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Research article

Some new generalizations of $F-$contraction type mappings that weaken certain conditions on Caputo fractional type differential equations

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Abstract: In this paper, firstly, we introduce some new generalizations of $F-$contraction, $F-$Suzuki contraction, and $F-$expanding mappings. Secondly, we prove the existence and uniqueness of the fixed points for these mappings. Finally, as an application of our main result, we investigate the existence of a unique solution of an integral boundary value problem for scalar nonlinear Caputo fractional differential equations with a fractional order (1,2).

Keywords: generalized $F-$contraction; generalized $F-$Suzuki contraction; generalized F-expanding mapping; fixed point
Mathematics Subject Classification: 47H10, 47H19, 54H25

1. Introduction

The Banach contraction principle generally known as Banach fixed-point theorem emerged in 1922 [1], and due to its coherence and effectiveness, it has turned out to be a very popular tool in several branches of mathematical analysis for solving the existence problems. Numerous researchers studied the Banach fixed point theorem in different directions and established the extensions, generalizations and the applications of their findings. Among them, Wardowski [2] provided very interesting extension of Banach's fixed point theorem.

Definition 1.1. [2] Let a function $F : (0, \infty) \mapsto \mathbb{R}$ satisfy the following conditions:

$(F1)$ F is strictly increasing, i.e. for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$.

$(F2)$ For each sequence $\{\alpha_n\}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim\limits_{n\to\infty} F(\alpha_n) = -\infty$.

$(F3)$ There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

The set of all functions F satisfying $(F1) - (F3)$ will be denoted by $\mathcal{F}$.

Example 1.1. [2] Suppose that the functions $F_i : (0, \infty) \mapsto \mathbb{R}$, $i = 1, 2, 3, 4$ are defined by
(1) $F_1 : t \to \ln t$.
(2) $F_2 : t \to t + \ln t$.
(3) $F_3 : t \to \frac{-1}{\sqrt{t}}$.
(4) $F_4 : t \to \ln(t^2 + t)$.
Then, $F_1, F_2, F_3, F_4 \in \mathcal{F}$.
Definition 1.2. [2] Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be a $F-$contraction on $(X, \mathcal{D})$ if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $u, v \in X$,

$$Tu \neq Tv \Rightarrow \tau + F(\mathcal{D}(Tu, Tv)) \leq F(\mathcal{D}(u, v)).$$

Note that we have $\mathcal{D}(Tu, Tv) < \mathcal{D}(u, v)$ for all $u, v \in X$ with $Tu \neq Tv$ concluding that every $F-$contraction is a contractive mapping.

Wardowski [2] proved the existence of a unique fixed point in a complete metric space for every $F-$contraction mapping T. He also showed that $F-$contractions are the generalizations of Banach contractions.

Theorem 1.1. [2] Let T be a self-mapping on a complete metric space $(X, \mathcal{D})$. If T forms a $F-$contraction, then it possesses a unique fixed point u^*. Moreover, for any $u \in X$ the sequence $\{T^n u\}$ is convergent to u^*.

After the development of $F-$contractions, several authors looked into the necessity of the conditions $(F1)$ – $(F3)$ and presented some weaken conditions by replacing or removing some of them.

We briefly present some existing cases. Secelean [3] suggested that the condition $(F2)$ can be replaced by a simple one:
$(F2')$ $\inf F = -\infty$, or, also by
$(F2'')$ there exists a sequence $\{\alpha_n\}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Secelean [4] also removed condition $(F3)$ on the operator T by assuming some boundedness condition. Moreover he also proved that $(F3)$ can be dropped without any additional supposition on T. Piri and Kumam [5] replaced condition $(F3)$ by the continuity of F. Vetro [6] replaced the constant τ with a function and generalized the $F-$contraction. Secelean and Wardowski [7] introduced $\psi F-$contraction and weak $\psi F-$contraction by weakening condition $(F1)$ and introducing the class of increasing functions ψ. Further, Lukács and Kajántó [8] found some results in $b-$metric spaces of $F-$contraction by omitting condition $(F2)$. Alsulami [9] introduced generalized $F-$Suzuki contraction in $b-$metric spaces and established the existence of fixed points by using the conditions $(F1)$ and $(F2)$ only.

Definition 1.3. Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be a F-Suzuki contraction if there exist a real number $\tau > 0$ such that for all $u, v \in X$,

$$\frac{1}{2}\mathcal{D}(u, Tu) < \mathcal{D}(u, v) \quad \text{implies} \quad \tau + F(\mathcal{D}(Tu, Tv)) \leq F(\mathcal{D}(u, v)),$$

where, $F \in \mathcal{F}$.

On the other hand, in 2017, Gornicki [10] introduced a new type of mappings called F-expanding mappings and proved some new fixed point results for this new kind of mapping, especially on a complete $G-$metric space.

Definition 1.4. [10] Let $(X,\mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is called $F-$expanding if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $u, v \in X$

$$\mathcal{D}(u,v) > 0 \Rightarrow F(\mathcal{D}(Tu,Tv)) \geq F(\mathcal{D}(u,v)) + \tau.$$

Theorem 1.2. [10] Let $(X,\mathcal{D})$ be a complete metric space and $T : X \mapsto X$ be surjective and $F-$expanding. Then T has a unique fixed point.

In 2001, James Merryfield [11] established a generalization of Banach contraction principal by following conjecture named as generalized Banach contraction conjecture.

Theorem 1.3. [11] Let $(X,\mathcal{D})$ be a complete metric space. Suppose $T : X \to X$ satisfies the following condition:

There exists an integer p and a number $k \in [0,1)$ such that for all $u, v \in X$ we have

$$\min\left\{\mathcal{D}\left(T^i u, T^i v\right) : i = 1, \ldots p\right\} \leq k\mathcal{D}(u,v).$$

Then, T has exactly one fixed point.

Following the Wardowski's idea along with the conjecture presented by James Merryfield [11], in this paper, we introduce some new generalizations of $F-$contraction, F-Suzuki contraction and $F-$expanding mappings and prove the existence of their unique fixed points. Moreover, as an application of our main result, we investigate the existence of unique solution of the nonlinear Caputo fractional differential equations.

Recently Proinov [12] proved some fixed point theorems that extend earlier results of Moradi [13], Geraghty [14], Amini- Harandi and Petruşel [15], Li and Jiang [16], Jleli and Samet [17], Wardowski [2], Wardowski and Van Dung [18], Piri and Kumam [5], Secelean [7], Lukács and Kaj́anto [8] and others. He also proved that the fixed point theorems of Wardowski [2] and Jleli and Samet [17] are equivalent to a special case of the well-known fixed point theorem of Skof [19].

Proinov established the following fixed point theorem for a self-mapping T on a complete metric space $(X,\mathcal{D})$.

Theorem 1.4. [12] Let $(X,\mathcal{D})$ be a metric space and $T : X \mapsto X$ be a mapping such that $F_1(\mathcal{D}(Tu,Tv)) \leq F(\mathcal{D}(u,v))$ for all $u, v \in X$,

where, the functions $F_1, F : (0,\infty) \to \mathbb{R}$ satisfy the following conditions.

i) F_1 is nondecreasing;

ii) $F(t) < F_1(t)$ for $t > 0$;

iii) $\limsup_{t\to\varepsilon+} F(t) < F_1(\varepsilon+)$ for any $\varepsilon > 0$.

Then T has a unique fixed point $u^* \in X$ and the iterative sequence $\{T^n u\}$ converges to u^* for every $u \in X$.

Proinov also obtained the following improvement of Wardowski's fixed point theorem.

Theorem 1.5. Let $(X,\mathcal{D})$ be a metric space and $T : X \mapsto X$ be a mapping such that

$$F(\mathcal{D}(Tu,Tv)) \leq F(\mathcal{D}(u,v)) - \tau, \text{ for all } u, v \in X, \text{ with } \mathcal{D}(Tu,Tv) > 0, \tag{1.1}$$

where, $\tau > 0$ and the function $F : (0,\infty) \to \mathbb{R}$ is nondecreasing. Then T has a unique fixed point $u^* \in X$ and the sequence $\{T^n u\}$ converges to u^* for every $u \in X$.

In this paper, we establish a fixed point theorem using a certain condition that generalizes the main contractive-type conditions used by Wardowski and Proinov.

