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Research article

Some new identities involving Laguerre polynomials

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Abstract: In this paper, we use elementary method and some sort of a counting argument to show the equality of two expressions. That is, let $f(n)$ and $g(n)$ be two functions, k be any positive integer. Then $f(n) = \sum_{r=0}^{n}(-1)^r \cdot \frac{n!}{r!} \cdot \binom{n+k-1}{r+k-1} \cdot g(r)$ if and only if $g(n) = \sum_{r=0}^{n}(-1)^r \cdot \frac{n!}{r!} \cdot \binom{n+k-1}{r+k-1} \cdot f(r)$ for all integers $n \geq 0$. As an application of this formula, we obtain some new identities involving the famous Laguerre polynomials.

Keywords: inversion formula; Laguerre polynomials; exponential generating function; identity
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1. Introduction

For any integer $n \geq 0$, the famous Laguerre polynomial $L_n(x)$ is defined by

$$L_n(x) = e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right) = \sum_{r=0}^{n}(-1)^r \cdot \frac{n!}{r!} \cdot \binom{n}{r} \cdot x^r, \tag{1.1}$$

where $\binom{n}{m} = \frac{m!}{m!(n-m)!}$.

The exponential generating function of $L_n(x)$ is

$$\frac{1}{1-t} \cdot e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} \cdot t^n. \tag{1.2}$$

It is clear that $L_0(x) = 1$, $L_1(x) = -x + 1$, and $L_{n+1}(x) = (2n + 1 - x)L_n(x) - n^2 L_{n-1}(x)$ for all positive integer $n \geq 1$. Therefore $L_n(x)$ satisfy a 3-term recurrence relation. It satisfies the integral identity

$$\int_0^{+\infty} e^{-x} \cdot L_m(x) \cdot L_n(x) dx = \begin{cases} 0, & \text{if } m \neq n; \\ (n!)^2, & \text{if } m = n. \end{cases} \tag{1.3}$$

In recent years, many papers investigated the elementary properties of Laguerre polynomials and recurrence polynomials (see [3–8]).

In this paper, as a note, we prove a new inversion formula related to Laguerre polynomials. That is, we shall prove the following conclusion.

Theorem. Let $f(n)$ and $g(n)$ be two functions, k be any positive integer. For all integer $n \geq 0$, if

$$f(n) = \sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n+k-1}{r+k-1} \cdot g(r), \tag{1.4}$$

then we have the inversion formula

$$g(n) = \sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n+k-1}{r+k-1} \cdot f(r). \tag{1.5}$$

It is clear that if we take $k = 1$, $f(n) = L_n(x)$ and $g(r) = x^r$, then from (1.1) we have the identity

$$x^n = \sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n}{r} \cdot L_r(x).$$

2. Proof of the theorem

In this section, we shall complete the proof of our main result. Hereinafter, we shall use some elementary number theory contents and properties of power series, which can be found in references [1,2], and also be found in [3,4], so we will not repeat them here. First we need the following simple lemma.

Lemma. For any integer $n \geq 0$, let $f(n)$ and $g(n)$ be two number theoretic functions. If $f(n)$ and $g(n)$ satisfy the identity

$$f(n) = \sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n}{r} \cdot g(r),$$

then we have the inversion formula

$$g(n) = \sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n}{r} \cdot f(r).$$

Proof. It is clear that the lemma holds if $n = 0$. So without loss of generality, we can assume that $n \geq 1$. At this time, from the definition of $f(n)$, we have

$$\begin{aligned}
&\sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n}{r} \cdot f(r) \\
= \ &\sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n}{r} \left(\sum_{s=0}^{r} (-1)^s \cdot \frac{r!}{s!} \cdot \binom{r}{s} \cdot g(s) \right) \\
= \ &n! \cdot \sum_{s=0}^{n} (-1)^s \cdot \frac{g(s)}{s!} \cdot \sum_{r=s}^{n} (-1)^r \cdot \binom{r}{s} \cdot \binom{n}{r}
\end{aligned}$$

