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Research article

Uniqueness for meromorphic solutions of Schwarzian differential equation

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Abstract: Let f be a meromorphic function, R be a nonconstant rational function and k be a positive integer. In this paper, we consider the Schwarzian differential equation of the form

$$\left[\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]^k=R(z).$$

We investigate the uniqueness of meromorphic solutions of the above Schwarzian differential equation if the meromorphic solution f shares three values with any other meromorphic function.

Keywords: differential equation; meromorphic function; sharing value; uniqueness
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1. Introduction and main results

In this paper, we use the basic notions of Nevanlinna's theory, see, e.g., [3, 5, 10]. In addition, we use the notation

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

to denote the order of growth of the meromorphic function f. Let $S(r,f)$ denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside of a set with finite logarithmic measure.

Let f and g be two meromorphic functions, a be a small function relative to both f and g. We say that f and g share a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicities. f and g share a IM if $f-a$ and $g-a$ have the same zeros ignoring multiplicities. Nevanlinna's four values theorem (see [8, 9]) says that if two nonconstant meromorphic functions f and g share four values CM, then $f=g$ or f is a Möbius transformation of g. The condition "f and g share four values CM" has been weakened to "f and g share two values CM and two values IM" by Gundersen [1, 2], as well as by Mues [7].

For Schwarzian differential equation

$$\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]^k = R(z, f). \tag{1.1}$$

Ishizaki [4, Theorem 1] showed that if the Schwarzian Eq (1.1) possesses an admissible solution, then $d + 2k \sum_{j=1}^{l} \delta(\alpha_j, f) \leq 4k$, where α_j are distinct complex constants, and $d = \deg R(z, f)$. In particular, when $R(z, f)$ is independent of z, it is shown that if (1.1) possesses an admissible solution f, then by some Möbius transformation $w = (af + b)/(cf + d)(ad - bc \neq 0)$, $R(z, f)$ can be reduced to some special forms, see [4, Theorem 3]. Liao and Ye [6] considered differential equation, which is a simple type of the Schwarzian differential equation, and gave the order of meromorphic solutions as follows.

Theorem 1. *[6, Theorem 3] Let P and Q be polynomials with $\deg P = m$ and $\deg Q = n$ and let $R(z) = P(z)/Q(z)$ and k be a positive integer. If f is a transcendental meromorphic solution of the equation*

$$\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]^k = R(z), \tag{1.2}$$

then $m - n + 2k > 0$ and the order $\rho(f) = (m - n + 2k)/2k$.

In the follows, we apply the Nevanlinna theory and uniqueness of meromorphic functions to the Schwarzian differential Eq (1.2), and investigate the uniqueness of meromorphic solutions if the meromorphic solution f shares three values with any other meromorphic function. we obtain

Theorem 2. *Let R be a nonconstant rational function and k be a positive integer. Suppose that f is a transcendental meromorphic solution of the differential Eq (1.2) without single poles. If a meromorphic function g shares a, b, ∞ CM with f, then $f \equiv g$.*

2. Lemmas

We now give some preparations.

Lemma 1. *[9, Theorem 1.51] Suppose that $n \geq 2$ and let $f_1, \cdots, f_n$ be meromorphic functions and $g_1, \cdots, g_n$ be entire functions such that*

(i) $\sum_{j=1}^{n} f_j \exp\{g_j\} = 0$;
(ii) when $1 \leq j < k \leq n$, $g_j - g_k$ is not constant;
(iii) when $1 \leq j \leq n$, $1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\} \,(r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ has finite linear measure or logarithmic measure. Then $f_j \equiv 0$ $(j = 1, \cdots, n)$.

Lemma 2. *Let f be a meromorphic solution of Eq (1.2), then f' is a meromorphic solution of equation*

$$W' = QW,$$

where Q is a nonconstant rational function.

Proof. Set

$$Q = \frac{f''}{f'}. \tag{2.1}$$

We then prove that Q is a nonconstant rational function.

