AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 12225–12244.
DOI:10.3934/math.2021708
Received: 05 July 2021
Accepted: 20 August 2021
Published: 25 August 2021

Research article

Hopf bifurcation in a delayed predator-prey system with asymmetric functional response and additional food

Luoyi Wu[1,2] **and Hang Zheng**[1,2,*]

[1] Department of Mathematics and Computer, Wuyi University, Wuyishan, Fujian, 354300, China
[2] Digital Fujian Tourism Big Data Institute, Wuyishan, Fujian, 354300, China

* **Correspondence:** Email: zhenghwyxy@163.com.

Abstract: In this paper, a delayed predator-prey system with additional food and asymmetric functional response is investigated. We discuss the local stability of equilibria and the existence of local Hopf bifurcation under the influence of the time delay. By using the normal form theory and center manifold theorem, the explicit formulas which determine the properties of bifurcating periodic solutions are obtained. Further, we prove that global periodic solutions exist after the second critical value of delay via Wu's theory. Finally, the correctness of the previous theoretical analysis is demonstrated by some numerical cases.

Keywords: delayed predator-prey system; stability; periodic solution; local Hopf bifurcation; global Hopf bifurcation
Mathematics Subject Classification: 34K25, 34C27, 34D20, 92D25

1. Introduction

In the real world, we often need to control or eliminate the target prey in a prey-predator system. Biological control and chemical control are the two essential methods. The main means of chemical control is to spray pesticides on pests at different fixed times (see e.g. [1]). However, chemical control is said to be environmentally detrimental. There are various forms of biological control, which is the focus of current research, such as immunocontraceptive technology, introducing other predators, offering the predator additional food and so on. Saunders et al. [2] used the approaches of immunocontraception and daughterless genes to control the growth of the target pest species. Ghosh et al. [3] investigated the effectiveness of periodic impulsive releases of natural enemies into a two-patch environment.

Numerous studies had shown that the additional food (to predator) can be helpful to increase predators and the prey can be controlled. Tena et al. [4] showed that the melinus females were better able to forage and oviposit by providing sugar to them. Srinivasu et al. [5] studied a standard predator-prey model with additional food and presented the evidence that how to eliminate target pests. Sahoo et al. [6] investigated that the chaotic population dynamics can be controlled to obtain regular population dynamics only by supplying additional food to top predator. Research on the models with additional food, one can refer to the literature [7–11].

Recently, Basheer et al. [12] believed that the additional food can increase the carrying capacity and birth rate of the predators, so they studied the following Holling-Tanner model with additional food:

$$\begin{cases} \dfrac{du}{dt} &= u(t)(1-u(t)) - \frac{su(t)v(t)}{mv(t)+\alpha\beta+u(t)}, \\ \dfrac{dv}{dt} &= \delta v(t)\Big(n + \frac{\beta - v(t)}{\alpha\beta+u(t)}\Big), \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ denote the prey and predator density. $\alpha, \beta \in \mathbb{R}_+$, and $\frac{1}{\alpha}$ and β represent the quality and quantity of the additional food, respectively. The biological significance of other parameters refer to [12]. Their research indicated that a conditional stable prey-extinction equilibrium could be obtained in system (1.1), but it was nonexistent in the absence of additional food.

On the other hand, if the population maturation time is considered, the corresponding model should be delayed. Therefore, lots of delayed predator-prey systems were studied (see, e.g. [13–24]). It is interesting that Jiang et al. [25] applied a delay differential equation (DDE) to two-enterprise interaction mechanism. The dynamical properties of DDE are far more complex than those of ordinary differential equation (ODE). A time delay can cause an equilibrium undergoes from stable state to unstable, which leads to the complicated dynamics of DDE. Arising periodic solution through the Hopf bifurcation is one of the hot topics (see, e.g. [18–24, 26]). However, these bifurcating periodic solutions from Hopf bifurcation are generally local. Whether these local periodic solutions exist globally is an interesting subject. Erbe et al. [27] established the global Hopf bifurcation theorem and Wu [28] applied it to a neural networks with memory. Thereafter, some researchers employed the theorem in [28] to study the global existence of periodic solutions for DDE (see, e.g. [29–34]).

