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Research article

A new family of degenerate poly-Bernoulli polynomials of the second kind with its certain related properties

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Abstract: The main object of this article is to present type 2 degenerate poly-Bernoulli polynomials of the second kind and numbers by arising from modified degenerate polyexponential function and investigate some properties of them. Thereafter, we treat the type 2 degenerate unipoly-Bernoulli polynomials of the second kind via modified degenerate polyexponential function and derive several properties of these polynomials. Furthermore, some new identities and explicit expressions for degenerate unipoly polynomials related to special numbers and polynomials are obtained. In addition, certain related beautiful zeros and graphical representations are displayed with the help of *Mathematica*.

Keywords: modified degenerate polyexponential functions; type 2 degenerate poly-Bernoulli polynomials of the second kind; degenerate unipoly functions
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1. Introduction

In [1,2], Carlitz initiated study of the degenerate Bernoulli and Euler polynomials and obtained some arithmetic and combinatorial results on them. In recent years, many mathematicians have drawn their attention to various degenerate versions of some old and new polynomials and numbers, namely some degenerate versions of Bernoulli numbers and polynomials of the second kind, Changhee numbers of the second kind, Daehee numbers of the second kind, Bernstein polynomials, central Bell numbers and polynomials, central factorial numbers of the second kind, Cauchy numbers, Eulerian numbers and polynomials, Fubini polynomials, Stirling numbers of the first kind, Stirling polynomials of the second kind, central complete Bell polynomials, Bell numbers and polynomials, type 2 Bernoulli numbers and polynomials, type 2 Bernoulli polynomials of the second kind, poly-Bernoulli numbers and polynomials, poly-Cauchy polynomials, and of Frobenius-Euler polynomials, to name a few [3,14,16–18] and the references therein. They have studied those polynomials and numbers with their interest not only in combinatorial and arithmetic properties but also in differential equations and certain symmetric identities [4,5] and references therein, and found many interesting results related to them [12,19–28]. It is remarkable that studying degenerate versions is not only limited to polynomials but also extended to transcendental functions.

The Bernoulli polynomials of the second are defined by as follows (see [9,13])

$$\frac{z}{\log(1+z)}(1+z)^x = \sum_{q=0}^{\infty} b_q(x)\frac{z^q}{q!}. \tag{1.1}$$

When $x = 0$, $b_q(0) = b_q$ are called the Bernoulli numbers of the second kind.

The degenerate exponential function $e_\lambda^x(z)$ is defined by (see [6–19])

$$e_\lambda^x(z) = (1+\lambda z)^{\frac{x}{\lambda}}, \quad e_\lambda(z) = (1+\lambda z)^{\frac{1}{\lambda}}, \ \lambda \in \mathbb{C}\backslash\{0\}. \tag{1.2}$$

We note that

$$e_\lambda^x(z) = \sum_{q=0}^{\infty} (x)_{q,\lambda}\frac{z^q}{q!}, \text{(see [4,21])}, \tag{1.3}$$

where $(x)_{q,\lambda} = x(x-\lambda)\cdots(x-(q-1)\lambda), (q \geq 1), \ (x)_{0,\lambda} = 1$.

Note that

$$\lim_{\lambda\to 0} e_\lambda^x(z) = \sum_{q=0}^{\infty} x^q\frac{z^q}{q!} = e^{xz}.$$

The degenerate Bernoulli polynomials which are defined by Carlitz's as follows (see [1,2])

$$\frac{z}{e_\lambda(z)-1}e_\lambda^x(z) = \frac{z}{(1+\lambda z)^{\frac{1}{\lambda}}-1}(1+\lambda z)^{\frac{x}{\lambda}} = \sum_{q=0}^{\infty} \beta_q(x;\lambda)\frac{z^q}{q!}. \tag{1.4}$$

At the point $x = 0, \beta_q(\lambda) = \beta_q(0;\lambda)$ are called the degenerate Bernoulli numbers.

Note that

$$\lim_{\lambda\longrightarrow 0} \beta_q(x;\lambda) = B_q(x).$$

The polylogarithm function is defined by

$$\mathrm{Li}_k(x) = \sum_{q=1}^{\infty} \frac{x^q}{q^k} \quad (k \in \mathbb{Z}, \quad | x |< 1), \text{(see [7])}. \tag{1.5}$$

Note that

$$\mathrm{Li}_1(x) = \sum_{q=1}^{\infty} \frac{x^q}{q} = -\log(1-x). \tag{1.6}$$

The poly-Bernoulli polynomials of the second are defined by (see [13])

$$\frac{\mathrm{Li}_k(1-e^{-z})}{\log(1+z)}(1+z)^x = \sum_{q=0}^{\infty} b_q^{(k)}(x)\frac{z^q}{q!}. \tag{1.7}$$

In the case when $x = 0$, $b_q^{(k)} = b_q^{(k)}(0)$ are called the poly-Bernoulli numbers of the second kind.
The modified degenerate polyexponential function is defined by (see [14])

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{q=1}^{\infty} \frac{(1)_{q,\lambda}}{(q-1)!q^k}x^q. \tag{1.8}$$

It is noteworthy to mention that

$$\mathrm{Ei}_{1,\lambda}(x) = \sum_{q=1}^{\infty} \frac{(1)_{q,\lambda}}{q!}x^q = e_\lambda(x) - 1.$$

The degenerate poly-Genocchi polynomials which are defined by Kim *et al.* as follows (see [14])

$$\frac{2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{e_\lambda(z)+1}e_\lambda^x(z) = \sum_{q=0}^{\infty} G_{q,\lambda}^{(k)}(x)\frac{z^q}{q!} \quad (k \in \mathbb{Z}). \tag{1.9}$$

When $x = 0$, $G_{q,\lambda}^{(k)} = G_{q,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers.
For $\lambda \in \mathbb{R}$, Kim-Kim defined the degenerate version of the logarithm function, denoted by $\log_\lambda(1+t)$ as follows (see [11])

$$\log_\lambda(1+z) = \sum_{q=1}^{\infty} \lambda^{\lambda-1}(1)_{q,1/\lambda}\frac{z^q}{q!}, \tag{1.10}$$

being the inverse of the degenerate version of the exponential function $e_\lambda(z)$ as has been shown below

$$e_\lambda(\log_\lambda(z)) = \log_\lambda(e_\lambda(z)) = z.$$

It is noteworthy to mention that

$$\lim_{\lambda\to 0}\log_\lambda(1+z) = \sum_{q=1}^{\infty}(-1)^{q-1}\frac{z^q}{q!} = \log(1+z).$$

The degenerate Daehee polynomials are defined by (see [15])

$$\frac{\log_\lambda(1+z)}{z}(1+z)^x = \sum_{q=0}^{\infty} D_{q,\lambda}(x)\frac{z^q}{q!}. \tag{1.11}$$

In the case when $x = 0$, $D_{q,\lambda} = D_{q,\lambda}(0)$ denotes the degenerate Daehee numbers.