2. Main results

We start this section by introducing some new types of generalized $F-$contraction and generalized $F-$Suzuki contraction mappings.
Let $F : (0, \infty) \mapsto \mathbb{R}$ satisfy the following conditions:
(F_1^*) For some $t > 0$ and $0 < F(t) < t$, if $0 < \alpha < \beta$, we have $F(\alpha) < F(\beta)$ and $F(\beta) \leq F(\alpha) + F(t)$.
$(F_{1'}^*)$ For some $t > 0$ and $0 < F(t) < t$, if $0 < \alpha < \beta$, we have $F(\alpha) < F(\beta)$.
(F_2^*) $\lim_{n\to\infty} F(\alpha_n) = -\infty \Leftrightarrow \lim_{n\to\infty} \alpha_n = 0$, for each positive number sequence $\{\alpha_n\}$.
(F_3^*) F is continuous on $(0, \infty)$.
Let us denote by $\mathfrak{F}$ the set of all functions F satisfying the conditions $(F_1^*), (F_2^*), (F_3^*)$ and denote by $\mathfrak{T}$ the set of all functions F satisfying the conditions $(F_{1'}^*), (F_2^*), (F_3^*)$.
Example 2.1. Define $F : (0, \infty) \mapsto \mathbb{R}$ by
$$F(u) = \begin{cases} \ln u, & 0 < u \leq 1 \\ 1 - \frac{1}{u}, & u > 1 \end{cases}.$$
Note that F is strictly increasing with an upper bound $F(t) = 1$. Therefore, for some $t > 0$, $0 < F(t) < t$, and $0 < \alpha < \beta$, we have $F(\alpha) < F(\beta) \leq F(\alpha) + F(t)$ and $F \in \mathfrak{F}$. Further, we can easily check that the conditions $(F_{1'}^*)$, (F_2^*) and (F_3^*) are also satisfied, hence $F \in \mathfrak{T}$.
Definition 2.1. Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \to X$ is said to be a generalized F-contraction if there exist $\tau > 0$ and an integer $p > 1$ such that for all $u, v \in X$,
$$\mathcal{D}(T^i u, T^i v) > 0 \Rightarrow \tau + F(\min\{\mathcal{D}(T^i u, T^i v)\}) \leq F(\mathcal{D}(u, v)), \quad i = 1, \ldots, p, \tag{2.1}$$
where $F \in \mathfrak{F}$.
Definition 2.2. Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be a generalized F-Suzuki contraction if there exist $\tau > 0$ and an integer $p > 1$ such that for all $u, v \in X$, with $u \neq v$
$$\begin{aligned} &\frac{1}{2}\mathcal{D}(T^{i-1}u, T^i u) < \mathcal{D}(T^{i-1}u, v) \\ &\Rightarrow \tau + F(\min\{\mathcal{D}(T^i u, T^i v)\}) \leq F(\mathcal{D}(u, v)), \quad i = 1, ..., p, \end{aligned}$$
where, $F \in \mathfrak{T}$.
Theorem 2.1. Let $(X, \mathcal{D})$ be a complete metric space and $T : X \to X$ be continuous generalized F-contraction mapping with $p = 2$. Then T has a unique fixed point $u^* \in X$ and for every $u_0 \in X$ the sequence $\{T^n u_0\}_{n=1}^{\infty}$ converges to the fixed point.
Proof. Let $u_0 \in X$ be an arbitrary point and define a sequence $\{u_n\} \subseteq X$ by $u_{n+1} = Tu_n = T^{n+1}u_0$, for all $n \in \mathbb{N}$.
Now, we will prove that $\lim_{n\to\infty} \mathcal{D}(u_n, Tu_n) = 0$.
If $u_{n_0} = Tu_{n_0}$ for some $n_0 \in \mathbb{N}$, then $\mathcal{D}(u_n, Tu_n) = \mathcal{D}(u_{n+1}, Tu_{n+1}) = \cdots = 0$, for all $n \geq n_0$ and so $\mathcal{D}(u_n, Tu_n)$ converges to 0, as $n \to \infty$.
Assume that $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}$.
By the contraction assumption on T, there exits $\tau > 0$ such that
$$\tau + F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \leq F(\mathcal{D}(u_{n-1}, u_n)),$$
or
$$F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \leq F(\mathcal{D}(u_{n-1}, u_n)) - \tau. \tag{2.2}$$

If $\mathcal{D}(u_{n-1}, u_n) = \min\{\mathcal{D}(Tu_{n-1}, Tu_n), \mathcal{D}(u_{n-1}, u_n)\}$, the inequality (2.2) will take the form as follows

$$\begin{aligned} &F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \\ &\le F(\min\{\mathcal{D}(Tu_{n-1}, Tu_n), \mathcal{D}(u_{n-1}, u_n)\}) - \tau, \end{aligned}$$

which also can be written as

$$\begin{aligned} &F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \\ &\le F(\min\{\mathcal{D}(T^2u_{n-2}, T^2u_{n-1}), \mathcal{D}(Tu_{n-2}, Tu_{n-1})\}) - \tau. \end{aligned} \tag{2.3}$$

If $\mathcal{D}(Tu_{n-1}, Tu_n) = \min\{\mathcal{D}(Tu_{n-1}, Tu_n), \mathcal{D}(u_{n-1}, u_n)\}$.
Then, by condition (F_1^*), we have

$$F(\mathcal{D}(u_{n-1}, u_n)) \le F(\mathcal{D}(Tu_{n-1}, Tu_n)) + F(\tau).$$

The above inequality together with inequality (2.1) yields that

$$\begin{aligned} &F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \\ &\le F(\mathcal{D}(Tu_{n-1}, Tu_n)) + F(\tau) - \tau, \end{aligned} \tag{2.4}$$

which is equivalent to,

$$\begin{aligned} &F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \\ &\le F(\min\{\mathcal{D}(T^2u_{n-2}, T^2u_{n-1}), \mathcal{D}(Tu_{n-2}, Tu_{n-1})\}) + F(\tau) - \tau. \end{aligned} \tag{2.5}$$

Combining (2.3) and (2.5) we have,

$$\begin{aligned} &F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \\ &\le F(\min\{\mathcal{D}(T^2u_{n-2}, T^2u_{n-1}), \mathcal{D}(Tu_{n-2}, Tu_{n-1})\}) + \delta_n F(\tau) - \tau, \end{aligned}$$

where, $\delta_n = \begin{cases} 1, & \mathcal{D}(u_{n-1}, u_n) > \mathcal{D}(Tu_{n-1}, Tu_n) = \mathcal{D}(u_n, u_{n+1}) \\ 0, & \mathcal{D}(u_{n-1}, u_n) < \mathcal{D}(Tu_{n-1}, Tu_n) = \mathcal{D}(u_n, u_{n+1}) \end{cases}$.
Repeating this process we get,

$$\begin{aligned} &F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \\ &\le F(\min\{\mathcal{D}(T^2u_{n-3}, T^2u_{n-2}), \mathcal{D}(Tu_{n-3}, Tu_{n-2})\}) + (\delta_n + \delta_{n-1})F(\tau) - 2t \\ &\le \cdots \\ &\le F(\min\{\mathcal{D}(T^2u_1, T^2u_0), \mathcal{D}(Tu_1, Tu_0)\}) + (\delta_n + \delta_{n-1} + \cdots + \delta_1)F(\tau) - (n-1)\tau \\ &\le F(\mathcal{D}(u_1, u_0)) + (\sum_{k=1}^{n} \delta_k)F(\tau) - nt. \end{aligned}$$

where, $0 \le (\sum_{k=0}^{n} \delta_k) < \frac{n+1}{2}$.
Therefore,

$$\lim_{n\to\infty}(\sum_{k=0}^{n} \delta_k)F(\tau) - nt = -\infty.$$

So that,

$$\lim_{n\to\infty} F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) = -\infty. \tag{2.6}$$

Therefore, the results of (2.6) and condition (F_2^*) implies,

$$\lim_{n\to\infty} \min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\} = 0,$$

or,

$$\lim_{n\to\infty} \min\{\mathcal{D}(Tu_n, T^2u_n), \mathcal{D}(u_n, Tu_n)\} = 0. \tag{2.7}$$

If $\lim_{n\to\infty} \min\{\mathcal{D}(Tu_n, T^2u_n), \mathcal{D}(u_n, Tu_n)\} = \lim_{n\to\infty} \mathcal{D}(u_n, Tu_n) = 0$, then

$$\lim_{n\to\infty} \mathcal{D}(Tu_n, T^2u_n) = \lim_{n\to\infty} \mathcal{D}(u_{n+1}, Tu_{n+1}) = 0.$$

If $\lim_{n\to\infty} \min\{\mathcal{D}(Tu_n, T^2u_n), \mathcal{D}(u_n, Tu_n)\} = \lim_{n\to\infty} \mathcal{D}(Tu_n, T^2u_n) = 0$, then

$$\lim_{n\to\infty} \mathcal{D}(u_n, Tu_n) = \lim_{n\to\infty} \mathcal{D}(Tu_{n-1}, T^2u_{n-1}) = 0.$$

Therefore, Eq (2.7) implies

$$\lim_{n\to\infty} \mathcal{D}(u_n, Tu_n) = 0. \tag{2.8}$$

Next, we claim that $\{u_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$ and sequence $\{n(k)\}_{k=1}^{\infty}$ and $\{m(k)\}_{k=1}^{\infty}$ of natural numbers such that

$$n(k) > m(k) > k, \mathcal{D}(u_{n(k)}, u_{m(k)}) \geq \epsilon \text{ and } \mathcal{D}(u_{n(k)-1}, u_{m(k)}) < \epsilon, \text{ for all } k \in \mathbb{N}. \tag{2.9}$$

So we have

$$\begin{aligned}
\epsilon &\leq \mathcal{D}(u_{n(k)}, u_{m(k)}) \\
&\leq \mathcal{D}(u_{n(k)}, u_{n(k)-1}) + \mathcal{D}(u_{n(k)-1}, u_{m(k)}) \\
&\leq \mathcal{D}(u_{n(k)}, u_{n(k)-1}) + \epsilon \\
&= \mathcal{D}(Tu_{n(k)-1}, u_{n(k)-1}) + \epsilon.
\end{aligned}$$

It follows from (2.8) and the above inequality that

$$\lim_{k\to\infty} \mathcal{D}(u_{n(k)}, u_{m(k)}) = \epsilon. \tag{2.10}$$

On the other hand, from (2.8) there exits $N \in \mathbb{N}$, such that

$$\mathcal{D}(u_{n(k)}, Tu_{n(k)}) < \frac{\epsilon}{4} \quad \text{and} \quad \mathcal{D}(u_{m(k)}, Tu_{m(k)}) < \frac{\epsilon}{4}, \quad \text{for all } k \geq N. \tag{2.11}$$

Next, we claim that

$$\mathcal{D}(Tu_{n(k)}, Tu_{m(k)}) = \mathcal{D}(u_{n(k)+1}, u_{m(k)+1}) > 0, \text{ for all } k \geq N. \tag{2.12}$$

Arguing by contradiction, there exists $l \geq N$ such that

$$\mathcal{D}(u_{n(l)+1}, u_{m(l)+1}) = 0. \tag{2.13}$$

It follows from (2.9), (2.11) and (2.13) that

$$\begin{aligned}
\epsilon &\leq \mathcal{D}(u_{n(l)}, u_{m(l)}) \\
&\leq \mathcal{D}(u_{n(l)}, u_{n(l)+1}) + \mathcal{D}(u_{n(l)+1}, u_{m(l)}) \\
&\leq \mathcal{D}(u_{n(l)}, u_{n(l)+1}) + \mathcal{D}(u_{n(l)+1}, u_{m(l)+1}) + \mathcal{D}(u_{m(l)+1}, u_{m(l)}) \\
&= \mathcal{D}(u_{n(l)}, Tu_{n(l)}) + \mathcal{D}(u_{n(l)+1}, u_{m(l)+1}) + \mathcal{D}(u_{m(l)}, Tu_{m(l)}) \\
&< \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} \\
&= \frac{\epsilon}{2},
\end{aligned}$$

which is a contradiction.
Therefore, it follows from (2.12) and the assumptions of the theorem that

$$\begin{aligned}
&\tau + F(\min\{\mathcal{D}(T^2u_{n(k)}, T^2u_{m(k)}), \mathcal{D}(Tu_{n(k)}, Tu_{m(k)})\}) \\
&\leq F(\mathcal{D}(u_{n(k)}, u_{m(k)})), \text{ for all } k \geq N.
\end{aligned} \tag{2.14}$$