Inspiration based on model (1.1), we consider the maturation time of predator and standard Holling type II functional response in the model, then we improve model (1.1) as follows:

$$\begin{cases} \dfrac{du}{dt} &= u(t)(1-u(t)) - \frac{su(t)v(t)}{m+\alpha\beta+u(t)}, \\ \dfrac{dv}{dt} &= \delta v(t)\Big(n + \frac{\beta - v(t-\tau)}{\alpha\beta+u(t-\tau)}\Big), \end{cases} \tag{1.2}$$

where τ represent the maturation time of predator. The initial conditions are chosen as:

$$u(\theta) = \varphi_1(\theta) \geq 0, v(\theta) = \varphi_2(\theta) \geq 0, \theta \in [-\tau, 0), \varphi_1(0) > 0, \varphi_2(0) > 0$$

where $(\varphi_1(\theta), \varphi_2(\theta)) \in C\{[-\tau, 0], \mathbb{R}_+^2\}, \mathbb{R}_+^2 = \{u, v : u \geq 0, v \geq 0\}$.

The main purpose of our work are concluded as follows:

(I) We consider the feedback time delay in the maturation time of predator, which is more general than the works in [12]. We investigate the effects of feedback time on the stability of the equilibria and the conditions on occurring Hopf bifurcations.

(II) The delayed feedback is designed for studying bifurcating periodic solutions. We analyze the direction and stability of bifurcating periodic solutions on the center manifold.
(III) The global existence of bifurcating periodic solutions are studied mathematically.

The rest of this paper is organized as follows. We first discuss some properties of system (1.2) to prepare for the next section. In Section 3, we study the local stability of each feasible equilibrium of system (1.2) with the effect of the time delay τ. The formulas determining the direction and stability of bifurcating periodic solutions are obtained via the theory in Hassard et al. [35] in Section 4. In Section 5, the global existence of periodic solutions under the second critical value is proved by using the theory in Wu [28]. Finally, some examples are utilized to demonstrate the validity of the previous results.

2. Preliminaries

Clearly, system (1.2) can be calculated by

$$\begin{cases} u(t) &= u(0)\exp\{\int_0^t \Big(1-u(\xi)-\frac{sv(\xi)}{m+\alpha\beta+u(\xi)}\Big)d\xi\}, \\ v(t) &= v(0)\exp\{\int_0^t (\delta\Big(n+\frac{\beta-v(\xi-\tau)}{\alpha\beta+u(\xi-\tau)}\Big))d\xi\}. \end{cases} \tag{2.1}$$

According to positive initial values of system (1.2), it is not difficult to obtain the following lemma.
Lemma 2.1 Any of the solutions of system (1.2) are positive for $t \geq 0$ with positive initial values.
Theorem 2.2 System (1.2) is ultimately bounded when τ is bounded.
Proof. From the first equation of system (1.2) and by using the comparison theorem, it is easy to obtain

$$\lim_{t\to+\infty}\sup u(t) \leq 1.$$

Namely, there exists a time T_1 such that $u(t) \leq 1+\varepsilon$ for arbitrary $\varepsilon > 0$ and $t > T_1$. By the second equation of (1.2), we have

$$\frac{dv}{dt} \leq \sigma\Big(\delta+\frac{1}{\alpha}\Big)v.$$

Integrating both sides on the interval $[t-\tau, t]$, it produces

$$v(t) \leq v(t-\tau)\exp\Big\{\sigma\Big(\delta+\frac{1}{\alpha}\Big)\tau\Big\},$$

which implies

$$v(t-\tau) \geq v(t)\exp\Big\{-\sigma\Big(\delta+\frac{1}{\alpha}\Big)\tau\Big\}.$$

