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Research article

New Hermite-Hadamard inequalities in fuzzy-interval fractional calculus via exponentially convex fuzzy interval-valued function

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Abstract: In the present note, we develop Hermite-Hadamard type inequality and He's inequality for exponential type convex fuzzy interval-valued functions via fuzzy Riemann-Liouville fractional integral and fuzzy He's fractional integral. Moreover, we establish Hermite-Fejér inequality via fuzzy Riemann-Liouville fractional integral.

Keywords: convex function; exponential type convex function; convex fuzzy interval-valued function; Exponential type convex fuzzy interval-valued function; fuzzy fractional integral operator; He's fuzzy fractional derivative; Hermite-Hadamard type inequality; Hermite-Hadamard Fejér type inequality
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1. Introduction

Convex functions play an important role in solving many problems of pure and applied mathematics, for example the all the developments in optimization theory are based on convex function. A real valued function $\text{ß} : \aleph \rightarrow \Re$ is said to be convex on the interval $\aleph$ if

$$\text{ß}(\top \ltimes_1 + (1 - \top)\ltimes_2) \leq \top\text{ß}(\ltimes_1) + (1 - \top)\text{ß}(\ltimes_2)$$

holds for all $\ltimes_1, \ltimes_2 \in \aleph$ and $\top \in [0, 1]$.

The Hermite-Hadamard (H-H) inequality [1, 2] for convex function $\text{ß} : \aleph \rightarrow \Re$ on $\aleph = [\ltimes_1, \ltimes_2]$ is

$$\text{ß}\left(\frac{\ltimes_1 + \ltimes_2}{2}\right) \leqslant \frac{1}{\ltimes_2 - \ltimes_1} \int_{\ltimes_1}^{\ltimes_2} \text{ß}(n)dn \leqslant \frac{\text{ß}(\ltimes_1) + \text{ß}(\ltimes_2)}{2}.$$

In 2013, Sarika et al. [3] gave the following fractional H-H-inequality for convex function:

$$ß\left(\frac{\ltimes_1 + \ltimes_2}{2}\right) \leqslant \frac{\Gamma(\measuredangle + 1)}{2(\ltimes_2 - \ltimes_1)^{\measuredangle}}[\dashv^{\measuredangle}_{\ltimes_1^+} ß(\ltimes_2) + \dashv^{\measuredangle}_{\ltimes_2^-} ß(\ltimes_1)] \leq \frac{ß(\ltimes_1) + ß(\ltimes_2)}{2},$$

where $ß : \aleph = [\ltimes_1, \ltimes_2] \to \Re$ supposed to be a non-negative function on $[\ltimes_1, \ltimes_2]$, $ß \in L_1(\ltimes_1, \ltimes_2)$ with $\ltimes_1 < \ltimes_2$ and also $\dashv^{\measuredangle}_{\ltimes_1^+}, \dashv^{\measuredangle}_{\ltimes_2^-}$ are the left and right sided Riemann-Liouville fractional integrals of order $0 \leq \measuredangle$ as follows [27]:

$$\dashv^{\measuredangle}_{\ltimes_1^+} ß(\ltimes) = \frac{1}{\Gamma(\measuredangle)} \int_{\ltimes_1}^{\ltimes} (\ltimes - \top)^{\measuredangle - 1} ß(\top) d\top, \ (\ltimes > \ltimes_1);$$

$$\dashv^{\measuredangle}_{\ltimes_2^-} ß(\ltimes) = \frac{1}{\Gamma(\measuredangle)} \int_{\ltimes}^{\ltimes_2} (\top - \ltimes)^{\measuredangle - 1} ß(\top) d\top, \ (\ltimes < \ltimes_2).$$

Definition 1.1. *[4] A real valued function $ß : \aleph \to \Re$ is said to be exponential type convex on the interval $\aleph$ if*

$$ß(\top \ltimes_1 + (1 - \top)\ltimes_2) \leq (\varrho^{\top} - 1)ß(\ltimes_1) + (\varrho^{(1-\top)} - 1)ß(\ltimes_2) \tag{1.1}$$

holds for all $\ltimes_1, \ltimes_2 \in \aleph$ and $\top \in [0, 1]$.

In 2020, Kadakal et al. [4] obtained the following new refinement of the classical H-H-inequality on the exponential type convex function:

$$\frac{1}{2(\varrho^{\frac{1}{2}} - 1)} ß\left(\frac{\ltimes_1 + \ltimes_2}{2}\right) \leqslant \frac{1}{\ltimes_2 - \ltimes_1} \int_{\ltimes_1}^{\ltimes_2} ß(n)dn \leqslant (\varrho - 2)[ß(\ltimes_1) + ß(\ltimes_2)].$$

Zhao et al. [5] gave the following result for interval valued function (I-V-F):

Let $ß : [\ltimes_1, \ltimes_2] \subset \Re \to \Pi_c^+$ be a convex I-V-F given by $ß(\ltimes) = [ß_*(\ltimes), ß^*(\ltimes)]$ for all $\ltimes \in [\ltimes_1, \ltimes_2]$ where $ß_*(\ltimes)$ is a convex function and $ß^*(\ltimes)$ is a concave function. If $ß(\ltimes)$ is a Riemann integrable function, then

$$ß\left(\frac{\ltimes_1 + \ltimes_2}{2}\right) \supseteq \frac{1}{\ltimes_2 - \ltimes_1} \ {}^{(IR)}\!\!\int_{\ltimes_1}^{\ltimes_2} ß(n)dn \supseteq \frac{ß(\ltimes_1) + ß(\ltimes_2)}{2}. \tag{1.2}$$

Budek et al. [6] gave the following strong connection between convex I-V-F and interval H-H-inequality as a counter part of (1.2):

$$ß\left(\frac{\ltimes_1 + \ltimes_2}{2}\right) \supseteq \frac{\Gamma(\measuredangle + 1)}{2(\ltimes_2 - \ltimes_1)^{\measuredangle}}[\dashv^{\measuredangle}_{\ltimes_1^+} ß(\ltimes_2) + \dashv^{\measuredangle}_{\ltimes_2^-} ß(\ltimes_1)] \supseteq \frac{ß(\ltimes_1) + ß(\ltimes_2)}{2},$$

where $ß : \aleph = [\ltimes_1, \ltimes_2] \to \Pi_c^+$ be a non-negative convex I-V-F on $[\ltimes_1, \ltimes_2]$, $ß \in L_1((\ltimes_1, \ltimes_2), \Pi_c^+)$ with $\ltimes_1 < \ltimes_2$ also $\dashv^{\measuredangle}_{\ltimes_1^+}$ and $\dashv^{\measuredangle}_{\ltimes_2^-}$ are the left sided and right sided Riemann-Liouville fractional of order $0 \leq \measuredangle$.

Allahviranloo et al. [7] introduced the following fuzzy interval Riemann-Liouville fractional integral operators.

Let $0 < \measuredangle$ and $L((\ltimes_1, \ltimes_2), \Pi_0)$ be the collection of all Lebesgue measurable fuzzy-I-V-Fs on $[\ltimes_1, \ltimes_2]$. Then, the fuzzy interval left and right Riemann-Liouville fractional integrals of $ß \in L((\ltimes_1, \ltimes_2), \Pi_0)$ with order $0 < \measuredangle$ are:

$$\dashv^{\measuredangle}_{\ltimes_1^+} ß(\ltimes) = \frac{1}{\Gamma(\measuredangle)} \int_{\ltimes_1}^{\ltimes} (\ltimes - \top)^{\measuredangle - 1} ß(\top) d\top \quad , (\ltimes > \ltimes_1);$$

$$\dashv^{\curlywedge}_{\ltimes_2^-} ß(\ltimes) = \frac{1}{\Gamma(\curlywedge)} \int_{\ltimes}^{\ltimes_2} (\top - \ltimes)^{\curlywedge - 1} ß(\top) d\top \quad , (\ltimes < \ltimes_2);$$

respectively, where $\Gamma(\ltimes) = \int_0^{\infty} \top^{\ltimes - 1} s^{-\top} d(\top)$ is the Euler ς function. The fuzzy interval left and right Riemann-Liouville fractional integral $\ltimes$ based on left and right end point functions can be given as

$$\begin{aligned}
[\dashv^{\curlywedge}_{\ltimes_1^+} ß(\ltimes)]^{\varsigma} &= \frac{1}{\Gamma(\curlywedge)} \int_{\ltimes_1}^{\ltimes} (\ltimes - \top)^{\curlywedge - 1} ß_{\varsigma}(\top) d\top \\
&= \frac{1}{\Gamma(\curlywedge)} \int_{\ltimes_1}^{\ltimes} (\ltimes - \top)^{\curlywedge - 1} [ß_*(\top, \varsigma), ß^*(\top, \varsigma)] d\top, \ (\ltimes > \ltimes_1),
\end{aligned}$$

where

$$\dashv^{\curlywedge}_{\ltimes_1^+} ß_*(\ltimes, \varsigma) = \frac{1}{\Gamma(\curlywedge)} \int_{\ltimes_1}^{\ltimes} (\ltimes - \top)^{\curlywedge - 1} ß_*(\top, \varsigma) d\top, \ (\ltimes > \ltimes_1),$$

and

$$\dashv^{\curlywedge}_{\ltimes_1^+} ß^*(\ltimes, \varsigma) = \frac{1}{\Gamma(\curlywedge)} \int_{\ltimes_1}^{\ltimes} (\ltimes - \top)^{\curlywedge - 1} ß^*(\top, \varsigma) d\top, \ (\ltimes > \ltimes_1).$$

Similarly, we can give right Riemann-Liouville fractional integral ß of $\ltimes$ based on left and right end point functions. For the applications of fractional order integral operators, we refer [8–12] to the readers. There are several papers with different integrals and fractional operators in the literature, for example in [13], Ekinci and Özdemir presented some new interesting integral inequalities via Riemann-Liouville integral operators. Grübüz and Özdemir in 2020 [14] established some inequalities for product of different kinds of convex functions. Grüss type inequalities for fractional integral operator involving the extended generalized Mittag-Leffler function were presented in [15]. For more advances in this direction, we recommend [16, 17] and references therein.

In this paper, we established Hermite-Hadamard type inequality and He's inequality for exponential type convex fuzzy interval-valued functions via fuzzy Riemann-Liouville fractional integral and fuzzy He's fractional integral. We also established Hermite-Fejér inequality via fuzzy Riemann-Liouville fractional integral.

2. Preliminaries

In this section, we firstly give some preliminary notations, definitions and results which will be helpful in our work. Then, we give new definitions and analyze some properties of exponential type convex fuzzy-I-V-Fs.

