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Research article

A basic study of a fractional integral operator with extended Mittag-Leffler kernel

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Abstract: In this present paper, the basic properties of an extended Mittag-Leffler function are studied. We present some fractional integral and differential formulas of an extended Mittag-Leffler function. In addition, we introduce a new extension of Prabhakar type fractional integrals with an extended Mittag-Leffler function in the kernel. Also, we present certain basic properties of the generalized Prabhakar type fractional integrals.

Keywords: Mittag-Leffler function; fractional integral; Prabhakar fractional integral
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1. Introduction

In the last few years, many mathematical models were developed for various real-world problems by utilizing the field of fractional calculus with boundary conditions. Therefore, this field has received more attention in a variety of fields in diverse domains see [13, 14, 18, 32, 33]. In the development of fractional calculus, many researchers focused on developing certain new fractional integral operators and their applications in diverse fields. In practical applications, certain various types of fractional operators such as Riemann-Liouville, Caputo, Riesz [11, 35] and Hilfer [12] fractional operators are introduced. Freshly, the researchers have studied certain new fractional integral and derivative operators and their possible applications in various disciplines of sciences. Very recently Khalil et al. [9] have introduced the notion of fractional conformable derivative (FCD) operators with some shortcomings. Abdeljawad [1] investigated the properties of the fractional conformable derivative operators. In [8], Jarad et al. introduced the fractional conformable integral and derivative operators. Anderson and Unless [4] developed the idea of conformable derivative by employing local proportional derivatives. Abdeljawad and Baleanu [2] investigated certain monotonicity results for fractional difference operators with discrete exponential kernels. Abdeljawad and Baleanu [3] have established fractional derivative operators with exponential kernel and their discrete versions. In [5], Atangana and Baleanu defined a new fractional derivative operator with the non-local and non-singular kernel. Caputo and Fabrizio [6] defined fractional derivative without a singular kernel. Certain properties of fractional derivative without a singular kernel can be found in the work of Losada and Nieto [10]. In [7], Jarad et al. defined generalized fractional derivatives generated by a class of local proportional derivatives.

On the other side, the researchers studied the fractional operators and have investigated and studied fractional integral inequalities to examine the existence and uniqueness of such type of fractional boundary value problems [46–52].

The special functions such as Mittag-Leffler function and its extensions appear as solution of fractional order differential and integral equations. Various interesting applications of such functions can be found in the study of the telegraph equations, kinetic equation, Levy flights, random walks, complex system, non-equilibrium statistical mechanics, super diffuse transport and quantum mechanics. The interested readers may refer to the work [15, 26–28] and the references cited therein.

2. Preliminaries

In this section, we present some basic and well known results.

In [53], the Gosta Mittag-Leffler function $E_\rho(z)$ is defined by

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}, \quad (z \in \mathbb{C};\ \Re(\rho) > 0). \tag{2.1}$$

The Wiman Mittag-Leffler function $E_{\rho,\varphi}(z)$ is defined in [54] by

$$E_{\rho,\varphi}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \varphi)}, \quad (z, \varphi \in \mathbb{C};\ \Re(\rho) > 0). \tag{2.2}$$

The researchers extensively studied $E_\rho(z)$ and $E_{\rho,\varphi}(z)$ [17, 20–22, 24, 25, 34]. The numerical investigation of such functions in the complex plane can be found in the work of [23, 36]. Prabhakar [31] has proposed the following generalization of Mittag-Leffler function by

$$E^{\delta}_{\rho,\varphi}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\rho n + \varphi)} \frac{z^n}{n!}, \quad (z, \varphi \in \mathbb{C};\ \Re(\rho) > 0), \tag{2.3}$$

where $(\delta)_n$ denotes the well-known Pochhammer symbol.

Rahman et al. [29] recently defined an extension of Mittag-Leffler function by

$$E^{\delta}_{\rho,\varphi;p,v}(z) = \sum_{n=0}^{\infty} \frac{(\delta; p, v)_n}{\Gamma(\rho n + \varphi)} \frac{z^n}{n!}, \quad (z, \rho, \varphi \in \mathbb{C};\ \Re(\rho) > 0, \Re(\varphi) > 0, \Re(p) > 0, p, v \geq 0), \tag{2.4}$$

where $(\delta; p, v)_n$ is an extended Pochhammer symbol recently defined by Srivastava et al. [44] by

$$(\delta; p, v)_n = \begin{cases} \frac{\Gamma_v(\delta+n;p)}{\Gamma(\delta)}, (\Re(p) > 0, p, v \geq 0, \delta \in \mathbb{C}), \\ (\delta; p)_n, (v = 0, \delta \in \mathbb{C}). \end{cases} \tag{2.5}$$