Since f is of finite order by Theorem 1, (2.1) shows Q is also of finite order and

$$\begin{cases} f'' = Qf', \\ f''' = Q'f' + Q^2 f' = (Q' + Q^2) f'. \end{cases} \tag{2.2}$$

We see from (1.2) that

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = R_1, \tag{2.3}$$

where R_1 is some nonconstant rational function. Thus (2.2) and (2.3) show that

$$\frac{f'(Q' + Q^2)}{f'} - \frac{3}{2}\left(\frac{Qf'}{f'}\right)^2 = R_1,$$

that is,

$$Q' - \frac{1}{2}Q^2 = R_1. \tag{2.4}$$

Since R_1 is a nonconstant rational function, we deduce from (2.4) that Q cannot be a constant. We then prove Q cannot be transcendental.

By comparing the poles of the both sides of (1.2), we get that $f^{'}$ has only finitely many zeros and poles under the assumption that f without single poles. Thus Q has only finitely many single poles. If Q is transcendental, then

$$T\left(r, Q' - \frac{1}{2}Q^2\right) = T(r, Q) + O(\log r),$$

contradicting to (2.4). Hence, Q is a nonconstant rational function. □

Lemma 3. *Let a, b be two distinct constants, β, γ be nonconstant polynomials with $\deg \beta \neq \deg \gamma$, and*

$$f = a + (b - a)\frac{e^{\beta} - 1}{e^{\gamma} - 1}. \tag{2.5}$$

Then f cannot be a meromorphic solution of Eq (1.2).

Proof. Assume that f is a meromorphic solution of Eq (1.2). Lemma 2 shows

$$f'' = Qf'. \tag{2.6}$$

Without loss of generality, we assume Q is a nonconstant polynomial. Otherwise, we just multiply the dominator of Q to both sides of (2.6). We now divide our proof into two cases.
Case 2.1. $\deg\beta > \deg\gamma$. Rewriting (2.5) as

$$f = a_{01}e^{\beta} + a_{00}, \tag{2.7}$$

where

$$a_{01} = \frac{b-a}{e^{\gamma}-1}, \quad a_{00} = a - \frac{b-a}{e^{\gamma}-1}. \tag{2.8}$$

Obviously,

$$\rho(a_{01}) = \rho(a_{00}) = \deg\gamma < \deg\beta. \tag{2.9}$$

Since e^{β} is of regular growth order $\deg\beta$, we see a_{01}, a_{00} are small functions of e^{β}. We conclude from (2.7) that

$$f' = a_{11}e^{\beta} + a_{10}, \tag{2.10}$$

where

$$\begin{cases} a_{11} = a'_{01} + a_{01}\beta', \\ a_{10} = a'_{00}, \end{cases} \tag{2.11}$$

and

$$f'' = a_{21}e^{\beta} + a_{20}, \tag{2.12}$$

where

$$\begin{cases} a_{21} = a'_{11} + a_{11}\beta', \\ a_{20} = a'_{10}. \end{cases} \tag{2.13}$$

We deduce from (2.9), (2.11) and $\rho(a'_{01}) = \rho(a_{01}), \rho(a'_{00}) = \rho(a_{00})$ that

$$\begin{cases} \rho(a_{11}) \le \max\{\rho(a'_{01}), \rho(a_{01}), \rho(\beta')\} \le \rho(a_{01}) < \deg\beta; \\ \rho(a_{10}) = \rho(a'_{00}) = \rho(a_{00}) < \deg\beta. \end{cases} \tag{2.14}$$

We assert that $a_{11} \not\equiv 0$. Otherwise, (2.11) shows

$$a'_{01} + a_{01}\beta' = 0. \tag{2.15}$$

Applying the method of separating variables to Eq (2.15), we have $a_{01} = ce^{-\beta}$, thus $\rho(a_{01}) = \deg\beta$, contradicting to (2.9). Similarly, we also get

$$\rho(a_{21}) < \deg\beta, \quad \rho(a_{20}) < \deg\beta \tag{2.16}$$

and $a_{21} \not\equiv 0$. Substituting (2.10), (2.12) into (2.6), we obtain

$$A_{11}e^{\beta} + A_{10} = 0, \tag{2.17}$$

where

$$\begin{cases} A_{11} = a_{21} - Qa_{11}, \\ A_{10} = a_{20} - Qa_{10}. \end{cases} \tag{2.18}$$

By (2.14), (2.16), (2.18) , we have

$$\begin{cases} \rho(A_{11}) \le \max\{\rho(a_{21}), \rho(a_{11})\} < \deg\beta, \\ \rho(A_{10}) \le \max\{\rho(a_{20}), \rho(a_{10})\} < \deg\beta. \end{cases} \tag{2.19}$$