Let Π_c be the space of all closed and bounded intervals of $\Re$, and $\nu \in \Pi_c$ given by

$$\nu = [\nu_*, \nu^*] = \{\ltimes \in \Re \big| \nu_* < \ltimes < \nu^*\}, \quad \nu_*, \nu^* \in \Re.$$

If $\nu_* = \nu^*$, then ν is said to be degenerate. In our work, all intervals are non-degenerate intervals. If $0 \leq \nu_*$, then ν is said to be a non-negative interval. The set of all non-negative intervals is denoted by Π_c^+ and given as

$$\Pi_c^+ = \{[\nu_*, \nu^*] \ \big| \ [\nu_*, \nu^*] \in \Pi_c, \ 0 \leq \nu_*\}.$$

Some binary operations are given as follows:

Scalar multiplication $\top \in \Re$

$$\top.\nu = \begin{cases} [\top\nu_*, \top\nu^*] & if \;\; 0 \leq \top; \\ [\top\nu^*, \top\nu_*] & if \;\; \top \leq 0. \end{cases}$$

Minkowski difference, addition, product, relation $\subset$ and $\leq_I$ for $\nu_1, \nu_2 \in \Pi_c$ are respectively given by [38]:

$$\nu_1 - \nu_2 = [\nu_{1*}, \nu_1^*] - [\nu_{2*}, \nu_2^*] = [\nu_{1*} - \nu_{2*}, \nu_1^* - \nu_2^*,];$$

$$\nu_1 + \nu_2 = [\nu_{1*}, \nu_1^*] + [\nu_{2*}, \nu_2^*] = [\nu_{1*} + \nu_{2*}, \nu_1^* + \nu_2^*,];$$

$$\nu_1 \times \nu_2 = [min\{\nu_{1*}\nu_{2*}, \nu_1^*\nu_{2*}, \nu_{1*}\nu_2^*, \nu_1^*\nu_2^*\}, max\{\nu_{1*}\nu_{2*}, \nu_1^*\nu_{2*}, \nu_{1*}\nu_2^*, \nu_1^*\nu_2^*\}];$$

$$\nu_1 \subseteq \nu_2 \Longleftrightarrow [\nu_{1*}, \nu_1^*] \subseteq [\nu_{2*}, \nu_2^*] \Longleftrightarrow \nu_{2*} \leq \nu_{1*}, \nu_1^* \leq \nu_2^*;$$

$$[\nu_{1*}, \nu_1^*] \leq_I [\nu_{2*}, \nu_2^*] \Longleftrightarrow \nu_{1*} \leq \nu_{2*}, \nu_1^* \leq \nu_2^*.$$

Definition 2.1. *A fuzzy subset $\pitchfork$ of $\Re$ is characterized by a mapping $\Cup : \Re \to [0, 1]$ is said to be the membership function for each fuzzy set and $\varsigma \in (0, 1]$, then $\varsigma -$ level sets of $\Cup$ is denoted and given as follows:*

$$\Cup_\varsigma = \{\ltimes \in \Re \big| \; \varsigma \leq \Cup(\ltimes)\}.$$

If $\varsigma = 0$, then

$$supp(\Cup) = \{\ltimes \in \Re \big| \; \Cup(\ltimes) > 0\}$$

is said to be support of $\Cup$ and $[\Cup]^o$ is said to be closure of $supp(\Cup)$.

Let $\prod(\Re)$ be the family of all fuzzy sets and $\Cup \in \prod(\Re)$ denote the family of all nonempty sets. $\Cup \in \prod(\Re)$ is a fuzzy set. Then, we define the following:

Case 2.2. *$\Cup$ is said to be normal if there exists $\ltimes \in \Re$ such that $\Cup(\ltimes) = 1$.*

Case 2.3. *$\Cup$ is said to be upper semi-continuous on $\Re$ if for given $\ltimes \in \Re$, there exists $\Im > 0$ there exists $\Im_1 > 0$ such that $\Cup(\ltimes) - \Cup(\ltimes_1) < \Im$ for all $\ltimes_1 \in \Re$ with $| \ltimes - \ltimes_1 | < \Im_1$.*

Case 2.4. *$\Cup$ is said to be fuzzy convex if $\Cup_\varsigma$ is convex for every $\varsigma \in [0, 1]$.*

Case 2.5. *$\Cup$ is compactly supported if $supp(\Cup)$ is compact.*

A fuzzy set is said to be a fuzzy number or fuzzy interval if it has properties given above in Cases 2.2–2.5. We denote by $\prod_0$ the family of all intervals [34–43].

Definition 2.6. *The $\Cup \in \prod_0$ be a fuzzy interval if and only if $\varsigma -$ levels $[\Cup]^\varsigma$ is a nonempty compact convex set of $\Re$. From these definitions, we have*

$$[\Cup]^\varsigma = [\Cup_*(\varsigma), \Cup^*(\varsigma)]$$

where

$$\Cup_*(\varsigma) = inf\{\ltimes \in \Re \big| \varsigma \leq \Cup(\ltimes)\}$$

and

$$\Cup^*(\varsigma) = sup\{\ltimes \in \Re \big| \varsigma \leq \Cup(\ltimes)\}.$$

The relations scalar multiplication$\odot$, addition$\oplus$, product$\otimes$, scalar addition and partial order relation $\preceq$ for $\mathbb{U}_1, \mathbb{U}_2 \in \prod_0$ and $\intercal \in \mathfrak{R}$ are respectively given by:

$$[\intercal \odot \mathbb{U}_1]^{\varsigma} = \intercal.[\mathbb{U}_1]^{\varsigma};$$

$$[\mathbb{U}_1 \oplus \mathbb{U}_2]^{\varsigma} = [\mathbb{U}_1]^{\varsigma} + [\mathbb{U}_2]^{\varsigma};$$

$$[\mathbb{U}_1 \otimes \mathbb{U}_2]^{\varsigma} = [\mathbb{U}_1]^{\varsigma} \times [\mathbb{U}_2]^{\varsigma};$$

$$[\intercal \oplus \mathbb{U}_1]^{\varsigma} = \intercal + [\mathbb{U}_1]^{\varsigma};$$

$$\mathbb{U}_1 \preceq \mathbb{U}_2 \Longleftrightarrow [\mathbb{U}_1]^{\varsigma} \leq_I [\mathbb{U}_2]^{\varsigma} \;\; ([18])$$

for all $\varsigma \in [0, 1]$.

For $\mathbb{U} \in \prod_0$ such that $\mathbb{U}_1 = \mathbb{U}_2 \oplus \mathbb{U}$, then Hukuhara difference [29] $\ominus$ is given by:

$$(\mathbb{U})^*(\varsigma) = (\mathbb{U}_1 \ominus \mathbb{U}_2)^*(\varsigma) = (\mathbb{U}_1)^*(\varsigma) - (\mathbb{U}_1)^*(\varsigma);$$

$$(\mathbb{U})_*(\varsigma) = (\mathbb{U}_1 \ominus \mathbb{U}_2)_*(\varsigma) = (\mathbb{U}_1)_*(\varsigma) - (\mathbb{U}_1)_*(\varsigma).$$

Definition 2.7. *[19] A fuzzy mapping* $ß : \aleph \subset \mathfrak{R} \to \prod_0$ *is said to be fuzzy I-V-F for all* $\varsigma \in [0, 1]$, *whose* ς*−levels define the family of I-V-Fs* $ß_{\varsigma} : \aleph \subset \mathfrak{R} \to \Pi_c$ *are given by* $ß_{\varsigma}(\ltimes) = [ß_*(\ltimes, \varsigma), ß^*(\ltimes, \varsigma)]$ *for all* $\ltimes \in \aleph$, $ß_*(\ltimes, \varsigma), ß^*(\ltimes, \varsigma) : \aleph \to \mathfrak{R}$ *are said to be lower and upper functions of* ß.

Definition 2.8. *[20] The fuzzy I-V-F* $ß : [\ltimes_1, \ltimes_2] \to \prod_0$ *is said to be convex fuzzy I-V-F on* $[\ltimes_1, \ltimes_2]$ *if,*

$$ß(\intercal \ltimes_{11} + (1 - \intercal)\ltimes_{12}) \preceq \intercal ß(\ltimes_{11}) \oplus (1 - \intercal)ß(\ltimes_{12})$$

for all $\ltimes_{11}, \ltimes_{12} \in [\ltimes_1, \ltimes_2]$, $\intercal \in [0, 1]$ *where* $0 \preceq ß(\ltimes)$ *for all* $\ltimes \in [\ltimes_1, \ltimes_2]$.

Now, we introduce exponential type convex fuzzy I-V-F.

Definition 2.9. *The fuzzy I-V-F* $ß : [\ltimes_1, \ltimes_2] \to \prod_0$ *is said to be exponential type convex fuzzy I-V-F on* $[\ltimes_1, \ltimes_2]$ *if,*

$$ß(\intercal \ltimes_{11} + (1 - \intercal)\ltimes_{12}) \preceq (\varrho^{\intercal} - 1)ß(\ltimes_{11}) \oplus (\varrho^{(1-\intercal)} - 1)ß(\ltimes_{12}) \tag{2.1}$$

for all $\ltimes_{11}, \ltimes_{12} \in [\ltimes_1, \ltimes_2]$, $\intercal \in [0, 1]$ *where* $0 \preceq ß(\ltimes)$ *for all* $\ltimes \in [\ltimes_1, \ltimes_2]$.

Remark 2.10. *If* (2.1) *is reversed, then* ß *is said to be exponential type concave fuzzy I-V-F on* $[\ltimes_1, \ltimes_2]$.

Remark 2.11. *If* $ß_*(\ltimes, \varsigma) = ß^*(\ltimes, \varsigma)$ *and* $\varsigma = 1$ *then we get* (1.1).

Remark 2.12. *Every nonnegative convex fuzzy I-V-F is exponential type convex fuzzy I-V-F, since* $\intercal \leq \varrho^{\intercal} - 1$ *and* $1 - \intercal \leq \varrho^{1-\intercal} - 1$ *for all* $\intercal \in [0, 1]$.

$$ß(\intercal \ltimes_{11} + (1 - \intercal)\ltimes_{12}) \preceq \intercal ß(\ltimes_{11}) \oplus (1 - \intercal)ß(\ltimes_{12}) \preceq (\varrho^{\intercal} - 1)ß(\ltimes_{11}) \oplus (\varrho^{(1-\intercal)} - 1)ß(\ltimes_{12})$$

3. Fuzzy-interval fractional H-H-inequalities

In this section, we use the following fractional H-H-inequality for convex fuzzy-I-V-Fs. We also give fractional H-H inequality for exponential type convex fuzzy-I-V-F through fuzzy order relation. The family of Lebesgue measurable fuzzy-I-V-Fs denoted by $L([\ltimes_1, \ltimes_2], \prod_0)$.