An extension of gamma function $\Gamma_v(\delta; p)$ is defined by

$$\Gamma_v(\delta; p) = \sqrt{\frac{2p}{\pi}} \int_0^{\infty} t^{\delta-\frac{3}{2}} e^{-t} K_{v+\frac{1}{2}}(\frac{p}{t}) dt,$$

where $K_{v+\frac{1}{2}}(\frac{\delta}{t})$ is the modified Bessel function of the first kind (see, e.g., Chaudhry and Zubair [16]). If we consider $v = 0$ in (2.5), then we get the following extension of Pochhammer symbol defined by Srivastava et al. [45] as

$$(\delta; p)_n = \begin{cases} \frac{\Gamma(\delta+n;p)}{\Gamma(\delta)}, (\Re(p) > 0, p \geq 0, \delta \in \mathbb{C}), \\ (\delta)_n, (, \delta \in \mathbb{C}), \end{cases} \tag{2.6}$$

where $\Gamma(\delta; p)$ is defined by

$$\Gamma(\delta; p) = \int_0^{\infty} t^{\delta-1} e^{-t-\frac{p}{t}} dt.$$

Similarly, if we consider $p = 0$ in (2.6), then we get the following classical Pochhammer symbol defined by

$$(\delta)_n = \frac{\Gamma(\delta + n)}{\Gamma(\delta)} = \begin{cases} \delta(\delta + 1)(\delta + 2) \cdots (\delta + n - 1), (n \in \mathbb{N}, \delta \in \mathbb{C}), \\ 1, (n = 0, \delta \in \mathbb{C} \setminus 0), \end{cases}$$

where $\Gamma(\delta)$ is defined by

$$\Gamma(\delta) = \int_0^{\infty} t^{\delta-1} e^{-t} dt.$$

The extensions and applications of these Mittag-Leffler functions were extensively studied by the researchers ([19, 28, 37–41, 43]).

The Lebesgue measurable space for real (complex) valued functions is defined by

$$\mathsf{L}(x_1, x_2) = \left\{ h : \|h\|_1 = \int_{x_1}^{x_2} |h(x)| dx < \infty \right\}. \tag{2.7}$$

The Riemann-Liouville type (left and right sided) fractional integral operators are respectively defined in [11, 35] by

$$(\Im^{\kappa}_{x_1+} h)(x) = \frac{1}{\Gamma(\kappa)} \int_{x_1}^{x} \frac{h(v)}{(x - v)^{1-\kappa}} dv, \quad (x > x_1), \tag{2.8}$$

and

$$(\mathfrak{I}^{\kappa}_{x_2-}h)(x) = \frac{1}{\Gamma(\kappa)}\int\limits_{x}^{x_2}\frac{h(\nu)}{(\nu-x)^{1-\kappa}}d\nu, \quad (x < x_2),$$

where $h \in \mathsf{L}(x_1, x_2)$, $\kappa \in \mathbb{C}$ and $\Re(\kappa) > 0$.

The Riemann-Liouville fractional derivatives (left and right sided) for $h(x) \in \mathsf{L}(x_1, x_2)$, $\kappa \in \mathbb{C}$, $\Re(\kappa) > 0$ and $n = [\Re(\kappa)] + 1$ are defined in [11,35] by

$$(\mathfrak{D}^{\kappa}_{x_1+}h)(x) = (\frac{d}{dx})^n(\mathfrak{I}^{n-\kappa}_{x_1+}h)(x), \tag{2.9}$$

$$(\mathfrak{D}^{\kappa}_{x_2-}h)(x) = (-\frac{d}{dx})^n(\mathfrak{I}^{n-\kappa}_{x_2-}h)(x),$$

respectively. The generalized differential operator $\mathfrak{D}^{\kappa,\nu}_{x_1+}$ of order $0 < \kappa < 1$ and type $0 < \nu < 1$ with respect to x can be found in ([25,35]) as;

$$(\mathfrak{D}^{\kappa,\nu}_{x_1+}h) = \left(\ \mathfrak{I}^{\nu(1-\lambda)}_{x_1+}\tfrac{d}{dx}(\mathfrak{I}^{(1-\nu)(1-\lambda)}_{x_1+}h)\ \right)(x). \tag{2.10}$$

Clearly, if we put $\nu = 0$ then (2.10) reduces to the operator $\mathfrak{D}^{\kappa}_{x_1+}$ defined in (2.9).

Moreover, we consider the following well known results.