$$
\begin{aligned}
&= n! \cdot \sum_{s=0}^{n} (-1)^s \cdot \frac{g(s)}{s!} \cdot \sum_{r=0}^{n-s} (-1)^{r+s} \cdot \binom{r+s}{s} \cdot \binom{n}{r+s} \\
&= n! \cdot \sum_{s=0}^{n} \frac{g(s)}{s!} \cdot \sum_{r=0}^{n-s} (-1)^r \cdot \frac{n!}{(n-s)! \cdot s!} \cdot \frac{(n-s)!}{r! \cdot (n-r-s)!} \\
&= n! \cdot \sum_{s=0}^{n} \frac{g(s)}{s!} \cdot \frac{n!}{(n-s)! \cdot s!} \cdot \sum_{r=0}^{n-s} (-1)^r \cdot \binom{n-s}{r} \\
&= n! \cdot \sum_{s=0}^{n} \frac{g(s)}{s!} \cdot \frac{n!}{(n-s)! \cdot s!} \cdot (1-1)^{n-s} \\
&= n! \cdot \frac{g(n)}{n!} \cdot \frac{n!}{0! \cdot n!} + n! \cdot \sum_{s=0}^{n-1} \frac{g(s)}{s!} \cdot \frac{n!}{(n-s)! \cdot s!} \cdot (1-1)^{n-s} \\
&= g(n).
\end{aligned}
$$

That is,

$$
g(n) = \sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n}{r} \cdot f(r).
$$

The lemma is proved.

Now for any positive integer $k \geq 1$, let $f(n)$ and $g(n)$ be defined as (1.4) and (1.5). If $k = 1$, then the theorem follows from the lemma. So without loss of generality we can assume $k \geq 2$. This time we have

$$
\begin{aligned}
&\sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n+k-1}{r+k-1} \cdot f(r) \\
= &\sum_{r=0}^{n} (-1)^r \cdot \frac{n!}{r!} \cdot \binom{n+k-1}{r+k-1} \left(\sum_{s=0}^{r} (-1)^s \cdot \frac{r!}{s!} \cdot \binom{r+k-1}{s+k-1} \cdot g(s) \right) \\
= &\ n! \cdot \sum_{s=0}^{n} (-1)^s \cdot \frac{g(s)}{s!} \cdot \sum_{r=s}^{n} (-1)^r \cdot \binom{r+k-1}{s+k-1} \cdot \binom{n+k-1}{r+k-1} \\
= &\ n! \cdot \sum_{s=0}^{n} (-1)^s \cdot \frac{g(s)}{s!} \cdot \sum_{r=0}^{n-s} (-1)^{r+s} \cdot \binom{r+s+k-1}{s+k-1} \cdot \binom{n+k-1}{r+s+k-1} \\
= &\ n! \cdot \sum_{s=0}^{n} \frac{g(s)}{s!} \cdot \sum_{r=0}^{n-s} (-1)^r \cdot \frac{(n+k-1)!}{(n-s)! \cdot (s+k-1)!} \cdot \frac{(n-s)!}{r! \cdot (n-r-s)!} \\
= &\ n! \cdot \sum_{s=0}^{n} \frac{g(s)}{s!} \cdot \binom{n+k-1}{s+k-1} \cdot \sum_{r=0}^{n-s} (-1)^r \cdot \binom{n-s}{r} \\
= &\ n! \cdot \sum_{s=0}^{n} \frac{g(s)}{s!} \cdot \binom{n+k-1}{s+k-1} \cdot (1-1)^{n-s} = g(n).
\end{aligned}
$$

The theorem is proved.

This theorem can also be proved by manipulating the exponential generating function in (1.2). Here we give an alternative proof for the theorem by using the well known method of the exponential generating functions. Consider

$$F(x)=\sum_{n=0}^{\infty} f(n)\frac{x^n}{n!} \quad \text{and} \quad G(x)=\sum_{n=0}^{\infty} g(n)\frac{x^n}{n!},$$

where $f(n)$ and $g(n)$ defined as in the theorem.

Then multiplying on both sides of (1.4) by $\frac{x^n}{n!}$ and summing over $n \geq 0$, we obtain

$$F(x)=(1-x)^{-k}G(-x(1-x)^{-1}). \tag{2.1}$$

Now let $y=-x(1-x)^{-1}$, so $x=-y(1-y)^{-1}$ and $1-x=(1-y)^{-1}$, and expressing (2.1) in terms of y gives

$$F\left(-y(1-y)^{-1}\right)=(1-y)^k G(y),$$

or rearranging slightly,

$$G(y)=(1-y)^{-k}F\left(-y(1-y)^{-1}\right). \tag{2.2}$$

Equating coefficients of $\frac{y^n}{n!}$ in (2.2) we obtain (1.5). This completes the second proof of the theorem.

3. Conclusions

The main result of this paper is a theorem, which proved a new reciprocal formula for some arithmetical functions, it revealed some essential properties of the Laguerre polynomials. The result is actually new contribution to the study of the properties of Laguerre polynomials. Of course, the methods adopted in this paper have some good reference for the further study of the Laguerre polynomials.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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