Lemma 3.1. *[21] Let* $ß : [\ltimes_1, \ltimes_2] \to \prod_0$ *be an convex fuzzy I-V-F on* $[\ltimes_1, \ltimes_2]$*, whose* $\varsigma-$*levels define the family of I-V-Fs* $ß_\varsigma : [\ltimes_1, \ltimes_2] \subset \Re \to \Pi_c^+$ *are given by* $ß_\varsigma(\ltimes) = [ß_*(\ltimes, \varsigma), ß^*(\ltimes, \varsigma)]$ *for all* $\ltimes \in [\ltimes_1, \ltimes_2]$ *and* $\varsigma \in [0, 1]$*. If* $ß \in L([\ltimes_1, \ltimes_2], \prod_0)$*, then*

$$ß\Big(\frac{\ltimes_1 + \ltimes_2}{2}\Big) \leq \frac{\Gamma(\leftthreetimes + 1)}{2(\ltimes_2 - \ltimes_1)^{\leftthreetimes}}[\dashv_{\ltimes_1^+}^{\leftthreetimes} ß(\ltimes_2) \oplus \dashv_{\ltimes_2^-}^{\leftthreetimes} ß(\ltimes_1)] \leq \frac{ß(\ltimes_1) \oplus ß(\ltimes_2)}{2}.$$

For concave fuzzy I-V-F, we have

$$\frac{ß(\ltimes_1) \oplus ß(\ltimes_2)}{2} \leq \frac{\Gamma(\leftthreetimes + 1)}{2(\ltimes_2 - \ltimes_1)^{\leftthreetimes}}[\dashv_{\ltimes_1^+}^{\leftthreetimes} ß(\ltimes_2) \oplus \dashv_{\ltimes_2^-}^{\leftthreetimes} ß(\ltimes_1)] \leq ß\Big(\frac{\ltimes_1 + \ltimes_2}{2}\Big).$$

Theorem 3.2. *Let* $ß : [\ltimes_1, \ltimes_2] \to \prod_0$ *be an exponential type convex fuzzy I-V-F on* $[\ltimes_1, \ltimes_2]$*, whose* $\varsigma-$*levels define the family of I-V-Fs* $ß_\varsigma : [\ltimes_1, \ltimes_2] \subset \Re \to \Pi_c^+$ *are given by* $ß_\varsigma(\ltimes) = [ß_*(\ltimes, \varsigma), ß^*(\ltimes, \varsigma)]$ *for all* $\ltimes \in [\ltimes_1, \ltimes_2]$ *and* $\varsigma \in [0, 1]$*. If* $ß \in L([\ltimes_1, \ltimes_2], \prod_0)$*, then*

$$\frac{1}{(\varrho^{\frac{1}{2}} - 1)} ß\Big(\frac{\ltimes_1 + \ltimes_2}{2}\Big) \leq \frac{\Gamma(\leftthreetimes + 1)}{(\ltimes_2 - \ltimes_1)^{\leftthreetimes}}[\dashv_{\ltimes_1^+}^{\leftthreetimes} ß(\ltimes_2) \oplus \dashv_{\ltimes_2^-}^{\leftthreetimes} ß(\ltimes_1)] \leq K[ß(\ltimes_1) \oplus ß(\ltimes_2)]$$

where

$$K = \leftthreetimes\Big(- \leftthreetimes\Gamma(\leftthreetimes, 1) - \frac{\Gamma(\leftthreetimes, -1)}{(-1)^{\leftthreetimes}} + \frac{\Gamma(\leftthreetimes)}{(-1)^{\leftthreetimes}} + \mathrm{e}\,\Gamma(\leftthreetimes) - \frac{2}{\leftthreetimes}\Big) \tag{3.1}$$

for all $\leftthreetimes > 0$*,* e *is Euler's number and* $\Gamma(a, b)$ *is incomplete Gamma function [22].*

Proof. Let $ß : [\ltimes_1, \ltimes_2] \to \prod_0$ be an exponential type convex fuzzy I-V-F on $[\ltimes_1, \ltimes_2]$, then

$$\frac{1}{(\varrho^{\frac{1}{2}} - 1)} ß\Big(\frac{\ltimes_1 + \ltimes_2}{2}\Big) \leq [ß(\top \ltimes_1 + (1 - \top)\ltimes_2) \oplus ß((1 - \top) \ltimes_1 + \top \ltimes_2)]$$

for every $\varsigma \in [0, 1]$, we get

$$\frac{1}{(\varrho^{\frac{1}{2}} - 1)} ß_*\Big(\frac{\ltimes_1 + \ltimes_2}{2}, \varsigma\Big) \leq [ß_*(\top \ltimes_1 + (1 - \top)\ltimes_2, \varsigma) + ß_*((1 - \top) \ltimes_1 + \top \ltimes_2, \varsigma)].$$

Multiplying by $\top^{\leftthreetimes - 1}$ and then integrating

$$\begin{aligned}
&\frac{1}{(\varrho^{\frac{1}{2}} - 1)} ß_*\Big(\frac{\ltimes_1 + \ltimes_2}{2}, \varsigma\Big) \int_0^1 \top^{\leftthreetimes - 1} d\top \\
&\leq \int_0^1 \top^{\leftthreetimes - 1} ß_*(\top \ltimes_1 + (1 - \top)\ltimes_2, \varsigma) d\top \\
&+ \int_0^1 \top^{\leftthreetimes - 1} ß_*((1 - \top) \ltimes_1 + \top \ltimes_2, \varsigma) d\top
\end{aligned}$$

taking $\ltimes_{11} = \top \ltimes_1 + (1-\top)\ltimes_2$ and $\ltimes_{12} = ((1-\top)\ltimes_1 + \top \ltimes_2$

$$\begin{aligned}
&\frac{1}{\leftthreetimes(\varrho^{\frac{1}{2}}-1)}\text{ß}_*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)\\
&\leq \int_{\ltimes_1}^{\ltimes_2}\Big(\frac{\ltimes_2-\ltimes_{11}}{\ltimes_2-\ltimes_1}\Big)^{\leftthreetimes-1}\text{ß}_*(\ltimes_{11},\varsigma)\frac{d\ltimes_{11}}{\ltimes_2-\ltimes_1}\\
&+\int_{\ltimes_1}^{\ltimes_2}\Big(\frac{\ltimes_{12}-\ltimes_1}{\ltimes_2-\ltimes_1}\Big)^{\leftthreetimes-1}\text{ß}_*(\ltimes_{12},\varsigma)\frac{d\ltimes_{12}}{\ltimes_2-\ltimes_1}\\
&\leq \frac{1}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}\Big[\int_{\ltimes_1}^{\ltimes_2}(\ltimes_2-\ltimes_{11})^{\leftthreetimes-1}\text{ß}_*(\ltimes_{11},\varsigma)d\ltimes_{11}\\
&+\int_{\ltimes_1}^{\ltimes_2}(\ltimes_{12}-\ltimes_1)^{\leftthreetimes-1}\text{ß}_*(\ltimes_{12},\varsigma)d\ltimes_{12}\Big].
\end{aligned}$$

So, we get

$$\frac{1}{\leftthreetimes(\varrho^{\frac{1}{2}}-1)}\text{ß}_*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big) \leq \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\daleth^{\leftthreetimes}_{\ltimes_1^+}\text{ß}_*(\ltimes_2,\varsigma)+\daleth^{\leftthreetimes}_{\ltimes_2^-}\text{ß}_*(\ltimes_1,\varsigma)].$$

Similarly,

$$\frac{1}{\leftthreetimes(\varrho^{\frac{1}{2}}-1)}\text{ß}^*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big) \leq \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\daleth^{\leftthreetimes}_{\ltimes_1^+}\text{ß}^*(\ltimes_2,\varsigma)+\daleth^{\leftthreetimes}_{\ltimes_2^-}\text{ß}^*(\ltimes_1,\varsigma)].$$

Thus, we can write

$$\begin{aligned}
&\frac{1}{\leftthreetimes(\varrho^{\frac{1}{2}}-1)}\Big[\text{ß}_*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big),\text{ß}^*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)\Big]\\
&\leq_I \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}\Big[[\daleth^{\leftthreetimes}_{\ltimes_1^+}\text{ß}_*(\ltimes_2,\varsigma)+\daleth^{\leftthreetimes}_{\ltimes_2^-}\text{ß}_*(\ltimes_1,\varsigma)],[\daleth^{\leftthreetimes}_{\ltimes_1^+}\text{ß}^*(\ltimes_2,\varsigma)+\daleth^{\leftthreetimes}_{\ltimes_2^-}\text{ß}^*(\ltimes_1,\varsigma)]\Big]
\end{aligned}$$

which is

$$\frac{1}{\leftthreetimes(\varrho^{\frac{1}{2}}-1)}\text{ß}\Big(\frac{\ltimes_1+\ltimes_2}{2}\Big) \leq \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\daleth^{\leftthreetimes}_{\ltimes_1^+}\text{ß}(\ltimes_2)\oplus\daleth^{\leftthreetimes}_{\ltimes_2^-}\text{ß}(\ltimes_1)]$$

$$\frac{1}{(\varrho^{\frac{1}{2}}-1)}\text{ß}\Big(\frac{\ltimes_1+\ltimes_2}{2}\Big) \leq \frac{\Gamma(\leftthreetimes+1)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\daleth^{\leftthreetimes}_{\ltimes_1^+}\text{ß}(\ltimes_2)\oplus\daleth^{\leftthreetimes}_{\ltimes_2^-}\text{ß}(\ltimes_1)].$$

Similarly,

$$\frac{\Gamma(\leftthreetimes+1)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\daleth^{\leftthreetimes}_{\ltimes_1^+}\text{ß}(\ltimes_2)\oplus\daleth^{\leftthreetimes}_{\ltimes_2^-}\text{ß}(\ltimes_1)] \leq K[\text{ß}(\ltimes_1)\oplus\text{ß}(\ltimes_2)].$$

From (3.1) and combining above inequalities,

$$\frac{1}{(\varrho^{\frac{1}{2}}-1)}\text{ß}\Big(\frac{\ltimes_1+\ltimes_2}{2}\Big) \leq \frac{\Gamma(\leftthreetimes+1)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\daleth^{\leftthreetimes}_{\ltimes_1^+}\text{ß}(\ltimes_2)\oplus\daleth^{\leftthreetimes}_{\ltimes_2^-}\text{ß}(\ltimes_1)] \leq K[\text{ß}(\ltimes_1)\oplus\text{ß}(\ltimes_2)].$$

□

Now, we deal with the H-H inequality in the sense of He's fractional derivatives as introduced in [24–27]. Let us recall the following fractional derivative introduced by He.