Theorem 2.1. *[26] The following results holds true:*

$$\mathfrak{I}^{\kappa}_{x_1+}(\nu - x_1)^{\varphi-1} = \frac{\Gamma(\varphi)}{\Gamma(\kappa+\varphi)}(x - x_1)^{\kappa+\varphi-1}, \tag{2.11}$$

where κ, $\varphi \in \mathbb{C}$, $\Re(\kappa) > 0$, $\Re(\varphi) > 0$,

Theorem 2.2. *[42] Suppose that the function* $h(z)$ *has a power series expansion* $h(z) = \sum\limits_{k=0}^{\infty} k_n z^k$ *and analytic in the disc* $|z| < R$, *then the following result holds:*

$${}_0\mathfrak{D}^{\kappa}_{z}\{z^{\varphi-1}h(z)\} = \frac{\Gamma(\varphi)}{\Gamma(\kappa+\varphi)}\sum_{n=0}^{\infty}\frac{a_n(\varphi)_n}{(\kappa+\varphi)_n}z^n.$$

Lemma 2.1. *[43] Suppose that* $x > x_1$, $\kappa \in (0, 1)$, $\nu \in [0, 1]$, $\Re(\varphi) > 0$ *and* $\Re(\kappa) > 0$, *then we have*

$$\mathfrak{D}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1}](x) = \frac{\Gamma(\varphi)}{\Gamma(\varphi-\kappa)}(x - x_1)^{\varphi-\kappa-1}. \tag{2.12}$$

3. Fractional integral and differential formulas of an extended Mittag-Leffler function

In this section, we present some fractional integration and differentiation formulas of an extended Mittag-Leffler function (2.4). We consider the following well-known result recently presented by Rahman et al. [29].

Theorem 3.1. *The following differentiation holds for* δ, κ, $\varphi \in \mathbb{C}$, $\Re(\varphi) > 0$, $\Re(\kappa) > 0$ *and* $n \in \mathbb{N}$,

$$(\frac{d}{dz})^n\left(z^{\kappa-1}E^{\delta}_{\rho,\varphi;p,\nu}(\omega z^{\rho})\right) = z^{\kappa-n-1}E^{\delta}_{\rho,\varphi-n;p,\nu}(\omega z^{\rho}). \tag{3.1}$$

Next, we present the following fractional calculus approach.

Theorem 3.2. *Suppose* $x > x_1 (x_1 \in \mathbb{R}_+ = [0, \infty))$*,* δ*,* κ*,* φ*,* $\omega \in \mathbb{C}$*,* $\Re(\varphi) > 0$ *and* $\Re(\kappa) > 0$*, then*

$$\mathfrak{I}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(\nu - x_1)^{\rho})](x) = (x - x_1)^{\kappa+\varphi-1}E^{\delta}_{\rho,\varphi+\kappa;p,v}(\omega(x - x_1)^{\rho}), \tag{3.2}$$

$$\mathfrak{D}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(x - x_1)^{\rho})] = (x - x_1)^{\varphi-\kappa-1}E^{\delta}_{\rho,\varphi-\kappa;p,v}(\omega(x - x_1)^{\rho}) \tag{3.3}$$

and

$$\mathfrak{D}^{\kappa,\nu}_{x_1+}[(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(x - x_1)^{\rho})](x) = (x - x_1)^{\varphi-\kappa-1}E^{\delta}_{\rho,\varphi-\kappa;p,v}(\omega(x - x_1)^{\rho}). \tag{3.4}$$

Proof. Consider

$$\begin{aligned}
&\mathfrak{I}^{\kappa}_{x_1+}[(\nu - x - 1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(x - x_1)^{\rho})] \\
=&\frac{1}{\Gamma(\kappa)}\int_{x_1}^{x}\frac{(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(\nu - x_1)^{\rho})}{(x - \nu)^{1-\kappa}}d\nu \\
=&\frac{1}{\Gamma(\kappa)}\sum_{n=0}^{\infty}\frac{(\delta; p, v)_n\omega^n}{\Gamma(\rho n + \varphi)n!}\int_{x_1}^{x}(\nu - x_1)^{\varphi+\rho n-1}(x - \nu)^{\kappa-1}d\nu \\
=&\sum_{n=0}^{\infty}\frac{(\delta; p, v)_n\omega^n}{\Gamma(\rho n + \varphi)n!}\left(\mathfrak{I}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi+\rho n-1}] \right).
\end{aligned}$$

By the use of (2.11), we have

$$\begin{aligned}
&\mathfrak{I}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(x - x_1)^{\rho})] \\
=&\sum_{n=0}^{\infty}\frac{(\delta; p, v)_n\omega^n}{\Gamma(\rho n + \varphi)n!}(x - x_1)^{\varphi+\kappa+\rho n-1}.\frac{\Gamma(\rho n + \varphi)}{\Gamma(\rho n + \varphi + \kappa)} \\
=&(x - x_1)^{\varphi+\kappa-1}\sum_{n=0}^{\infty}\frac{(\delta; p, v)_n}{\Gamma(\rho n + \varphi + \kappa)}\frac{[\omega(x - x_1)^{\rho}]^n}{n!} \\
=&(x - x_1)^{\varphi+\kappa-1}E^{\delta}_{\rho,\varphi+\kappa;p,v}(\omega(x - x_1)^{\rho}),
\end{aligned}$$

which completes the proof of (3.2).

