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Research article

Existence results of nonlinear generalized proportional fractional differential inclusions via the diagonalization technique

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Abstract: The present paper is concerned with the existence of solutions of a new class of nonlinear generalized proportional fractional differential inclusions with the right-hand side contains a Carathèodory-type multi-valued nonlinearity on infinite intervals. The investigation of the proposed inclusion problem relies on the multi-valued form of Leray-Schauder nonlinear alternative incorporated with the diagonalization technique. By specializing the parameters involved in the problem at hand, an illustrated example is proposed.

Keywords: fractional differential inclusions; generalized proportional fractional derivatives; diagonalization method; Carathèodory multi-valued maps
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1. Introduction

Differential systems of fractional-order have recently acquired a great reputation and abundant superiority due to their large applications in numerous fields of science, engineering and the utilization of real-world models; see, for example, the books [4, 12, 15, 16, 18, 20, 21].

Analogous to the expansion of the theory of fractional-order systems, fractional differential inclusions were also extensively studied. Numerous contributions concerning the existence, uniqueness and stability results related to the fractional differential inclusions are available in the literature. Benchohra et al. [5], studied the following differential inclusion:

$$\begin{cases} {}^{H}_{C}\mathfrak{D}^{r} y(t) \in F(t, y(t)), & \text{for a.e. } t \in [1, \infty), \quad 1 < r \leq 2, \\ y(1) = y_1 \in \mathbb{R}, \end{cases}$$

where ${}^{H}_{C}\mathfrak{D}^{r}$ is the Caputo-Hadamard fractional derivative and $F : [1,\infty)\times\mathbb{R}\to\mathfrak{P}(\mathbb{R})$ ($\mathfrak{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) is a multi-valued map. They investigated the existence results when F has convex or non-convex values using suitable fixed point theorems and a diagonalization method.

Nyamoradi et al. [17], discussed the existence results of a multi-point BVP for a fractional differential inclusion of the form

$$\begin{cases}\mathfrak{D}^{\alpha}u(t)+F\left(t,u(t),u'(t)\right)\ni 0, \quad 0<t<+\infty,\ \ 2<\alpha<3,\\ u(0)=u'(0)=0, \qquad \mathfrak{D}^{\alpha-1}u(+\infty)=\sum_{i=1}^{m-2}\beta_i u(\xi_i),\end{cases}$$

where $\mathfrak{D}^{\alpha}$ is the Riemann-Liouville fractional derivative, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<+\infty$, and $F : [0,+\infty)\times\mathbb{R}\times\mathbb{R}\to\mathfrak{P}(\mathbb{R})$ is a set-valued map. For more recent papers related to fractional differential inclusions, see [2, 19, 22] and the references existing therein.

Motivated by the aforesaid papers and looking forward to considering generalized fractional derivative that encompasses the classical derivatives as special cases, we shall study the following fractional differential inclusions with generalized proportional fractional derivatives of Caputo type:

$$\begin{cases}{}^{C}\mathfrak{D}_a^{\delta,\varrho}u(t)\in\Phi\left(t,u(t)\right), \ \text{ for } \ \text{a.e. } t\in\mathcal{J}:=[a,\infty),\ \ 0<\delta<1,\\ u(a)=u_a\in\mathbb{R},\end{cases}\tag{1.1}$$

where $\varrho\in(0,1]$, ${}^{C}\mathfrak{D}_a^{\delta,\varrho}$ denotes the δ-order generalized proportional fractional derivatives of Caputo type, $\mathfrak{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\Phi:\mathcal{J}\times\mathbb{R}\to\mathfrak{P}(\mathbb{R})$ is a multi-valued map. In the main proofs we will use the ideas of [5] but with corrections of the subscripts and the used sets.

Very recently, Jarad et al. [10] introduced a new more general fractional derivative operator so-called the generalized proportional fractional derivative. The new fractional derivative operator $\mathfrak{D}_a^{\delta,\varrho}$ contains two parameters and has features, including maintaining the semi-group property and convergence to the initial function as δ tends to zero. Additionally, it is well behaved and has fundamental features over the classical derivatives in the sense that it generalizes previously defined derivatives in the literature. It is useful to note that the authors in [7] proposed an important equivalence between the tempered and the generalized proportional fractional integrals and derivatives. For some recent papers which have been detailed within the generalized proportional fractional derivative, see [3, 9, 11, 13] and the references existing therein. To the best of the authors' knowledge, there are no studies that dealt with the fractional differential inclusions with the generalized proportional fractional derivatives.

The main contributions of this note could be summarized as:

- We shall give the concept of a mild solution to the inclusion problem (1.1).
- With the aid of the nonlinear alternative of Leray-Schauder type for multivalued maps, the existence result is established.
- Due to the proposed inclusion problem is on an infinite interval, a diagonalization technique was needed to complete the proofs.
- An example is proposed to explain the suitability of the obtained findings.

2. Preliminaries

We start this section with some definitions and lemmas of the generalized proportional fractional derivatives and integrals.

Definition 2.1. ([10, 13]) Let $\varrho \in (0,1]$, $\delta > 0$, and let $g \in L^1[a,\infty)$. The generalized proportional fractional integral of a function g of order δ is defined by

$$(\mathfrak{I}_a^{\delta,\varrho} g)(t) = \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} g(s)\, ds = \frac{1}{\varrho^{\delta}} e^{\frac{\varrho-1}{\varrho}t} \left(\mathfrak{I}_a^{\delta}\left(e^{-\frac{\varrho-1}{\varrho}t} g(t)\right)\right),$$

where $\mathfrak{I}_a^{\delta}$ denotes the Riemann-Liouville fractional integral of order δ: $(\mathfrak{I}_a^{\delta} g)(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} g(s)\, ds$.

