ as follows:

$$\eta = dz - y_1dx_1 - x_1dy_1, \quad \nu = \frac{\partial}{\partial z},$$
$$\widetilde{\varphi}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_2},\ \widetilde{\varphi}(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z},\ \widetilde{\varphi}(\frac{\partial}{\partial y_1}) = -\frac{\partial}{\partial y_2},$$
$$\widetilde{\varphi}(\frac{\partial}{\partial y_2}) = \frac{\partial}{\partial y_1} + x_1\frac{\partial}{\partial z},\ \widetilde{\varphi}(\frac{\partial}{\partial z}) = 0.$$

It is easy to check (4.1) and (4.2). Then, $(\widetilde{\varphi}, \nu, \eta)$ is an almost contact structure on R^5. Now, we define metric $\widetilde{g}$ on R^5 by

$$\begin{aligned}\widetilde{g} &= (y_1^2 - 1)dx_1^2 - dx_2^2 + (x_1^2 + 1)dy_1^2 + dy_2^2 + dz^2 - y_1dx_1 \otimes dz - y_1dz \otimes dx_1 \\ &+ x_1y_1dx_1 \otimes dy_1 + x_1y_1dy_1 \otimes dx_1 - x_1dy_1 \otimes dz - x_1dz \otimes dy_1,\end{aligned}$$

with respect to the natural field of frames. Then we can easily see that $(\widetilde{\varphi}, \nu, \eta, \widetilde{g})$ is an indefinite Sasakian structure on R^5. We set the difference tensor field $\overline{\mathbb{K}}$ as

$$\overline{\mathbb{K}}(X,Y) = \lambda\widetilde{g}(Y,\nu)\widetilde{g}(X,\nu)\nu,$$

where $\lambda \in C^\infty(\widetilde{M})$. Then, $(\widetilde{D} = \widetilde{\nabla} + \overline{\mathbb{K}}, \widetilde{g}, \widetilde{\varphi}, \nu)$ is an indefinite Sasakian statistical structure on $\widetilde{M}$.

5. Lightlike hypersurfaces of indefinite Sasakian statistical manifolds

Definition 4. Let (M, g, D, D^*) be a hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. The quadruplet (M, g, D, D^*) is called lightlike hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$ if the induced metric g is degenerate.

Let $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$ be a $(2m+1)-$ dimensional Sasakian statistical manifold and (M, g) be a lightlike hypersurface of $\widetilde{M}$, such that the structure vector field ν is tangent to M. For any $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltrTM)$, in view of (4.1)–(4.3), we have

$$\widetilde{g}(\xi, \nu) = 0, \ \widetilde{g}(N, \nu) = 0, \tag{5.1}$$

$$\widetilde{\varphi}^2\xi = -\xi, \ \widetilde{\varphi}^2 N = -N. \tag{5.2}$$

Also using (3.1) and (4.8) we obtain

$$B(\xi, \nu) = 0, \ B(\nu, \nu) = 0, \tag{5.3}$$

$$B^*(\xi, \nu) = 0, \ B^*(\nu, \nu) = 0. \tag{5.4}$$

Proposition 5.1. *Let $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$ be a $(2m + 1)-$ dimensional Sasakian statistical manifold and (M, g, D, D^*) be its lightlike hypersurface such that the structure vector field ν is tangent to M. Then we have*

$$g(\widetilde{\varphi}\xi, \xi) = 0, \tag{5.5}$$

$$g(\widetilde{\varphi}\xi, N) = -g(\xi, \widetilde{\varphi}N) = -g(A^*_N\xi, \nu), \tag{5.6}$$

$$g(\widetilde{\varphi}\xi, \widetilde{\varphi}N) = 1, \tag{5.7}$$

where ξ is a local section of $RadTM$ and N is a local section of $ltrTM$.
Proof. *Using (4.8) and (3.1), we have*

$$\begin{aligned} g(\widetilde{\varphi}\xi, \xi) &= g(-\widetilde{D}_\xi\nu + g(\widetilde{D}_\xi\nu, \nu)\nu, \xi) \\ &= g(-D_\xi\nu - B(\xi, \nu)N, \xi), \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} g(\widetilde{\varphi}\xi, N) &= g(-\widetilde{D}_\xi\nu + g(\widetilde{D}_\xi\nu, \nu)\nu, N) \\ &= g(\nu, \widetilde{D}^*_\xi N), \\ &= -g(A^*_N\xi, \nu). \end{aligned}$$

From (4.3) and (5.1), we have (5.7).