From condition (F_3^*), (2.10) and (2.14), we get

$$\tau + F(\epsilon) \leq F(\epsilon),$$

which shows that $\{u_n\}$ is a Cauchy sequence. From the completeness of $(X, \mathcal{D})$, $\{u_n\}$ converges to some point $u^* \in X$.
Finally, the continuity of T yields that

$$\mathcal{D}(Tu^*, u^*) = \lim_{n\to\infty} \mathcal{D}(Tu_n, u_n) = \lim_{n\to\infty} \mathcal{D}(u_{n+1}, u_n) = \mathcal{D}(u^*, u^*) = 0.$$

For uniqueness, we assume that u' is another fixed point such that $Tu^* = u^* \neq u' = Tu'$. Then we have

$$\begin{aligned}
F(\mathcal{D}(u^*, u')) &= F(\min\{\mathcal{D}(T^2u^*, T^2u'), \mathcal{D}(Tu^*, Tu')\}) \\
&< \tau + F(\min\{\mathcal{D}(T^2u^*, T^2u'), \mathcal{D}(Tu^*, Tu')\}) \\
&\leq F(\mathcal{D}(u^*, u')),
\end{aligned}$$

which is a contradiction.
Therefore, T has a unique fixed point in X.
Theorem 2.2. Let $(X, \mathcal{D})$ be a complete metric space and $T : X \to X$ be a generalized F-Suzuki contraction mapping with $p = 2$. Then T has a unique fixed point in X and for every $u_0 \in X$ the sequence $\{T^n u_0\}_{n=1}^{\infty}$ converges to the fixed point.
Proof. Let $u_0 \in X$ be an arbitrary point and define a sequence $\{u_n\} \subseteq X$ by $u_{n+1} = Tu_n = T^{n+1}u_0$, for all $n \in \mathbb{N}$.
Now, we will prove that $\lim_{n\to\infty} \mathcal{D}(u_n, Tu_n) = 0$.
If $u_{n_0} = Tu_{n_0}$ for some $n_0 \in \mathbb{N}$, then $\mathcal{D}(u_n, Tu_n) = \mathcal{D}(u_{n+1}, Tu_{n+1}) = \cdots = 0$, for all $n \geq n_0$ and so

$\mathcal{D}(u_n, Tu_n)$ converges to 0, as $n \to \infty$.
Assume that $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}$.
Since, for all $n \in \mathbb{N}$,

$$\frac{1}{2}\mathcal{D}(u_{n-1}, Tu_{n-1}) < \mathcal{D}(u_{n-1}, u_n),$$

and

$$\frac{1}{2}\mathcal{D}(Tu_{n-1}, T^2u_{n-1}) < \mathcal{D}(Tu_{n-1}, Tu_n),$$

so, from the contraction assumption on T, there exits $\tau > 0$ such that

$$\tau + F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \leq F(\mathcal{D}(u_{n-1}, u_n)),$$

or

$$F(\min\{\mathcal{D}(T^2u_{n-1}, T^2u_n), \mathcal{D}(Tu_{n-1}, Tu_n)\}) \leq F(\mathcal{D}(u_{n-1}, u_n)) - \tau.$$

As in the proof of Theorem 2.1, the above inequality gives that

$$\lim_{n\to\infty} \mathcal{D}(u_n, Tu_n) = 0.$$

Moreover, analysis similar to that in the proof of Theorem 2.1 shows the sequence $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
Since, $(X, \mathcal{D})$ is complete, then the sequence $\{u_n\}_{n=1}^{\infty}$ converges to some point $u^* \in X$, that is,

$$\lim_{n\to\infty} \mathcal{D}(u_n, u^*) = 0. \tag{2.15}$$

Now, we will claim that

$$\frac{1}{2}\mathcal{D}(u_n, Tu_n) < \mathcal{D}(u_n, u^*) \text{ or } \frac{1}{2}\mathcal{D}(T^2u_n, Tu_n) < \mathcal{D}(Tu_n, u^*) \\ \text{or } \frac{1}{2}\mathcal{D}(T^2u_n, T^3u_n) < \mathcal{D}(T^2u_n, u^*). \tag{2.16}$$

Suppose, on the contrary, that there exists a $m \in \mathbb{N}$ satisfying the following three inequalities,

$$\frac{1}{2}\mathcal{D}(u_m, Tu_m) \geq \mathcal{D}(u_m, u^*), \tag{2.17}$$

$$\frac{1}{2}\mathcal{D}(Tu_m, T^2u_m) \geq \mathcal{D}(Tu_m, u^*), \tag{2.18}$$

$$\frac{1}{2}\mathcal{D}(T^3u_m, T^2u_m) \geq \mathcal{D}(T^2u_m, u^*). \tag{2.19}$$

Now, (2.17) along with triangular inequality gives

$$2\mathcal{D}(u_m, u^*) \leq \mathcal{D}(u_m, Tu_m) \leq \mathcal{D}(u_m, u^*) + \mathcal{D}(Tu_m, u^*),$$

which implies that

$$\mathcal{D}(u_m, u^*) \le \mathcal{D}(Tu_m, u^*). \tag{2.20}$$

Also, (2.20) along with (2.18) gives

$$\mathcal{D}(u_m, u^*) \le \mathcal{D}(Tu_m, u^*) \le \frac{1}{2}\mathcal{D}(Tu_m, T^2u_m). \tag{2.21}$$

Similarly, (2.18) yields

$$2\mathcal{D}(Tu_m, u^*) \le \mathcal{D}(Tu_m, T^2u_m) \le \mathcal{D}(Tu_m, u^*) + \mathcal{D}(T^2u_m, u^*),$$

which implies

$$\mathcal{D}(Tu_m, u^*) \le \mathcal{D}(T^2u_m, u^*). \tag{2.22}$$

From (2.19) and (2.22), we have

$$\mathcal{D}(Tu_m, u^*) \le \mathcal{D}(T^2u_m, u^*) \le \frac{1}{2}\mathcal{D}(T^2u_m, T^3u_m). \tag{2.23}$$

Now, by the contraction assumption on T, there exists $\tau > 0$ such that

$$\tau + F(\min\{\mathcal{D}(Tu_m, T^2u_m), \mathcal{D}(T^2u_m, T^3u_m)\}) \le F(\mathcal{D}(u_m, Tu_m)).$$

If $\min\{\mathcal{D}(Tu_m, T^2u_m), \mathcal{D}(T^2u_m, T^3u_m)\} = \mathcal{D}(Tu_m, T^2u_m)$, we have

$$\tau + F(\mathcal{D}(Tu_m, T^2u_m)) \le F(\mathcal{D}(u_m, Tu_m)),$$

which yields,

$$F(\mathcal{D}(Tu_m, T^2u_m)) < F(\mathcal{D}(u_m, Tu_m)).$$

From condition $(F^*_{1'})$, we have

$$\begin{aligned}
\mathcal{D}(Tu_m, T^2u_m) &< \mathcal{D}(u_m, Tu_m) \\
&\le \mathcal{D}(u_m, u^*) + \mathcal{D}(u^*, Tu_m) \\
&\le \frac{1}{2}\mathcal{D}(Tu_m, T^2u_m) + \frac{1}{2}\mathcal{D}(Tu_m, T^2u_m) \\
&= \mathcal{D}(Tu_m, T^2u_m),
\end{aligned}$$

which is a contradiction.
Likewise, if $\min\{\mathcal{D}(Tu_m, T^2u_m), \mathcal{D}(T^2u_m, T^3u_m)\} = \mathcal{D}(T^2u_m, T^3u_m)$, we have

$$\begin{aligned}
\mathcal{D}(T^2u_m, T^3u_m) &< \mathcal{D}(Tu_m, T^2u_m) \\
&\le \mathcal{D}(Tu_m, u^*) + \mathcal{D}(u^*, T^2u_m) \\
&\le \frac{1}{2}\mathcal{D}(T^2u_m, T^3u_m) + \frac{1}{2}\mathcal{D}(T^2u_m, T^3u_m)
\end{aligned}$$

$$= \mathcal{D}(T^2u_m, T^3u_m),$$

which is a contradiction. Hence, (2.16) holds true.
Again, from (2.16), we have

$$\tau + F(\min\{\mathcal{D}(Tu_n, u^*), \mathcal{D}(T^2u_n, u^*)\}) \leq F(\mathcal{D}(u_n, u^*)).$$

Using (2.15) and condition (F_2^*), we can write

$$\lim_{n\to\infty} F(\mathcal{D}(Tu_n, u^*)) = -\infty \quad \text{or} \quad \lim_{n\to\infty} F(\mathcal{D}(T^2u_n, Tu^*)) = -\infty,$$

and

$$\lim_{n\to\infty} \mathcal{D}(Tu_n, u^*) = 0 \quad \text{or} \quad \lim_{n\to\infty} \mathcal{D}(T^2u_n, Tu^*) = 0.$$

Hence, u^* is a fixed point of T.
For uniqueness, let us suppose that T has another fixed point u', such that $Tu^* = u^* \neq u' = Tu'$.
Since, $0 = \frac{1}{2}\mathcal{D}(u', Tu') < \mathcal{D}(u', u^*)$ and $0 = \frac{1}{2}\mathcal{D}(Tu', T^2u') < \mathcal{D}(Tu', Tu^*)$, from the contraction assumption on T, there exists $\tau > 0$ such that,

$$\begin{aligned} F(\mathcal{D}(u', u^*)) &= F(\min\{\mathcal{D}(T^2u', T^2u^*), \mathcal{D}(Tu', Tu^*)\}) \\ &\leq F(\mathcal{D}(u', u^*)) - \tau \\ &< F(\mathcal{D}(u', u^*)), \end{aligned}$$

which is a contradiction. Therefore, T has a unique fixed point.