The degenerate Bernoulli polynomials of the second kind which are defined by Kim et al. as follows (see [9])

$$\frac{z}{\log_\lambda(1+z)}(1+z)^x = \sum_{q=0}^{\infty} b_{q,\lambda}(x)\frac{z^q}{q!}. \tag{1.12}$$

When $x = 0$, $b_{q,\lambda} = b_{q,\lambda}(0)$ are called the degenerate Bernoulli numbers of the second kind.

Note here that $\lim_{\lambda\to 0} b_{q,\lambda}(x) = b_q(x), (q \geq 0)$.

The degenerate Stirling numbers of the first kind are defined by

$$\frac{1}{k!}(\log_\lambda(1+z))^k = \sum_{q=k}^{\infty} S_{1,\lambda}(q,k)\frac{z^q}{q!} \ \ (k \geq 0), \text{(see [11,12])}. \tag{1.13}$$

It is noticed that

$$\lim_{\lambda\to 0} S_{1,\lambda}(q,k) = S_1(q,k),$$

are the Stirling numbers of the first kind presented by

$$\frac{1}{k!}(\log(1+z))^k = \sum_{q=k}^{\infty} S_1(q,k)\frac{z^q}{q!} \ \ (k \geq 0), \text{(see [7, 17])}.$$

The degenerate Stirling numbers of the second kind are defined by (see [8])

$$\frac{1}{k!}(e_\lambda(z)-1)^k = \sum_{q=k}^{\infty} S_{2,\lambda}(q,k)\frac{z^q}{q!} \ \ (k \geq 0). \tag{1.14}$$

It is clear that

$$\lim_{\lambda\to 0} S_{2,\lambda}(q,k) = S_2(q,k),$$

are the Stirling numbers of the second kind specified by

$$\frac{1}{k!}(e^z-1)^k = \sum_{q=k}^{\infty} S_2(q,k)\frac{z^q}{q!} \ \ (k \geq 0), \text{(see [1–28])}.$$

Motivated by the works of Kim et al. [11,14], in this paper, we study the type 2 degenerate poly-Bernoulli polynomials of the second kind arising from modified degenerate polyexponential function and obtain some related identities and explicit expressions. Also, we establish the type 2 degenerate unipoly-Bernoulli polynomials of the second kind attached to an arithmetic function by using modified degenerate polyexponential function and discuss some properties of them.

2. Type 2 degenerate poly-Bernoulli polynomials of the second kind

Here, the type 2 degenerate poly-Bernoulli polynomials of the second kind are defined by using the modified degenerate polyexponential function which is called the degenerate poly-Bernoulli polynomials of the second kind as

$$\frac{\mathrm{Ei}_{k,\lambda}\left(\log_{\lambda}(1+z)\right)}{\log_{\lambda}(1+z)}(1+z)^{x}=\sum_{j=0}^{\infty}Pb_{j,\lambda}^{(k)}(x)\frac{z^{j}}{j!},(k\in\mathbb{Z}). \tag{2.1}$$

When $x=0$, $Pb_{j,\lambda}^{(k)}=Pb_{j,\lambda}^{(k)}(0)$ are called the type 2 degenerate poly-Bernoulli numbers of the second kind.

Note that

$$\begin{aligned}\lim_{\lambda\to0}\frac{\mathrm{Ei}_{k,\lambda}\left(\log_{\lambda}(1+z)\right)}{\log_{\lambda}(1+z)}(1+z)^{x}&=\sum_{j=0}^{\infty}\lim_{\lambda\to0}Pb_{j,\lambda}^{(k)}(x)\frac{z^{j}}{j!}\\&=\frac{\mathrm{Ei}_{k}\left(\log(1+z)\right)}{\log(1+z)}(1+z)^{x}=\sum_{j=0}^{\infty}Pb_{j}^{(k)}(x)\frac{z^{j}}{j!},(k\in\mathbb{Z}),\end{aligned} \tag{2.2}$$

where $Pb_{j}^{(k)}(x)$ are called the type 2 poly-Bernoulli polynomials of the second kind (see [9]).

First, we note that

$$\begin{aligned}\mathrm{Ei}_{k,\lambda}\left(\log_{\lambda}(1+z)\right)&=\sum_{q=1}^{\infty}\frac{(1)_{q,\lambda}(\log_{\lambda}(1+z))^{q}}{(q-1)!q^{k}}\\&=\sum_{q=0}^{\infty}\frac{(1)_{q+1,\lambda}(\log_{\lambda}(1+z))^{q+1}}{(q+1)^{k}q!}\\&=\sum_{q=0}^{\infty}\frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}}\frac{1}{(q+1)!}(\log_{\lambda}(1+z))^{q+1}\\&=\sum_{q=0}^{\infty}\frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}}\sum_{r=q+1}^{\infty}S_{1,\lambda}(r,q+1)\frac{z^{r}}{r!}.\end{aligned} \tag{2.3}$$

By making use of (2.1) and (2.3), we see that

$$\begin{aligned}&\frac{z}{\log_{\lambda}(1+z)}(1+z)^{x}\mathrm{Ei}_{k,\lambda}\left(\log_{\lambda}(1+z)\right)\\&=\frac{z}{\log_{\lambda}(1+z)}(1+z)^{x}\sum_{q=0}^{\infty}\frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}}\sum_{r=q}^{\infty}\frac{S_{1,\lambda}(r+1,q+1)}{r+1}\frac{z^{r}}{r!}\\&=\sum_{j=0}^{\infty}b_{j,\lambda}(x)\frac{z^{j}}{j!}\sum_{q=0}^{\infty}\frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}}\sum_{r=q}^{\infty}\frac{S_{1,\lambda}(r+1,q+1)}{r+1}\frac{z^{r}}{r!}\\&=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}\sum_{q=0}^{r}\frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}}\frac{S_{1,\lambda}(r+1,q+1)}{r+1}b_{j-r,\lambda}(x)\right)\frac{z^{j}}{j!}.\end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. For $k \in \mathbb{Z}$ and $j \geq 0$, we have

$$Pb_{j,\lambda}^{(k)}(x) = \sum_{r=0}^{j} \binom{j}{r} \sum_{q=0}^{r} \frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}} \frac{S_{1,\lambda}(r+1,q+1)}{r+1} b_{j-r,\lambda}(x).$$