Thus, $\rho(A_{1j}) < \deg\beta \ \ (j = 0, 1)$. Since e^{β} is of regular growth order $\deg\beta$, we obtain

$$T(r, A_{1j}) = o\{T(r, e^{\beta})\}, \quad j = 0, 1.$$

Applying Lemma 1 to (2.17), we have

$$A_{11} \equiv 0, \quad A_{10} \equiv 0.$$

Thus, we obtain from (2.13), (2.18) that

$$a'_{11} + a_{11}\beta' - Qa_{11} = 0, \quad a'_{10} - Qa_{10} = 0.$$

Applying the method of separating variables to the above equations, we have

$$a_{11} = c_1 e^{-\beta + \int Q dz}, \quad a_{10} = c_2 e^{\int Q dz},$$

that is

$$a_{11} = ce^{-\beta}a_{10}. \tag{2.20}$$

By (2.7), we also have

$$a'_{01} = (b - a)\frac{-\gamma' e^{\gamma}}{(e^{\gamma} - 1)^2}, \quad a'_{00} = (b - a)\frac{\gamma' e^{\gamma}}{(e^{\gamma} - 1)^2}. \tag{2.21}$$

Substituting (2.11), (2.21) into (2.20), we obtain

$$(\beta' - \gamma' - \beta' e^{-\gamma})e^{\beta} - c\gamma' = 0.$$

Applying Lemma 1 and $\deg\beta > \deg\gamma$, we have $-c\gamma' \equiv 0$, then $c = 0$, thus $a_{11} = 0$, a contradiction.
Case 2.2. $\deg\beta < \deg\gamma$. Rewriting (2.5) as

$$f = a + \frac{b_{00}}{e^{\gamma} - 1}, \tag{2.22}$$

where

$$b_{00} = (b-a)(e^{\beta}-1). \tag{2.23}$$

Obviously,

$$\rho(b_{00}) = \deg \beta < \deg \gamma. \tag{2.24}$$

Thus, we conclude from (2.22) that

$$f' = \frac{b_{11}e^{\gamma} + b_{10}}{(e^{\gamma}-1)^2}, \tag{2.25}$$

where

$$\begin{cases} b_{11} = b_{00}' - b_{00}\gamma', \\ b_{10} = -b_{00}', \end{cases} \tag{2.26}$$

and

$$f'' = \frac{b_{22}e^{2\gamma} + b_{21}e^{\gamma} + b_{20}}{(e^{\gamma}-1)^3}, \tag{2.27}$$

where

$$\begin{cases} b_{22} = b_{11}' - b_{11}\gamma', \\ b_{21} = b_{10}' - 2\gamma' b_{10} - b_{11}' - b_{11}'\gamma', \\ b_{20} = -b_{10}'. \end{cases} \tag{2.28}$$

We deduce from (2.24), (2.26) that

$$\begin{cases} \rho(b_{11}) \leq \max\{\rho(b_{00}'), \rho(b_{00}), \rho(\gamma')\} \leq \rho(b_{00}) < \deg \gamma, \\ \rho(b_{10}) = \rho(b_{00}') = \rho(b_{00}) < \deg \gamma. \end{cases} \tag{2.29}$$

We assert that $b_{11} \not\equiv 0$. Otherwise, (2.26) shows

$$b_{00}' - b_{00}\gamma'(z) = 0. \tag{2.30}$$

Applying the method of separating variables to Eq (2.30), we have $b_{00} = ce^{\gamma}$, and $\rho(b_{00}) = \deg \gamma$. contradicting to (2.24). Similarly, we also get

$$\rho(b_{22}) < \deg \gamma, \quad \rho(b_{21}) < \deg \gamma, \quad \rho(b_{20}) < \deg \gamma, \tag{2.31}$$

and $b_{22} \not\equiv 0$.