Proposition 5.1 makes it possible to make the following decompositions:

$$S(TM) = \{\widetilde{\varphi}RadTM \oplus \widetilde{\varphi}ltr(TM)\}\perp L_0 \perp \langle\nu\rangle, \tag{5.8}$$

where L_0 is non-degenerate and $\widetilde{\varphi}-$ invariant distribution of rank $2m-4$ on M. If we denote the following distributions on M

$$L = RadTM \perp \widetilde{\varphi}RadTM \perp L_0, \; L^{'} = \widetilde{\varphi}ltr(TM), \tag{5.9}$$

then L is invariant and $L^{'}$ is anti-invariant distributions under $\widetilde{\varphi}$. Also we have

$$TM = L \oplus L^{'} \perp \langle \nu \rangle. \tag{5.10}$$

Now, we consider two null vector field U and W and their $1-$ forms u and w as follows:

$$U = -\widetilde{\varphi}N, \; u(X) = \widetilde{g}(X, W), \tag{5.11}$$

$$W = -\widetilde{\varphi}\xi, \; w(X) = \widetilde{g}(X, U). \tag{5.12}$$

Then, for any $X \in \Gamma(T\widetilde{M})$, we have

$$X = SX + u(X)U, \tag{5.13}$$

where S projection morphism of $T\widetilde{M}$ on the distribution L. Applying $\widetilde{\varphi}$ to last equation, we obtain

$$\begin{aligned} \widetilde{\varphi}X &= \widetilde{\varphi}SX + u(X)\widetilde{\varphi}U, \\ \widetilde{\varphi}X &= \varphi X + u(X)N, \end{aligned} \tag{5.14}$$

where φ is a tensor field of type $(1,1)$ defined on M by $\varphi X = \widetilde{\varphi}SX$.

Again, we apply $\widetilde{\varphi}$ to (5.14) and using (4.1)–(4.3) we have

$$\begin{aligned} \widetilde{\varphi}^2 X &= \widetilde{\varphi}\varphi X + u(X)\widetilde{\varphi}N, \\ -X + g(X,\nu)\nu &= \varphi^2 X - u(X)U. \end{aligned}$$

which means that

$$\varphi^2 X = -X + g(X,\nu)\nu + u(X)U. \tag{5.15}$$

Now applying φ to the Eq (5.15) and since $\varphi U = 0$, we have $\varphi^3 + \varphi = 0$ which gives that φ is an f–structure.

Definition 5. Let (M, g, D, D^*) be a hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. The quadruplet (M, g, D, D^*) is called screen semi-invariant lightlike hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$ if

$$\widetilde{\varphi}(ltrTM) \subset S(TM),$$
$$\widetilde{\varphi}(RadTM) \subset S(TM).$$

We remark that a hypersurface of indefinite Sasakian statistical manifold is screen semi-invariant lightlike hypersurface.

Example 4. Let us recall the Example 2. Suppose that M is a hypersurface of R^5_2 defined by

$$x_1 = y_2,$$

Then $RadTM$ and $ltr(TM)$ are spanned by $\xi = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2}$ and $N = \frac{1}{2}\{-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2}\}$, respectively. Applying $\widetilde{\varphi}$ to this vector fields, we have

$$\widetilde{\varphi}\xi = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_1},\ \widetilde{\varphi}N = \frac{1}{2}\left\{\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1}\right\}.$$

Thus M is a screen semi-invariant lightlike hyperfurface of indefinite Sasakian statistical manifold R^5_2.