Definition 2.2. ([10, 13]) Let $\varrho \in (0,1]$ and $0 < \delta < 1$. The generalized proportional fractional derivative of Caputo type of order δ of a function $g \in C^1[a,\infty)$ is defined by

$$\begin{aligned}({}^{C}\mathfrak{D}^{\delta,\varrho} g)(t) &= \mathfrak{I}_a^{1-\delta,\varrho}\left(\mathfrak{D}^{1,\varrho} g\right)(t)\\ &= \frac{1}{\varrho^{\delta}\Gamma(1-\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}\,(t-s)^{-\delta}\,(\mathfrak{D}^{1,\varrho} g)(s)\, ds,\end{aligned}$$

where $(\mathfrak{D}^{1,\varrho} g)(t) = (\mathfrak{D}^{\varrho} g)(t) \equiv (1-\varrho) g(t) + \varrho g'(t)$.

Lemma 2.3. *([10, 13]) Let* $\varrho \in (0,1]$ *and* $0 < \delta < 1$ *and* $\gamma > 0$. *Then we have the following properties:*

$$\left(\mathfrak{I}_a^{\delta,\varrho}\, \mathfrak{I}_a^{\gamma,\varrho} g\right)(t) = \left(\mathfrak{I}_a^{\gamma,\varrho}\, \mathfrak{I}_a^{\delta,\varrho} g\right)(t) = \left(\mathfrak{I}_a^{\delta+\gamma,\varrho} g\right)(t), \quad \text{with } g \in L^1[a,\infty); \tag{2.1}$$

$$\left({}^{C}\mathfrak{D}_a^{\delta,\varrho}\, \mathfrak{I}_a^{\delta,\varrho} g\right)(t) = g(t), \quad \text{with } g \in L^1[a,\infty); \tag{2.2}$$

$$\left(\mathfrak{I}_a^{\delta,\varrho}\, {}^{C}\mathfrak{D}_a^{\delta,\varrho} g\right)(t) = g(t) - e^{\frac{\varrho-1}{\varrho}(t-a)}\, g(a), \qquad \text{with } g \in C^1[a,\infty). \tag{2.3}$$

2.1. Multi-valued maps analysis

Let $\mathbb{X}$ be a Banach space. We use the notations

$$\mathfrak{P}(\mathbb{X}) = \{Z \in \mathfrak{P}(\mathbb{X}) : Z \neq \emptyset\},$$

$$\mathfrak{P}_{\mathbf{cl}}(\mathbb{X}) = \{Z \in \mathfrak{P}(\mathbb{X}) : Z \text{ is closed}\},$$

$$\mathfrak{P}_{\mathbf{bd}}(\mathbb{X}) = \{Z \in \mathfrak{P}(\mathbb{X}) : Z \text{ is bounded}\},$$

$$\mathfrak{P}_{\mathbf{cp}}(\mathbb{X}) = \{Z \in \mathfrak{P}(\mathbb{X}) : Z \text{ is compact}\},$$

$$\mathfrak{P}_{\mathbf{cvx}}(\mathbb{X}) = \{Z \in \mathfrak{P}(\mathbb{X}) : Z \text{ is convex}\}.$$

We will use the following definitions:

- A multi-valued map $\mathfrak{U} : \mathbb{X} \to \mathfrak{P}(\mathbb{X})$ is convex (closed) valued, if $\mathfrak{U}(x)$ is convex (closed) for all $x \in \mathbb{X}$.
- $\mathfrak{U}$ is bounded on bounded sets if $\mathfrak{U}(B) = \cup_{x\in B}\mathfrak{U}(x)$ is bounded in $\mathbb{X}$ for any $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbb{X})$, i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in \mathfrak{U}(x)\}\} < \infty$.

- $\mathfrak{U}$ is called upper semi-continuous (u.s.c.) on $\mathbb{X}$ if for each $x^o \in \mathbb{X}$, the set $\mathfrak{U}(x^o)$ is nonempty, closed subset of $\mathbb{X}$, and if for each open set N of $\mathbb{X}$ containing $\mathfrak{U}(x^o)$, there exists an open neighborhood N^o of x^o such that $\mathfrak{U}(N^o) \subset N$.
- $\mathfrak{U}$ is completely continuous if $\mathfrak{U}(B)$ is relatively compact for each $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbb{X})$.
- The graph of $\mathfrak{U}$ will be defined as the set $Gr(\mathfrak{U}) := \{(x, y) \in \mathbb{X} \times \mathbb{Y} : y \in \mathfrak{U}(x)\}$.
- We say that $x \in \mathbb{X}$ is a fixed point of $\mathfrak{U}$ if $x \in \mathfrak{U}(x)$.

Consider the sequence $\{b_k\}_{k=1}^{\infty}$ such that $a < b_1 < b_2 < \cdots < b_k < \cdots$ and $\lim_{k\to\infty} b_k = \infty$.
For any $k \in \mathbb{N}$ we denote $\mathcal{J}_k := [a, b_k] \subset \mathcal{J}$.