Substituting (2.25), (2.27) into (2.6), we obtain

$$A_{22}e^{2\gamma} + A_{21}e^{\gamma} + A_{20} = 0, \tag{2.32}$$

where

$$\begin{cases} A_{22} = b_{22} - Qb_{11}, \\ A_{21} = b_{21} + Qb_{11} - Qb_{10}, \\ A_{20} = b_{20} + Qb_{10}. \end{cases} \tag{2.33}$$

By (2.29), (2.31) , we have

$$\begin{cases} \rho(A_{22}) \leq \max\{\rho(b_{22}), \rho(b_{11})\} < \deg\gamma, \\ \rho(A_{21}) \leq \max\{\rho(b_{21}), \rho(b_{11}), \rho(b_{10})\} < \deg\gamma, \\ \rho(A_{20}) \leq \max\{\rho(b_{20}), \rho(b_{10})\} < \deg\gamma. \end{cases} \tag{2.34}$$

Thus, $\rho(A_{2j}) < \deg\gamma$ ($j = 0, 1, 2$). Since e^{γ} is of regular growth order $\deg\gamma$, we obtain

$$T(r, A_{2j}) = o\{T(r, e^{2\gamma})\} = o\{T(r, e^{\gamma})\}, \quad j = 0, 1, 2.$$

Applying Lemma 1 to (2.32), we have

$$A_{22} \equiv 0, \qquad A_{21} \equiv 0, \qquad A_{20} \equiv 0.$$

Thus, we obtain from (2.28), (2.33) that

$$b'_{11} - b_{11}\gamma' - Qa_{11} = 0, \quad -b'_{10} + Qb_{10} = 0.$$

Applying the method of separating variables to equation ,we have

$$b_{11} = c_1 e^{\gamma - \int Q dz}, \quad b_{10} = c_2 e^{\int Q dz},$$

that is

$$b_{11} = ce^{\gamma} b_{10}. \tag{2.35}$$

By (2.23), we have

$$b'_{00} = (b - a)\beta' e^{\beta}. \tag{2.36}$$

Substituting (2.26), (2.36) into (2.35), we obtain

$$c\beta' e^{\gamma} + (\beta' - \gamma'\beta')e^{0} = 0.$$

Applying Lemma 1 and $\deg\beta < \deg\gamma$, we have $c\beta' \equiv 0$, then $c = 0$, thus $b_{11} = 0$, a contradiction. In conclusion, f of the form (2.5) cannot be a meromorphic solution of Eq (1.2).

□

3. Proof of Theorem 2

Proof. Since f and g share a, b, ∞ CM, we can get

$$N\left(r, \frac{1}{f-a}\right) = N\left(r, \frac{1}{g-a}\right),\ N\left(r, \frac{1}{f-b}\right) = N\left(r, \frac{1}{g-b}\right),\ N(r, f) = N(r, g).$$

By applying the second fundamental theorem to function g, we obtain

$$T(r, g) \leq N(r, g) + N(r, \frac{1}{g-a}) + N(r, \frac{1}{g-b}) + S(r, g)$$

$$= N(r,f) + N(r,\frac{1}{f-a}) + N(r,\frac{1}{f-b}) + S(r,g)$$
$$\leq 3T(r,f) + S(r,g).$$

Similarly, we can get $T(r,f) \leq 3T(r,g) + S(r,f)$, so $T(r,g) = O(T(r,f) + S(r,f))$, and then $\rho(g) = \rho(f) < \infty$ by Theorem 1.

Furthermore, there exist polynomials α and β such that

$$\frac{g-a}{f-a} = e^{\alpha}, \tag{3.1}$$

and

$$\frac{g-b}{f-b} = e^{\beta}. \tag{3.2}$$

Assume, to the contrary, that $f \not\equiv g$. Then from (3.1) and (3.2), we obtain

$$e^{\alpha} \not\equiv 1, \quad e^{\beta} \not\equiv 1, \quad e^{\alpha} \not\equiv e^{\beta}, \quad \alpha \not\equiv \beta.$$

Again by (3.1) and (3.2), we get

$$f = a + (b-a)\frac{e^{\beta}-1}{e^{\beta-\alpha}-1}, \tag{3.3}$$

or

$$f = a + (b-a)\frac{e^{\beta}-1}{e^{\gamma}-1}, \tag{3.4}$$

where $\gamma = \beta - \alpha$ is a nonzero polynomial.

We now suppose that α and β are not all constants. Otherwise, we easily obtain a contradiction by (3.3). Thus, we split our proofs into two cases.

Case 1. α is a constant and β is non-constant polynomial.