Definition 3.3. *For any L_1 function* ß *on an interval* $[0, \mathfrak{I}]$*, the* $\Bbbk$*-th He's fractional derivative of* ß$(\mathfrak{I})$ *is defined by*

$$ð_{\mathfrak{I}}^{\Bbbk} ß(\mathfrak{I}) = \frac{1}{\Gamma(n-\Bbbk)} \frac{d^n}{d\mathfrak{I}^n} \int_0^{\mathfrak{I}} (\top - \mathfrak{I})^{n-\Bbbk-1} ß(\top) d\top .$$

Now we introduce, fuzzy interval He's fractional derivative ⋉ *based on left and right end point functions can be given as*

$$[\eth^{\Bbbk}_{\ltimes}ß(\ltimes)]^{\varsigma} = \frac{1}{\Gamma(n-\Bbbk)}\frac{d^n}{d\ltimes^n}\int_0^{\ltimes}(\top-\ltimes)^{n-\Bbbk-1}ß_{\varsigma}(\top)d\top$$
$$= \frac{1}{\Gamma(n-\Bbbk)}\frac{d^n}{d\ltimes^n}\int_0^{\ltimes}(\top-\ltimes)^{n-\Bbbk-1}[ß_*(\top,\varsigma), ß^*(\top,\varsigma)]d\top, \quad (\ltimes > \ltimes_1)$$

where

$$\eth^{\Bbbk}_{\ltimes}ß_*(\ltimes,\varsigma) = \frac{1}{\Gamma(n-\Bbbk)}\frac{d^n}{d\ltimes^n}\int_0^{\ltimes}(\top-\ltimes)^{n-\Bbbk-1}ß_*(\top,\varsigma)d\top, \ (\ltimes > \ltimes_1) \tag{3.2}$$

and

$$\eth^{\Bbbk}_{\ltimes}ß^*(\ltimes,\varsigma) = \frac{1}{\Gamma(n-\Bbbk)}\frac{d^n}{d\ltimes^n}\int_0^{\ltimes}(\top-\ltimes)^{n-\Bbbk-1}ß^*(\top,\varsigma)d\top, \ (\ltimes > \ltimes_1)$$

where, $\varsigma \in [0,1]$.

Theorem 3.4. *Let* $ß : [\ltimes_1,\ltimes_2] \to \prod_0$ *be an exponential type convex fuzzy I-V-F on* $[\ltimes_1,\ltimes_2]$*, whose* $\varsigma-$*levels define the family of I-V-Fs* $ß_{\varsigma} : [\ltimes_1,\ltimes_2] \subset \Re \to \Pi_c^+$ *are given by* $ß_{\varsigma}(\ltimes) = [ß_*(\ltimes,\varsigma), ß^*(\ltimes,\varsigma)]$ *for all* $\ltimes \in [\ltimes_1,\ltimes_2]$ *and* $\varsigma \in [0,1]$*. If* $ß \in L_1([\ltimes_1,\ltimes_2],\prod_0)$*, then*

$$(-1)^{n-\Bbbk-1}ß\left(\frac{\ltimes_2}{2}\right) \leq \frac{(\varrho^{\frac{1}{2}}-1)\Im^{\Bbbk}}{\ltimes_2^{n-\Bbbk}}[(-1)^{n-\Bbbk-1}\eth^{\Bbbk}_{(1-\Im)b}ß((1-\Im)b) + \eth^{\Bbbk}_{\Im b}ß(\Im b)].$$

Proof. Let $ß : [\ltimes_1,\ltimes_2] \to \prod_0$ be an exponential type convex fuzzy I-V-F on $[\ltimes_1,\ltimes_2]$, then

$$ß\left(\frac{\ltimes_1+\ltimes_2}{2}\right) \leq (\varrho^{\frac{1}{2}}-1)[ß(\top\ltimes_1+(1-\top)\ltimes_2) \oplus ß((1-\top)\ltimes_1+\top\ltimes_2)].$$

For every $\varsigma \in [0,1]$, we get

$$ß_*\left(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\right) \leq (\varrho^{\frac{1}{2}}-1)[ß_*(\top\ltimes_1+(1-\top)\ltimes_2,\varsigma) + ß_*((1-\top)\ltimes_1+\top\ltimes_2,\varsigma)].$$

Taking $\ltimes_1 = 0$ and $0 \leq \ltimes_2$ and multiplying by $\dfrac{(\top-\Im)^{n-\Bbbk-1}}{\Gamma(n-\Bbbk)}$, we get

$$\frac{(\top-\Im)^{n-\Bbbk-1}}{\Gamma(n-\Bbbk)}ß_*\left(\frac{\ltimes_2}{2},\varsigma\right) \leq (\varrho^{\frac{1}{2}}-1)\frac{(\top-\Im)^{n-\Bbbk-1}}{\Gamma(n-\Bbbk)}[ß_*((1-\top)\ltimes_2,\varsigma) + ß_*(\top\ltimes_2,\varsigma)].$$

Integrating with respect to $\top$ over $[0,\Im]$, we get

$$ß_*\left(\frac{\ltimes_2}{2},\varsigma\right)\frac{1}{\Gamma(n-\Bbbk)}\int_0^{\Im}(\top-\Im)^{n-\Bbbk-1}d\top$$
$$\leq \frac{(\varrho^{\frac{1}{2}}-1)}{\Gamma(n-\Bbbk)}\int_0^{\Im}(\top-\Im)^{n-\Bbbk-1}ß_*((1-\top)\ltimes_2,\varsigma)d\top$$
$$+ \frac{(\varrho^{\frac{1}{2}}-1)}{\Gamma(n-\Bbbk)}\int_0^{\Im}(\top-\Im)^{n-\Bbbk-1}ß_*(\top\ltimes_2,\varsigma)d\top,$$

$$
\begin{aligned}
&ß_*\Big(\frac{\ltimes_2}{2},\varsigma\Big)\frac{(-1)^{n-\Bbbk-1}\Im^{n-\Bbbk}}{\Gamma(n-\Bbbk)}\\
&\le \frac{(\varrho^{\frac{1}{2}}-1)}{\Gamma(n-\Bbbk)}\int_0^{\Im}(\intercal-\Im)^{n-\Bbbk-1}ß_*((1-\intercal)\ltimes_2,\varsigma)d\intercal\\
&+\frac{(\varrho^{\frac{1}{2}}-1)}{\Gamma(n-\Bbbk)}\int_0^{\Im}(\intercal-\Im)^{n-\Bbbk-1}ß_*(\intercal\ltimes_2,\varsigma)d\intercal .
\end{aligned}
$$

Taking n-th derivative on both sides and using (3.2), we get

$$
(-1)^{n-\Bbbk-1}ß_*\Big(\frac{\ltimes_2}{2},\varsigma\Big)\le\frac{(\varrho^{\frac{1}{2}}-1)\Im^{\Bbbk}}{\ltimes_2^{n-\Bbbk}}[(-1)^{n-\Bbbk-1}ð^{\Bbbk}_{(1-\Im)b}ß_*((1-\Im)b,\varsigma)+ð^{\Bbbk}_{\Im b}ß_*(\Im b,\varsigma)].
$$

Similarly,

$$
(-1)^{n-\Bbbk-1}ß^*\Big(\frac{\ltimes_2}{2},\varsigma\Big)\le\frac{(\varrho^{\frac{1}{2}}-1)\Im^{\Bbbk}}{\ltimes_2^{n-\Bbbk}}[(-1)^{n-\Bbbk-1}ð^{\Bbbk}_{(1-\Im)b}ß^*((1-\Im)b,\varsigma)+ð^{\Bbbk}_{\Im b}ß^*(\Im b,\varsigma)].
$$

Thus, we can write

$$
\begin{aligned}
&(-1)^{n-\Bbbk-1}\Big[ß_*(\frac{\ltimes_2}{2},\varsigma\Big),ß^*\Big(\frac{\ltimes_2}{2},\varsigma\Big)\Big]\\
&\le\frac{(\varrho^{\frac{1}{2}}-1)\Im^{\Bbbk}}{\ltimes_2^{n-\Bbbk}}\Big[(-1)^{n-\Bbbk-1}ð^{\Bbbk}_{(1-\Im)b}[ß_*((1-\Im)b,\varsigma),ß^*((1-\Im)b,\varsigma)]+ð^{\Bbbk}_{\Im b}[ß_*(\Im b,\varsigma),ß^*(\Im b,\varsigma)]\Big].
\end{aligned}
$$

So,

$$
(-1)^{n-\Bbbk-1}ß\Big(\frac{\ltimes_2}{2}\Big)\le\frac{(\varrho^{\frac{1}{2}}-1)\Im^{\Bbbk}}{\ltimes_2^{n-\Bbbk}}[(-1)^{n-\Bbbk-1}ð^{\Bbbk}_{(1-\Im)b}ß((1-\Im)b)+ð^{\Bbbk}_{\Im b}ß(\Im b)].
$$

□

4. Fuzzy-interval fractional H-H-Fejér inequalities

In this section, we will purpose fractional H-H-F inequality for exponential type convex fuzzy-I-V-F, to generalize the classical fractional H-H and H-H-F inequalities. First, we will purpose the following inequality linked with the right part of the classical H-H-F inequality for exponential type convex fuzzy-I-V-F through fuzzy order relation, which is said to be 2nd fuzzy fractional H-H-F inequality.