Now, we have

$$\mathfrak{D}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(\nu - x_1)^{\rho})] = (\frac{d}{dx})^n\left\{ \mathfrak{I}^{n-\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(\nu - x_1)^{\rho})] \right\},$$

which on using (3.2) takes the following form:

$$\mathfrak{D}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(\nu - x_1)^{\rho})] = (\frac{d}{dx})^n\left\{ (x - x_1)^{\varphi-\kappa+n-1}E^{\delta}_{\rho,\varphi-\kappa+n;p,v}(\omega(x - x_1)^{\rho}) \right\}.$$

Applying (3.1), we have

$$\mathfrak{D}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(\nu - x_1)^{\rho})](x) = \left\{ (x - x_1)^{\varphi-\kappa-1}E^{\delta}_{\rho,\varphi-\kappa;p,v}(\omega(\nu)^{\rho}) \right\}.$$

This completes the proof of (3.3).

To prove (3.4), we have

$$\begin{aligned}
&\left(\mathfrak{D}^{\kappa,\nu}_{x_1+}[(\nu - x_1)^{\varphi-1} E^{\delta}_{\rho,\varphi;p,\nu}(\omega(\nu - x_1)^{\rho})] \right)(x)\\
=&\left(\mathfrak{D}^{\kappa,\nu}_{x_1+}[\sum_{n=0}^{\infty} \tfrac{(\delta;p,\nu)_n}{\Gamma(\rho n+\varphi)} \tfrac{\omega^n}{n!} (\nu - x_1)^{\rho n+\varphi-1}] \right)(x)\\
=&\sum_{n=0}^{\infty} \frac{(\delta; p, \nu)_n}{\Gamma(\rho n + \varphi)} \frac{\omega^n}{n!} \left(\mathfrak{D}^{\kappa,\nu}_{x_1+}[(\nu - x_1)^{\rho n+\varphi-1}] \right)(x).
\end{aligned}$$

By applying (2.12), we get

$$\begin{aligned}
&\left(\mathfrak{D}^{\kappa,\nu}_{x_1+}[(\nu - x_1)^{\varphi-1} E^{\delta}_{\rho,\varphi;p,\nu}(\omega(\nu - x_1)^{\rho})] \right)(x)\\
=&\sum_{n=0}^{\infty} \frac{(\delta; p, \nu)_n}{\Gamma(\rho n + \varphi)} \frac{\omega^n}{n!} \frac{\Gamma(\rho n + \varphi)}{\Gamma(\rho n + \varphi - \kappa)} (x - x_1)^{\rho n+\varphi-\kappa-1}\\
=&(x - x_1)^{\varphi-\kappa-1} \sum_{n=0}^{\infty} \frac{(\delta; p, \nu)_n}{\Gamma(\rho n + \varphi - \kappa)} \frac{[\omega(x - x_1)^{\rho}]^n}{n!}\\
=&(x - x_1)^{\varphi-\kappa-1} E^{\delta}_{\rho,\varphi-\kappa;p,\nu}(\omega(x - x_1)^{\rho}),
\end{aligned}$$

which completes the required proof. □

If we consider $\nu = 0$, then we get the following result of Parmar et al. [30].

Corollary 3.1. *Suppose* $x > x_1 (x_1 \in \mathbb{R}_+ = [0, \infty))$, δ, κ, φ, $\omega \in \mathbb{C}$, $\Re(\varphi) > 0$ *and* $\Re(\kappa) > 0$, *then*

$$\mathfrak{I}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1} E^{\delta}_{\rho,\varphi;p}(\omega(\nu - x_1)^{\rho})](x) = (x - x_1)^{\kappa+\varphi-1} E^{\delta}_{\rho,\varphi+\kappa;p}(\omega(x - x_1)^{\rho}),$$

$$\mathfrak{D}^{\kappa}_{x_1+}[(\nu - x_1)^{\varphi-1} E^{\delta}_{\rho,\varphi;p}(\omega(x - x_1)^{\rho})] = (x - x_1)^{\varphi-\kappa-1} E^{\delta}_{\rho,\varphi-\kappa;p}(\omega(x - x_1)^{\rho})$$

and

$$\mathfrak{D}^{\kappa,\nu}_{x_1+}[(\nu - x_1)^{\varphi-1} E^{\delta}_{\rho,\varphi;p}(\omega(x - x_1)^{\rho})](x) = (x - x_1)^{\varphi-\kappa-1} E^{\delta}_{\rho,\varphi-\kappa;p}(\omega(x - x_1)^{\rho}).$$

Remark 3.1. *Put* $p = 0$ *in Corollary 3.1, then we lead to the work presented by Prabhakar [31].*

4. Extension of Prabhakar type fractional integral and its properties

In this section, we estimate a new extension of Prabhakar type fractional integral and its properties.