Denote $e^{-\alpha} = B(\neq 0)$. then $B \not\equiv 1$, (3.3) shows

$$f = a + (b-a)\frac{e^{\beta}-1}{Be^{\beta}-1}, \tag{3.5}$$

which yields

$$f' = (b-a)(1-B)\frac{-\beta' e^{\beta}}{(Be^{\beta}-1)^2}, \tag{3.6}$$

$$f'' = (b-a)(1-B)\frac{B(\beta'^2 - \beta'')e^{2\beta} + (\beta'' + \beta'^2)e^{\beta}}{(Be^{\beta}-1)^3}. \tag{3.7}$$

Substituting (3.6), (3.7) into Eq (2.6), we conclude that

$$A_{31}e^{2\beta} + A_{30}e^{\beta} = 0, \tag{3.8}$$

where

$$\begin{cases} A_{31} = B(1-B)(\beta'^2 - \beta'') + QB(1-B)\beta', \\ A_{30} = (1-B)\beta'' + (1-B)\beta'^2 - (1-B)Q\beta'. \end{cases} \tag{3.9}$$

By (3.9), we have

$$\rho(A_{31}) < \deg\beta, \ \ \rho(A_{30}) < \deg\beta.$$

Thus, $\rho(A_{3j}) < \deg\gamma (j = 0, 1)$. Since e^{β} is of regular growth order $\deg\beta$, we obtain

$$T(r, A_{3j}) = o\{T(r, e^{2\gamma})\} = o\{T(r, e^{\gamma})\}, \quad j = 0, 1, 2.$$

Applying Lemma 1 to (3.8), we have

$$A_{31} \equiv 0, \qquad A_{30} \equiv 0.$$

By $A_{30} \equiv 0$, we obtain

$$\beta'' + \beta'^2 - \beta' Q = 0.$$

Applying the method of separating variables to equation, we have

$$\beta' = c_3 e^{-\beta + \int Q dz}. \tag{3.10}$$

By (2.6), we have

$$f' = c_4 e^{\int Q dz}. \tag{3.11}$$

Substituting (3.6), (3.11) into Eq (3.10), we conclude that

$$(Be^{\beta} - 1)^2 = c. \tag{3.12}$$

Thus, we have β is a constant, a contradiction.

Case 2. β is a constant and α is non-constant polynomial. Similar to the proof of Case 1, we also get a contradiction. We deduce from (3.4) and Lemma 3 that $\deg\beta = \deg\gamma$, and

$$\begin{aligned} f' &= (b-a)\frac{(\beta' - \gamma')e^{\beta+\gamma} - \beta' e^{\beta} + \gamma' e^{\gamma}}{(e^{\gamma} - 1)^2}, \\ f'' &= (b-a)\frac{(\beta'' - \gamma'' + (\beta' - \gamma')^2)e^{\beta+2\gamma}}{(e^{\gamma} - 1)^3} \\ &\quad + (b-a)\frac{(-2\beta'' - 2\beta'^2 + \gamma'' + \gamma'^2 + 2\beta'\gamma')e^{\beta+\gamma}}{(e^{\gamma} - 1)^3} \\ &\quad + (b-a)\frac{(\gamma'' - \gamma'^2)e^{2\gamma} + (\beta'' + \beta'^2)e^{\beta}}{(e^{\gamma} - 1)^3} \\ &\quad - (b-a)\frac{(\gamma'' + \gamma'^2)e^{\gamma}}{(e^{\gamma} - 1)^3}. \end{aligned} \tag{3.13}$$

Substituting (3.13) into Eq (2.6), we conclude that

$$A_{44}e^{\beta+2\gamma}+A_{43}e^{\beta+\gamma}+A_{42}e^{2\gamma}+A_{41}e^{\beta}+A_{40}e^{\gamma}=0, \tag{3.14}$$

where

$$\begin{cases} A_{44}=\beta''-\gamma''+(\beta'-\gamma')^2-Q(\beta'-\gamma'),\\ A_{43}=-2\beta''+\gamma''-2\beta'^2+\gamma'^2+2\beta'\gamma'+Q(2\beta'-\gamma'),\\ A_{42}=\gamma''-\gamma'^2-Q\gamma',\\ A_{41}=\beta''+\beta'^2-Q\beta',\\ A_{40}=-\gamma''-\gamma'^2+Q\gamma'. \end{cases} \tag{3.15}$$