In the next definition, we introduce some new notions of generalized $F-$expanding mappings.

Definition 2.3. Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be a generalized $F-$expanding mapping of type (A), if there exists $\tau > 0$ such that for all $u, v \in X$,

$$\mathcal{D}(u, v) > 0 \Rightarrow F(\min\{\mathcal{D}(T^2u, T^2v), \mathcal{D}(Tu, Tv)\}) \geq F(\mathcal{D}(u, v)) + \tau,$$

where $F \in \mathfrak{F}$.

Definition 2.4. Let $(X, \mathcal{D})$ be a metric space. A mapping $T : X \mapsto X$ is said to be a generalized $F-$expanding mapping of type (B), if there exists $\tau \geq 0$ such that for all $u, v \in X$,

$$\begin{aligned} &\mathcal{D}(T^2u, T^2v) \neq \mathcal{D}(u, v), \mathcal{D}(u, v) > 0 \\ &\Rightarrow F(\min\{\mathcal{D}(T^2u, T^2v), \mathcal{D}(Tu, Tv)\}) \geq F(\mathcal{D}(u, v)) + \tau, \end{aligned}$$

where $F \in \mathfrak{F}$.

Theorem 2.3. Let $(X, \mathcal{D})$ be a complete metric space and let $T : X \mapsto X$ be continuous surjective and generalized $F-$expanding of type (A). Then T has a unique fixed point in X and for every $u_0 \in X$ the sequence $\{T^n u_0\}_{n=1}^{\infty}$ converges to the fixed point.

Proof. Firstly, we will show that T is bijective, which only needs to show that

$$u \neq v \quad (\text{or} \quad \mathcal{D}(u, v) > 0) \Rightarrow Tu \neq Tv \quad (\text{or} \quad \mathcal{D}(Tu, Tv) > 0). \tag{2.24}$$

If $\mathcal{D}(u, v) > 0$, from the assumption on T, there exits $\tau > 0$ such that

$$F(\min\{\mathcal{D}(T^2u, T^2v), \mathcal{D}(Tu, Tv)\}) \geq F(\mathcal{D}(u, v)) + \tau,$$

or

$$F(\min\{\mathcal{D}(T^2u, T^2u), \mathcal{D}(Tu, Tu)\}) \geq F(\mathcal{D}(u, v)) + \tau.$$

The second above inequality together with condition (F_1^*) implies,

$$F(0) \geq F(\mathcal{D}(u, v)) + \tau,$$

which yields that $0 \geq \mathcal{D}(u, v)$, a contradiction.
So, we have $Tu \neq Tv$ or $\mathcal{D}(Tu, Tv) > 0$, hence T is injective and then bijective. Consider a mapping S such that $TS = ST = I_u$, where I_u is identity mapping on X.
Let $u = S^2u, v = S^2v$, so that $Tu = Su, Tv = Sv$ and $T^2u = u, T^2v = v$.
If $\min\{\mathcal{D}(T^2u, T^2v), \mathcal{D}(Tu, Tv)\} = \mathcal{D}(Tu, Tv)$, there exits $\tau > 0$ such that

$$F(\mathcal{D}(Tu, Tv)) \geq F(\mathcal{D}(u, v)) + \tau. \tag{2.25}$$

Condition (F_1^*) yields

$$\mathcal{D}(Tu, Tv) > \mathcal{D}(u, v) > 0, \tag{2.26}$$

which together with (2.25) implies that

$$F(\mathcal{D}(T^2u, T^2v)) > F(\mathcal{D}(u, v)) + \tau.$$

Using the inverse mapping S, the above inequality takes the form

$$F(\mathcal{D}(u, v)) \geq F(\mathcal{D}(S^2u, S^2v)) + \tau. \tag{2.27}$$

Moreover, we have

$$F(\mathcal{D}(u, v)) \geq F(\mathcal{D}(Su, Sv)) + \tau. \tag{2.28}$$

Combining (2.27) and (2.28), we have

$$\mathcal{D}(u, v) > 0 \Rightarrow F(\mathcal{D}(u, v)) \geq F(\min\{\mathcal{D}(Su, Sv), \mathcal{D}(S^2u, S^2v)\}) + \tau. \tag{2.29}$$

Again, if $\min\{\mathcal{D}(T^2u, T^2v), \mathcal{D}(Tu, Tv)\} = \mathcal{D}(T^2u, T^2v)$, there exists $\tau > 0$ such that

$$F(\mathcal{D}(T^2u, T^2v)) \geq F(\mathcal{D}(u, v)) + \tau,$$

and

$$F(\mathcal{D}(u, v)) \geq F(\mathcal{D}(S^2u, S^2v)) + \tau. \tag{2.30}$$

From condition (F_1^*) we have,

$$\mathcal{D}(u,v) > \mathcal{D}(S^2u, S^2v) > 0.$$

Combining (2.29) and (2.30), together with the assumption on T, we have

$$F(\mathcal{D}(u,v)) \geq F(\min\{\mathcal{D}(Su,Sv), \mathcal{D}(S^2u,S^2v)\}) + \tau,$$

which shows that S is the generalized $F-$contraction defined in Theorem 2.1. From the conclusion of Theorem 2.1, S has a unique fixed point, so does T.

Theorem 2.4. Let $(X,\mathcal{D})$ be a complete metric space and let $T : X \mapsto X$ be continuous surjective and generalized $F-$expanding of type (B). Then T has a unique fixed point in X and for every $u_0 \in X$ the sequence $\{T^n u_0\}_{n=1}^{\infty}$ converges to the fixed point.

Proof. Firstly, we will show that T is bijective, which only needs to show that

$$u \neq v \quad (\text{or } \mathcal{D}(u,v) > 0) \Rightarrow Tu \neq Tv \quad (\text{or } \mathcal{D}(Tu,Tv) > 0). \tag{2.31}$$

If $\mathcal{D}(u,v) > 0$, from the assumption on T, there exits $\tau \geq 0$ such that

$$F(\min\{\mathcal{D}(T^2u,T^2v), \mathcal{D}(Tu,Tv)\}) \geq F(\mathcal{D}(u,v)) + \tau,$$

or

$$F(\min\{\mathcal{D}(T^2u,T^2u), \mathcal{D}(Tu,Tu)\}) \geq F(\mathcal{D}(u,v)) + \tau.$$

The second inequality together with condition (F_1^*) implies,

$$F(0) \geq F(\mathcal{D}(u,v)) + \tau,$$

which yields that $0 \geq \mathcal{D}(u,v)$, a contradiction.
So, we have $Tu \neq Tv$ or $\mathcal{D}(Tu,Tv) > 0$, hence T is injective and then bijective.
Consider a mapping S such that $TS = ST = I_u$, where I_u is identity mapping on X.
Let $u = S^2u, v = S^2v$, so that $Tu = Su, Tv = Sv$ and $T^2u = u, T^2v = v$.
If $\min\{\mathcal{D}(T^2u,T^2v), \mathcal{D}(Tu,Tv)\} = \mathcal{D}(Tu,Tv)$, there exists $\tau \geq 0$ such that

$$F(\mathcal{D}(Tu,Tv)) \geq F(\mathcal{D}(u,v)) + \tau,$$

so that

$$F(\mathcal{D}(Su,Sv)) \geq F(\mathcal{D}(S^2u,S^2v)) + \tau. \tag{2.32}$$

From condition (F_1^*), we have

$$\mathcal{D}(Su,Sv) \geq \mathcal{D}(S^2u,S^2v).$$

Again, if $\min\{\mathcal{D}(T^2u,T^2v), \mathcal{D}(Tu,Tv)\} = \mathcal{D}(T^2u,T^2v)$, there exists $\tau \geq 0$ such that

$$F(\mathcal{D}(T^2u,T^2v)) \geq F(\mathcal{D}(u,v)) + \tau,$$

and

$$F(\mathcal{D}(u,v)) \geq F(\mathcal{D}(S^2u, S^2v)) + \tau. \tag{2.33}$$

From condition (F_1^*), we can write

$$\mathcal{D}(u,v) \geq \mathcal{D}(S^2u, S^2v).$$

Since T is bijective, we have

$$\mathcal{D}(Su, Sv) > \mathcal{D}(S^2u, S^2v) \quad \text{and} \quad \mathcal{D}(u,v) > \mathcal{D}(S^2u, S^2v).$$

Combining (2.32) and (2.33), together with the assumption on T, we have

$$F(\mathcal{D}(u,v)) > F(\min\{\mathcal{D}(Su,Sv), \mathcal{D}(S^2u, S^2v)\}) + \tau,$$

which is equivalently stated as there exists $\tau' > 0$ such that

$$F(\mathcal{D}(u,v)) \geq F(\min\{\mathcal{D}(Su,Sv), \mathcal{D}(S^2u, S^2v)\}) + \tau',$$

which shows that S is the generalized $F-$contraction defined in Theorem 2.1. From the conclusion of Theorem 2.1, S has a unique fixed point, so does T.

Theorem 2.5. Let $(X, \mathcal{D})$ be a complete metric space. Suppose a continuous mapping $T : X \mapsto X$ satisfy

$$F_1\left(\min\{(\mathcal{D}(T^2u_{m-1}, T^2u_m)), (\mathcal{D}(Tu_{m-1}, Tu_m))\}\right) \leq F(\mathcal{D}(u_{m-1}, u_m)) - \tau, \tag{2.34}$$

where, non-decreasing functions $F, F_1 \in \mathfrak{F}$ and for all $t, t_1 \in \mathbb{R}_+$, there exist $\upsilon > 0, \tau > 2\upsilon$, such that

$$F_1(t_1) < F(t_2) \leq F_1(t_1) + \upsilon. \tag{2.35}$$

Then T has a unique fixed point in X and for every $u_0 \in X$, the sequence $\{T^m u_0\}_{m=1}^{+\infty}$ converges to the fixed point.