Corollary 2.1. Putting $k = 1$ in Theorem 2.1 yields

$$Pb_{j,\lambda}(x) = \sum_{r=0}^{j} \binom{j}{r} \sum_{q=0}^{r} \frac{(1)_{q+1,\lambda} S_{1,\lambda}(r+1,q+1)}{r+1} b_{j-r,\lambda}(x).$$

Let $1 \leq k \in \mathbb{Z}$. For $s \in \mathbb{C}$, the function $\chi_{k,\lambda}(s)$ is given as

$$\chi_{k,\lambda}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z^{s-1}}{\log_\lambda(1+z)} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz. \tag{2.5}$$

From Eq (2.5), we have

$$\begin{aligned} \chi_{k,\lambda}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{z^{s-1}}{\log_\lambda(1+z)} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz \\ &= \frac{1}{\Gamma(s)} \int_0^1 \frac{z^{s-1}}{\log_\lambda(1+z)} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz \\ &+ \frac{1}{\Gamma(s)} \int_1^\infty \frac{z^{s-1}}{\log_\lambda(1+z)} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz. \end{aligned} \tag{2.6}$$

For any $s \in \mathbb{C}$, the second integral is absolutely convergent and thus, the second term on the r.h.s. vanishes at non-positive integers. That is,

$$\lim_{s \to -m} \left| \frac{1}{\Gamma(s)} \int_1^\infty \frac{z^{s-1}}{\log_\lambda(1+z)} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz \right| \leq \frac{1}{\Gamma(-m)} M = 0. \tag{2.7}$$

On the other hand, the first integral in Eq (2.7), for $\Re(s) > 0$ can be written as

$$\frac{1}{\Gamma(s)} \sum_{r=0}^{\infty} \frac{Pb_{r,\lambda}^{(k)}}{r!} \frac{1}{s+r},$$

which defines an entire function of s. Thus, we may include that $\chi_{k,\lambda}(s)$ can be continued to an entire function of s.

Further, from (2.6) and (2.7), we obtain

$$\begin{aligned} \chi_{k,\lambda}(-m) &= \lim_{s \to -m} \frac{1}{\Gamma(s)} \int_0^1 \frac{z^{s-1}}{\log_\lambda(1+z)} \mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) dz \\ &= \lim_{s \to -m} \frac{1}{\Gamma(s)} \int_0^1 z^{s-1} \sum_{r=0}^{\infty} \frac{Pb_{r,\lambda}^{(k)} z^r}{r!} dz = \lim_{s \to -m} \frac{1}{\Gamma(s)} \sum_{r=0}^{\infty} \frac{Pb_{r,\lambda}^{(k)}}{s+r} \frac{1}{r!} \end{aligned}$$

$$= \cdots + 0 + \cdots + 0 + \lim_{s \to -m} \frac{1}{\Gamma(s)} \frac{1}{s+m} \frac{Pb_{m,\lambda}^{(k)}}{m!} + 0 + 0 + \cdots \tag{2.8}$$

$$= \lim_{s \to -m} \frac{\left(\frac{\Gamma(1-s)\sin \pi s}{\pi}\right)}{s+m} \frac{Pb_{m,\lambda}^{(k)}}{m!} = \Gamma(1+m)\cos(\pi m) \frac{Pb_{m,\lambda}^{(k)}}{m!}$$

$$= (-1)^m Pb_{m,\lambda}^{(k)}.$$

In view of (2.8), we obtain the following theorem.

Theorem 2.2. Let $k \geq 1$ and $m \in \mathbb{N} \bigcup \{0\}$, $s \in \mathbb{C}$, we have

$$\chi_{k,\lambda}(-m) = (-1)^m Pb_{m,\lambda}^{(k)}.$$

Using (1.8), we observe that

$$\frac{\mathrm{d}}{\mathrm{dx}} \mathrm{Ei}_{k,\lambda} \left(\log_\lambda(1+x)\right) = \frac{\mathrm{d}}{\mathrm{dx}} \sum_{j=1}^{\infty} \frac{(1)_{j,\lambda}(\log_\lambda(1+x))^j}{j^k (j-1)!}$$

$$= \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)} \sum_{j=1}^{\infty} \frac{(1)_{j,\lambda}(\log_\lambda(1+x))^j}{j^{k-1}(j-1)!} = \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)} \mathrm{Ei}_{k-1,\lambda}(\log_\lambda(1+x)). \tag{2.9}$$

Thus, by (2.9), for $k \geq 2$, we get

$$\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+x)) = \int_0^x \frac{(1+z)^{\lambda-1}}{\log(1+z)} \mathrm{Ei}_{k-1,\lambda}(\log_\lambda(1+z)) dz$$

$$= \int_0^x \underbrace{\frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)} \int_0^z \cdots \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)} \int_0^z \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}}_{(k-2)\text{-times}} dz \cdots dz$$

$$\times \mathrm{Ei}_{1,\lambda}(\log_\lambda(1+z)) dz \cdots dz$$

$$= \int_0^x \underbrace{\frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)} \int_0^z \cdots \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)} \int_0^z \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)}}_{(k-2)\text{-times}} z dz \cdots dz. \tag{2.10}$$