Obviously, we obtain that

$$\begin{cases} \rho(A_{44})<\deg\beta=\deg\gamma,\\ \rho(A_{43})<\deg\beta=\deg\gamma,\\ \rho(A_{42})<\deg\beta=\deg\gamma,\\ \rho(A_{41})<\deg\beta=\deg\gamma,\\ \rho(A_{40})<\deg\beta=\deg\gamma. \end{cases}$$

Thus,

$$\rho(A_{4j})<\deg\beta=\deg\gamma (j=0,1,2,3,4). \tag{3.16}$$

Therefore, Eq (3.14) can be rewritten as

$$A_{44}e^{\beta+\gamma}+A_{43}e^{\beta}+A_{42}e^{\gamma}+A_{41}e^{\beta-\gamma}+A_{40}=0. \tag{3.17}$$

In the following, we divide our proof into four cases.
Case A. $\deg(\beta+\gamma)<\deg\gamma$. Combining this with $\deg\beta=\deg\gamma$, we get $\deg(\beta-\gamma)=\deg\gamma, \deg(\beta-2\gamma)=\deg\gamma$. Thus, $e^{\beta}, e^{\gamma}, e^{\beta-\gamma}, e^{\beta-2\gamma}$ are of regular growth order $\deg\gamma$. Equation (3.17) shows that

$$A_{43}e^{\beta}+A_{42}e^{\gamma}+A_{41}e^{\beta-\gamma}+B_{00}=0, \tag{3.18}$$

where

$$B_{00}=A_{44}e^{\beta+\gamma}+A_{40}.$$

By this and (3.16), we obtain $\rho(B_{00})\le\max\{\rho(A_{44}),\rho(A_{40}),\deg(\beta+\gamma)\}<\deg\gamma=\deg\beta$. Then

$$\begin{cases} T(r,A_{4j})=o\{T(r,e^{\beta})\}=o\{T(r,e^{\gamma})\}=o\{T(r,e^{\beta-\gamma})\}=o\{T(r,e^{\beta-2\gamma})\} \quad (j=0,1,2,3),\\ T(r,B_{00})=o\{T(r,e^{\beta})\}=o\{T(r,e^{\gamma})\}=o\{T(r,e^{\beta-\gamma})\}=o\{T(r,e^{\beta-2\gamma})\}. \end{cases}$$

Together with (3.18) and Lemma 1, we have

$$B_{00}\equiv 0, \quad A_{4j}\equiv 0 \quad (j=1,2,3).$$

By $A_{42}\equiv 0$ and (3.15), we have

$$\gamma'' - \gamma'^2 - Q\gamma' = 0. \tag{3.19}$$

In Case A, we again split two subcases.
Subcase A.1. If $\deg \gamma \geq 2$. Applying the method of separating variables to equation, we have

$$\gamma' = c_5 e^{\gamma + \int Q dz}. \tag{3.20}$$

Substituting (3.11), (3.13) into Eq (3.20), we conclude that

$$c(\beta' - \gamma')e^{\beta+2\gamma} + (c-1)\gamma' e^{2\gamma} + 2\gamma' e^{\gamma} - (\gamma' + c\beta' e^{\beta+\gamma})e^0 = 0. \tag{3.21}$$

By (3.21), $\deg(\beta+\gamma) < \deg \gamma$ and Lemma 1, we obtain $\gamma'(z) \equiv 0$, thus $\gamma(z)$ is a constant , a contradiction.
Subcase A.2. If $\deg \gamma = 1$. Let $\gamma(z) = mz + n_1$, where $m \neq 0, n_1$ are constants. Hence, $\gamma' = m, \gamma'' = 0$. Substituting these into Eq (3.19), we get

$$-m^2 - Qm = 0,$$

that is $Q = -m$, a contradiction.
Case B. $\deg(\beta - \gamma) < \deg \gamma$. Equation (3.17) shows that

$$A_{44}e^{\beta-\gamma}e^{2\gamma} + (A_{43}e^{\beta-\gamma} + A_{42})e^{\gamma} + (A_{41}e^{\beta-\gamma} + A_{40})e^0 = 0. \tag{3.22}$$

Together with (3.16), (3.22), $\deg(\beta - \gamma) < \deg \gamma$ and Lemma 1, we have

$$A_{44}e^{\beta-\gamma} \equiv 0, A_{43}e^{\beta-\gamma} + A_{42} \equiv 0, A_{41}e^{\beta-\gamma} + A_{40} \equiv 0.$$