Proof. As, $F, F_1 : (0, \infty) \to \mathbb{R}_+$ are non-decreasing functions, so that we can write

$$\begin{aligned} &F_1\left(\min\{(\mathcal{D}(T^2u_{m-1}, T^2u_m)), (\mathcal{D}(Tu_{m-1}, Tu_m))\}\right) \\ &\quad = \min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\} \leq F(\mathcal{D}(u_{m-1}, u_m)) - \tau. \end{aligned} \tag{2.36}$$

If, $F(\mathcal{D}(u_{m-1}, u_m)) \leq F_1(\mathcal{D}(u_{m-1}, u_m))$, we have

$$\min\left(F_1\left(\mathcal{D}\left(T^2u_{m-1}, T^2u_m\right)\right), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\right) \leq F_1(\mathcal{D}(u_{m-1}, u_m)) - \tau.$$

If, $F(\mathcal{D}(u_{m-1}, u_m)) > F_1(\mathcal{D}(u_{m-1}, u_m))$,
Using condition (2.35) , we can write (2.36) as follows

$$\min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\} \leq F_1(\mathcal{D}(u_{m-1}, u_m)) + \upsilon - \tau. \tag{2.37}$$

Then, we have either

$$F_1(\mathcal{D}(u_{m-1}, u_m)) = \min\{F_1(\mathcal{D}(Tu_{m-1}, Tu_m)), F_1(\mathcal{D}(u_{m-1}, u_m))\}, \tag{2.38}$$

or

$$F_1(\mathcal{D}(Tu_{m-1}, Tu_m)) = \min\{F_1(\mathcal{D}(Tu_{m-1}, Tu_m)), F_1(\mathcal{D}(u_{m-1}, u_m))\}. \tag{2.39}$$

If inequality (2.38) holds true, the inequality (2.37) will take the form

$$\begin{aligned}&\min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\}\\&\leq \min\{F_1(\mathcal{D}(Tu_{m-1}, Tu_m)), F_1(\mathcal{D}(u_{m-1}, u_m))\} + \upsilon - \tau.\end{aligned} \tag{2.40}$$

If inequality (2.39) is true, we have $F_1(\mathcal{D}(Tu_{m-1}, Tu_m)) < F_1(\mathcal{D}(u_{m-1}, u_m))$.
From condition (A), we have

$$F_1(\mathcal{D}(Tu_{m-1}, Tu_m)) < F_1(\mathcal{D}(u_{m-1}, u_m)) \leq F_1(\mathcal{D}(Tu_{m-1}, Tu_m)) + \upsilon. \tag{2.41}$$

Using inequality (2.41) in (2.37), we can write

$$\min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m)) \leq F_1(\mathcal{D}(Tu_{m-1}, Tu_m)) + 2\upsilon - \tau.$$

Moreover, from (2.39), we have

$$\begin{aligned}&\min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\\&\leq \min\{F_1(\mathcal{D}(Tu_{m-1}, Tu_m)), F_1(\mathcal{D}(u_{m-1}, u_m)) + 2\upsilon - \tau.\end{aligned} \tag{2.42}$$

Combining both inequalities (2.40) and (2.42), we have

$$\begin{aligned}&\min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\\&\leq \min\{F_1(\mathcal{D}(Tu_{m-1}, Tu_m)), F_1(\mathcal{D}(u_{m-1}, u_m)) + \delta_m\upsilon - \tau,\end{aligned} \tag{2.43}$$

where $\delta_m = \begin{cases}1 & \text{if} \quad F_1(t_2) > F_1(t_1)\\ 2 & \text{if} \quad F_1(t_2) < F_1(t_1)\end{cases}, t_1, t_2 \in \mathbb{R}_+, t_1 \neq t_2.$
The above inequality can be written as

$$\begin{aligned}&\min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\}\\&\leq \min\{F_1(\mathcal{D}(T^2u_{m-2}, T^2u_{m-1})), F_1(\mathcal{D}(u_{m-2}, u_{m-1}))\} + \delta_m\upsilon - \tau.\end{aligned}$$

Repeating this process, we have

$$\begin{aligned}&\min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\}\\&\leq \min\{F_1(\mathcal{D}(T^2u_{m-3}, T^2x_{m-2})), F_1(\mathcal{D}(u_{m-3}, u_{m-2}))\}\\&+ \delta_m\upsilon + \delta_{m-1}\upsilon - 2\tau\\&\leq \cdots\end{aligned}$$

$$\leq \min\{F_1(\mathcal{D}(T^2u_1, T^2x_0)), F_1(\mathcal{D}(Tx_1, Tx_0))\} + \sum_{j=1}^{m} \delta_j \upsilon - m\tau.$$

So that

$$\begin{aligned}&\min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\}\\ &\leq F_1(\mathcal{D}(u_1, u_0)) + \sum_{j=1}^{m} \delta_j \upsilon - (m+1)\tau.\end{aligned}$$

Since, $\tau > 2\upsilon$ and $\sum\limits_{j=1}^{m} \delta_j < m + 1$, we have

$$\lim_{m\to+\infty} \sum_{j=1}^{m} \delta_j \upsilon - (m+1)\tau = -\infty.$$

So that we can write

$$\lim_{m\to+\infty} \min\{F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)), F_1(\mathcal{D}(Tu_{m-1}, Tu_m))\} = -\infty. \tag{2.44}$$

Now, Eq (2.44) further has two possible cases.

$$\lim_{m\to+\infty} F_1(\mathcal{D}(T^2u_{m-1}, T^2u_m)) = -\infty. \tag{G}$$

$$\lim_{m\to+\infty} F_1(\mathcal{D}(Tu_{m-1}, Tu_m)) = -\infty. \tag{H}$$

Condition (F_2^*) among case (G) yields

$$\lim_{m\to+\infty} \mathcal{D}(T^2u_{m-1}, T^2u_m) = 0.$$

or equivalently,

$$\lim_{m\to+\infty} \mathcal{D}(T^2u_{m-1}, T^2u_m) = \lim_{m\to+\infty} \mathcal{D}(u_{m+1}, Tu_{m+1}) = \lim_{m\to+\infty} \mathcal{D}(u_m, Tu_m) = 0.$$

Condition (F_2^*) among case (H) yields

$$\lim_{m\to+\infty} \mathcal{D}(Tu_{m-1}, Tu_m) = \lim_{m\to+\infty} \mathcal{D}(u_m, Tu_m) = 0.$$

Therefore, from (2.44), we get

$$\lim_{m\to+\infty} \mathcal{D}(u_m, Tu_m) = 0. \tag{2.45}$$

Now, we will prove that the sequence $\{u_m\}_{m=1}^{+\infty}$ is a Cauchy sequence.
Suppose, on the contrary, that there exist $\varepsilon > 0$ and sequences $\{\mathfrak{g}(m)\}_{m=1}^{+\infty}$ and $\{\mathfrak{h}(m)\}_{m=1}^{+\infty}$ of natural numbers such that for all $m \in \mathbb{N}$,

$$\mathfrak{g}(m) > \mathfrak{h}(m) > m, \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}) \geq \varepsilon, \mathcal{D}(u_{\mathfrak{g}(m)-1}, u_{\mathfrak{h}(m)}) < \varepsilon, \tag{2.46}$$

So that we can write

$$\begin{aligned}\varepsilon &\le \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)})\\ &\le \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{g}(m)-1}) + \mathcal{D}(u_{\mathfrak{g}(m)-1}, u_{\mathfrak{h}(m)})\\ &< \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{g}(m)-1}) + \varepsilon\\ &= \mathcal{D}(u_{\mathfrak{g}(m)-1}, Tu_{\mathfrak{g}(m)-1}) + \varepsilon.\end{aligned}$$

That is,

$$\varepsilon \le \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}) < \mathcal{D}(u_{\mathfrak{g}(m)-1}, Tu_{\mathfrak{g}(m)-1}) + \varepsilon. \tag{2.47}$$

Inequalities (2.45) and (2.47) yield

$$\lim_{m\to+\infty} \mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}) = \varepsilon.$$

Further, from (2.45) there exists $N \in \mathbb{N}$ such that for all $m \ge N$,

$$\mathcal{D}(u_{\mathfrak{g}(m)}, Tu_{\mathfrak{g}(m)}) < \frac{\varepsilon}{4}, \quad \mathcal{D}(u_{\mathfrak{h}(m)}, Tu_{\mathfrak{h}(m)}) < \frac{\varepsilon}{4}, \tag{2.48}$$

Next we claim that for all $m \ge N$,

$$\mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}) = \mathcal{D}(u_{\mathfrak{g}(m)+1}, u_{\mathfrak{h}(m)+1}) > 0. \tag{2.49}$$

Suppose, on the contrary, that there exist $r \ge N$, such that

$$\mathcal{D}(u_{\mathfrak{g}(r)+1}, u_{\mathfrak{h}(r)+1}) = 0. \tag{2.50}$$

It follows from (2.45), (2.46) and (2.50) that

$$\begin{aligned}\varepsilon &\le \mathcal{D}(u_{\mathfrak{g}(r)}, u_{\mathfrak{h}(r)})\\ &\le \mathcal{D}(u_{\mathfrak{g}(r)}, u_{\mathfrak{g}(r)+1}) + \mathcal{D}(u_{\mathfrak{g}(r)+1}, u_{\mathfrak{h}(r)})\\ &\le \mathcal{D}(u_{\mathfrak{g}(r)}, u_{\mathfrak{g}(r)+1}) + \mathcal{D}(u_{\mathfrak{g}(r)+1}, u_{\mathfrak{h}(r)+1}) + \mathcal{D}(u_{\mathfrak{h}(r)+1}, u_{\mathfrak{h}(r)})\\ &= \mathcal{D}(u_{\mathfrak{g}(r)}, Tu_{\mathfrak{g}(r)}) + \mathcal{D}(u_{\mathfrak{g}(r)+1}, u_{\mathfrak{h}(r)+1}) + \mathcal{D}(u_{\mathfrak{h}(r)}, Tu_{\mathfrak{h}(r)})\\ &< \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4}\\ &= \frac{\varepsilon}{2}.\end{aligned}$$

Which is a contradiction. Therefore, (2.49) together with the assumption of the theorem gives

$$\begin{aligned}&\tau + \min\{F_1(\mathcal{D}(T^2u_{\mathfrak{g}(m)}, T^2u_{\mathfrak{h}(m)})), F_1(\mathcal{D}(Tu_{\mathfrak{g}(m)}, Tu_{\mathfrak{h}(m)}))\}\\ &\le F(\mathcal{D}(u_{\mathfrak{g}(m)}, u_{\mathfrak{h}(m)}).\end{aligned} \tag{2.51}$$

From (F_3^*), (2.45), (2.51) and the assumption of F-contraction, we get $\tau + \min\{F_1(\varepsilon), F_1(\varepsilon)\} \le F(\varepsilon)$, which yields $\tau + F_1(\varepsilon) \le F(\varepsilon)$. Then the condition (2.35) allows us to write $\tau + F_1(\varepsilon) \le F_1(\varepsilon) + \upsilon$.