Theorem 4.1. *Let* $ß:[\ltimes_1,\ltimes_2]\to\prod_0$ *be an exponential type convex fuzzy I-V-F with* $\ltimes_1<\ltimes_2$*, whose* $\varsigma-$*levels define the family of I-V-Fs* $ß_\varsigma:[\ltimes_1,\ltimes_2]\subset\Re\to\Pi_c^+$ *are given by* $ß_\varsigma(\ltimes)=[ß_*(\ltimes,\varsigma),ß^*(\ltimes,\varsigma)]$ *for all* $\ltimes\in[\ltimes_1,\ltimes_2]$ *and* $\varsigma\in[0,1]$*. If* $ß\in L([\ltimes_1,\ltimes_2],\prod_0)$ *and* $\perp:[\ltimes_1,\ltimes_2]\to\Re$ *,* $0\le\perp(\ltimes)$ *symmetric with respect to* $\frac{\ltimes_1+\ltimes_2}{2}$*, then*

$$
[\daleth^{\curlywedge}_{\ltimes_1^+}ß\perp(\ltimes_2)\oplus\daleth^{\curlywedge}_{\ltimes_2^-}ß\perp(\ltimes_1)]\le M\,\frac{(\ltimes_2-\ltimes_1)^{\curlywedge}}{\Gamma(\curlywedge)}[ß(\ltimes_1)\oplus ß(\ltimes_2)], \tag{4.1}
$$

where

$$
M=\int_0^1\intercal^{\curlywedge-1}[\varrho^{\intercal}+\varrho^{1-\intercal}-2]\perp((1-\intercal)\ltimes_1+\intercal\ltimes_2)d\intercal . \tag{4.2}
$$

Proof. Since, $0 \leq \mathsf{T}^{\measuredangle-1} \perp (\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2)$ and ß is and exponential type convex fuzzy I-V-F, then for all $\varsigma \in [0,1]$, we have

$$\begin{aligned}
&\mathsf{T}^{\measuredangle-1}\, ß_*(\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2, \varsigma) \perp (\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2) \\
&\leq \mathsf{T}^{\measuredangle-1}((\varrho^{\mathsf{T}} - 1)ß(\ltimes_1, \varsigma) + (\varrho^{(1-\mathsf{T})} - 1)ß(\ltimes_2, \varsigma)) \perp (\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2)
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{T}^{\measuredangle-1}\, ß_*((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2, \varsigma) \perp ((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2) \\
&\leq \mathsf{T}^{\measuredangle-1}((\varrho^{(1-\mathsf{T})} - 1)ß_*(\ltimes_1, \varsigma) + (\varrho^{\mathsf{T}} - 1)ß_*(\ltimes_2, \varsigma)) \perp ((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2).
\end{aligned}$$

So

$$\begin{aligned}
&\int_0^1 \mathsf{T}^{\measuredangle-1} ß_*(\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2, \varsigma) \perp (\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2) d\mathsf{T} \\
&+ \int_0^1 \mathsf{T}^{\measuredangle-1} ß_*((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2, \varsigma) \perp ((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2) d\mathsf{T} \\
&\leq \int_0^1 \Big[\, \mathsf{T}^{\measuredangle-1} ((\varrho^{\mathsf{T}} - 1)ß_*(\ltimes_1, \varsigma) + (\varrho^{(1-\mathsf{T})} - 1)ß_*(\ltimes_2, \varsigma)) \perp (\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2) \\
&+ \mathsf{T}^{\measuredangle-1}((\varrho^{(1-\mathsf{T})} - 1)ß_*(\ltimes_1, \varsigma) + (\varrho^{\mathsf{T}} - 1)ß_*(\ltimes_2, \varsigma)) \perp ((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2)\Big] d\mathsf{T} \\
&= ß_*(\ltimes_1, \varsigma) \int_0^1 \mathsf{T}^{\measuredangle-1}[\varrho^{\mathsf{T}} + \varrho^{1-\mathsf{T}} - 2] \perp (\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2) d\mathsf{T} \\
&+ ß_*(\ltimes_2, \varsigma) \int_0^1 \mathsf{T}^{\measuredangle-1}[\varrho^{\mathsf{T}} + \varrho^{1-\mathsf{T}} - 2] \perp ((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2) d\, \mathsf{T}\, .
\end{aligned}$$

Since $\perp$ is symmetric, so

$$\begin{aligned}
&= [ß_*(\ltimes_1, \varsigma) + ß_*(\ltimes_2, \varsigma)] \int_0^1 \mathsf{T}^{\measuredangle-1}[\varrho^{\mathsf{T}} + \varrho^{1-\mathsf{T}} - 2] \perp ((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2) d\mathsf{T} \\
&= M\, [ß_*(\ltimes_1, \varsigma) + ß_*(\ltimes_2, \varsigma)].
\end{aligned}$$

Now, taking $\ltimes = ((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2, \varsigma)$, we have

$$\begin{aligned}
&\int_0^1 \mathsf{T}^{\measuredangle-1} ß_*(\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2, \varsigma) \perp (\mathsf{T} \ltimes_1 + (1-\mathsf{T})\ltimes_2) d\mathsf{T} \\
&+ \int_0^1 \mathsf{T}^{\measuredangle-1} ß_*((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2, \varsigma) \perp ((1-\mathsf{T}) \ltimes_1 + \mathsf{T} \ltimes_2) d\mathsf{T} \\
&= \int_{\ltimes_1}^{\ltimes_2} \left(\frac{\ltimes - \ltimes_1}{\ltimes_2 - \ltimes_1}\right)^{\measuredangle-1} ß_*(\ltimes_1 + \ltimes_2 - \ltimes, \varsigma) \perp (\ltimes) \frac{d\ltimes}{\ltimes_2 - \ltimes_1} + \int_{\ltimes_1}^{\ltimes_2} \left(\frac{\ltimes - \ltimes_1}{\ltimes_2 - \ltimes_1}\right)^{\measuredangle-1} ß_*(\ltimes, \varsigma) \perp (\ltimes) \frac{d\ltimes}{\ltimes_2 - \ltimes_1} \\
&= \frac{1}{(\ltimes_2 - \ltimes_1)^{\measuredangle}} \Big[\int_{\ltimes_1}^{\ltimes_2} (\ltimes_2 - \ltimes)^{\measuredangle-1} ß_*(\ltimes, \varsigma) \perp (\ltimes_1 + \ltimes_2 - \ltimes) d\ltimes \int_{\ltimes_1}^{\ltimes_2} (\ltimes - \ltimes_1)^{\measuredangle-1} ß_*(\ltimes, \varsigma) \perp (\ltimes) d\ltimes \Big] \\
&= \frac{\Gamma(\measuredangle)}{(\ltimes_2 - \ltimes_1)^{\measuredangle}} [\dashv^{\measuredangle}_{\ltimes_1^+} ß_* \perp (\ltimes_2) + \dashv^{\measuredangle}_{\ltimes_2^-} ß_* \perp (\ltimes_1)].
\end{aligned}$$

So,

$$\frac{\Gamma(\measuredangle)}{(\ltimes_2-\ltimes_1)^{\measuredangle}}[\dashv^{\measuredangle}_{\ltimes_1^+}ß_*\perp(\ltimes_2)+\dashv^{\measuredangle}_{\ltimes_2^-}ß_*\perp(\ltimes_1)]\leq M\,[ß_*(\ltimes_1,\varsigma)+ß_*(\ltimes_2,\varsigma)].$$

Similarly,

$$\frac{\Gamma(\measuredangle)}{(\ltimes_2-\ltimes_1)^{\measuredangle}}[\dashv^{\measuredangle}_{\ltimes_1^+}ß^*\perp(\ltimes_2)+\dashv^{\measuredangle}_{\ltimes_2^-}ß^*\perp(\ltimes_1)]\leq M\,[ß^*(\ltimes_1,\varsigma)+ß^*(\ltimes_2,\varsigma)].$$

Combining both left and right end point functions,

$$\frac{\Gamma(\measuredangle)}{(\ltimes_2-\ltimes_1)^{\measuredangle}}[\dashv^{\measuredangle}_{\ltimes_1^+}ß_*\perp(\ltimes_2)+\dashv^{\measuredangle}_{\ltimes_2^-}ß_*\perp(\ltimes_1),\dashv^{\measuredangle}_{\ltimes_1^+}ß^*\perp(\ltimes_2)+\dashv^{\measuredangle}_{\ltimes_2^-}ß^*\perp(\ltimes_1)]$$
$$\leq_I\ M\,[ß_*(\ltimes_1,\varsigma)+ß_*(\ltimes_2,\varsigma),ß^*(\ltimes_1,\varsigma)+ß^*(\ltimes_2,\varsigma)].$$

Thus, we get

$$[\dashv^{\measuredangle}_{\ltimes_1^+}ß\perp(\ltimes_2)\oplus\dashv^{\measuredangle}_{\ltimes_2^-}ß\perp(\ltimes_1)]\leq\ M\,\frac{(\ltimes_2-\ltimes_1)^{\measuredangle}}{\Gamma(\measuredangle)}[ß(\ltimes_1)\oplus ß(\ltimes_2)].$$

□

Now, we will purpose the following inequality linked with the left part of classical H-HF inequality for exponential type convex fuzzy-I-V-F through fuzzy order relation, which is said to be 1st fuzzy fractional H-HF inequality.

Theorem 4.2. *Let* $ß:[\ltimes_1,\ltimes_2]\to\prod_0$ *be an exponential type convex fuzzy I-V-F with* $\ltimes_1<\ltimes_2$*, whose* ς*−levels define the family of I-V-Fs* $ß_\varsigma:[\ltimes_1,\ltimes_2]\subset\Re\to\Pi_c^+$ *are given by* $ß_\varsigma(\ltimes)=[ß_*(\ltimes,\varsigma),ß^*(\ltimes,\varsigma)]$ *for all* $\ltimes\in[\ltimes_1,\ltimes_2]$ *and* $\varsigma\in[0,1]$*. If* $ß\in L([\ltimes_1,\ltimes_2],\prod_0)$ *and* $\perp:[\ltimes_1,\ltimes_2]\to\Re$ *,* $0\leq\perp(\ltimes)$ *symmetric with respect to* $\frac{\ltimes_1+\ltimes_2}{2}$*, then*

$$\frac{1}{(\varrho^{\frac{1}{2}}-1)}ß\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)[\dashv^{\measuredangle}_{\ltimes_1^+}\perp(\ltimes_2)+\dashv^{\measuredangle}_{\ltimes_2^-}\perp(\ltimes_1)]\leq[\dashv^{\measuredangle}_{\ltimes_1^+}ß\perp(\ltimes_2)\oplus\dashv^{\measuredangle}_{\ltimes_2^-}ß\perp(\ltimes_1)].\tag{4.3}$$

Proof. Since, $ß:[\ltimes_1,\ltimes_2]\to\prod_0$ is an exponential type convex fuzzy I-V-F on $[\ltimes_1,\ltimes_2]$, then

$$\frac{1}{(\varrho^{\frac{1}{2}}-1)}ß\Big(\frac{\ltimes_1+\ltimes_2}{2}\Big)\leq[ß(\top\ltimes_1+(1-\top)\ltimes_2)\oplus ß((1-\top)\ltimes_1+\top\ltimes_2)].$$

For every $\varsigma\in[0,1]$, we get

$$\frac{1}{(\varrho^{\frac{1}{2}}-1)}ß_*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)\leq[ß_*(\top\ltimes_1+(1-\top)\ltimes_2,\varsigma)+ß_*((1-\top)\ltimes_1+\top\ltimes_2,\varsigma)].$$

Since, $\perp((1-\top)\ltimes_1+\top\ltimes_2)=\perp(\top\ltimes_1+(1-\top)\ltimes_2)$, so multiplying by $\top^{\measuredangle-1}\perp((1-\top)\ltimes_1+\top\ltimes_2)$ and then integrating, we get

$$\frac{1}{(\varrho^{\frac{1}{2}}-1)}ß_*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)\int_0^1\top^{\measuredangle-1}\perp((1-\top)\ltimes_1+\top\ltimes_2)d\top$$
$$\leq\int_0^1\top^{\measuredangle-1}ß_*(\top\ltimes_1+(1-\top)\ltimes_2,\varsigma)\perp((1-\top)\ltimes_1+\top\ltimes_2)d\top$$

$$+\int_0^1 \top^{\leftthreetimes-1}\text{ß}_*((1-\top)\ltimes_1+\top\ltimes_2,\varsigma)\perp((1-\top)\ltimes_1+\top\ltimes_2)d\top.$$