Example 5. Let M be a hypersurface of $(\widetilde{\varphi}, \nu, \eta, \widetilde{g})$ on $\widetilde{M} = R^5$ in Example 3. Suppose that M is a hypersurface of R^5_2 defined by

$$x_2 = y_2,$$

Then the tangent space TM is spanned by $\{U_i\}_{1\leq i\leq 4}$, where $U_1 = \frac{\partial}{\partial x_1}$, $U_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}$, $U_3 = \frac{\partial}{\partial y_1}$, $U_4 = \nu$. $RadTM$ and $ltr(TM)$ are spanned by $\xi = U_2$ and $N = \frac{1}{2}\{-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}\}$, respectively. Applying $\widetilde{\varphi}$ to this vector fields, we have

$$\widetilde{\varphi}\xi = U_1 + U_3 + (x_1 + y_1)U_4,\ \widetilde{\varphi}N = \frac{1}{2}\{-U_1 + U_3 + (x_1 - y_1)U_4\}.$$

Thus M is a screen semi-invariant lightlike hyperfurface of indefinite Sasakian statistical manifold $\widetilde{M}$.

In view of (5.11) and (5.12), we have

$$\widetilde{g}(U, W) = 1.$$

Thus $\langle U\rangle \oplus \langle W\rangle$ is non-degenerate vector budle of $S(TM)$ with rank 2. If we consider (5.8) and (5.9), we get

$$S(TM) = \{U \oplus W\}\perp L_0 \perp \langle \nu\rangle, \tag{5.16}$$

and

$$L = RadTM \perp \langle W\rangle \perp L_0,\ L^{'} = \langle U\rangle. \tag{5.17}$$

Thus, for any $X \in \Gamma(TM)$, we can write

$$X = PX + QX + g(X, \nu)\nu, \tag{5.18}$$

where P and Q are projections of TM into L and $L^{'}$. Thus, we can write $QX = u(X)U$. Using (4.1)–(4.3), (5.14) and (5.18), we have

$$\varphi^2 X = -X + g(X, \nu)\nu + u(X)U,$$

where $\widetilde{\varphi}PX = \varphi X$. We can easily see that

$$g(\varphi X, \varphi Y) = g(X, Y) - g(X, \nu)g(Y, \nu) - u(X)w(Y) - u(Y)w(X), \tag{5.19}$$

for any $X, Y \in \Gamma(TM)$. Also we have the following identities:

$$g(\varphi X, Y) = g(X, \varphi Y) - u(X)\eta(Y) - u(Y)\eta(X), \tag{5.20}$$

$$\varphi\nu = 0, \ g(\varphi X, \nu) = 0. \tag{5.21}$$

Thus, we have the following proposition.

Proposition 5.2. *Let (M, g, D, D^*) be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. Then φ need not be an almost contact structure.*

Lemma 5.3. *Let (M, g, D, D^*) be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. For any $X, Y \in \Gamma(TM)$, we have the following identities:*

$$D_X\varphi Y - \varphi D^*_X Y = u(Y)A_N X - B^*(X, Y)U + g(X, Y)\nu - g(\nu, Y)X, \tag{5.22}$$

$$D_X(u(Y)) - u(D^*_X Y) = -B(X, \varphi Y) - u(Y)\tau(X) \tag{5.23}$$

Proof. *Using Gauss and Weingarten formulas in (4.7) we obtain*

$$D_X\varphi Y + B(X, \widetilde{\varphi}Y) + D_X(u(Y))N - u(Y)A_N X + u(Y)\tau(X)N - \varphi\nabla^*_X Y + B^*(X, Y)U = g(X, Y)\nu - g(\nu, Y)X. \tag{5.24}$$

If we take tangential and transversal parts of this last equation, we have (5.22) and (5.23).

Similarly, we have the following lemma.

Lemma 5.4. *Let (M, g, D, D^*) be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. For any $X, Y \in \Gamma(TM)$, we have the following identities:*

$$D^*_X\varphi Y - \varphi D_X Y = u(Y)A^*_N X - B(X, Y)U + g(X, Y)\nu - g(\nu, Y)X, \tag{5.25}$$

$$D^*_X(u(Y)) - u(D_X Y) = -B^*(X, \varphi Y) - u(Y)\tau^*(X). \tag{5.26}$$

Lemma 5.3 and Lemma 5.4 are give us the following theorem.