That yields a contradiction as $\tau > 2\upsilon$. The completeness of $(X, \mathcal{D})$ proves that $\{u_m\}_{m=1}^{+\infty}$ converges to some point u^* in X. Now, the continuity of T implies

$$\mathcal{D}(Tu, u) = \lim_{m\to+\infty} \mathcal{D}(Tu_m, u_m) = \lim_{m\to+\infty} \mathcal{D}(u_{m+1}, u_m) = \mathcal{D}(u^*, u^*) = 0.$$

Therefore, T has a unique fixed point u^*.

Here is an example to show the validity of Theorem 2.1.

Example 2.2. [20] Let B be closed unit ball in l_1 space of all absolutely summable sequence $u = (u_1, u_2, \cdots)$ with a metric inherited from the standard norm $\|u\| = \sum\limits_{i=1}^{\infty} \|u_i\|$.

Consider a function $h : [-1, 1] \mapsto [-1, 1]$ given by

$$h(w) = \begin{cases} 1 + 2w, & -1 \le w \le -1/2 \\ 0, & -1/2 \le w \le 1/2 \\ -1 + 2w, & 1/2 \le w \le 1 \end{cases}.$$

It is easy to observe that, for all $w_1, w_2 \in [-1, 1]$, we have

$$|h(w_2) - h(w_1)| \le 2|w_2 - w_1|,$$

and

$$|h(w)| \le |w|.$$

Further, let us define a surjective mapping $T : B \mapsto B$ by

$$Tu = T(u_1, u_2, \cdots) = (h(u_2), \frac{2}{3}u_3, u_4, u_5, \cdots).$$

Then for $i \ge 2$, we have

$$T^i u = (h(\frac{2}{3}u_{i+1}), \frac{2}{3}u_{i+2}, u_{i+3}, u_{i+4}, \cdots).$$

For each $u = (u_1, u_2, \cdots), v = (v_1, v_2, \cdots) \in B$, we have

$$\begin{aligned} &\|Tu - Tv\| \\ &= |h(u_2) - h(v_2)| + \frac{2}{3}|u_3 - v_3| + \sum_{k=4}^{\infty} |u_4 - v_4| \\ &\le 2|u_2 - v_2| + \frac{2}{3}|u_3 - v_3| + \sum_{k=4}^{\infty} |u_k - v_k| \\ &\le 2\|u - v\|. \end{aligned}$$

and for $i \ge 2$,

$$\begin{aligned} &\|T^i u - T^i v\| \\ &= |h(\frac{2}{3}u_{i+1}) - h(\frac{2}{3}v_{i+1})| + \frac{2}{3}|u_{i+2} - v_{i+2}| \end{aligned}$$

$$\begin{aligned}
&+\sum_{k=i+3}^{\infty}|u_k-v_k| \\
&\leq \frac{4}{3}|u_{i+1}-v_{i+1}|+\frac{2}{3}|u_{i+2}-v_{i+2}|+\sum_{k=i+3}^{\infty}|u_k-v_k| \\
&\leq \frac{4}{3}||u-v||.
\end{aligned}$$

Then, we have for all $u=(u_1,u_2,\cdots), v=(v_1,v_2,\cdots)\in B$,

$$\begin{aligned}
&\frac{1}{2}||Tu-Tv||+\frac{1}{2}||T^2u-T^2v|| \\
&\leq |u_2-v_2|+|u_3-v_3|+\frac{5}{6}|u_4-v_4|+\sum_{k=5}^{\infty}|u_k-v_k| \\
&\leq ||u-v||,
\end{aligned}$$

which implies that,

$$||Tu-Tv||+||T^2u-T^2v||\leq 2||u-v||.$$

If $||Tu-Tv||\neq||T^2u-T^2v||$, there exists $\tau>0$, such that

$$e^{\tau}\min\{||Tu-Tv||,||T^2u-T^2v||\}\leq||u-v||,$$

or

$$\tau+\ln(\min\{||Tu-Tv||,||T^2u-T^2v||\})\leq\ln(||u-v||).$$

Therefore, $F(\alpha)=\ln\alpha$ represents generalized F-contraction mapping, hence Theorem 2.1 guarantees the existence of a unique fixed point of T.

Note that, $F(\alpha)=\ln\alpha$ does not contract, whenever $\max\{||Tu-Tv||,||T^2u-T^2v||)>||u-v||$.

That is, $\tau+\ln(||Tu-Tv||)\geq\ln(||u-v||)$, for all $u,v\in B$.

Therefore, T does not represent F-contraction mapping defined in [2].

Hence Theorem 1.1 does not guarantee the existence of a fixed point.

Similarly, for $F'(\alpha)=\ln\alpha+\alpha$, we can write

$$\begin{aligned}
&\tau+\ln(\min\{||Tu-Tv||,||T^2u-T^2v||\})+(\min\{||Tu-Tv||,||T^2u-T^2v||\}) \\
&\leq\ln(||u-v||)+||u-v||,
\end{aligned}$$

so,

$$\begin{aligned}
&e^{(\min\{||Tu-Tv||,||T^2u-T^2v||\})+\tau}(\min\{||Tu-Tv||,||T^2u-T^2v||\}) \\
&\leq e^{||u-v||}||u-v||.
\end{aligned}$$

This can be written as,

$$\begin{aligned}
&\min\{||Tu-Tv||,||T^2u-T^2v||\} \\
&\leq e^{||u-v||-(\min\{||Tu-Tv||,||T^2u-T^2v||\}-\tau}||u-v||. \qquad (2.52)
\end{aligned}$$

Therefore, for

$$||u-v||\leq\min\{||Tu-Tv||,||T^2u-T^2v||\}+\tau,$$

inequality (2.52) shows that T is a generalized F-contraction mapping.

3. Applications to Caputo fractional differential equations

As an application of our work, we will study the existence of solutions to Caputo fractional differential equations of the fractional order in (1,2) and the integral boundary condition. The main condition in the problems studied in [21, 22] is associated with sufficient small Lipschitz constant. We will use a less restrictive condition than the Lipschitz condition by applying our obtained fixed point theorems.

For, $1 < l < 2$, and a Caputo fractional derivative ${}^{C}_{\vartheta_1}D^l_t z(t) = \frac{1}{\Gamma(2-l)} \int\limits_{\vartheta_1}^{t} (t-s)^{1-l} z''(s)ds$, consider a nonlinear Caputo fractional differential equation:

$$ {}^{C}_{\vartheta_1}D^l_t(z(t)) = u(t, z(t)), \text{ for } t \in (\vartheta_1, \vartheta_2), \tag{3.1} $$

with an integral boundary condition:

$$ z(\vartheta_1) = 0, z(\vartheta_2) = \int\limits_{\vartheta_1}^{\lambda} z(s)ds, \quad (\vartheta_1 < \lambda < \vartheta_2), \tag{3.2} $$

where $z \in \mathbb{R}$, ϑ_1, ϑ_2 are the given real numbers such that $0 \leq \vartheta_1 < \vartheta_2$.

Let $\Omega = C([\vartheta_1, \vartheta_2], \mathbb{R})$ with a norm $\|z\|_{[\vartheta_1,\vartheta_2]} = \sup_{s\in[\vartheta_1,\vartheta_2]}|z(s)|$.

For any $z, v \in \Omega$, we define $\mathcal{D}(z, v) = \|z - v\|_{[\vartheta_1,\vartheta_2]}$.

Consider the linear fractional differential equation:

$$ {}^{C}_{\vartheta_1}D^l_t(z(t) = g(t) \text{ for}, t \in (\vartheta_1, \vartheta_2), \tag{3.3} $$

with the integral boundary condition (3.2) where $g \in \Omega$.

Lemma 3.1. For $g \in \Omega$, following function represents the solution of boundary value problem (3.1), (3.2).

$$ \begin{aligned} z(t) &= \frac{1}{\Gamma(l)} \int_{\vartheta_1}^{t} (t-s)^{l-1} g(s)ds \\ &+ \frac{2(t-\vartheta_1)}{((\lambda-\vartheta_1)^2 - 2(\vartheta_2-\vartheta_1))\Gamma(l)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{l-1} g(s)ds \\ &- \frac{2(t-\vartheta_1)}{((\lambda-\vartheta_1)^2 - 2(\vartheta_2-\vartheta_1))\Gamma(l)} \int\limits_{\vartheta_1}^{\lambda} \int_{\vartheta_1}^{s} (s-\xi)^{l-1} g(\xi)\mathcal{D}\xi ds. \end{aligned} \tag{3.4} $$

The proof of Lemma 3.1 is based on the presentation of the solution

$$ z(t) = \frac{1}{\Gamma(l)} \int_{\vartheta_1}^{t} (t-s)^{l-1} g(s)ds - \mathcal{D}_1 - \mathcal{D}_2(t - \vartheta_1), $$

given in [23].

Next, we will define a mild solution of (3.1) and (3.2).