Let $C(\mathcal{J}_k, \mathbb{R})$ be a Banach space of continuous functions from $\mathcal{J}_k$ into $\mathbb{R}$ with the norm $\|u\|_k = \sup_{t\in\mathcal{J}_k} |u(t)|$. Let $L^1(\mathcal{J}_k, \mathbb{R})$ be the Banach space of Lebesgue integrable functions $u : \mathcal{J}_k \to \mathbb{R}$ and normed by $\|u\|_{L^1,k} = \int_{\mathcal{J}_k} |y(t)|\, dt$.

Suppose that $u \in C(\mathcal{J}_k, \mathbb{R})$. We define the set of the selections of Φ by

$$\mathcal{S}_{\Phi,u}^{\mathcal{J}_k} := \{v \in L^1(\mathcal{J}_k, \mathbb{R}) : v(t) \in \Phi(t, u(t)) \text{ for a.e. } t \in \mathcal{J}_k\}.$$

Definition 2.4. A multi-valued map $\Phi : \mathcal{J} \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto \Phi(t, u)$ is measurable for each $u \in \mathbb{R}$;

(ii) $u \mapsto \Phi(t, u)$ is upper semi-continuous for almost all $t \in \mathcal{J}$.

Remark 2.5. Note if the multi-valued map $\Phi : \mathcal{J} \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is Carathéodory then the condition (ii) is satisfied on $\mathcal{J}_k$, $k = 1, 2, \ldots$ and the multi-valued map $\Phi : \mathcal{J}_k \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is Carathéodory on $\mathcal{J}_k \times \mathbb{R}$.

Lemma 2.6. *[6] If $\mathfrak{U} : \mathbb{X} \to \mathfrak{P}_{\mathbf{cl}}(\mathbb{Y})$ is upper semi-continuous, then $Gr(\mathfrak{U})$ is a closed subset of $\mathbb{X} \times \mathbb{Y}$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{X}$ and $\{y_n\}_{n\in\mathbb{N}} \in \mathbb{Y}$, if $\lim_{n\to\infty} x_n = x_*$, $\lim_{n\to\infty} y_n = y_*$ and $y_n \in \mathfrak{U}(x_n)$, then $y_* \in \mathfrak{U}(x_*)$. Conversely, if $\mathfrak{U}$ is completely continuous and has a closed graph, then $\mathfrak{U}$ is upper semi-continuous.*

Lemma 2.7. *[14] Let $\mathbb{X}$ be a Banach space. Let $\Phi : \mathcal{J}_k \times \mathbb{X} \to \mathfrak{P}(\mathbb{X})$ be a multi-valued map satisfying the Carathèodory conditions with $\mathcal{S}_{\Phi}^{\mathcal{J}_k} \neq \emptyset$, and let $\Theta : L^1(\mathcal{J}_k, \mathbb{X}) \to C(\mathcal{J}_k, \mathbb{X})$ be a linear continuous mapping. Then the multi-valued map $\Theta \circ \mathcal{S}_{\Phi}^{\mathcal{J}_k} : C(\mathcal{J}_k, \mathbb{X}) \to \mathfrak{P}(C(\mathcal{J}_k, \mathbb{X}))$ defined by*

$$(\Theta \circ \mathcal{S}_{\Phi}^{\mathcal{J}_k})(u) : C(\mathcal{J}_k, \mathbb{X}) \to \mathfrak{P}_{\mathbf{bd,cl,cvx}}(C(\mathcal{J}_k, \mathbb{X}), \quad u \mapsto (\Theta \circ \mathcal{S}_{\Phi}^{\mathcal{J}_k})(u) = \Theta(\mathcal{S}_{\Phi,u}^{\mathcal{J}_k}),$$

is a closed graph operator in $C(\mathcal{J}_k, \mathbb{X}) \times C(\mathcal{J}_k, \mathbb{X})$.

Lemma 2.8. *[8] Let $\mathbb{E}$ be a Banach space and C a nonempty closed convex subset of $\mathbb{E}$. Let U be a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to \mathfrak{P}_{\mathbf{cp,cvx}}(C)$ be an upper semi-continuous compact map. Then either*

(i) *T has a fixed points in $\overline{U}$,*

or

(ii) *there is a $u \in \partial U$ and $v \in (0, 1)$ with $u \in vT(u)$.*

3. Main results

Let us start with giving the concept of the mild solution of the inclusion problem (1.1).

Definition 3.1. A function $u \in C(\mathcal{J}, \mathbb{R})$ is said to be a mild solution of the inclusion problem (1.1) if there exists a function $\phi \in L^1(\mathcal{J}, \mathbb{R})$ with $\phi(t) \in \Phi(t, u(t))$ for a.e. $t \in \mathcal{J}$ such that ${}_a^C\mathfrak{D}^{\delta,\varrho}u(t) = \phi(t)$ and $u(a) = u_a$.

The following lemma plays an essential role in the forthcoming discussions.