Theorem 5.5. *A lightlike hypersurface M of an indefinite Sasakian statistical manifold $\widetilde{M}$ need not be a statistical manifold.*

Proposition 5.6. *Let (M, g, D, D^*) be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. For any $X, Y \in \Gamma(TM)$, we have the following expressions:*
(i) *If the vector field U is parallel with respect to ∇^*, then we have*

$$A_N X = u(A_N X)U + \tau(A_N X)\nu. \ \tau(X) = 0. \tag{5.27}$$

(ii) *If the vector field U is parallel with respect to ∇, then we have*

$$A^*_N X = u(A^*_N X)U + \tau(A^*_N X)\nu. \ \tau^*(X) = 0. \tag{5.28}$$

Proof. *Replacing Y in (5.22) by U, we obtain*

$$-\varphi D^*_X Y = A_N X - B^*(X, U)U + g(X, U)\nu.$$

Applying φ to this equation and using (5.15), we get

$$D^*_X U - g(D^*_X U, \nu)\nu - u(D^*_X U)U = \varphi A_N X.$$

If U is parallel with respect to ∇^* then $\varphi A_N X = 0$. From (5.14), we have $\widetilde{\varphi}(A_N X) = u(A_N X)N$. If $\widetilde{\varphi}$ is applied to the last equation and using (4.2), we obtain $A_N X = u(A_N X)U + \tau(A_N X)\nu$. Also, if we write U instead of Y in the Eq (5.23), we have $\tau(X) = 0$.

We can easily obtain the Eq (5.28) with a similar method.

Proposition 5.7. *Let (M, g, D, D^*) be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. For any $X, Y \in \Gamma(TM)$, we have the following expressions:*
***(i)** If the vector field W is parallel with respect to ∇^*, then we have*

$$\overline{A}^*_\xi X = g(\overline{A}^*_\xi X, \nu)\nu + u(\overline{A}^*_\xi X)U, \ \tau^*(X) = 0. \tag{5.29}$$

***(ii)** If the vector field W is parallel with respect to ∇, then we have*

$$\overline{A}_\xi X = g(\overline{A}_\xi X, \nu)\nu + u(\overline{A}_\xi X)U, \ \tau(X) = 0. \tag{5.30}$$

Proof. *If we write ξ instead of Y in the Eq (5.22), we obtain*

$$D_X\varphi\xi - \varphi D^*_X \xi = -B^*(X, \xi)U.$$

If W is parallel with respect to D, using (3.14) and (5.12) in this equation, we obtain

$$\varphi\overline{A}^*_\xi X - \tau^*(X)W = -B^*(X, \xi)U.$$

Applying $\widetilde{\varphi}$ this and using (5.15) we have

$$-\overline{A}^*_\xi X + g(\overline{A}^*_\xi X, \nu)\nu + u(\overline{A}^*_\xi X)U = \tau^*(X)\xi.$$

If we take screen and radical parts of this last equation, we have (5.29).

Similarly, we can easily obtain the Eq (5.30).

Definition 6. ([17,23]) Let (M, g) be a hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$.

- **(i)** M is called *totally geodesic with respect to $\widetilde{D}$* if $B = 0$.
- **(ii)** M is called *totally geodesic with respect to $\widetilde{D}^*$* if $B^* = 0$.