Definition 3.1. The function $z \in \Omega$ is a mild solution of the boundary value problem (3.1) and (3.2) if it satisfies:

$$\begin{aligned} z(t) &= \frac{1}{\Gamma(l)} \int_{\vartheta_1}^{t} (t-s)^{l-1} u(s, z(s)) ds \\ &+ \frac{2(t-\vartheta_1)}{((\lambda-\vartheta_1)^2 - 2(\vartheta_2-\vartheta_1))\Gamma(l)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{l-1} u(s, z(s)) ds \\ &- \frac{2(t-\vartheta_1)}{((\lambda-\vartheta_1)^2 - 2(\vartheta_2-\vartheta_1)\Gamma(l)} \int_{\vartheta_1}^{\lambda} \int_{\vartheta_1}^{s} (s-\xi)^{l-1} u(\xi, z(\xi)) \mathcal{D}\xi ds, t \in [\vartheta_1, \vartheta_2]. \end{aligned} \tag{3.5}$$

For any function $u \in \Omega$, we define a mapping $\Upsilon : \Omega \to \Omega$ by

$$\begin{aligned} \Upsilon(u)(t) &= \frac{1}{\Gamma(l)} \int_{\vartheta_1}^{t} (t-s)^{l-1} u(s, u(s)) ds \\ &+ \frac{2(t-\vartheta_1)}{((\lambda-\vartheta_1)^2 - 2(\vartheta_2-\vartheta_1)\Gamma(l)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{l-1} u(s, u(s)) ds \\ &- \frac{2(t-\vartheta_1)}{((\lambda-\vartheta_1)^2 - 2(\vartheta_2-\vartheta_1))\Gamma(l)} \int_{\vartheta_1}^{\lambda} \int_{\vartheta_1}^{s} (s-\xi)^{l-1} u(\xi, u(\xi)) \mathcal{D}\xi ds, \end{aligned} \tag{3.6}$$

for $t \in [\vartheta_1, \vartheta_2]$. Now, we establish the existence result as follows.

Theorem 3.1. Suppose that,

(i) There exists a constant $K > 0$, such that,

$$\frac{K(\vartheta_2-\vartheta_1)l}{\Gamma(1+l)} \left(1 + \frac{2K(\vartheta_2-\vartheta_1)}{(2(\vartheta_2-\vartheta_1) - (\lambda-\vartheta_1)^2)} \left(1 + \frac{\lambda-\vartheta_1}{1+l}\right)\right) \in (0, \infty), \tag{3.7}$$

and a function $u \in C([\vartheta_1, \vartheta_2] \times \mathbb{R}, \mathbb{R})$ such that

$$|u(t,z) - u(t,v)| \leq K|z-v|^r, z, v \in \mathbb{R}, t \in [\vartheta_1, \vartheta_2];$$

Where, $r \in (0, 1]$,

(ii) There exists a function $z_0 \in \Omega$ such that $\mathcal{D}(z_0, \Upsilon(z_0)) > 0$, where the operator Υ is defined by (3.6);

(iii) For any two functions u, $z \in \Omega$, such that $\mathcal{D}(u, z) > 0$, the inequality $\mathcal{D}(\Upsilon(u), \Upsilon(z)) > 0$ holds.

Then the boundary value problem (3.1), (3.2) has a mild solution.

Proof. Note that any fixed point of the mapping Υ is a mild solution of the boundary value problem (3.1) and (3.2). Now, let $z, v \in \Omega$ be such that $\mathcal{D}(z, v) > 0$. By condition (i) of the theorem, we obtain

$$\begin{aligned} |\Upsilon(z)(t) - \Upsilon(v)(t)| &\leq \frac{1}{\Gamma(l)} \int_{\vartheta_1}^{t} (t-s)^{l-1} |u(s, z(s)) - u(s, v(s))| Ts \\ &+ \frac{2(t-\vartheta_1)}{(2(\vartheta_2-\vartheta_1) - (\lambda-\vartheta_1)^2)\Gamma(l)} \int_{\vartheta_1}^{\vartheta_2} (1-s)^{l-1} |u(s, z(s)) - u(s, v(s))| Ts \\ &+ \frac{2(t-\vartheta_1)}{(2(\vartheta_2-\vartheta_1) - (\lambda-\vartheta_1)^2)\Gamma(l)} \int_{\vartheta_1}^{\lambda} \left(\int_{\vartheta_1}^{s} (s-t)^{l-1} |u(t, z(t)) - u(t, v(t))| Tt \right) Ts \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{K}{\Gamma(l)} \int_{\vartheta_1}^{t} (t-s)^{l-1} |z(s) - v(s)|^r ds \\
&+ \frac{2K(t-\vartheta_1)}{(2(\vartheta_2 - \vartheta_1) - (\lambda - \vartheta_1)^2)\Gamma(l)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{l-1} |z(s) - v(s)|^r ds \\
&+ \frac{2K(t-\vartheta_1)}{(2(\vartheta_2 - \vartheta_1) - (\lambda - \vartheta_1)^2)\Gamma(l)} \int_{\vartheta_1}^{\lambda} \left(\int_{\vartheta_1}^{s} (s-\xi)^{l-1} |z(\xi) - v(\xi)|^r \mathcal{D}\xi \right) ds \\
&\leq \left(\frac{K(t-\vartheta_1)^l}{l\Gamma(l)} + \frac{2K(t-\vartheta_1)}{(2(\vartheta_2 - \vartheta_1) - (\lambda - \vartheta_1)^2)\Gamma(l)} \left(\frac{(\vartheta_2 - \vartheta_1)^l}{l} + \frac{(\lambda - \vartheta_1)^{1+l}}{l(1+l)} \right) \right) \|z - v\|_\infty^r \\
&\leq \frac{K(\vartheta_2 - \vartheta_1)^l}{\Gamma(1+l)} \left(1 + \frac{2K(\vartheta_2 - \vartheta_1)}{(2(\vartheta_2 - \vartheta_1) - (\lambda - \vartheta_1)^2)} \left(1 + \frac{\lambda - \vartheta_1}{1+l} \right) \right) \|z - v\|_\infty^r \\
&= \Lambda \|z - v\|_\infty^r , t \in [\vartheta_1, \vartheta_2]
\end{aligned}
$$

where,

$$
\Lambda = \frac{K(\vartheta_2 - \vartheta_1)^l}{\Gamma(1+l)} \left(1 + \frac{2K(\vartheta_2 - \vartheta_1)}{(2(\vartheta_2 - \vartheta_1) - (\lambda - \vartheta_1)^2)} \left(1 + \frac{\lambda - \vartheta_1}{1+l} \right) \right) \in (0, \infty).
$$

Therefore,

$$
\|\Upsilon(z) - \Upsilon(v)\|_\infty \leq \Lambda \|z - v\|_\infty^r . \tag{3.8}
$$

Further, relation (3.8) yields

$$
\left\|\Upsilon^2(z) - \Upsilon^2(v)\right\|_\infty \leq \Lambda^{r+1} \|z - v\|_\infty^{r^2} . \tag{3.9}
$$

As, $\|z - v\|_\infty^{r^2} \leq \|z - v\|_\infty^r , \quad r \in (0, 1]$.

The inequality (3.9) can be written as

$$
\left\|\Upsilon^2(z) - \Upsilon^2(v)\right\|_\infty \leq \Lambda^{r+1} \|z - v\|_\infty^r . \tag{3.10}
$$

Relations (3.8), (3.10) can be combined in one of the following forms

$$
Min\left\{\mathcal{D}\left(\Upsilon^2(z), \Upsilon^2(v)\right), \mathcal{D}(\Upsilon(z), \Upsilon(v))\right\} \leq \Lambda^{r+1}(\mathcal{D}(z, v))^r, \text{ if } \Lambda < 1. \tag{3.11}
$$

$$
Min\left\{\mathcal{D}\left(\Upsilon^2(z), \Upsilon^2(v)\right), \mathcal{D}(\Upsilon(z), \Upsilon(v))\right\} \leq \Lambda(\mathcal{D}(z, v))^r, \quad \text{if } \Lambda > 1. \tag{3.12}
$$

If relation (3.12) holds, we can write

$$
\ln Min\left\{\mathcal{D}\left(\Upsilon^2(z), \Upsilon^2(v)\right), \mathcal{D}(\Upsilon(z), \Upsilon(v))\right\} \leq \ln \Lambda + r \ln \mathcal{D}(z, v) . \tag{3.13}
$$

Define, $F(t) = r \ln t - \ln p \in \mathfrak{F}$, and $F_1(t) = \ln t \in \mathfrak{F}$, where $p \in (0, 1)$, $p\Lambda < 1$ and $p\Lambda^{r+1} < 1$, so that the relation (3.13) can be written as

$$
F_1\left(Min\left\{\mathcal{D}\left(\Upsilon^2(z), \Upsilon^2(v)\right), \mathcal{D}(\Upsilon(z), \Upsilon(v))\right\}\right) \leq \ln P\Lambda + Fd(z, v) ,
$$

or,

$$
\ln\left(\frac{1}{p\Lambda}\right) + F_1\left(Min\left\{\mathcal{D}\left(\Upsilon^2(z), \Upsilon^2(v)\right), \mathcal{D}(\Upsilon(z), \Upsilon(v))\right\}\right) \leq Fd(z, v) .
$$

If relation (3.11) holds, we can write

$$\ln Min\left\{\mathcal{D}\left(\Upsilon^2(z),\Upsilon^2(v)\right),\mathcal{D}\left(\Upsilon(z),\Upsilon(v)\right)\right\}\leq(r+1)\ln\Lambda+r\ln\mathcal{D}(z,v)\,. \tag{3.14}$$

The relation (3.14) can be written as

$$F_1\left(Min\left\{\mathcal{D}\left(\Upsilon^2(z),\Upsilon^2(v)\right),\mathcal{D}\left(\Upsilon(z),\Upsilon(v)\right)\right\}\right)\leq\ln\Lambda^{r+1}p+Fd\,(z,v)\,.$$

So that,

$$\ln\left(\frac{1}{p\Lambda^{r+1}}\right)+F_1\left(Min\left\{\mathcal{D}\left(\Upsilon^2(z),\Upsilon^2(v)\right),\mathcal{D}\left(\Upsilon(z),\Upsilon(v)\right)\right\}\right)\leq Fd\,(z,v)\,.$$

Therefore, $\Upsilon:\Omega\to\Omega$ is a generalized F-contraction mapping and the operator Υ has a fixed point in Ω. That is, there exists a function $z^*\in C([\vartheta_1,\vartheta_2],\mathbb{R})$ such that $z^*=\Upsilon(z^*)$. The function z^* is a mild solution of the boundary value problem for (3.1) and (3.2).