Lemma 3.2. *([13], Lemma 4.1) For any $w \in C([a, b], \mathbb{R})$ the solution u of the linear generalized proportional fractional differential equation*

$$\begin{cases} {}_a^C\mathfrak{D}^{\delta,\varrho}u(t) = w(t), \quad \text{for} \quad \text{a.e. } t \in [a, b], \quad 0 < \delta < 1, \\ u(a) = u_a \in \mathbb{R}, \end{cases} \tag{3.1}$$

is given by the following integral equation

$$u(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} w(s)\, ds, \quad t \in [a, b]. \tag{3.2}$$

Theorem 3.3. *Suppose that for every $k \in \mathbb{N}$:*

(H1) *A multi-valued map $\Phi : \mathcal{J} \times \mathbb{R} \to \mathfrak{P}_{\mathbf{cp,cvx}}(\mathbb{R})$ is Carathèodory;*

(H2) *There exist $p, q \in C(\mathcal{J}, \mathbb{R}^+)$ such that the nonegative functions $P(t) = \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1}p(s)ds$ and $Q(s) = \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1}q(s)ds$ are bounded on $\mathcal{J}$, i.e. there exist constants $\mathcal{P} > 0$ and $0 < \mathcal{Q} < \varrho^\delta\Gamma(\delta + 1)$ such that $P(t) \leq \mathcal{P}$, $Q(t) \leq \mathcal{Q}$ for $t \in \mathcal{J}$ and*

$$\|\Phi(t, u)\|_{\mathfrak{P}} := \sup\{|v| : v \in \Phi(t, u)\} \leq p(t) + q(t)|u| \text{ for } (t, u) \in \mathcal{J} \times \mathbb{R}.$$

Then the inclusion problem (1.1) possesses at least one solution on $\mathcal{J}$.

Remark 3.4. Note that for $a = 0$ the function $q(t) = e^{-t}$ satisfies the condition (H2), i.e., for the nonegative function $Q(t) = \int_0^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1}q(s)ds$ there exists a positive constant $\mathcal{Q}$ (depending on ϱ and δ) such that $0 < \mathcal{Q} < \varrho^\delta\Gamma(\delta + 1)$ and $Q(t) \leq \mathcal{Q}$ for $t \in [0, \infty)$ (see Figure 1 for the graphs of $Q(t)$ and the corresponding bounds $\mathcal{A} = \varrho^\delta\Gamma(\delta + 1)$ for various values of δ and ϱ).

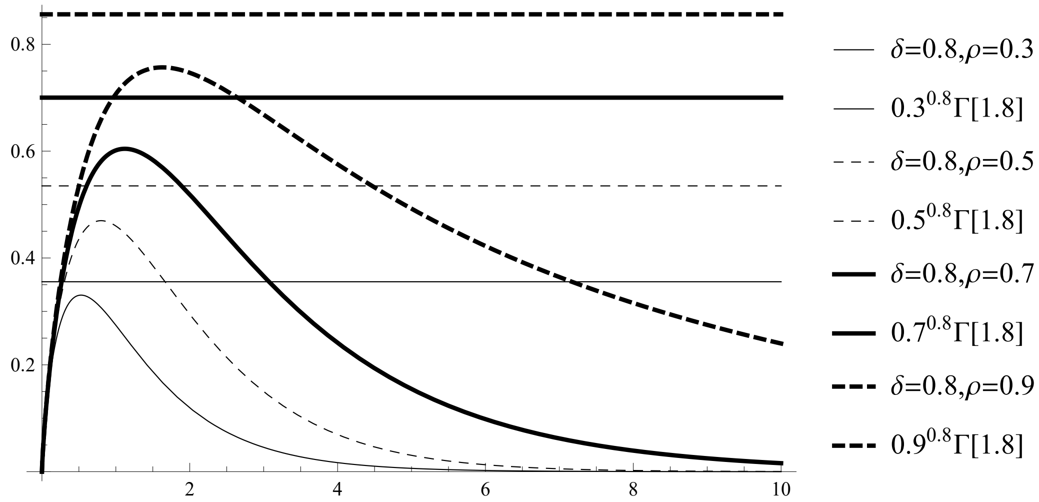

Figure 1. Graphs of Q(t) and A for different values of δ and ϱ.

Proof. We fix $k \in \mathbb{N}$ and consider

$$\begin{cases} {}^{C}_{a}\mathfrak{D}^{\delta,\varrho}u(t) \in \Phi(t, u(t)), & \text{for a.e. } t \in \mathcal{J}_k, \ \ 0 < \delta < 1, \\ u(a) = u_a. \end{cases} \tag{3.3}$$

We shall prove that (3.3) has a solution $u_k \in C(\mathcal{J}_k, \mathbb{R})$ with $|u_k(t)| \leq M$ for each $t \in \mathcal{J}_k$, where $M > 0$ is an arbitrary constant.

Define the multi-valued map $\mathcal{T}_k : C(\mathcal{J}_k, \mathbb{R}) \to \mathfrak{P}(C(\mathcal{J}_k, \mathbb{R}))$ by

$$\mathcal{T}_k(u) = \left\{ f \in C(\mathcal{J}_k, \mathbb{R}) : \ f(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \tfrac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} w(s)\, ds, \quad w \in \mathcal{S}^{\mathcal{J}_k}_{\Phi,u} \right\}.$$

In the light of Lemma 3.2, one can easily know that the fixed points of $\mathcal{T}_k$ are solutions of the problem (3.3). We shall show that $\mathcal{T}_k$ satisfies the assumptions of Lemma 2.8. The proof will be given in following steps.