Taking $\ltimes = ((1-\top)\ltimes_1 + \top\ltimes_2$, we get

$$\begin{aligned}
&\frac{1}{(\varrho^{\frac{1}{2}}-1)}\text{ß}_*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)\int_0^1 \top^{\leftthreetimes-1}\perp((1-\top)\ltimes_1+\top\ltimes_2)d\top\\
&\leq \int_{\ltimes_1}^{\ltimes_2}\Big(\frac{\ltimes-\ltimes_1}{\ltimes_2-\ltimes_1}\Big)^{\leftthreetimes-1}\text{ß}_*(\ltimes_1+\ltimes_2-\ltimes,\varsigma)\perp(\ltimes)\frac{d\ltimes}{\ltimes_2-\ltimes_1}\\
&+\int_{\ltimes_1}^{\ltimes_2}\Big(\frac{\ltimes-\ltimes_1}{\ltimes_2-\ltimes_1}\Big)^{\leftthreetimes-1}\text{ß}_*(\ltimes,\varsigma)\perp(\ltimes)\frac{d\ltimes}{\ltimes_2-\ltimes_1}\\
&\leq \int_{\ltimes_1}^{\ltimes_2}\Big(\frac{\ltimes_2-\ltimes}{\ltimes_2-\ltimes_1}\Big)^{\leftthreetimes-1}\text{ß}_*(\ltimes)\perp(\ltimes_1+\ltimes_2-\ltimes,\varsigma)\frac{d\ltimes}{\ltimes_2-\ltimes_1}\\
&+\int_{\ltimes_1}^{\ltimes_2}\Big(\frac{\ltimes-\ltimes_1}{\ltimes_2-\ltimes_1}\Big)^{\leftthreetimes-1}\text{ß}_*(\ltimes,\varsigma)\perp(\ltimes)\frac{d\ltimes}{\ltimes_2-\ltimes_1}\\
&\leq \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\dashv_{\ltimes_1^+}^{\leftthreetimes}\text{ß}_*\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\text{ß}_*\perp(\ltimes_1)].
\end{aligned}$$

Similarly,

$$\int_0^1 \top^{\leftthreetimes-1}\perp((1-\top)\ltimes_1+\top\ltimes_2)d\top = \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\dashv_{\ltimes_1^+}^{\leftthreetimes}\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\perp(\ltimes_1)].$$

So,

$$\frac{\Gamma(\leftthreetimes)}{(\varrho^{\frac{1}{2}}-1)(\ltimes_2-\ltimes_1)^{\leftthreetimes}}\text{ß}_*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)[\dashv_{\ltimes_1^+}^{\leftthreetimes}\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\perp(\ltimes_1)] \leq \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\dashv_{\ltimes_1^+}^{\leftthreetimes}\text{ß}_*\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\text{ß}_*\perp(\ltimes_1)].$$

Similarly,

$$\frac{\Gamma(\leftthreetimes)}{(\varrho^{\frac{1}{2}}-1)(\ltimes_2-\ltimes_1)^{\leftthreetimes}}\text{ß}^*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)[\dashv_{\ltimes_1^+}^{\leftthreetimes}\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\perp(\ltimes_1)] \leq \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\dashv_{\ltimes_1^+}^{\leftthreetimes}\text{ß}^*\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\text{ß}^*\perp(\ltimes_1)].$$

Combining both left and right end point functions, we get

$$\begin{aligned}
&\frac{\Gamma(\leftthreetimes)}{(\varrho^{\frac{1}{2}}-1)(\ltimes_2-\ltimes_1)^{\leftthreetimes}}\Big[\text{ß}_*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big),\text{ß}^*\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)\Big]\times[\dashv_{\ltimes_1^+}^{\leftthreetimes}\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\perp(\ltimes_1)]\\
&\leq_I \frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}\Big[[\dashv_{\ltimes_1^+}^{\leftthreetimes}\text{ß}_*\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\text{ß}_*\perp(\ltimes_1)],[\dashv_{\ltimes_1^+}^{\leftthreetimes}\text{ß}^*\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\text{ß}^*\perp(\ltimes_1)]\Big].
\end{aligned}$$

Thus, we get

$$\frac{1}{(\varrho^{\frac{1}{2}}-1)}\text{ß}\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)[\dashv_{\ltimes_1^+}^{\leftthreetimes}\perp(\ltimes_2)+\dashv_{\ltimes_2^-}^{\leftthreetimes}\perp(\ltimes_1)] \preceq [\dashv_{\ltimes_1^+}^{\leftthreetimes}\text{ß}\perp(\ltimes_2)\oplus\dashv_{\ltimes_2^-}^{\leftthreetimes}\text{ß}\perp(\ltimes_1)].$$

□

Corollary 4.3. *Combining* (4.1) *and* (4.3), *we get*

$$\frac{1}{(\varrho^{\frac{1}{2}}-1)}ß\Big(\frac{\ltimes_1+\ltimes_2}{2},\varsigma\Big)[\dashv^{\leftthreetimes}_{\ltimes_1^+}\perp(\ltimes_2)+\dashv^{\leftthreetimes}_{\ltimes_2^-}\perp(\ltimes_1)]$$
$$\leq[\dashv^{\leftthreetimes}_{\ltimes_1^+}ß\perp(\ltimes_2)\oplus\dashv^{\leftthreetimes}_{\ltimes_2^-}ß\perp(\ltimes_1)]$$
$$\leq\ M\ \frac{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}{\Gamma(\leftthreetimes)}[ß(\ltimes_1)\oplus ß(\ltimes_2)].$$

Theorem 4.4. *Let* $ß_1, ß_2 : [\ltimes_1,\ltimes_2] \to \prod_0$ *be exponential type convex fuzzy I-V-Fs with* $\ltimes_1 < \ltimes_2$, *whose* ς*−levels define the family of I-V-Fs* $ß_{1\varsigma}, ß_{2\varsigma} : [\ltimes_1,\ltimes_2] \subset \Re \to \Pi_c^+$ *are given by* $ß_{1\varsigma}(\ltimes) = [ß_{1*}(\ltimes,\varsigma), ß_1^*(\ltimes,\varsigma)]$ *and* $ß_{2\varsigma}(\ltimes) = [ß_{2*}(\ltimes,\varsigma), ß_1^*(\ltimes,\varsigma)]$ *for all* $\ltimes \in [\ltimes_1,\ltimes_2]$ *and* $\varsigma \in [0, 1]$. *If* $ß_1, ß_2, ß_1\otimes ß_2 \in L([\ltimes_1,\ltimes_2],\prod_0)$, *then*

$$\frac{\Gamma(\leftthreetimes)}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\dashv^{\leftthreetimes}_{\ltimes_1^+}ß_1(\ltimes_2)\otimes ß_2(\ltimes_2)\oplus\dashv^{\leftthreetimes}_{\ltimes_2^-}ß_1(\ltimes_1)\otimes ß_2(\ltimes_1)]\leq K_1\triangle_1((\ltimes_1,\ltimes_2))+2K_2\triangle_2((\ltimes_1,\ltimes_2))\tag{4.4}$$

where

$$K_1=\Big[\frac{2^{\leftthreetimes}-\mathrm{e}\leftthreetimes\big(\mathrm{e}\,\Gamma(\leftthreetimes,2)-2^{\leftthreetimes+1}\,\Gamma(\leftthreetimes,1)+2^{\leftthreetimes+1}\,\Gamma(\leftthreetimes)-\mathrm{e}\,\Gamma(\leftthreetimes)+2^{\leftthreetimes+1}\big)}{\leftthreetimes\cdot 2^{\leftthreetimes}}+\frac{\mathrm{e}^2+5}{2}\Big],\tag{4.5}$$

$$K_2=\mathrm{e}\,\Gamma(\leftthreetimes,1)+\frac{\Gamma(\leftthreetimes,-1)}{(-1)^{\leftthreetimes}}-\frac{\Gamma(\leftthreetimes)}{(-1)^{\leftthreetimes}}-\mathrm{e}\,\Gamma(\leftthreetimes)+\frac{\mathrm{e}}{\leftthreetimes}+\frac{1}{\leftthreetimes},\tag{4.6}$$

$$\triangle_1(\ltimes_1,\ltimes_2)=[\triangle_{1*}((\ltimes_1,\ltimes_2),\varsigma),\triangle_1^*((\ltimes_1,\ltimes_2),\varsigma)]\tag{4.7}$$
$$=[(ß_{1*}(\ltimes_1,\varsigma)\times ß_{2*}(\ltimes_1,\varsigma)+ß_{1*}(\ltimes_2,\varsigma)\times ß_{2*}(\ltimes_2,\varsigma)),\tag{4.8}$$

$$(ß_1^*(\ltimes_1,\varsigma)\times ß_2^*(\ltimes_1,\varsigma)+ß_1^*(\ltimes_2,\varsigma)\times ß_2^*(\ltimes_2,\varsigma))],\tag{4.9}$$

$$\triangle_2(\ltimes_1,\ltimes_2)=[\triangle_{2*}((\ltimes_1,\ltimes_2),\varsigma),\triangle_2^*((\ltimes_1,\ltimes_2),\varsigma)]\tag{4.10}$$
$$=[(ß_1^*(\ltimes_1,\varsigma)\times ß_2^*(\ltimes_2,\varsigma)+ß_1^*(\ltimes_2,\varsigma)\times ß_2^*(\ltimes_1,\varsigma)),\tag{4.11}$$

$$(ß_1^*(\ltimes_1,\varsigma)\times ß_2^*(\ltimes_2,\varsigma)+ß_1^*(\ltimes_2,\varsigma)\times ß_2^*(\ltimes_1,\varsigma))].\tag{4.12}$$

Proof. Since both $ß_1, ß_2$ are exponential type convex fuzzy I-V-Fs, then for all $\varsigma \in [0, 1]$, we have

$$ß_{1*}(\top\ltimes_1+(1-\top)\ltimes_2,\varsigma)\leq((\varrho^{\top}-1)ß_{1*}(\ltimes_1,\varsigma)+(\varrho^{(1-\top)}-1)ß_{1*}(\ltimes_2,\varsigma)),$$

and

$$ß_{2*}(\top\ltimes_1+(1-\top)\ltimes_2,\varsigma)\leq((\varrho^{\top}-1)ß_{2*}(\ltimes_1,\varsigma)+(\varrho^{(1-\top)}-1)ß_{2*}(\ltimes_2,\varsigma)).$$