Definition 4.1. *Let* $\varphi, \rho, \delta, \omega \in \mathbb{C}$, $\Re(\delta) > 0$, $\Re(\rho) > 0$, $\Re(\varphi) > 0$. *Then the new Prabhakar left and right sided fractional integral are defined by*

$$({}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi} h)(x) = \int_{x_1}^{x} (x - \nu)^{\varphi-1} E^{\delta}_{\rho,\varphi;p,\nu}(\omega(x - \nu)^{\rho}) h(\nu) d\nu, x > x_1 \tag{4.1}$$

and

$$({}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_2-;\rho,\varphi} h)(x) = \int_{x}^{x_2} (\nu - x)^{\varphi-1} E^{\delta}_{\rho,\varphi;p,\nu}(\omega(\nu - x_2)^{\rho}) h(\nu) d\nu, x < x_2, \tag{4.2}$$

respectively. Substituting $v = 0$, then (4.1) *and* (4.2) *reduce to the following operators*

$$({}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h)(x) = \int_{x_1}^{x}(x-\nu)^{\varphi-1}E^{\delta}_{\rho,\varphi;p}(\omega(x-\nu)^{\rho})h(\nu)d\nu, x > x_1 \tag{4.3}$$

and

$$({}_{p}\varepsilon^{\omega;\delta}_{x_2-;\rho,\varphi}h)(x) = \int_{x}^{x_2}(\nu-x)^{\varphi-1}E^{\delta}_{\rho,\varphi;p}(\omega(\nu-x_2)^{\rho})h(\nu)d\nu, x < x_2. \tag{4.4}$$

When we substitute $p = 0$, then (4.3) *and* (4.4) *reduce to the following Prabhakar operators*

$$(\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h)(x) = \int_{x_1}^{x}(x-\nu)^{\varphi-1}E^{\delta}_{\rho,\varphi}(\omega(x-\nu)^{\rho})h(\nu)d\nu, x > x_1 \tag{4.5}$$

and

$$(\varepsilon^{\omega;\delta}_{x_2-;\rho,\varphi}h)(x) = \int_{x}^{x_2}(\nu-x)^{\varphi-1}E^{\delta}_{\rho,\varphi}(\omega(\nu-x_2)^{\rho})h(\nu)d\nu, x < x_2. \tag{4.6}$$

If we consider $\omega = 0$, then the integral operators in (4.5) *and* (4.6) *reduce to the well-known Riemann-Liouville fractional integral operators.*

Next, we present the following results.

Theorem 4.1. *Suppose that $\varphi, \rho, \kappa, \delta, \omega \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\varphi) > 0$, $\Re(\kappa) > 0$, $p, v \geq 0$ and $\Re(\delta) > 0$, then the following result holds:*

$$\left({}^{v}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}[(\nu-x_1)^{\kappa-1}]\right)(x) = (x-x_1)^{\kappa+\varphi-1}\Gamma(\kappa)E^{\delta;p,v}_{\rho,\varphi+\kappa}(\omega(x-x_1)^{\rho}).$$

Proof. From definition of Prabhakar fractional integral (4.1)

$$\left({}^{v}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h\right)(x) = \int_{x_1}^{x}(x-\nu)^{\varphi-1}E^{\delta}_{\rho,\varphi}(\omega(x-\nu)^{\rho})h(\nu)d\nu.$$