Step 1. *$\mathcal{T}_k(u)$ is convex, for any $u \in C(\mathcal{J}_k, \mathbb{R})$.*

For $f_1, f_2 \in \mathcal{T}_k(u)$, there exist $w_1, w_2 \in \mathcal{S}^{\mathcal{J}_k}_{\Phi,u}$ such that

$$f_i(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} w_i(s)\, ds, \ \ i = 1, 2.$$

Let $0 \leq \mu \leq 1$. Then, for $t \in \mathcal{J}_k$, one has

$$(\mu f_1 + (1-\mu) f_2)(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} (\mu w_1 + (1-\mu) w_2)(s).$$

Since $\mathcal{S}^{\mathcal{J}_k}_{\Phi,u}$ is convex (because Φ has convex values), then $\mu f_1 + (1-\mu) f_2 \in \mathcal{T}_k(u)$. This implies that $\mathcal{T}_k(u)$ is convex.

Step 2. *$\mathcal{T}_k(u)$ maps bounded sets (balls) into bounded sets in $C(\mathcal{J}_k, \mathbb{R})$.*

For a positive number r, let $\Xi_r := \{u \in C(\mathcal{J}_k, \mathbb{R}) : \|u\|_k \leq r\}$ be a bounded ball in $C(\mathcal{J}_k, \mathbb{R})$. For each $u \in \Xi_r$, $f \in \mathcal{T}_k(u)$, there exists $w \in \mathcal{S}_{\Phi,u}^{\mathcal{J}_k}$ such that

$$f(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} w(s)\, ds.$$

In view of (H2) and for each $t \in \mathcal{J}_k$, one has

$$\begin{aligned} |f(t)| &\leq |u_a| \left| e^{\frac{\varrho-1}{\varrho}(t-a)} \right| + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} |w(s)|\, ds \\ &\leq |u_a| + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} (p(s) + q(s)|u(s)|)\, ds \\ &\leq |u_a| + \frac{1}{\varrho^\delta \Gamma(\delta+1)} (\mathcal{P} + \mathcal{Q} r). \end{aligned}$$

Therefore, we get

$$\|f\| \leq |u_a| + \frac{\mathcal{P} + \mathcal{Q} r}{\varrho^\delta \Gamma(\delta+1)}.$$

Step 3. *$\mathcal{T}_k(u)$ maps bounded sets into equicontinuous sets in $C(\mathcal{J}_k, \mathbb{R})$.*

Take $t_1, t_2 \in \mathcal{J}_k$, $t_1 < t_2$. For each $u \in \Xi_r$, $f \in \mathcal{T}_k(u)$, we obtain that

$$\begin{aligned} |\mathcal{T}_k(u)(t_2) - \mathcal{T}_k(u)(t_1)| &\leq |u_a| \left| e^{\frac{\varrho-1}{\varrho}(t_2-a)} - e^{\frac{\varrho-1}{\varrho}(t_1-a)} \right| \\ &\quad + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^{t_1} \left| e^{\frac{\varrho-1}{\varrho}(t_2-s)} (t_2-s)^{\delta-1} - e^{\frac{\varrho-1}{\varrho}(t_1-s)} (t_1-s)^{\delta-1} \right| |w(s)|\, ds \\ &\quad + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_{t_1}^{t_2} e^{\frac{\varrho-1}{\varrho}(t_2-s)} (t_2-s)^{\delta-1} |w(s)|\, ds \\ &\leq |u_a| \left| e^{\frac{\varrho-1}{\varrho}(t_2-a)} - e^{\frac{\varrho-1}{\varrho}(t_1-a)} \right| \\ &\quad + \frac{(\|p\|_k + \|q\|_k r)}{\varrho^\delta \Gamma(\delta)} \int_a^{t_1} \left| e^{\frac{\varrho-1}{\varrho}(t_2-s)} (t_2-s)^{\delta-1} - e^{\frac{\varrho-1}{\varrho}(t_1-s)} (t_1-s)^{\delta-1} \right| ds \\ &\quad + \frac{(\|p\|_k + \|q\|_k r)}{\varrho^\delta \Gamma(\delta)} \int_{t_1}^{t_2} e^{\frac{\varrho-1}{\varrho}(t_2-s)} (t_2-s)^{\delta-1}\, ds. \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1–3 together with the Arzelà-Ascoli theorem, we infer that $\mathcal{T}_k$ is completely continuous.

Step 4. *$\mathcal{T}_k$ has a closed graph.*

Consider the sequence $\{u_n\}_{n=1}^\infty$ with $u_n \in C(\mathcal{J}_k, \mathbb{R})$, $\lim_{n\to\infty} u_n = u_*$, $f_n \in \mathcal{T}_k(u_n)$, and $\lim_{n\to\infty} f_n = f_*$. We shall show that $f_* \in \mathcal{T}_k(u_*)$. Indeed, since $f_n \in \mathcal{T}_k(u_n)$, there exists $w_n \in \mathcal{S}_{\Phi,u_n}^{\mathcal{J}_k}$ in way that for $t \in \mathcal{J}_k$, one has

$$f_n(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} w_n(s)\, ds, \; n = 1, 2, \ldots.$$

It must be proving that there exists $w_* \in \mathcal{S}_{\Phi,u_*}^{\mathcal{J}_k}$ in ways that

$$f_*(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t-s)^{\delta-1} w_*(s)\, ds.$$

Define the linear operator $\Theta : L^1(\mathcal{J}_k, \mathbb{R}) \to C(\mathcal{J}_k, \mathbb{R})$ by

$$w \mapsto \Theta(w)(t) = \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} w(s)\, ds.$$

The operator Θ is continuous. Indeed, let $\{\tilde{w}_m\}_{m=1}^{\infty}$ be a sequence such that $\tilde{w}_m \in L^1(\mathcal{J}_k, \mathbb{R})$ with $\lim_{m\to\infty} \tilde{w}_m = \tilde{w}_* \in L^1(J_k, \mathbb{R})$. Then for each $t \in \mathcal{J}_k$, we get

$$|\Theta(\tilde{w}_m)(t) - \Theta(\tilde{w}_*)(t)| \le \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} |\tilde{w}_m(s) - \tilde{w}_*(s)|\, ds,$$

which implies that $\Theta(\tilde{w}_m) \to \Theta(\tilde{w}_*)$ as $m \to \infty$ in $C(\mathcal{J}_k, \mathbb{R})$.