From (2.1) and (2.10), we get

$$\sum_{j=0}^{\infty} Pb_{j,\lambda}^{(k)} \frac{x^j}{j!} = \frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+x)\right)}{\log_\lambda(1+x)} = \frac{1}{\log_\lambda(1+x)}$$

$$\times \int_0^x \underbrace{\frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)} \int_0^z \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)} \cdots \int_0^z}_{(k-2)\text{-times}} \frac{(1+z)^{\lambda-1}}{\log_\lambda(1+z)} z dz \cdots dz. \tag{2.11}$$

$$= \frac{x}{\log_\lambda(1+x)} \sum_{q=0}^{\infty} \sum_{q_1 + \cdots + q_{k-1} = q} \binom{q}{q_1, \cdots, q_{k-1}}$$

$$\times\frac{b_{q_1,\lambda}(\lambda-1)}{q_1+1}\frac{b_{q_2,\lambda}(\lambda-1)}{q_1+q_2+1}\cdots\frac{b_{q_{k-1},\lambda}(\lambda-1)}{q_1+\cdots+q_{k-1}+1}\frac{x^q}{q!}$$

$$=\sum_{j=0}^{\infty}\sum_{q=0}^{j}\binom{j}{q}\sum_{q_1+\cdots+q_{k-1}=q}\binom{q}{q_1,\cdots,q_{k-1}}b_{j-q,\lambda}$$

$$\times\frac{b_{q_1,\lambda}(\lambda-1)}{q_1+1}\frac{b_{q_2,\lambda}(\lambda-1)}{q_1+q_2+1}\cdots\frac{b_{q_{k-1},\lambda}(\lambda-1)}{q_1+\cdots+q_{k-1}+1}\frac{x^j}{j!}. \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.3. For $j\in\mathbb{N}$ and $k\in\mathbb{Z}$, we have

$$Pb_{j,\lambda}^{(k)}=\sum_{q=0}^{j}\binom{j}{q}\sum_{q_1+\cdots+q_{k-1}=q}\binom{q}{q_1,\cdots,q_{k-1}}b_{j-q,\lambda}$$

$$\times\frac{b_{q_1,\lambda}(\lambda-1)}{q_1+1}\frac{b_{q_2,\lambda}(\lambda-1)}{q_1+q_2+1}\cdots\frac{b_{q_{k-1},\lambda}(\lambda-1)}{q_1+\cdots+q_{k-1}+1}.$$

Corollary 2.2. Taking $k=2$ in Theorem 2.3 yields

$$Pb_{j,\lambda}^{(2)}=\sum_{q=0}^{j}\binom{j}{q}\frac{b_{q,\lambda}(\lambda-1)}{q+1}b_{j-q,\lambda}.$$

Replacing z by $e_\lambda(z)-1$ in (2.1), we get

$$\sum_{q=0}^{\infty}Pb_{q,\lambda}^{(k)}(x)\frac{(e_\lambda(z)-1)^q}{q!}=\frac{\mathrm{Ei}_{k,\lambda}(z)}{z}e_\lambda^x(z)$$

$$=\sum_{j=0}^{\infty}\frac{(x)_{j,\lambda}z^j}{j!}\sum_{r=0}^{\infty}\frac{(1)_{r+1,\lambda}z^r}{(r+1)^k r!}=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}\frac{(1)_{r+1,\lambda}(x)_{j-r,\lambda}}{(r+1)^k}\right)\frac{z^j}{j!}. \tag{2.13}$$

On the other hand,

$$\sum_{q=0}^{\infty}Pb_{q,\lambda}^{(k)}(x)\frac{(e_\lambda(z)-1)^q}{q!}=\sum_{q=0}^{\infty}Pb_{q,\lambda}^{(k)}(x)\sum_{j=q}^{\infty}S_{2,\lambda}(j,q)\frac{z^j}{j!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}Pb_{q,\lambda}^{(k)}(x)S_{2,\lambda}(j,q)\right)\frac{z^j}{j!}. \tag{2.14}$$

In view of (2.13) and (2.14), we get the following theorem.

Theorem 2.4. For $k\in\mathbb{Z}$ and $j\geq 0$, we have

$$\sum_{q=0}^{j}Pb_{q,\lambda}^{(k)}(x)S_{2,\lambda}(j,q)=\sum_{r=0}^{j}\binom{j}{r}\frac{(1)_{r+1,\lambda}(x)_{j-r,\lambda}}{(r+1)^k}.$$

By using (2.1), we get

$$\sum_{j=1}^{\infty}\left[Pb_{j,\lambda}^{(k)}(x+1)-Pb_{j,\lambda}^{(k)}(x)\right]\frac{z^j}{j!}=\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{\log_\lambda(1+z)}(1+z)^{x+1}-\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{\log_\lambda(1+z)}(1+z)^{x}$$

$$=\frac{z\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{\log_\lambda(1+z)}(1+z)^{x}=\left(\frac{z}{\log_\lambda(1+z)}(1+z)^{x}\right)\left(\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)\right)$$

$$=\left(\sum_{j=0}^{\infty}b_{j,\lambda}(x)\frac{z^j}{j!}\right)\left(\sum_{q=1}^{\infty}\frac{(1)_{q,\lambda}(\log_\lambda(1+z))^q}{(q-1)!q^k}\right)$$

$$=\left(\sum_{j=0}^{\infty}b_{j,\lambda}(x)\frac{z^j}{j!}\right)\left(\sum_{q=1}^{\infty}\frac{(1)_{q,\lambda}(\log_\lambda(1+z))^q}{(q-1)!q^k}\right)$$

$$=\left(\sum_{j=1}^{\infty}b_{j,\lambda}(x)\frac{z^j}{j!}\right)\left(\sum_{r=1}^{\infty}\sum_{q=1}^{r}\frac{(1)_{q,\lambda}}{q^{k-1}}S_{1,\lambda}(r,q)\frac{z^r}{r!}\right)$$

$$=\sum_{j=1}^{\infty}\left(\sum_{r=1}^{j}\binom{j}{r}\sum_{q=1}^{r}\frac{(1)_{q,\lambda}}{q^{k-1}}S_{1,\lambda}(r,q)b_{j-r,\lambda}(x)\right)\frac{z^j}{j!}. \tag{2.15}$$

Therefore, by comparing the coefficients on both sides of (2.15), we obtain the following theorem.

Theorem 2.5. For $j\geq 0$, we have

$$Pb_{j,\lambda}^{(k)}(x+1)-Pb_{j,\lambda}^{(k)}(x)=\sum_{r=1}^{j}\binom{j}{r}\sum_{q=1}^{r}\frac{(1)_{q,\lambda}}{q^{k-1}}S_{1,\lambda}(r,q)b_{j-r,\lambda}(x).$$

By making use of (1.3) and (2.1), we have

$$\sum_{j=0}^{\infty}Pb_{j,\lambda}^{(k)}(x+\eta)\frac{z^j}{j!}=\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{\log_\lambda(1+z)}(1+z)^{x+\eta}$$

$$=\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{\log_\lambda(1+z)}(1+z)^{x}(1+z)^{\eta}=\left(\sum_{j=0}^{\infty}Pb_{j,\lambda}^{(k)}(x)\frac{z^j}{j!}\right)\left(\sum_{q=0}^{\infty}(\eta)_q\frac{z^q}{q!}\right)$$

$$=\sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\binom{j}{q}Pb_{j-q,\lambda}^{(k)}(x)(\eta)_q\right)\frac{z^j}{j!}. \tag{2.16}$$

Therefore, by Eq (2.16), we obtain the following theorem.