Theorem 5.8. *Let (M, g, D, D^*) be a lightlike hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$.*
***(i)** M is totally geodesic with respect to $\widetilde{D}$ if and only if*

$$D_X\varphi Y - \varphi D^*_X Y = g(X, Y)\nu, \ \forall X \in \Gamma(TM), \ Y \in \Gamma(L), \tag{5.31}$$

$$A_N X = -\varphi D^*_X U - g(X, U)\nu, \ \forall X \in \Gamma(TM). \tag{5.32}$$

***(ii)** M is totally geodesic with respect to $\widetilde{D}^*$ if and only if*

$$D^*_X\varphi Y - \varphi D_X Y = g(X, Y)\nu, \ \forall X \in \Gamma(TM), \ Y \in \Gamma(L), \tag{5.33}$$

$$A^*_N X = -\varphi D_X U - g(X,U)\nu, \ \forall X \in \Gamma(TM). \tag{5.34}$$

Proof. *For any $Y \in \Gamma(L)$ we know that $u(Y) = 0$. Then the Eqs (5.22) and (5.25) are reduced to the equations, respectively*

$$D_X \varphi Y - \varphi D^*_X Y = -B^*(X,Y)U + g(X,Y)\nu, \tag{5.35}$$

$$D^*_X \varphi Y - \varphi D_X Y = -B(X,Y)U + g(X,Y)\nu. \tag{5.36}$$

On the other hand, replacing Y by U in (5.22) and (5.25), respectively, we also have

$$A_N X = -\varphi D^*_X U + B^*(X,U)U - g(X,U)\nu, \tag{5.37}$$

$$A^*_N X = -\varphi D_X U + B(X,U)U - g(X,U)\nu. \tag{5.38}$$

If taking into account (5.35)–(5.38), we can easily obtain our assertion.

The following two theorems give a characterization of the integrability of distributions $L\perp\langle\nu\rangle$ and $L^{'}\perp\langle\nu\rangle$, respectively.

Theorem 5.9. *Let (M, g, D, D^*) be a screen semi-invariant hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. The following assertions are equivalent:*
(i) *The distribution $L\perp\langle\nu\rangle$ is integrable.*
(ii) *$B^*(X, \varphi Y) = B^*(\varphi X, Y)$, for all $X, Y \in \Gamma(L\perp\langle\nu\rangle)$,*
(iii) *$B(X, \varphi Y) = B(\varphi X, Y)$, for all $X, Y \in \Gamma(L\perp\langle\nu\rangle)$.*
Proof. *We know that $X \in \Gamma(L\perp\langle\nu\rangle)$ if and only if $u(X) = \widetilde{g}(X, W) = 0$. For any $X, Y \in \Gamma(L\perp\langle\nu\rangle)$, using (3.1) and (5.14), we obtain*

$$u[X,Y] = -u(D_X Y) + u(D_Y X).$$

From (5.23), we have

$$u[X,Y] = B^*(Y, \varphi X) - B^*(\varphi Y, X).$$

This gives the equivalence between (i) and (ii). Similarly we can easily see that the relation (i) and (iii).

Theorem 5.10. *Let (M, g, D, D^*) be a screen semi-invariant hypersurface of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. The following assertions are equivalent:*
(i) *The distribution $L^{'}\perp\langle\nu\rangle$ is integrable.*
(ii) *$A^*_{\widetilde{\varphi}X} Y - A^*_{\widetilde{\varphi}Y} X = g(X,\nu)Y - g(Y,\nu)X$, for all $X, Y \in \Gamma(L^{'}\perp\langle\nu\rangle)$.*
(ii)*$A_{\widetilde{\varphi}X} Y - A_{\widetilde{\varphi}Y} X = g(X,\nu)Y - g(Y,\nu)X$, for all $X, Y \in \Gamma(L^{'}\perp\langle\nu\rangle)$.*
Proof. *$X \in \Gamma(L^{'}\perp\langle\nu\rangle)$ if and only if $\varphi X = 0$. For any $X, Y \in \Gamma(L\perp\langle\nu\rangle)$, using (3.2), (3.3) and (5.14) in (4.7), we have*

$$\varphi D^*_X Y = -g(X,Y)\nu + \widetilde{g}(Y,\nu)X - A_{\widetilde{\varphi}Y} X + B^*(X,Y)U.$$

Therefore, we can get

$$\varphi[X,Y] = -A_{\widetilde{\varphi}Y} X + A_{\widetilde{\varphi}X} Y + \widetilde{g}(Y,\nu)X - \widetilde{g}(X,\nu)Y.$$

This gives the equivalence between (i) and (ii). Similarly, the relationship between (i) and (iii) is easily seen.