Substituting (3.15) and $\beta = \alpha + \gamma$ into the last equality $A_{41}e^{\beta-\gamma} + A_{40} \equiv 0$, we have

$$(\beta'' + \beta'^2 - Q\beta')e^{\alpha} + (-\gamma'' - \gamma'^2 + Q\gamma')e^0 = 0. \tag{3.23}$$

Together with (3.23) and Lemma 1, we have

$$-\gamma'' - \gamma'^2 + Q\gamma' \equiv 0. \tag{3.24}$$

In Case B, we again split two subcases.
Subcase B.1. If $\deg \gamma \geq 2$. Applying the method of separating variables to equation , we have

$$\gamma' = c_6 e^{-\gamma + \int Q dz}. \tag{3.25}$$

Substituting (3.11), (3.13) into Eq (3.25), we conclude that

$$\gamma' e^{2\gamma} - (2\gamma' - c(\beta' - \gamma')e^{\beta-\gamma})e^{\gamma} + (c\beta' e^{\beta-\gamma} + (1-c)\gamma')e^0 = 0. \tag{3.26}$$

By (3.26), $\deg(\beta - \gamma) < \deg \gamma$ and Lemma 1, we obtain $\gamma' \equiv 0$, a contradiction.

Subcase B.2. If $\deg\gamma = 1$. Let $\gamma = mz + n_1$, where $m \neq 0, n_1$ are constants. Hence, $\gamma' = m, \gamma'' = 0$. Substituting these into Eq (3.24), we get

$$m^2 - Qm = 0,$$

that is $Q = m$, a contradiction.

Case C. $\deg(\beta - 2\gamma) < \deg\gamma$. Equation (3.17) shows that

$$A_{44}e^{\beta} + A_{43}e^{\beta-\gamma} + A_{40}e^{-\gamma} + (A_{42} + A_{41}e^{\beta-2\gamma}) = 0. \tag{3.27}$$

By $\deg\beta = \deg\gamma$ and $\deg(\beta-2\gamma) < \deg\gamma$, we have $\deg(\beta-\gamma) = \deg(\beta+\gamma) = \deg\gamma$. By this and (3.16), we have

$$\begin{cases} T(r, A_{4j}) = o\{T(r, e^{\beta})\} = o\{T(r, e^{\gamma})\} = o\{T(r, e^{\beta-\gamma})\} = o\{T(r, e^{\beta+\gamma})\} \quad (j = 0, 3, 4), \\ T(r, A_{42} + A_{41}e^{\beta-2\gamma}) = o\{T(r, e^{\beta})\} = o\{T(r, e^{\gamma})\} = o\{T(r, e^{\beta-\gamma})\} = o\{T(r, e^{\beta+\gamma})\}. \end{cases}$$

Together with (3.27) and Lemma 1, we have

$$A_{44} \equiv 0, \quad A_{43} \equiv 0, \quad A_{40} \equiv 0.$$

By $A_{40} \equiv 0$ and (3.15), we also have (3.24). Using the same method as the above Subcase B, we get a contradiction.

Case D. $\deg(\beta + \gamma) = \deg(\beta - \gamma) = \deg(\beta - 2\gamma) = \deg\gamma$. By this and (3.15), for $j = 0, 1, 2, 3, 4$, we have

$$T(r, A_{4j}) = o\{T(r, e^{\beta})\} = o\{T(r, e^{\gamma})\} = o\{T(r, e^{\beta-\gamma})\} = o\{T(r, e^{\beta+\gamma})\} = o\{T(r, e^{\beta-2\gamma})\}.$$

Combining this with Lemma 1, we have

$$A_{4j} \equiv 0, \quad j = 1, 2, 3, 4.$$

By $A_{40} \equiv 0$ and (3.15), we also have (3.24). Using the same method as the above Subcase B, we get a contradiction. □

4. Conclusions

Together with the Nevanlinna theory and uniqueness of meromorphic functions, this paper considers the certain type of Schwarzian differential equation, and investigate the uniqueness of meromorphic solutions if the meromorphic solution f shares three values with any other meromorphic function.

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Conflict of interest

The authors declare no conflict of interest.

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