Theorem 2.6. For $j\geq 0$, we have

$$Pb_{j,\lambda}^{(k)}(x+\eta)=\sum_{q=0}^{j}\binom{j}{q}Pb_{j-q,\lambda}^{(k)}(x)(\eta)_q.$$

By using (2.1), we have

$$\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{\log_\lambda(1+z)}=\sum_{j=0}^{\infty}Pb_{j,\lambda}^{(k)}\frac{z^j}{j!}$$

$$\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right) = \log_\lambda(1+z)\sum_{j=0}^{\infty} Pb_{j,\lambda}^{(k)}\frac{z^j}{j!}$$

$$\begin{aligned}
\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{z} &= \frac{\log_\lambda(1+z)}{z}\sum_{j=0}^{\infty} Pb_{j,\lambda}^{(k)}\frac{z^j}{j!} \\
&= \left(\sum_{q=0}^{\infty} D_{q,\lambda}\frac{t^q}{q!}\right)\left(\sum_{j=0}^{\infty} Pb_{j,\lambda}^{(k)}\frac{z^j}{j!}\right) \\
&= \sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\binom{j}{q} Pb_{j-q,\lambda}^{(k)} D_{q,\lambda}\right)\frac{z^j}{j!}. \qquad (2.17)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{z} &= \frac{1}{z}\sum_{q=1}^{\infty}\frac{(1)_{q,\lambda}(\log_\lambda(1+z))^q}{(q-1)!q^k} \\
&= \frac{1}{z}\sum_{q=0}^{\infty}\frac{(1)_{q+1,\lambda}(\log_\lambda(1+z))^{q+1}}{q^k!(q+1)} \\
&= \frac{1}{z}\sum_{q=0}^{\infty}\frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}}\frac{1}{(q+1)!}(\log_\lambda(1+z))^{q+1} \\
&= \sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}}\frac{S_{1,\lambda}(j+1,q+1)}{j+1}\right)\frac{z^j}{j!}. \qquad (2.18)
\end{aligned}$$

Thus, by equations (2.17) and (2.18), we get the following theorem.

Theorem 2.7. For $j \geq 0$, we have

$$\sum_{q=0}^{j}\binom{j}{q} Pb_{j-q,\lambda}^{(k)} D_{q,\lambda} = \sum_{q=0}^{j}\frac{(1)_{q+1,\lambda}}{(q+1)^{k-1}}\frac{S_{1,\lambda}(j+1,q+1)}{j+1}.$$

From (2.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} Pb_{j,\lambda}^{(k)}(x)\frac{z^j}{j!} &= \frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{\log_\lambda(1+z)}(1+z)^x \\
&= \frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+z)\right)}{\log_\lambda(1+z)} e_\lambda^x(\log_\lambda(1+z)) \\
&= \sum_{j=0}^{\infty} Pb_{j,\lambda}^{(k)}(x)\frac{z^j}{j!}\sum_{q=0}^{\infty}(x)_{q,\lambda}\sum_{r=q}^{\infty} S_{1,\lambda}(r,q)\frac{z^r}{r!} \\
&= \sum_{j=0}^{\infty} Pb_{j,\lambda}^{(k)}(x)\frac{z^j}{j!}\sum_{r=0}^{\infty}\sum_{q=0}^{r}(x)_{q,\lambda} S_{1,\lambda}(r,q)\frac{z^r}{r!}
\end{aligned}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}Pb_{j-r,\lambda}^{(k)}(x)_{q,\lambda}S_{1,\lambda}(r,q)\right)\frac{z^j}{j!}. \tag{2.19}$$

Therefore, by comparing the coefficients on both sides of (2.19), we obtain the following theorem.

Theorem 2.8. For $j \geq 0$, we have

$$Pb_{j,\lambda}^{(k)}(x) = \sum_{r=0}^{j}\binom{j}{r}Pb_{j-r,\lambda}^{(k)}(x)_{q,\lambda}S_{1,\lambda}(r,q).$$

3. The degenerate unipoly-Bernoulli polynomials of the second kind

Let p be any arithmetic real or complex valued function defined on $\mathbb{N}$. Kim-Kim [7] presented the unipoly function attached to polynomials $p(x)$ as

$$u_k(x|p) = \sum_{j=1}^{\infty}\frac{p(j)}{j^k}x^n, (k \in \mathbb{Z}). \tag{3.1}$$

Moreover,

$$u_k(x|1) = \sum_{j=1}^{\infty}\frac{x^j}{j^k} = \mathrm{Li}_k(x), (\text{see } [10,14]), \tag{3.2}$$

represent the known ordinary polylogarithm function.

Dolgy and Khan [3] introduced the degenerate unipoly function attached to polynomials $p(x)$ are considered as follows

$$u_{k,\lambda}(x|p) = \sum_{j=1}^{\infty}p(j)\frac{(1)_{j,\lambda}x^j}{j^k}. \tag{3.3}$$

We see that

$$u_{k,\lambda}\left(x|\frac{1}{\Gamma}\right) = \mathrm{Ei}_{k,\lambda}(x), (\text{see}[14]) \tag{3.4}$$

is the modified degenerate polyexponential function.

Now, we introduce the degenerate unipoly-Bernoulli polynomials of the second kind attached to polynomials $p(x)$ as

$$\frac{u_{k,\lambda}\left(\log_\lambda(1+z)|p\right)}{\log_\lambda(1+z)}(1+z)^x = \sum_{j=0}^{\infty}Pb_{j,\lambda,p}^{(k)}(x)\frac{z^j}{j!}. \tag{3.5}$$

When $x = 0$, $Pb_{j,\lambda,p}^{(k)} = Pb_{j,\lambda,p}^{(k)}(0)$ are called the degenerate unipoly-Bernoulli numbers of the second kind attached to p.