Therefore, we have

$$\begin{aligned}\left({}^{v}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}[(\nu-x_1)^{\kappa-1}]\right)(x) &= \int_{x_1}^{x}(x-\nu)^{\varphi-1}(\nu-x_1)^{\kappa-1}E^{\delta;p,v}_{\rho,\varphi}(\omega(x-\nu)^{\rho})d\nu \\ &= \sum_{n=0}^{\infty}\frac{(\delta;p,v)_n}{\Gamma(\rho n+\varphi))}\frac{\omega^n}{n!}\left(\int_{x_1}^{x}(\nu-x_1)^{\kappa-1}(x-\nu)^{\kappa+\rho n-1}d\nu\right) \\ &= \sum_{n=0}^{\infty}\frac{(\delta;p,v)_n}{\Gamma(\rho n+\varphi))}\frac{\omega^n}{n!}\Im^{\rho n+\varphi}_{x_1+}[(\nu-x_1)^{\kappa-1}] \\ &= (x-x_1)^{\varphi+\kappa-1}\left\{\sum_{n=0}^{\infty}\frac{(\delta;p,v)_n}{\Gamma(\rho n+\varphi)}\frac{[\omega(x-x_1)^{\rho}]^n}{n!}\frac{\Gamma(\kappa)\Gamma(\rho n+\varphi)}{\Gamma(\rho n+\varphi+\kappa)}\right\} \\ &= (x-x_1)^{\varphi+\kappa-1}\Gamma(\kappa)E^{\delta;p,v}_{\rho,\varphi+\kappa}(\omega(x-x_1)^{\rho}),\end{aligned}$$

which completes the desired proof. □

Theorem 4.2. *Suppose that $\delta, \rho, \varphi, \omega, \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\varphi) > 0$ $v, p \geq 0$, then the following result holds:*

$$\|{}_p^v\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}\Phi\|_1 \leq M\|\Phi\|_1,$$

where

$$M = (x_2 - x_1)^{\Re e(\varphi)} \sum_{n=0}^{\infty} \frac{|(\delta; p, v)_n|}{\Gamma(\rho n + \varphi)(\Re(\varphi) + \Re(\rho)n)} \frac{|\omega(x_2 - x_1)^{\rho}|^n}{n!}.$$

Proof. Consider (4.1) and (2.7) and interchanging the order of integrations, we obtain

$$\begin{aligned}
\|{}_p^v\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}\Phi\|_1 &= \int_{x_1}^{x_2} |\int_{x_1}^{x} (x-v)^{\varphi-1} E_{\rho,\varphi}^{\delta}(\omega(x-v)^{\rho})\Phi(v)dv|dx \\
&\leq \int_{x_1}^{x_2} \left[\int_{v}^{x} (x-v)^{\Re e(\varphi)-1} |E_{\rho,\varphi;p,v}^{\delta}(\omega(x-v)^{\rho})|dx \right] |\Phi(v)|dv \\
&= \int_{x_1}^{x_2} \left[\int_{0}^{x-v} (u)^{\Re e(\varphi)-1} |E_{\rho,\varphi;p,v}^{\delta}(\omega(u)^{\rho})|du \right] |\Phi(v)|dv, \text{(By setting } (x-v) = u) .
\end{aligned}$$

After simplification, we obtain

$$\|{}_p^v\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}\Phi\|_1 \leq \int_{x_1}^{x_2} \left[\sum_{n=0}^{\infty} \frac{|(\delta;p,v)_n|}{\Gamma(\rho n+\varphi)} \frac{|\omega^n|}{n!} \left(\frac{(u)^{\Re(\varphi)+\Re(\rho)n}}{(\Re(\varphi)+\Re(\rho)n)} \right)_0^{x_2-x_1} \right] |\Phi(v)|dv.$$

This can further be expressed as

$$\begin{aligned}
\|{}_p^v\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}\phi\|_1 &\leq \left\{ (x_2 - x_1)^{\Re e(\varphi)} \sum_{n=0}^{\infty} \frac{|(\delta;p,v)_n|}{\Gamma(\rho n+\varphi)(\Re(\varphi)+\Re(\rho)n)} \frac{|\omega(x_2-x_1)^{\rho}|^n}{n!} \right\} . \int_{x_1}^{x_2} |\Phi(v)|dv \\
&= M\|\Phi\|_1,
\end{aligned}$$

where

$$M = (x_2 - x_1)^{\Re e(\varphi)} \sum_{n=0}^{\infty} \frac{|(\delta; p, v)_n|}{\Gamma(\rho n + \varphi)(\Re(\varphi) + \Re(\rho)n)} \frac{|\omega(x_2 - x - 1)^{\rho}|^n}{n!},$$

which completes the desired result. □

Corollary 4.1. *The following result holds for $\rho, \varphi, \delta \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\varphi) > 0$, $\Re(\delta) > 0$,*

$$\left({}_p^v\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}1\right)(x) = (x - x_1)^{\varphi} E_{\rho,\varphi+1;p,v}^{\delta}(\omega(x - x_1)^{\rho}).$$