In the light of Lemma 2.7, we infer that $\Theta \circ \mathcal{S}_{\Phi,u}^{\mathcal{J}_k}$ is closed graph operator.

Additionally, we have $f_n(t) - u_a e^{\frac{\varrho-1}{\varrho}(t-a)} \in \mathcal{S}_{\Phi,u_n}^{\mathcal{J}_k}$. Since $\lim_{n\to\infty} u_n = u_*$, it follows that

$$f_*(t) - u_a e^{\frac{\varrho-1}{\varrho}(t-a)} = \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} w_*(s)\, ds,$$

for some $w_* \in \mathcal{S}_{\Phi,u_*}^{\mathcal{J}_k}$.

Hence, by virtue of Lemma 2.6, the multi-valued operator $\mathcal{T}_k$ is an upper semi-continuous operator on $C(\mathcal{J}_k, \mathbb{R})$.

Step 5. *We show there exists an open set $U \subseteq C(\mathcal{J}_k, \mathbb{R})$ with $u \notin \mathcal{T}_k(u)$ for any $\nu \in (0,1)$ and all $u \in \partial U$.*

Let $\nu \in (0,1)$ and $u \in \nu\mathcal{T}_k(u)$. Thus, there exists $w \in \mathcal{S}_{\Phi,u}^{\mathcal{J}_k}$ in ways that for each $t \in \mathcal{J}_k$, we have

$$u(t) = \nu u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{\nu}{\varrho^{\delta}\Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} w(s)\, ds.$$

From (H2) and for each $t \in \mathcal{J}_k$, one has

$$\begin{aligned}
|u(t)| &\le |u_a| \left| e^{\frac{\varrho-1}{\varrho}(t-a)} \right| + \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} |w(s)|\, ds \\
&\le |u_a| + \frac{1}{\varrho^{\delta}\Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} (p(s) + q(s)|u(s)|)\, ds \\
&\le |u_a| + \frac{1}{\varrho^{\delta}\Gamma(\delta+1)}(P(t) + Q(t)\|u\|) \le |u_a| + \frac{1}{\varrho^{\delta}\Gamma(\delta+1)}(\mathcal{P} + \mathcal{Q}\|u\|).
\end{aligned}$$

Therefore, we get

$$\frac{|u(t)|}{|u_a| + \frac{1}{\varrho^{\delta}\Gamma(\delta+1)}(\mathcal{P} + \mathcal{Q}\|u\|_k)} \le 1.$$

Note the function $h(u) = \mathcal{P} + \mathcal{Q}u : (0,,\infty) \to (0,\infty)$ is non-decreasing.

In view of assumption (H2) we have $1 - \frac{\mathcal{Q}}{\varrho^{\delta}\Gamma(\delta+1)} > 0$ and choose the constant L such that

$$L > \frac{|u_a| + \frac{\mathcal{P}}{\varrho^{\delta}\Gamma(\delta+1)}}{1 - \frac{\mathcal{Q}}{\varrho^{\delta}\Gamma(\delta+1)}}.$$

Therefore,

$$\frac{L}{|u_a| + \frac{\mathcal{P}+QL}{\varrho^\delta \Gamma(\delta+1)}} > 1.$$

Then $\|u\|_k \neq L$. Let us define the set

$$U_k = \{u \in C(\mathcal{J}_k, \mathbb{R}) : \|u\|_k < L\}.$$

The operator $\mathcal{T}_k : \overline{U}_k \to \mathfrak{P}(C(J_k, \mathbb{R}))$ is upper semi-continuous and completely continuous. From the definition of U_k , there exists no $u \in \partial U_k$ such that $u \in \nu\mathcal{T}_k(u)$ for some $\nu \in (0, 1)$. As a consequence of Leray-Schauder nonlinear alternative (Lemma 2.8), it follows that $\mathcal{T}_k$ possesses a fixed point $u_k \in \overline{U}_k$, which is a solution of the problem (3.3).

Step 6. A diagonalization process.

First, we set $\mathbb{N}_k = \mathbb{N}^* - \{k\}$, i.e. $\mathbb{N}_k = \{j = k + 1, k + 2, \dots\}$. For any $k \in \mathbb{N}$, let

$$x_k(t) = \begin{cases} u_k(t); & t \in [a, b_k] \\ u_k(b_k); & t \in [b_k, \infty), \end{cases}$$

where $u_k(t)$ is the fixed point of $\mathcal{T}_k$, which is a solution of the problem (3.3) and which existence if proved in Step 5.

Note that $\mathbb{N}_{k+1} \subset \mathbb{N}_k$ for any $k \in \mathbb{N}$.