Remark 3.1. In comparison with the result of [24], we used a weaker condition $\Lambda\in(0,\infty)$ to prove the existence of solution to (3.1) and (3.2) instead of $\Lambda\in(0,1)$.

Moreover, one can easily observe that the use of multiple functions in the generalized F-contraction also allows us to define a function $u\in C([\vartheta_1,\vartheta_2]\times\mathbb{R},\mathbb{R})$ in Theorem 3.1 with a weaker condition $|u(\tau,z)-u(\tau,v)|^m\leq K|z-v|^r, z,v\in\mathbb{R},\tau\in[\vartheta_1,\vartheta_2]$, where $m,r\in(0,\infty)$.

Example 3.1. Consider the nonlinear Caputo fractional differential equation

$${}_2^CD_\tau^{1.75}(z(\tau))=\frac{1}{\sqrt{\tau+14}}\arctan(\sqrt{|z(\tau)|}+e^\tau\cos\tau)+\sin\tau,\ \text{ for }\tau\in(2,3) \tag{3.15}$$

with the integral boundary condition:

$$z(2)=0,\quad z(3)=\int_0^{2.5}z(s)ds. \tag{3.16}$$

In this case,

$$u(\tau,u)=\frac{1}{\sqrt{\tau+14}}\arctan\,(\sqrt{|u|}+e^\tau\cos\tau)+\sin\tau,$$

and

$$|u(\tau,z)-u(\tau,u)|\ \leq\left(\frac{\pi}{4}+1\right)\sqrt{|z-v|}.$$

where,

$$\begin{aligned}\Lambda&=\frac{K(\vartheta_2-\vartheta_1)^l}{\Gamma(1+l)}\left(1+\frac{2K(\vartheta_2-\vartheta_1)}{(2(\vartheta_2-\vartheta_1)-(\lambda-\vartheta_1)^2)}\left(1+\frac{\lambda-\vartheta_1}{1+l}\right)\right)\\&=\frac{\left(\frac{\pi}{4}+1\right)}{\Gamma(2.75)}\left(1+\frac{2\left(\frac{\pi}{4}+1\right)}{1.75}\frac{3.25}{2.75}\right)\in(0,\infty),\qquad\text{such that},\Lambda>1.\end{aligned}$$

Therefore, Theorem 3.1 guarantees the solution of boundary value problem (3.15) and (3.16).

Remark 3.2. Note that the boundary value problem (3.15) and (3.16) are also studied in [22] (see Example 5 therein) and [24] (see Example 3.3 therein). Based on the obtained fixed points theorems we used the weaker conditions for the right hand side part of the equation and found the existence of fixed point for $K > 0$ and $\Lambda > 1$.

Remark 3.3. Wardowski obtained some fixed point theorems (see; Theorem 1.1) assuming that T satisfies the following contractive-type condition

$$\tau + F(\mathcal{D}(Tx, Ty)) \leq F(\mathcal{D}(x, y)), \tag{3.17}$$

where, $F : (0, \infty) \mapsto \mathbb{R}$ is nondecreasing. Whereas, the condition that we used in Theorem 2.1 is of the following form

$$\tau + F\left(\min\left\{(\mathcal{D}(Tx, Ty)), \left(\mathcal{D}(T^2x, T^2y)\right)\right\}\right) \leq F\left(\mathcal{D}(x, y)\right). \tag{3.18}$$

One can easily observe that relation (3.18) represents a generalization of (3.17).

Moreover, the following Proinov's condition represents a generalization of Wardowski's contraction condition.

$$F_1(\mathcal{D}(Tx, Ty)) \leq F(\mathcal{D}(x, y)). \tag{3.19}$$

The main condition we used in Theorem 2.5 is of the form

$$F_1\left(\min\left\{(\mathcal{D}(Tx, Ty)), \left(\mathcal{D}(T^2x, T^2y)\right)\right\}\right) \leq F(\mathcal{D}(x, y)), \tag{3.20}$$

where, $F(\mathcal{D}(x, y)) = F'(\mathcal{D}(x, y)) - \tau$. In (3.19), function F_1 cannot exceed F. Whereas, the condition in (3.20) allows F_1 to exceed F for different iterates.

4. Conclusions

Although, Proinov [12] claimed that being a special case of Skof's result [19], the $F-$ contraction type mappings and their generalizations do not add a valuable work in the literature anymore, we found some new generalizations that extend Wardowski [2], Skof [19] as well as Proinov's idea [12] of $F-$contraction type mappings. Moreover, with the use of multiple functions and the idea of generalized Banach contraction principal [11], we applied less restrictive conditions on Caputo fractional differential equations than the sufficient small Lipschitz constant studied by Mehmood [22] and Hanadi [24]. The new generalizations of F-contraction, F-expanding type mappings and the corresponding results will break open new grounds for the research workers as they will be able to find the existence of solution to an extensive range of differential equations (see [25–31]) with some weaker conditions.

5. Questions

In this research, the new generalizations of F-contraction mapping, F-Suzuki contraction mapping, F-expanding mapping and the corresponding results will provide a new direction of metric fixed point theory for the research workers. They may try to find the existence of fixed point for the further extensions of certain generalized mappings.

1) One may find the above results with $p > 2$, for the generalizations of F-contraction, F-Suzuki contraction and F-expanding mappings.
2) One may work on the idea of introducing new generalizations of F-contraction, F-Suzuki contraction and F-expanding mappings.

3) There may exist the possibility of finding fixed points for these generalized mappings in other generalized metric spaces.

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Conflicts of interest

The authors declare that they have no competing interests.

References

1. S. Banach, Sur les operations dans les ensembles abstraits et leur application aux quations intgrales, *Fund. Math.*, **3** (1922), 133–183.
2. D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 94.
3. N. A. Secelean, Iterated function systems consisting of F-contractions, *Fixed Point Theory Appl.*, **2013** (2013), 277.
4. N. A. Secelean, Weak $F-$contractions and some fixed point results, *B. Iran. Math. Soc.*, **42** (2016), 779–798.
5. H. Piri, P. Kumam, Some fixed point theorems concerning $F-$contraction in complete metric spaces, *Fixed Point Theory Appl.*, **2014** (2014), 210.
6. F. Vetro, $F-$contractions of Hardy-Rogers type and application to multistage decesion process, *Nonlinear Anal. Model. Control*, **21** (2016), 531–546.
7. N. A. Secelean, D. Wardowski, $\psi F-$contractions: Not necessarily nonexspansive Picard operators, *Results Math.*, **70** (2016), 415–431.
8. A. Lukács, S. Kajántó, Fixed point results for various type $F-$contractions in completes $b-$metric spaces, *Fixed Point Theory*, **19** (2018), 321–334.
9. H. H. Alsulami, E. Karapınar, H. Piri, Fixed points of generalized $F-$Suzuki type contraction in complete $b-$metric spaces, *Discrete Dyn. Nat. Soc.*, **2015** (2015), 969726.
10. J. Gornicki, Fixed point theorems for $F-$expanding mappings, *Fixed Point Theory Appl.*, **2017** (2016), 9.
11. J. Merryfield, J. D. Stein Jr., A generalization of the Banach contraction principle, *J. Math. Anal. Appl.*, **273** (2002), 112–120.
12. P. D. Proinov, Fixed point theorems for generalized contractive mappings in metric spaces, *J. Fixed Point Theory Appl.* **22** (2020), 21.
13. S. Moradi, Fixed point of single-valued cyclic weak ϕF-contraction mappings, *Filomat*, **28** (2014), 1747–1752.
14. M. A. Geraghty, On contractive mappings, *P. Am. Math. Soc.*, **40** (1973), 604–608.

15. A. Amini-Harandi, A. Petruşel, A fixed point theorem by altering distance technique in complete metric spaces, *Miskolc Math. Notes*, **14** (2013), 11-–17.
16. Z. L. Li, S. J. Jiang, Fixed point theorems of JS-quasi-contractions, *Fixed Point Theory Appl.*, **2016** (2016), 40.
17. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 38.
18. D. Wardowski, N. V. Dung, Fixed points of F-weak contractions on complete metric spaces, *Demonstratio Math.*, **47** (2014), 146—155.
19. F. Skof, Teoremi di punto fisso per applicazioni negli spazi metrici, *Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **111** (1977), 323–329.
20. K. Goeblel, B. Sims, Mean Lipschitzian mappings, *Contemp. Math.*, **513** (2010), 157–167.
21. W. J. Liu, H. F. Zhuang, Existence of solutions for Caputo fractional boundary value problems with integral conditions, *Carpathian J. Math.*, **33** (2017), 207–217.
22. N. Mehmood, N. Ahmad, Existence results for fractional order boundary value problem with nonlocal non-separated type multi-point integral boundary conditions, *AIMS Mathematics*, **5** (2019), 385–398.
23. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam, The Netherlands: Elsevier, 2006.
24. Z. Hanadi, A. Hoda, S. H. Fouad, A. Jamshaid, Generalized Fixed point results with application to nonlinear fractional differential equations, *Mathematics*, **8** (2020), 1168.
25. N. Saleem, I. Iqbal, B. Iqbal, S. Radenović, Coincidence and fixed points of multivalued $F-$contractions in generalized metric space with application, *J. Fixed Point Theory Appl.*, **22** (2020), 81.
26. Y. C. Guo, M. Q. Chen, X. B. Shu, F. Xu, The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm, *Stoch. Anal. Appl.*, **39** (2021), 643–666.
27. X. Ma, X. B. Shu, J. Z. Mao, Existence of almost periodic solutions for fractional impulsive neutral stochastic differential equations with infinite delay, *Stoch. Dynam.*, **20** (2020), 2050003.
28. X. B. Shu, Y.J. Shi, A study on the mild solution of impulsive fractional evolution equations, *Appl. Math. Comput.*, **273** (2016), 465–476.
29. X. B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal. Theor.*, **74** (2011), 2003–2011.
30. L. C. Ceng, M. J. Shang, Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings, *Optimization*, **70** (2021), 715–740.
31. L. C. Ceng, A. Petrusel, X. Qin, J. C. Yao, Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints, *Optimization*, **70** (2021), 1337–1358.

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