If we take $p(j) = \frac{1}{\Gamma(j)}$. Then, we have

$$\sum_{j=0}^{\infty}Pb_{j,\lambda,\frac{1}{\Gamma}}^{(k)}(x)\frac{z^j}{j!} = \frac{1}{\log_\lambda(1+z)}(1+z)^x u_{k,\lambda}\left(\log_\lambda(1+z)|\frac{1}{\Gamma}\right)$$

$$= \frac{1}{\log_\lambda(1+z)}(1+z)^x\sum_{q=1}^{\infty}\frac{(1)_{q,\lambda}(\log_\lambda(1+z))^q}{q^k(q-1)!}. \tag{3.6}$$

For $k = 1$, we have

$$\sum_{j=0}^{\infty} Pb^{(1)}_{j,\lambda,\frac{1}{\Gamma}}(x)\frac{z^j}{j!} = \frac{1}{\log_\lambda(1+z)}(1+z)^x \sum_{q=1}^{\infty} \frac{(1)_{q,\lambda}(\log_\lambda(1+z))^q}{q!} = \frac{z}{\log_\lambda(1+z)}(1+z)^x. \tag{3.7}$$

Thus, we have

$$Pb^{(1)}_{j,\lambda,\frac{1}{\Gamma}}(x) = b_{j,\lambda}(x), (j \geq 0). \tag{3.8}$$

By making use of (1.12) and (3.3), we note that

$$\begin{aligned}
u_{k,\lambda}(\log_\lambda(1+z)|p) &= \sum_{q=1}^{\infty} p(q)\frac{(1)_{q,\lambda}(\log_\lambda(1+z))^q}{q^k} \\
&= \sum_{q=1}^{\infty} \frac{p(q)(1)_{q,\lambda}(\log_\lambda(1+z))^q}{q^k}\frac{q!}{q!} \\
&= \sum_{q=1}^{\infty} \frac{p(q)(1)_{q,\lambda}q!}{q^k}\frac{(\log_\lambda(1+z))^q}{q!} \\
&= \sum_{q=1}^{\infty} \frac{p(q)(1)_{q,\lambda}q!}{q^k}\sum_{r=q}^{\infty} S_{1,\lambda}(r,q)\frac{z^r}{r!} \\
&= \sum_{r=1}^{\infty}\left(\sum_{q=1}^{r} \frac{p(q)(1)_{q,\lambda}q!}{q^k}S_{1,\lambda}(r,q)\right)\frac{z^r}{r!}.
\end{aligned}$$

Thus, we have the required result.

Lemma 3.1. For $k \in \mathbb{Z}$, we have

$$u_{k,\lambda}(\log_\lambda(1+z)|p) = \sum_{r=1}^{\infty}\left(\sum_{q=1}^{r} \frac{p(q)(1)_{q,\lambda}q!}{q^k}S_{1,\lambda}(r,q)\right)\frac{z^r}{r!}.$$

Recalling from (3.5), we have

$$\begin{aligned}
\sum_{j=0}^{\infty} Pb^{(k)}_{j,\lambda,p}(x)\frac{z^j}{j!} &= \frac{1}{\log_\lambda(1+z)}(1+z)^x u_{k,\lambda}(\log_\lambda(1+z)|p) \\
&= \frac{1}{\log_\lambda(1+z)}(1+z)^x \sum_{q=1}^{\infty} \frac{(1)_{q,\lambda}p(q)}{q^k}(\log_\lambda(1+z))^q \\
&= \frac{1}{\log_\lambda(1+z)}(1+z)^x \sum_{q=0}^{\infty} \frac{(1)_{q+1,\lambda}p(q+1)}{(q+1)^k}(\log_\lambda(1+z))^{q+1} \\
&= \frac{1}{\log_\lambda(1+z)}(1+z)^x \sum_{q=0}^{\infty} \frac{(1)_{q+1,\lambda}p(q+1)(q+1)!}{(q+1)^k}\sum_{r=q+1}^{\infty} S_{r,\lambda}(r,q+1)\frac{z^r}{r!}
\end{aligned}$$

$$= \frac{z}{\log_\lambda(1+z)}(1+z)^x \sum_{q=0}^{\infty} \frac{(1)_{q+1,\lambda} p(q+1)(q+1)!}{(q+1)^k} \sum_{r=q}^{\infty} \frac{S_{1,\lambda}(r+1,q+1)}{r+1} \frac{z^r}{r!}$$

$$= \sum_{j=0}^{\infty} b_{j,\lambda}(x)\frac{z^j}{j!} \sum_{r=0}^{\infty}\left(\sum_{q=0}^{r} \frac{(1)_{q+1,\lambda} p(q+1)(q+1)!}{(q+1)^k} \frac{S_{1,\lambda}(r+1,q+1)}{r+1}\right)\frac{z^r}{r!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\sum_{q=0}^{r}\binom{j}{r} \frac{(1)_{q+1,\lambda} p(q+1)(q+1)!}{(q+1)^k} \frac{S_{1,\lambda}(r+1,q+1)}{r+1} b_{j-r,\lambda}(x)\right)\frac{z^j}{j!}. \tag{3.9}$$

Therefore, by comparing the coefficients on both sides of (3.9), we obtain the following theorem.
Theorem 3.1. For $j \geq 0$ and $k \in \mathbb{Z}$. Then

$$Pb^{(k)}_{j,\lambda,p}(x) = \sum_{r=0}^{j}\sum_{q=0}^{r}\binom{j}{r} \frac{(1)_{q+1,\lambda} p(q+1)(q+1)!}{(q+1)^k} \frac{S_{1,\lambda}(r+1,q+1)}{r+1} b_{j-r,\lambda}(x).$$

Moreover,

$$Pb^{(k)}_{j,\lambda,\frac{1}{\Gamma}}(x) = \sum_{r=0}^{j}\sum_{q=0}^{r}\binom{j}{r} \frac{b_{j-r,\lambda}(x)}{(q+1)^{k-1}} \frac{S_{1,\lambda}(r+1,q+1)}{r+1}.$$

Using (3.5), we have

$$\sum_{j=0}^{\infty} Pb^{(k)}_{j,\lambda,p}(x)\frac{z^j}{j!} = \frac{1}{\log_\lambda(1+z)} u_{k,\lambda}(\log_\lambda(1+z)|p)(1+z)^x$$

$$= \frac{u_{k,\lambda}\left(\log_\lambda(1+z)|p\right)}{\log_\lambda(1+z)} \sum_{j=0}^{\infty}(x)_j\frac{z^j}{j!}$$

$$= \sum_{i=0}^{\infty} Pb^{(k)}_{i,\lambda,p}\frac{z^i}{i!} \sum_{j=0}^{\infty}(x)_j\frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\binom{j}{i} Pb^{(k)}_{i,\lambda,p}(x)_{j-i}\right)\frac{z^j}{j!}. \tag{3.10}$$