Since $0 \leq ß_1(\ltimes), ß_2(\ltimes)$, we get

$$
\begin{aligned}
& ß_{1*}(\top \ltimes_1 + (1-\top)\ltimes_2, \varsigma) \times ß_{2*}(\top \ltimes_1 + (1-\top)\ltimes_2, \varsigma) \\
& \leq ((\varrho^{\top}-1)ß_{1*}(\ltimes_1, \varsigma) + (\varrho^{(1-\top)}-1)ß_{1*}(\ltimes_2, \varsigma)) \\
& \times ((\varrho^{\top}-1)ß_{2*}(\ltimes_1, \varsigma) + (\varrho^{(1-\top)}-1)ß_{2*}(\ltimes_2, \varsigma)) \\
& \leq (\varrho^{\top}-1)^2 ß_{1*}(\ltimes_1, \varsigma) \times ß_{2*}(\ltimes_1, \varsigma) + (\varrho^{1-\top}-1)^2 ß_{1*}(\ltimes_2, \varsigma) \times ß_{2*}(\ltimes_2, \varsigma) \\
& + (\varrho^{\top}-1)(\varrho^{1-\top}-1)ß_{1*}(\ltimes_1, \varsigma) \times ß_{2*}(\ltimes_2, \varsigma) \\
& + (\varrho^{\top}-1)(\varrho^{1-\top}-1)ß_{1*}(\ltimes_2, \varsigma) \times ß_{2*}(\ltimes_1, \varsigma).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& ß_{1*}((1-\top)\ltimes_1 + \top \ltimes_2, \varsigma) \times ß_{2*}((1-\top)\ltimes_1 + \top \ltimes_2, \varsigma) \\
& \leq (\varrho^{1-\top}-1)^2 ß_{1*}(\ltimes_1, \varsigma) \times ß_{2*}(\ltimes_1, \varsigma) + (\varrho^{\top}-1)^2 ß_{1*}(\ltimes_2, \varsigma) \times ß_{2*}(\ltimes_2, \varsigma) \\
& + (\varrho^{1-\top}-1)(\varrho^{\top}-1)ß_{1*}(\ltimes_1, \varsigma) \times ß_{2*}(\ltimes_2, \varsigma) \\
& + (\varrho^{1-\top}-1)(\varrho^{\top}-1)ß_{1*}(\ltimes_2, \varsigma) \times ß_{2*}(\ltimes_1, \varsigma).
\end{aligned}
$$

By adding, we get

$$
\begin{aligned}
& ß_{1*}(\top \ltimes_1 + (1-\top)\ltimes_2, \varsigma) \times ß_{2*}(\top \ltimes_1 + (1-\top)\ltimes_2, \varsigma) \\
& + ß_{1*}((1-\top)\ltimes_1 + \top \ltimes_2, \varsigma) \times ß_{2*}((1-\top)\ltimes_1 + \top \ltimes_2, \varsigma) \\
& \leq [(\varrho^{1-\top}-1)^2 + (\varrho^{\top}-1)^2](ß_{1*}(\ltimes_1, \varsigma) \times ß_{2*}(\ltimes_1, \varsigma) + ß_{1*}(\ltimes_2, \varsigma) \times ß_{2*}(\ltimes_2, \varsigma)) \\
& + 2(\varrho^{1-\top}-1)(\varrho^{\top}-1)(ß_{1*}(\ltimes_1, \varsigma) \times ß_{2*}(\ltimes_2, \varsigma) + ß_{1*}(\ltimes_2, \varsigma) \times ß_{2*}(\ltimes_1, \varsigma)).
\end{aligned}
$$

Multiplying by $\top^{\curlywedge-1}$, integrating and using (4.7) and (4.10) give

$$
\begin{aligned}
& \int_0^1 \top^{\curlywedge-1} ß_{1*}(\top \ltimes_1 + (1-\top)\ltimes_2, \varsigma) \times ß_{2*}(\top \ltimes_1 + (1-\top)\ltimes_2, \varsigma) d\top \\
& + \int_0^1 \top^{\curlywedge-1} ß_{1*}((1-\top)\ltimes_1 + \top \ltimes_2, \varsigma) \times ß_{2*}((1-\top)\ltimes_1 + \top \ltimes_2, \varsigma) d\top \\
& \leq \triangle_{1*}((\ltimes_1, \ltimes_2), \varsigma) \int_0^1 \top^{\curlywedge-1}[(\varrho^{1-\top}-1)^2 + (\varrho^{\top}-1)^2] d\top \\
& + 2\triangle_{2*}((\ltimes_1, \ltimes_2), \varsigma) \int_0^1 \top^{\curlywedge-1}(\varrho^{1-\top}-1)(\varrho^{\top}-1) d\top .
\end{aligned}
$$

From (4.5) and (4.6), we have

$$
\begin{aligned}
& \frac{\Gamma(\curlywedge)}{(\ltimes_2-\ltimes_1)^{\curlywedge}}[\dashv^{\curlywedge}_{\ltimes_1^+} ß_{1*}(\ltimes_2, \varsigma) \times ß_{2*}(\ltimes_2, \varsigma) + \dashv^{\curlywedge}_{\ltimes_2^-} ß_{1*}(\ltimes_1, \varsigma) \times ß_{2*}(\ltimes_1, \varsigma)] \\
& \leq K_1 \triangle_{1*}((\ltimes_1, \ltimes_2), \varsigma) + 2K_2 \triangle_{2*}((\ltimes_1, \ltimes_2), \varsigma).
\end{aligned}
$$

Similarly,

$$\frac{\Gamma(\curlywedge)}{(\rtimes_2-\ltimes_1)^{\curlywedge}}[\daleth^{\curlywedge}_{\ltimes_1^+}ß_1^*(\rtimes_2,\varsigma)\times ß_2^*(\rtimes_2,\varsigma)+\daleth^{\curlywedge}_{\rtimes_2^-}ß_1^*(\ltimes_1,\varsigma)\times ß_2^*(\ltimes_1,\varsigma)]$$
$$\leq K_1\triangle_1^*((\ltimes_1,\rtimes_2),\varsigma)+2K_2\triangle_2^*((\ltimes_1,\rtimes_2),\varsigma).$$

Combining both left and right end point functions, we attain

$$\frac{\Gamma(\curlywedge)}{(\rtimes_2-\ltimes_1)^{\curlywedge}}[\daleth^{\curlywedge}_{\ltimes_1^+}ß_{1*}(\rtimes_2,\varsigma)\times ß_{2*}(\rtimes_2,\varsigma)+\daleth^{\curlywedge}_{\rtimes_2^-}ß_{1*}(\ltimes_1,\varsigma)\times ß_{2*}(\ltimes_1,\varsigma),$$
$$\daleth^{\curlywedge}_{\ltimes_1^+}ß_1^*(\rtimes_2,\varsigma)\times ß_2^*(\rtimes_2,\varsigma)+\daleth^{\curlywedge}_{\rtimes_2^-}ß_1^*(\ltimes_1,\varsigma)\times ß_2^*(\ltimes_1,\varsigma)]$$
$$\leq IK_1[\triangle_{1*}((\ltimes_1,\rtimes_2),\varsigma),\triangle_1^*((\ltimes_1,\rtimes_2),\varsigma)]+2K_2[\triangle_{2*}((\ltimes_1,\rtimes_2),\varsigma),\triangle_2^*((\ltimes_1,\rtimes_2),\varsigma)].$$

Thus

$$\frac{\Gamma(\curlywedge)}{(\rtimes_2-\ltimes_1)^{\curlywedge}}[\daleth^{\curlywedge}_{\ltimes_1^+}ß_1(\rtimes_2)\otimes ß_2(\rtimes_2)\oplus\daleth^{\curlywedge}_{\rtimes_2^-}ß_1(\ltimes_1)\otimes ß_2(\ltimes_1)]\leq K_1\triangle_1((\ltimes_1,\rtimes_2))+2K_2\triangle_2((\ltimes_1,\rtimes_2)).$$

□

Theorem 4.5. *Let* $ß_1, ß_2 : [\ltimes_1, \rtimes_2] \to \prod_0$ *be exponential type convex fuzzy I-V-Fs with* $\ltimes_1 < \rtimes_2$*, whose* ς*-levels define the family of I-V-Fs* $ß_{1\varsigma}, ß_{2\varsigma} : [\ltimes_1, \rtimes_2] \subset \mathfrak{R} \to \Pi_c^+$ *are given by* $ß_{1\varsigma}(\ltimes) = [ß_{1*}(\ltimes,\varsigma), ß_1^*(\ltimes,\varsigma)]$ *and* $ß_{2\varsigma}(\ltimes) = [ß_{2*}(\ltimes,\varsigma), ß_1^*(\ltimes,\varsigma)]$ *for all* $\ltimes \in [\ltimes_1, \rtimes_2]$ *and* $\varsigma \in [0, 1]$*. If* $ß_1, ß_2, ß_1 \otimes ß_2 \in L([\ltimes_1, \rtimes_2], \prod_0)$*, then*

$$\frac{1}{\curlywedge}ß_1\Big(\frac{\ltimes_1+\rtimes_2}{2}\Big)\otimes ß_2\Big(\frac{\ltimes_1+\rtimes_2}{2}\Big)$$
$$\leq\frac{\Gamma(\curlywedge+1)(\varrho^{\frac{1}{2}}-1)^2}{(\rtimes_2-\ltimes_1)^{\curlywedge}}[\daleth^{\curlywedge}_{\ltimes_1^+}ß_1(\rtimes_2)\otimes ß_2(\rtimes_2)\oplus\daleth^{\curlywedge}_{\rtimes_2^-}ß_1(\ltimes_1)\otimes ß_2(\ltimes_1)]$$
$$+(\varrho^{\frac{1}{2}}-1)^2\Big[K_1\triangle_1((\ltimes_1,\rtimes_2))+2K_2\triangle_2((\ltimes_1,\rtimes_2))\Big]. \quad (4.13)$$

Proof. Since both $ß_1, ß_2$ are exponential type convex fuzzy I-V-Fs, then for all $\varsigma \in [0, 1]$, we have

$$ß_{1*}\Big(\frac{\ltimes_1+\rtimes_2}{2},\varsigma\Big)\leq(\varrho^{\frac{1}{2}}-1)[ß_{1*}(\top\ltimes_1+(1-\top)\rtimes_2,\varsigma)+ß_{1*}((1-\top)\ltimes_1+\top\rtimes_2,\varsigma)],$$

and

$$ß_{2*}\Big(\frac{\ltimes_1+\rtimes_2}{2},\varsigma\Big)\leq(\varrho^{\frac{1}{2}}-1)[ß_{2*}(\top\ltimes_1+(1-\top)\rtimes_2,\varsigma)+ß_{2*}((1-\top)\ltimes_1+\top\rtimes_2,\varsigma)].$$