Proof. Consider

$$\left({}_{p}^{v}\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}1\right)(x) = \int_{x_1}^{x}(x-\nu)^{\varphi-1}E_{\rho,\varphi;p,v}^{\delta}(\omega(x-\nu)^{\rho})d\nu$$
$$= \int_{x_1}^{x}(x-\nu)^{\varphi-1}\left(\sum_{n=0}^{\infty}\frac{(\delta;p,v)_n\omega^n(x-\nu)^{n\rho}}{\Gamma(n\rho+\varphi)n!}\right)d\nu.$$

Interchanging the order of summation and integration, we have

$$\left({}_{p}^{v}\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}1\right)(x) = \sum_{n=0}^{\infty}\frac{(\delta;p,v)_n\omega^n}{\Gamma(n\rho+\varphi)n!}\int_{x_1}^{x}(x-\nu)^{\varphi+n\rho-1}d\nu$$
$$= (x-x_1)^{\varphi}\sum_{n=0}^{\infty}\frac{(\delta;p,v)_n\omega^n(x-x_1)^{n\rho}}{\Gamma(n\rho+\varphi)(n\rho+\varphi)n!}$$
$$= (x-x_1)^{\varphi}E_{\rho,\varphi+1;p,v}^{\delta}(\omega(x-x_1)^{\rho}),$$

which completes the desired proof. □

Theorem 4.3. *The extended Prabhakar operator can be expressed in the following series of Riemann-Liouville fractional integrals for $\delta,\ \rho,\ \varphi,\ \omega \in \mathbb{C}$ with $\Re(\rho) > 0,\ \Re(\varphi) > 0,\ \Re(\delta) > 0,\ p, v \geq 0$ and $x > x_1$,*

$$\left({}_{p}^{v}\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}h\right)(x) = \sum_{n=0}^{\infty}\frac{\Gamma_v(\delta+n;p)\omega^n}{\Gamma(\delta)}\Im_{x_1+}^{\rho n+\varphi}h(x).$$

Proof. By using (2.4) in (4.1) and then interchanging the order of summation and integration, we obtain

$$\left({}_{p}^{v}\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}h\right)(x) = \int_{x_1}^{x}(x-\nu)^{\varphi-1}E_{\rho,\varphi;p,v}^{\delta}(\omega(x-\nu)^{\rho})h(\nu)d\nu$$
$$= \int_{x_1}^{x}(x-\nu)^{\varphi-1}\sum_{n=0}^{\infty}\frac{\Gamma_v(\delta+n;p)\omega^n(x-\nu)^{\rho n}}{\Gamma(\rho n+\varphi)}h(\nu)d\nu$$
$$= \sum_{n=0}^{\infty}\frac{\Gamma_v(\delta+n;p)\omega^n}{\Gamma(\delta)}\frac{1}{\Gamma(\rho n+\varphi)}\int_{x_1}^{x}(x-\nu)^{\rho n+\varphi-1}h(\nu)d\nu$$
$$= \sum_{n=0}^{\infty}\frac{\Gamma_v(\delta+n;p)\omega^n}{\Gamma(\delta)}\Im_{x_1+}^{\rho n+\varphi}h(x),$$

which completes the desired proof. □

Theorem 4.4. *The following result satisfies for $\kappa,\ \delta,\ \rho,\ \varphi,\ \omega \in \mathbb{C}$ with $\Re(\rho) > 0,\ \Re(\varphi) > 0,\ \Re(\delta) > 0,\ \Re(\kappa) > 0,\ p, v \geq 0$ and $x > x_1$,*

$$(\Im_{x_1+}^{\kappa}[{}_{p}^{v}\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}h])(x) = ({}_{p}^{v}\varepsilon_{x_1+;\rho,\varphi+\kappa}^{\omega;\delta}h)(x) = ({}_{p}^{v}\varepsilon_{x_1+;\rho,\varphi}^{\omega;\delta}[\Im_{x_1+}^{\kappa}h])(x), \tag{4.7}$$

for arbitrary function $h \in L(x_1, x_2)$.

Proof. By utilizing (4.1) and (2.8), we have

$$
\begin{aligned}
(\Im^{\kappa}_{x_1+}[{}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h])(x) =& \frac{1}{\Gamma(\kappa)}\int_{x_1}^{x}\frac{[({}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h)(\nu)]}{(x-\nu)^{1-\kappa}}d\nu\\
=& \frac{1}{\Gamma(\kappa)}\int_{x_1}^{x}(x-\nu)^{\kappa-1}\left[\ \int_{x_1}^{\nu}(\nu-u)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,\nu}(\omega(\nu-u)^{\rho})h(u)du\ \right]d\nu.
\end{aligned}
$$

By interchanging the order of integrations, we have

$$
(\Im^{\kappa}_{x_1+}[\,{}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h])(x) = \int_{x_1}^{x}\left[\frac{1}{\Gamma(\kappa)}\int_{u}^{x}(x-\nu)^{\kappa-1}(\nu-u)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,\nu}(\omega(\nu-u)^{\rho})d\nu\right]h(u)du.
$$