Upon comparing the coefficients on both sides of Eq (3.10), we get the following theorem.
Theorem 3.2. For $j \geq 0$ and $k \in \mathbb{Z}$. Then

$$Pb^{(k)}_{j,\lambda,p}(x) = \sum_{i=0}^{j}\binom{j}{i} Pb^{(k)}_{i,\lambda,p}(x)_{j-i}.$$

By making use of (1.11), (1.12) and (3.5), we have

$$\sum_{j=0}^{\infty} Pb^{(k)}_{j,\lambda,p}\frac{z^j}{j!} = \frac{1}{\log_\lambda(1+z)} u_k\left(\log_\lambda(1+z)|p\right)$$

$$
\begin{aligned}
&= \frac{1}{\log_\lambda(1+z)} \sum_{q=1}^{\infty} \frac{(1)_{q,\lambda} p(q)}{q^k} (\log_\lambda(1+z))^q \\
&= \sum_{q=0}^{\infty} \frac{(1)_{q,\lambda} p(q+1)}{(q+1)^k} (\log_\lambda(1+z))^{q+1} \\
&= \frac{z}{\log_\lambda(1+z)} \frac{\log_\lambda(1+z)}{z} \sum_{q=0}^{\infty} \frac{(1)_{q+1,\lambda} p(q+1) q!}{(q+1)^k} \frac{(\log_\lambda(1+z))^q}{q!} \\
&= \sum_{j=0}^{\infty} D_{j,\lambda} \frac{z^j}{j!} \sum_{i=0}^{\infty} b_{i,\lambda} \frac{z^i}{i!} \sum_{q=0}^{\infty} \frac{(1)_{q+1,\lambda} p(q+1) q!}{(q+1)^k} \sum_{r=q}^{\infty} S_{1,\lambda}(r,q) \frac{z^r}{r!} \\
&= \sum_{j=0}^{\infty} D_{j,\lambda} \frac{z^j}{j!} \sum_{i=0}^{\infty} b_{i,\lambda} \frac{z^i}{i!} \sum_{r=0}^{\infty} \sum_{q=0}^{r} \frac{(1)_{q+1,\lambda} p(q+1) q!}{(q+1)^k} S_{1,\lambda}(r,q) \frac{z^r}{r!} \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i} D_{j-i,\lambda} b_{i,\lambda} \frac{z^j}{j!} \sum_{r=0}^{\infty} \sum_{q=0}^{r} \frac{(1)_{q+1,\lambda} p(q+1) q!}{(q+1)^k} S_{1,\lambda}(r,q) \frac{z^r}{r!} \\
&= \sum_{j=0}^{\infty} \left(\sum_{r=0}^{j} \sum_{q=0}^{r} \sum_{i=0}^{j-r} \binom{j-r}{i} \binom{j}{r} D_{j-i-r,\lambda} b_{i,\lambda} \frac{(1)_{q+1,\lambda} p(q+1) q!}{(q+1)^k} S_{1,\lambda}(r,q) \right) \frac{z^j}{j!}. \qquad (3.11)
\end{aligned}
$$

Thus, by comparing the coefficients on both sides of (3.11), we obtain the following theorem.

Theorem 3.3. For $j \geq 0$ and $k \in \mathbb{Z}$. Then

$$
Pb_{j,\lambda,p}^{(k)} = \sum_{r=0}^{j} \sum_{q=0}^{r} \sum_{i=0}^{j-r} \binom{j-r}{i} \binom{j}{r} D_{j-i-r,\lambda} b_{i,\lambda} \frac{(1)_{q+1,\lambda} p(q+1) q!}{(q+1)^k} S_{1,\lambda}(r,q).
$$

4. Numerical computations

In this section, certain numerical computations are done to calculate certain zeros of the degenerate poly-Bernoulli polynomials of the second kind and show some graphical representations. The first five members of $Pb_{j,\lambda}^{(k)}(x)$ are calculated and given as:

$$
\begin{aligned}
Pb_{0,\lambda}^{(k)}(x) &= 1, \\
Pb_{1,\lambda}^{(k)}(x) &= \frac{1}{2} + x - \frac{1}{8\log 3} - \frac{\log 81}{8\log 3}, \\
Pb_{2,\lambda}^{(k)}(x) &= \frac{1}{2} + x^2 + \frac{10}{81(\log 3)^2} + \frac{1}{8\log 3} - \frac{x}{4\log 3} - \frac{\log 81}{8\log 3} - \frac{x\log 81}{4\log 3}, \\
Pb_{3,\lambda}^{(k)}(x) &= -\frac{1}{4} + 2x - \frac{3x^2}{2} + x^3 - \frac{5}{16(\log 3)^3} - \frac{10}{27(\log 3)^2} + \frac{10x}{27(\log 3)^2} \\
&\quad - \frac{1}{4\log 3} + \frac{3x}{4\log 3} - \frac{3x^2}{8\log 3} + \frac{\log 81}{16\log 3} - \frac{3x^2\log 81}{8\log 3},
\end{aligned}
$$

$$Pb_{4,\lambda}^{(k)}(x) = \frac{1}{2} - 6x + 8x^2 - 4x^3 + x^4 + \frac{176}{125(\log 3)^4} + \frac{15}{8(\log 3)^3} - \frac{5x}{4(\log 3)^3} + \frac{110}{81(\log 3)^2} - \frac{20x}{9(\log 3)^2} + \frac{20x^2}{27(\log 3)^2} + \frac{3}{4\log 3} - \frac{11x}{4\log 3} + \frac{9x^2}{4\log 3} - \frac{x^3}{2\log 3} - \frac{\log 81}{8\log 3} + \frac{3x^2\log 81}{4\log 3} - \frac{x^3\log 81}{2\log 3}.$$

To show the behavior of $Pb_{j,\lambda}^{(k)}(x)$, we display the graph $Pb_{j,\lambda}^{(k)}(x)$ for $k = 4$ and $\lambda = 3$, this graph is presented in Figure 1.