Consider $\mathbb{N}_1$. Then the sequence $\{x_m(t)\}_{m=2}^{\infty}$ is defined for $t \in \mathcal{J}_1$. There exists $w_{1,m} \in \mathcal{S}_{\Phi,u}^{\mathcal{J}_1}$ such that

$$x_m(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t - s)^{\delta-1} w_{1,m}(s)\, ds,$$

and $|x_m(t)| \leq M$ for $t \in \mathcal{J}_1 = [a, b_1]$.

Thus, for $t_1, t_2 \in \mathcal{J}_1$, $t_1 < t_2$, one has

$$\begin{aligned} |x_m(t_2) - x_m(t_1)| \leq{}& |u_a| \left| e^{\frac{\varrho-1}{\varrho}(t_2-a)} - e^{\frac{\varrho-1}{\varrho}(t_1-a)} \right| \\ &+ \frac{(\|p\|_1 + \|q\|_1 M)}{\varrho^\delta \Gamma(\delta)} \int_a^{t_1} \left| e^{\frac{\varrho-1}{\varrho}(t_2-s)} (t_2 - s)^{\delta-1} - e^{\frac{\varrho-1}{\varrho}(t_1-s)} (t_1 - s)^{\delta-1} \right| ds \\ &+ \frac{(\|p\|_1 + \|q\|_1 M)}{\varrho^\delta \Gamma(\delta)} \int_{t_1}^{t_2} e^{\frac{\varrho-1}{\varrho}(t_2-s)} (t_2 - s)^{\delta-1}\, ds. \end{aligned}$$

Thanks to the Arzelà-Ascoli theorem, the sequence $\{x_m(t)\}$ has an uniformly convergent subsequence in $\mathcal{J}_1$, so there is a subset $\mathbb{N}_1 = \{2, 3, \dots\}$ of $\mathbb{N}$ and a function $y_1 \in C(\mathcal{J}_1, \mathbb{R})$ in ways that $\{x_m(t)\} \to y_1(t)$ uniformly in $\mathcal{J}_1$ as $m \to \infty$ through $\mathbb{N}_1$. Additionally, the integral equality

$$y_1(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t - s)^{\delta-1} w_1(s)\, ds, \quad t \in \mathcal{J}_1,$$

holds with $w_1 \in \mathcal{S}_{\Phi,u}^{\mathcal{J}_1}$ and $|y_1(t)| \leq M$ for $t \in \mathcal{J}_1$.

Consider $\mathbb{N}_2$. Then the sequence $\{x_m(t)\}_{m=3}^{\infty}$ is defined for $t \in \mathcal{J}_2$. There exists $w_{2,m} \in \mathcal{S}_{\Phi,u}^{\mathcal{J}_2}$ such that

$$x_m(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)} (t - s)^{\delta-1} w_{2,m}(s)\, ds,$$

and $|x_m(t)| \le M$ for $t \in \mathcal{J}_2 = [a, b_2]$. Therefore, for $t_1, t_2 \in \mathcal{J}_2$, $t_1 < t_2$, one obtain that

$$\begin{aligned}|x_m(t_2) - x_m(t_1)| \le |u_a| \left| e^{\frac{\varrho-1}{\varrho}(t_2-a)} - e^{\frac{\varrho-1}{\varrho}(t_1-a)} \right| \\ + \frac{(\|p\|_2 + \|q\|_2 M)}{\varrho^\delta \Gamma(\delta)} \int_a^{t_1} \left| e^{\frac{\varrho-1}{\varrho}(t_2-s)}(t_2-s)^{\delta-1} - e^{\frac{\varrho-1}{\varrho}(t_1-s)}(t_1-s)^{\delta-1} \right| ds \\ + \frac{(\|p\|_2 + \|q\|_2 M)}{\varrho^\delta \Gamma(\delta)} \int_{t_1}^{t_2} e^{\frac{\varrho-1}{\varrho}(t_2-s)}(t_2-s)^{\delta-1} \, ds.\end{aligned}$$

Again, by the Arzelà-Ascoli theorem, the sequence $\{x_m(t)\}$ has an uniformly convergent subsequence, so there is a subset $\mathbb{N}_2 = \{3, 4, \dots\}$ of $\mathbb{N}_1$ and a function $y_2 \in C(\mathcal{J}_2, \mathbb{R})$ in ways that $x_m(t) \to y_2(t)$ uniformly on $\mathcal{J}_2$ as $m \to \infty$ through $\mathbb{N}_2$. Additionally, the integral equality

$$y_2(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} w_2(s) \, ds, \quad t \in \mathcal{J}_2,$$

holds with $w_2 \in \mathcal{S}_{\Phi,u}^{\mathcal{J}_2}$ and $|y_2(t)| \le M$ for $t \in \mathcal{J}_2$.

Note that $y_1(t) \equiv y_2(t)$ and $w_1(t) \equiv w_2(t)$ on $\mathcal{J}_1$, since $\mathbb{N}_2 \subset \mathbb{N}_1$.

Inductively, considering the set $\mathbb{N}_k$ for $k = 3, 4, \dots$ and the sequence $\{x_m(t)\}_{m=k+1}^{\infty}$ defined on $\mathcal{J}_k$ we obtain the limit function $y_k(t) \in C(\mathcal{J}_k, \mathbb{R})$ such that $x_m(t) \to y_k(t)$ uniformly on $\mathcal{J}_k$ as $m \to \infty$ through $\mathbb{N}_k$. Also, $y_k(t) \equiv y_{k+1}(t)$and $w_k(t) \equiv w_{k+1}(t)$ on $\mathcal{J}_k$, $k = 1, 2, \dots$. Additionally, the integral equality

$$y_k(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} w_k(s) \, ds, \quad t \in \mathcal{J}_k,$$

holds with $w_k \in \mathcal{S}_{\Phi,u}^{\mathcal{J}_k}$ and $|y_k(t)| \le M$ for $t \in \mathcal{J}_k$.