By putting $\nu - u = \theta$, we obtain

$$
\begin{aligned}
(\Im^{\kappa}_{x_1+}[{}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h])(x) &= \int_{x_1}^{x}\left[\ \frac{1}{\Gamma(\kappa)}\int_{0}^{x-u}(x-u-\theta)^{\kappa-1}(\theta)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,\nu}(\omega(\theta)^{\rho})d\theta\ \right]h(u)du\\
&= \int_{x_1}^{x}\left[\ \frac{1}{\Gamma(\kappa)}\int_{0}^{x-u}\frac{(\theta)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,\nu}(\omega(\theta)^{\rho})}{(x-u-\theta)^{1-\kappa}}d\theta\ \right]h(u)du.
\end{aligned}
$$

Thus by the use of (2.8) and applying (3.2), we obtain

$$
\begin{aligned}
(\Im^{\kappa}_{x_1+}[{}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h])(x) &= \int_{x_1}^{x}\left[\ (\theta)^{\kappa+\varphi-1}E^{\delta}_{\rho,\varphi+\kappa;p,\nu}(\omega(\theta)^{\rho})\ \right]h(u)du\\
&= \int_{x_1}^{x}(x-u)^{\kappa+\varphi-1}E^{\delta}_{\rho,\varphi+\kappa}(\omega(x-u)^{\rho})h(u)du.
\end{aligned}
$$

Thus, we obtain

$$
(\Im^{\kappa}_{x_1+}[{}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}h])(x) = \left({}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi+\kappa}h\right)(x). \tag{4.8}
$$

Next, consider the right hand side of (4.7) and utilizing (4.1) to derive the second part, we have

$$
\begin{aligned}
({}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}[\Im^{\kappa}_{x_1+}h])(x) &= \int_{x_1}^{x}(x-\nu)^{\varphi-1}E^{\delta}_{\rho,\varphi;p,\nu}(\omega(x-\nu)^{\rho})[\Im^{\kappa}_{x_1+}h](\nu)d\nu\\
&= \int_{x_1}^{x}E^{\delta}_{\rho,\varphi;p,\nu}(\omega(x-\nu)^{\rho})\left(\ \frac{1}{\Gamma(\kappa)}\int_{x_1}^{\nu}\frac{f(u)}{(\nu-u)^{1-\kappa}}du\ \right)d\nu.
\end{aligned}
$$

By interchanging the order of integrations, we obtain

$$
({}^{\nu}_{p}\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}[\Im^{\kappa}_{x_1+}h])(x) = \int_{x_1}^{x}\frac{1}{\Gamma(\kappa)}\left[\ \int_{u}^{x}(x-\nu)^{\varphi-1}(\nu-u)^{\kappa-1}E^{\delta}_{\rho,\varphi;p,\nu}(\omega(x-\nu)^{\rho})d\nu\ \right]h(u)du.
$$

By setting $x-\tau=\theta$, we have

$$\begin{aligned}({}_p^v\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}[\Im^{\kappa}_{x_1+}h])(x) &= \int_{x_1}^{x} \frac{1}{\Gamma(\kappa)}\left[\int_{x-u}^{0}(\theta)^{\varphi-1}(x-\theta-u)^{\kappa-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(\theta)^{\rho})(-d\theta)\right]h(u)du \\ &= \int_{a}^{x} \frac{1}{\Gamma(\kappa)}\left[\int_{0}^{x-u}(\theta)^{\varphi-1}(x-\theta-u)^{\kappa-1}E^{\delta}_{\rho,\varphi;p,v}(\omega(\theta)^{\rho})d\theta\right]h(u)du.\end{aligned}$$

Further, by using (2.8) and applying (3.2), we obtain

$$({}_p^v\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi}[\Im^{\kappa}_{x_1+}h])(x) = \left({}_p^v\varepsilon^{\omega;\delta}_{x_1+;\rho,\varphi+\kappa}h\right)(x). \tag{4.9}$$

Thus (4.8) and (4.9) complete the desired proof. □

5. Conclusions

In this article, we have investigated fractional integral and differential formulas for the extended Mittag-Leffler function. Also, we defined a further extension of Prabhakar fractional operator and investigated its various properties. The special cases of our result can be found in the literature cited therein. One can easily investigate many of its applications and develop various fractional integral inequalities by utilizing the newly defined Prabhakar fractional operator.

Authors contributions

All the authors contributed equally and they read and approved the final manuscript for publication.

Conflict of interest

The authors declare that they have no competing interests.

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