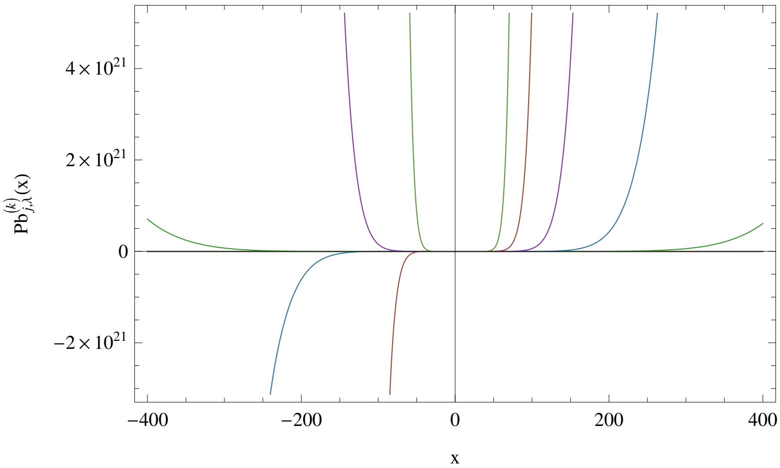

Figure 1. Graph of $Pb_{j,\lambda}^{(k)}(x)$.

Next, the approximate solutions of $Pb_{j,\lambda}^{(k)}(x) = 0$ when $k = 4$ and $\lambda = 3$, are calculated and listed in Table 1.

The zeros of $Pb_{j,\lambda}^{(k)}(x)$ for $\lambda \in \mathbb{C}$, $j = 12$ are plotted in Figure 2.

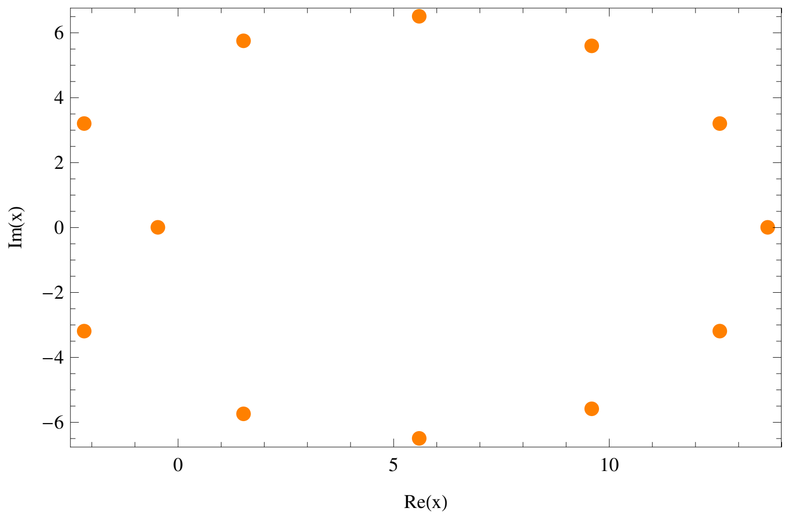

Figure 2. Zeros of $Pb_{12,\lambda}^{(k)}(x)$.

Table 1. Approximate solutions of $Pb_{j,\lambda}^{(k)}(x) = 0$.

j	Real zeros	Complex zeros
1	0.11378	–
2	0.212959, 1.0146	–
3	0.468628, 0.788431, 2.08428	–
4	2.27482, 3.00114	$0.589582 - 0.515659\,i, 0.589582 + 0.515659\,i$
5	4.09322	$0.470967 - 0.872952\,i,\ 0.470967 + 0.872952\,i,$ $2.76687 - 0.464588\,i,\ 2.76687 + 0.464588\,i$
6	4.47754, 4.94352	$0.270509 - 1.2071\,i,\ 0.270509 + 1.2071\,i$ $2.8603 - 1.06554\,i,\ 2.8603 + 1.06554\,i$
7	6.12953	$-0.00407237 - 1.52417\,i,\ -0.00407237 + 1.52417\,i,$ $2.8544 - 1.67974\,i,\ 2.8544 + 1.67974\,i$ $4.98314 - 0.749479\,i,\ 4.98314 + 0.749479\,i$
8	–	$-0.344872 - 1.82511\,i,\ -0.344872 + 1.82511\,i,$ $2.7537 - 2.30093\,i,\ 2.7537 + 2.30093\,i,$ $5.21262 - 1.46596\,i,\ 5.21262 + 1.46596\,i,$ $6.83367 - 0.248836\,i,\ 6.83367 + 0.248836\,i$

The stacking structure of approximate zeros of $Pb_{j,\lambda}^{(k)}(x) = 0$ for $\lambda = 4, j = 1, 2, ..., 12$ is given in Figure 3.

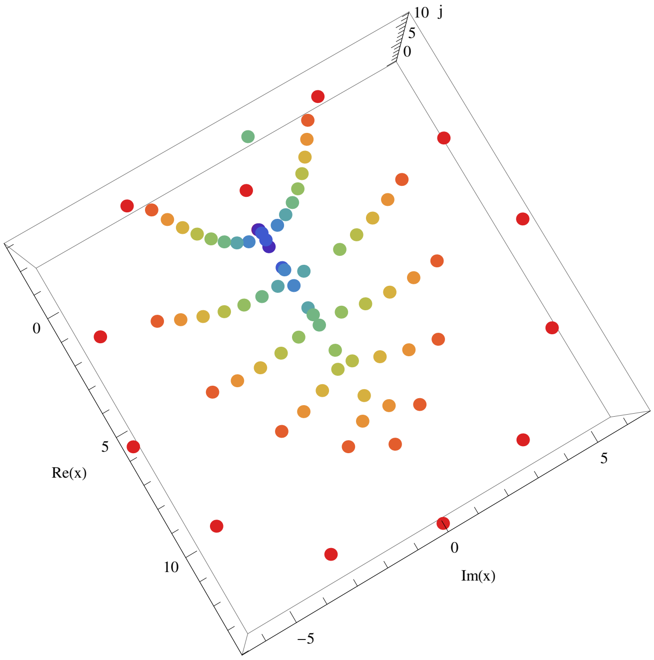

Figure 3. Stacking structure of zeros $Pb_{j,\lambda}^{(k)}(x)$.

5. Conclusions

In this article, we introduced the type 2 degenerate poly-Bernoulli polynomials of the second kind and derived many related interesting properties. Furthermore, we defined the degenerate unipoly

Bernoulli polynomials of the second kind and established some considerable results. Finally, certain related beautiful zeros and graphs are shown.

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Conflict of interest

The authors declare no conflict of interest.

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