So

$$ß_{1*}\Big(\frac{\ltimes_1+\rtimes_2}{2},\varsigma\Big)\times ß_{2*}\Big(\frac{\ltimes_1+\rtimes_2}{2},\varsigma\Big)$$
$$\leq(\varrho^{\frac{1}{2}}-1)^2\Big[ß_{1*}(\top\ltimes_1+(1-\top)\rtimes_2,\varsigma)\times ß_{2*}(\top\ltimes_1+(1-\top)\rtimes_2,\varsigma)$$
$$+ß_{1*}(\top\ltimes_1+(1-\top)\rtimes_2,\varsigma)\times ß_{2*}((1-\top)\ltimes_1+\top\rtimes_2,\varsigma)\Big]$$
$$+(\varrho^{\frac{1}{2}}-1)^2\Big[ß_{1*}((1-\top)\ltimes_1+\top\rtimes_2,\varsigma)\times ß_{2*}(\top\ltimes_1+(1-\top)\rtimes_2,\varsigma)$$

$$+ ß_{1*}((1-\top)\ltimes_1 + \top\ltimes_2, \varsigma) \times ß_{2*}((1-\top)\ltimes_1 + \top\ltimes_2, \varsigma)\Big]$$

$$\leq (\varrho^{\frac{1}{2}}-1)^2\Big[ß_{1*}(\top\ltimes_1 + (1-\top)\ltimes_2, \varsigma) \times ß_{2*}(\top\ltimes_1 + (1-\top)\ltimes_2, \varsigma)$$

$$+ ß_{1*}((1-\top)\ltimes_1 + \top\ltimes_2, \varsigma) \times ß_{2*}((1-\top)\ltimes_1 + \top\ltimes_2, \varsigma)\Big]$$

$$+ (\varrho^{\frac{1}{2}}-1)^2\Big[\Big((\varrho^{\top}-1)ß_{1*}(\ltimes_1, \varsigma) + (\varrho^{1-\top}-1)ß_{1*}(\ltimes_2, \varsigma)\Big)$$

$$\times\Big((\varrho^{1-\top}-1)ß_{2*}(\ltimes_1, \varsigma) + (\varrho^{\top}-1)ß_{2*}(\ltimes_2, \varsigma)\Big)$$

$$+\Big((\varrho^{1-\top}-1)ß_{1*}(\ltimes_1, \varsigma) + (\varrho^{\top}-1)ß_{1*}(\ltimes_2, \varsigma)\Big)$$

$$\times\Big((\varrho^{\top}-1)ß_{2*}(\ltimes_1, \varsigma) + (\varrho^{1-\top}-1)ß_{2*}(\ltimes_2, \varsigma)\Big)\Big].$$

From (4.7) and (4.10), we have

$$\leq (\varrho^{\frac{1}{2}}-1)^2\Big[ß_{1*}(\top\ltimes_1 + (1-\top)\ltimes_2, \varsigma) \times ß_{2*}(\top\ltimes_1 + (1-\top)\ltimes_2, \varsigma)$$

$$+ ß_{1*}((1-\top)\ltimes_1 + \top\ltimes_2, \varsigma) \times ß_{2*}((1-\top)\ltimes_1 + \top\ltimes_2, \varsigma)\Big]$$

$$+ (\varrho^{\frac{1}{2}}-1)^2\Big[\Big((\varrho^{\top}-1)^2 + (\varrho^{1-\top}-1)^2\Big)\triangle_{2*}((\ltimes_1, \ltimes_2), \varsigma)$$

$$+ 2\Big((\varrho^{\top}-1)(\varrho^{1-\top}-1)\Big)\triangle_{1*}((\ltimes_1, \ltimes_2), \varsigma)\Big].$$

From (4.5) and (4.6), multiplying by $\top^{\leftthreetimes-1}$ and then integrating over [0,1], we achieve

$$\frac{1}{\leftthreetimes}ß_{1*}\Big(\frac{\ltimes_1+\ltimes_2}{2}, \varsigma\Big) \times ß_{2*}\Big(\frac{\ltimes_1+\ltimes_2}{2}, \varsigma\Big)$$

$$\leq \frac{(\varrho^{\frac{1}{2}}-1)^2}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}\Big[\int_{\ltimes_1}^{\ltimes_2}(\ltimes_2-\ltimes_{11})^{\leftthreetimes-1}ß_{1*}(\ltimes_{11}, \varsigma) \times ß_{2*}(\ltimes_{11}, \varsigma)d\ltimes$$

$$+ \int_{\ltimes_1}^{\ltimes_2}(\ltimes_{12}-\ltimes_1)^{\leftthreetimes-1}ß_{1*}(\ltimes_{12}, \varsigma) \times ß_{2*}(\ltimes_{12}, \varsigma)d\ltimes\Big]$$

$$+ (\varrho^{\frac{1}{2}}-1)^2\Big[K_1\triangle_{2*}((\ltimes_1, \ltimes_2), \varsigma) + 2K_2\triangle_{1*}((\ltimes_1, \ltimes_2), \varsigma)\Big]$$

$$= \frac{\Gamma(\leftthreetimes+1)(\varrho^{\frac{1}{2}}-1)^2}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\dashv^{\leftthreetimes}_{\ltimes_1^+}ß_{1*}(\ltimes_2, \varsigma) \times ß_{2*}(\ltimes_2, \varsigma) + \dashv^{\leftthreetimes}_{\ltimes_2^-}ß_{1*}(\ltimes_1, \varsigma) \times ß_{2*}(\ltimes_1, \varsigma)]$$

$$+ (\varrho^{\frac{1}{2}}-1)^2\Big[K_1\triangle_{2*}((\ltimes_1, \ltimes_2), \varsigma) + 2K_2\triangle_{1*}((\ltimes_1, \ltimes_2), \varsigma)\Big].$$

Similarly,

$$\frac{1}{\leftthreetimes}ß_1^*\Big(\frac{\ltimes_1+\ltimes_2}{2}, \varsigma\Big) \times ß_2^*\Big(\frac{\ltimes_1+\ltimes_2}{2}, \varsigma\Big)$$

$$\leq \frac{\Gamma(\leftthreetimes+1)(\varrho^{\frac{1}{2}}-1)^2}{(\ltimes_2-\ltimes_1)^{\leftthreetimes}}[\dashv^{\leftthreetimes}_{\ltimes_1^+}ß_1^*(\ltimes_2, \varsigma) \times ß_2^*(\ltimes_2, \varsigma) + \dashv^{\leftthreetimes}_{\ltimes_2^-}ß_1^*(\ltimes_1, \varsigma) \times ß_2^*(\ltimes_1, \varsigma)]$$

$$+ (\varrho^{\frac{1}{2}} - 1)^2\Big[K_1\triangle_2^*((\ltimes_1, \ltimes_2), \varsigma) + 2K_2\triangle_1^*((\ltimes_1, \ltimes_2), \varsigma)\Big].$$

Combining both left and right end point functions, we have

$$\frac{1}{\curlywedge}\Big[\text{ß}_{1*}\Big(\frac{\ltimes_1 + \ltimes_2}{2}, \varsigma\Big) \times \text{ß}_{2*}\Big(\frac{\ltimes_1 + \ltimes_2}{2}, \varsigma\Big), \text{ß}_1^*\Big(\frac{\ltimes_1 + \ltimes_2}{2}, \varsigma\Big) \times \text{ß}_2^*\Big(\frac{\ltimes_1 + \ltimes_2}{2}, \varsigma\Big)\Big]$$
$$\leq I\frac{\Gamma(\curlywedge + 1)(\varrho^{\frac{1}{2}} - 1)^2}{(\ltimes_2 - \ltimes_1)^{\curlywedge}}\Big[[\exists^{\curlywedge}_{\ltimes_1^+}\text{ß}_{1*}(\ltimes_2, \varsigma) \times \text{ß}_{2*}(\ltimes_2, \varsigma) + \exists^{\curlywedge}_{\ltimes_2^-}\text{ß}_{1*}(\ltimes_1, \varsigma) \times \text{ß}_{2*}(\ltimes_1, \varsigma)],$$

$$[\exists^{\curlywedge}_{\ltimes_1^+}\text{ß}_1^*(\ltimes_2, \varsigma) \times \text{ß}_2^*(\ltimes_2, \varsigma) + \exists^{\curlywedge}_{\ltimes_2^-}\text{ß}_1^*(\ltimes_1, \varsigma) \times \text{ß}_2^*(\ltimes_1, \varsigma)]\Big]$$
$$+ (\varrho^{\frac{1}{2}} - 1)^2\Big[K_1[\triangle_{2*}((\ltimes_1, \ltimes_2), \varsigma), \triangle_2^*((\ltimes_1, \ltimes_2), \varsigma)] + 2K_2[\triangle_{1*}((\ltimes_1, \ltimes_2), \varsigma), \triangle_1^*((\ltimes_1, \ltimes_2), \varsigma)]\Big].$$

Thus, we get

$$\frac{1}{\curlywedge}\text{ß}_1\Big(\frac{\ltimes_1 + \ltimes_2}{2}\Big) \otimes \text{ß}_2\Big(\frac{\ltimes_1 + \ltimes_2}{2}\Big)$$
$$\leq \frac{\Gamma(\curlywedge + 1)(\varrho^{\frac{1}{2}} - 1)^2}{(\ltimes_2 - \ltimes_1)^{\curlywedge}}[\exists^{\curlywedge}_{\ltimes_1^+}\text{ß}_1(\ltimes_2) \otimes \text{ß}_2(\ltimes_2) \oplus \exists^{\curlywedge}_{\ltimes_2^-}\text{ß}_1(\ltimes_1) \otimes \text{ß}_2(\ltimes_1)]$$
$$+ (\varrho^{\frac{1}{2}} - 1)^2\Big[K_1\triangle_2((\ltimes_1, \ltimes_2)) + 2K_2\triangle_1((\ltimes_1, \ltimes_2))\Big].$$

□

5. Conclusions

In this paper, we gave some new approaches for Hermite-Hadamard type integral inequalities for exponentially convex fuzzy interval-valued functions. The main findings include some new bounds with error estimations via fuzzy Riemann-Liouville fractional integrals and Hes fractional integrals. We also proved some Fejér type inequalities with the same argument. There are several papers with classical integrals, fundamental concepts of time sclaes and fractional operators in the literature. All of these papers aim to provide new estimations and optimal approaches. But, the main motivation of this paper is that we obtained new method by using fuzzy fractional integrals and derivatives for exponentially convex fuzzy interval-valued functions fractional calculus.

Conflict of interest

The authors declare that they do not have any conflict of interests.

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