6. Invariant submanifolds

Let (M, g, D, D^*) be a lightlike submanifold of an indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. if M is tangent to the structure vector field ν, then ν belongs to $S(TM)$ (see [13]). Using this, we say that M is an invariant lightlike submanifold of $\widetilde{M}$ if M is tangent to the structure vector field ν and

$$\widetilde{\varphi}(S(TM)) = S(TM),\ \widetilde{\varphi}(RadTM) = RadTM. \tag{6.1}$$

Proposition 6.1. *Let (M, g, D, D^*) be an invariant lightlike submanifold of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. For any $X, Y \in \Gamma(TM)$, we have the following identities:*

$$D_X\varphi Y - \varphi D^*_X Y = g(X, Y)\nu - g(\nu, Y)X, \tag{6.2}$$
$$h(X, \widetilde{\varphi}Y) = \widetilde{\varphi}h^*(X, Y), \tag{6.3}$$

where h and h^* are second fundemental forms for affine dual connections $\widetilde{D}$ and $\widetilde{D}^*$, respectively.
Proof. *Using (5.14) and Gauss formula in (4.7), we obtain*

$$D_X\varphi Y + h(X, \widetilde{\varphi}Y) - \varphi D^*_X Y - \widetilde{\varphi}h^*(X, Y) = g(X, Y)\nu - g(\nu, Y)X.$$

If we take tangential and transversal parts of this last equation, our claim is proven.

Similarly to the above proposition, the following proposition is given for dual connection D^*.

Proposition 6.2. *Let (M, g, D, D^*) be an invariant lightlike submanifold of indefinite Sasakian statistical manifold $(\widetilde{M}, \widetilde{D}, \widetilde{g}, \widetilde{\varphi}, \nu)$. For any $X, Y \in \Gamma(TM)$, we have the following identities:*

$$D^*_X\varphi Y - \varphi D_X Y = g(X, Y)\nu - g(\nu, Y)X, \tag{6.4}$$
$$h^*(X, \widetilde{\varphi}Y) = \widetilde{\varphi}h(X, Y), \tag{6.5}$$

where h and h^* are second fundemental forms for affine dual connections $\widetilde{D}$ and $\widetilde{D}^*$, respectively.

From the Eqs (6.3) and (6.5), we have

$$h(X, \nu) = 0,\ h^*(X, \nu) = 0. \tag{6.6}$$

A lightlike submanifold may not be an indefinite Sasakian statistical manifold. The following theorem gives a situation where this can happen.

Theorem 6.3. *An invariant lightlike submanifold of indefinite Sasakian statistical manifold is an indefinite Sasakian statistical manifold.*
Proof. *In a invariant lightlike submanifold, $u(X) = 0$, for any $X \in \Gamma(TM)$. Then from (5.14) we have*

$$\varphi^2 X = -X + g(X, \nu)\nu.$$

Since $\widetilde{\varphi}X = \varphi X$, using (4.1)–(4.3), we obtain

$$\varphi\nu = 0,\ \eta(\varphi X) = 0, \tag{6.7}$$

$$\widetilde{g}(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{6.8}$$

Then (g, φ, ν) is an almost contact metric structure.

Using (3.5), we get

$$Xg(\varphi Y, \varphi Z) = g(D_X\varphi Y, \varphi Z) + g(\varphi Y, D^*_X\varphi Z). \tag{6.9}$$

This equation says that D and D^* are dual connections. Moreover torsion tensor of the connection D is equal zero. Then, the Eqs (3.5) and (3.6) tell us that (D, g) is a statistical structure.

If we consider Gauss formula and (4.8) we obtain

$$D_X\nu = -\varphi X + g(D_X\nu, \nu)\nu. \tag{6.10}$$

If we consider (6.2) and (6.10) in the theorem 4.1, our assertion are proven.