For any $t \in [a, \infty)$ we consider the smallest number $j \in \mathbb{N}: \ t \le b_j$, i.e. $t \in \mathcal{J}_j$. Then according the above proved there exists a function $y_j(t) \in C(\mathcal{J}_j, \mathbb{R})$ and the integral equality

$$y_j(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} w_j(s) \, ds, \quad t \in \mathcal{J}_j,$$

holds with $w_j \in \mathcal{S}_{\Phi,u}^{\mathcal{J}_j}$ and $|y_j(t)| \le M$ for $t \in \mathcal{J}_j$.

For any $t \in [a, \infty)$ we define the function $y \in C([a, \infty), \mathbb{R})$ by $y(t) = y_j(t)$ and $w(s) = w_j(s)$ for $s \in [0, t]$ where j is the smallest number such that $t \le b_j$. Then $y(a) = u_a$ and the equality

$$y(t) = u_a e^{\frac{\varrho-1}{\varrho}(t-a)} + \frac{1}{\varrho^\delta \Gamma(\delta)} \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1} w(s) \, ds, \quad t \in \mathcal{J},$$

holds.

Thus,

$${}_a^C\mathfrak{D}^{\delta,\varrho} y(t) \in \Phi(t, y(t)), \quad \text{for a.e. } t \in \mathcal{J}, \ \ 0 < \delta < 1,$$

for each $k \in \mathbb{N}$. This completes the proof of the theorem. □

Example: Consider the following fractional differential inclusions with generalized proportional fractional derivatives of Caputo type:

$$\begin{cases} {}^C\mathfrak{D}_0^{0.8,\varrho} u(t) \in \Phi(t, u(t)), & \text{for a.e. } t \in \mathcal{J} := [0, \infty), \ \ 0 < \delta < 1, \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{3.4}$$

where $\Phi : [0,\infty) \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is Carathéodory and such that $\|\Phi(t,u)\|_{\mathfrak{P}} \le t^{-0.1} + e^{-t}|u|$ for $t \ge 0,\ u \in \mathbb{R}$. In this case $p(t) = t^{-0.1}$ and $q(t) = e^{-t}$. The function $p(t)$ satisfies the condition (H2) (see Figure 2 for the the graph of the function $P(t) = \int_a^t e^{\frac{\varrho-1}{\varrho}(t-s)}(t-s)^{\delta-1}s^{-0.1}ds$ for $\delta = 0.8$ and different values of ϱ). According to Remark 3.4 the function $q(t)$ satisfies the condition (H2). Therefore, according to Theorem 3.3 the inclusion problem (3.4) possesses at least one solution on $[0,\infty)$ for any initial value $u_0 \in \mathbb{R}$ and any $\delta \in (0,1)$.

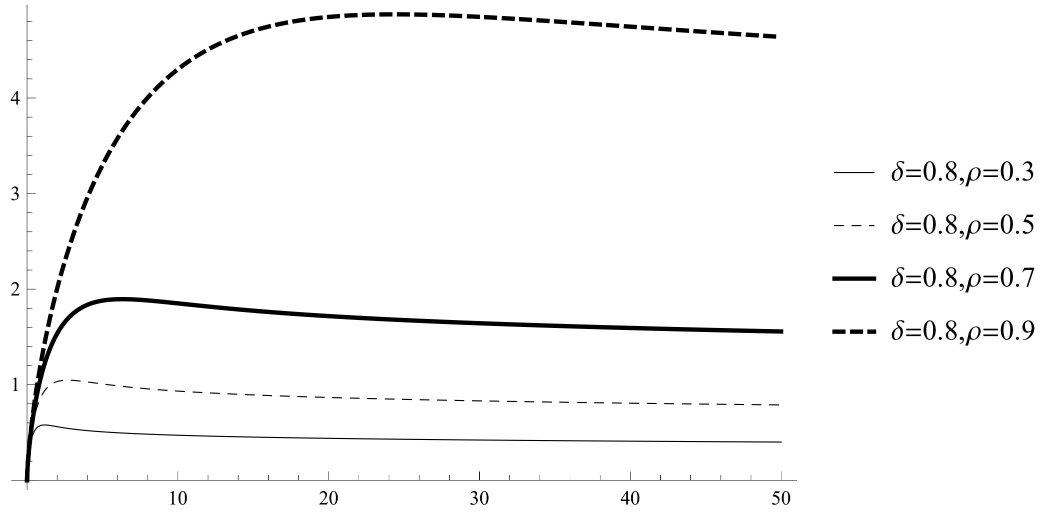

Figure 2. Graphs of $P(t)$ for $\delta = 0.8$ and different values of ϱ.

4. Conclusions

Through the present work, we investigate the existence theorems of mild solutions for fractional differential inclusions involving the generalized fractional derivatives of Caputo-type on unbounded domain. By means of a suitable fixed point theorem for multi-valued maps together with a diagonlization process, the desired result is obtained. Finally, an example is proposed to explain the suitability of the obtained findings.

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Conflict of interest

The authors declare that they have no competing interests.

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