Example 6. Let $\widetilde{M} = \left(\mathbb{R}^7_2, \widetilde{g}, \widetilde{\phi}, \nu\right)$ be the manifold endowed with the usual Sasakian structure (see, for example, [[13], p. 321] for such a structure), in which $\widetilde{g}$ has signature $(-, +, +, -, +, +, +)$, with respect to the canonical basis $\left\{\partial x^1, \partial x^2, \partial x^3, \partial y^1, \partial y^2, \partial y^3, \partial z\right\}$. By choosing the difference tensor $\overline{\mathbb{K}}(X, Y) = \widetilde{g}(Y, \nu)\widetilde{g}(X, \nu)\nu$, we can easily see that $(\widetilde{D} = \widetilde{\nabla} + \overline{\mathbb{K}}, \widetilde{g}, \widetilde{\varphi}, \nu)$ is an indefinite Sasakian statistical structure on $\widetilde{M}$. Now, we recall the example in [27] as follows:

Suppose that M is a submanifold of $\widetilde{M}$ given by

$$\begin{aligned}
&x^1 = v^1 \cos h\theta, y^1 = v^2 \cos h\theta, x^2 = v^1 \sin h\theta - v^2,\\
&y^2 = v^1 + v^2 \sin h\theta, x^3 = \sin v^3 \sin hv^4, y^3 = \cos v^3 \cos hv^4, z = v^5.
\end{aligned}$$

It is easy to see that the vector fields $\xi_1, \xi_2, \nu, Z_1, Z_2$, and given by

$$\begin{aligned}
\xi_1 &= \cos h\theta\partial x^1 + \sin h\theta\partial x^2 + \partial y^2 + \left(y^1 \cos h\theta + y^2 \sin h\theta\right)\partial z,\\
\xi_2 &= -\partial x^2 + \cos h\theta\partial y^1 + \sin h\theta\partial y^2 - y^2\partial z, \quad \nu = 2\partial z,\\
Z_1 &= \cos v^3 \sin h^4\partial x^3 - \sin v^3 \cos hv^4\partial y^3 + y^3 \cos v^3 \sin hv^4\partial z,\\
Z_2 &= \sin v^3 \cos hv^4\partial x^3 + \cos v^3 \sin hv^4\partial y^3 + y^3 \sin v^3 \cos hv^4\partial z,
\end{aligned}$$

spans TM. Moreover, one can see that $\mathrm{Rad}\,TM = \mathrm{Span}\{\xi_1, \xi_2\}$ and $S(TM) = \mathrm{Span}\ \{Z_1, Z_2, \nu\}$. Furthermore, we note that $\widetilde{\phi}\xi_2 = \xi_1$ and $\widetilde{\phi}Z_2 = Z_1$. It follows that $\mathrm{Rad}\,TM$ and $S(TM)$ are invariant under $\widetilde{\phi}$. On the other hand, $l\,\mathrm{tr}(TM)$ is spanned by N_1 and N_2, where

$$\begin{aligned}
N_1 &= 2\left\{-\cos h\theta\partial x^1 - \sin h\theta\partial x^2 + \partial y^2 - \left(y^1 \cos h\theta + y^2 \sin h\theta\right)\partial z\right\}\\
N_2 &= 2\left\{-\partial x^2 - \cosh\theta\partial y^1 - \sin h\theta\partial y^2 - y^2\partial z\right\}
\end{aligned}$$

Note that $\widetilde{\phi}N_2 = N_1$; hence, $l\,\mathrm{lr}(TM)$ is invariant under $\widetilde{\phi}$. Therefore, M is a five-dimensional invariant lightlike submanifold of indefinite Sasakian statistical manifold $\widetilde{M}$ and M is an indefinite Sasakian statistical manifold.

7. Conclusions and future work

In this paper, we expanded the Sasakian statistical manifold concept to indefinite Sasakian statistical manifolds and introduced lightlike hypersurfaces of an indefinite Sasakian statistical manifold. Some relations among induced geometrical objects with respect to dual connections in a lightlike hypersurface of an indefinite statistical manifold are obtained. We also give some original examples in this context.. We hope that, this introductory study will bring a new perspective for researchers and researchers will further work on it focusing on new results not available so far on lightlike geometry

Conflict of interest

The author declares that there is no competing interest.

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