Meanwhile, it follows from the second equation of (1.2) that

$$\begin{aligned} \frac{dv}{dt} &\leq \delta v\Big(n+\frac{1}{\alpha}-\frac{\exp\Big\{-\sigma\Big(\delta+\frac{1}{\alpha}\Big)\tau\Big\}}{\alpha\beta+1+\varepsilon}v\Big), \\ &= \frac{\delta\exp\Big\{-\sigma(\delta+\frac{1}{\alpha})\tau\Big\}}{\alpha\beta+1+\varepsilon}v\Big((\alpha\beta+1+\varepsilon)(n+\frac{1}{\alpha})\exp\Big\{\sigma(n+\frac{1}{\alpha})\tau\Big\}-v\Big). \end{aligned}$$

Using the comparison theorem, we obtain

$$\lim_{t\to+\infty}\sup v(t) \leq (\alpha\beta+1)(n+\frac{1}{\alpha})\exp\Big\{\sigma(n+\frac{1}{\alpha})\tau\Big\}.$$

Therefore, Theorem 2.2 is confirmed.

Theorem 2.3 In the absence of delay, if $s < \delta$, then there is no closed loop in the first quadrant of system (1.2).

Proof. Let $B(u,v) = \frac{1}{uv}$, $f_1 = u(1-u) - \frac{suv}{m+\alpha\beta+u}$, $f_2 = \delta v\Big(n + \frac{\beta - v}{\alpha\beta+u)}\Big)$, then

$$
\begin{aligned}
&\frac{\partial(Bf_1)}{\partial u} + \frac{\partial(Bf_2)}{\partial v} \\
&= -\frac{1}{v} + \frac{s}{(m+\alpha\beta+u)^2} - \frac{\delta}{u(\alpha\beta+u)}, \\
&\leq -\frac{1}{v} + \frac{s}{u(\alpha\beta+u)} - \frac{\delta}{u(\alpha\beta+u)}, \\
&< 0 \ \ (provided \ \ s < \delta).
\end{aligned}
$$

By using the Bendixson-Dulac criterion, the proof is completed.

3. Equilibrium, local stability and Hopf bifurcation

In the paper, we use the following representations for the sake of simplicity:

$$
\begin{aligned}
&A \equiv m + \alpha\beta + sn - 1, B \equiv sn\alpha\beta + s\beta - m - \alpha\beta, C \equiv \frac{\bar{u}}{m+\alpha\beta+\bar{u}}, \\
&D \equiv \frac{\delta\bar{v}}{\alpha\beta+\bar{u}}, \qquad G \equiv \bar{u} - C(1-\bar{u}), \qquad F \equiv G + snC,
\end{aligned}
$$

where $(\bar{u}, \bar{v})$ stands for the positive interior equilibrium and is defined in this section.

In order to obtain the equilibria, we discuss the algebraic equations:

$$
\begin{cases}
u(1-u) - \frac{suv}{m+\alpha\beta+u} = 0, \\
\delta v(n + \frac{\beta - v}{\alpha\beta+u}) = 0.
\end{cases}
\tag{3.1}
$$

Let $(\bar{u}, \bar{v})$ stands for the interior equilibrium, where $\bar{u}$ is the positive root of the equation $u^2 + Au + B = 0$. Thus, we have $\bar{u}_{\pm} = \frac{-A \pm \sqrt{A^2-4B}}{2}$ and $\bar{v} = n(\alpha\beta + \bar{u}) + \beta$. The equilibria are as follows:

(i) Trivial equilibrium $E_0 = (0,0)$.

(ii) Predator-extinction equilibrium $E_1 = (1,0)$.

(iii) Prey-extinction equilibrium $E_2 = (0, n\alpha\beta + \beta)$.

(iv) A unique coexisting equilibrium $E_+ = (\bar{u}_+, \bar{v}_+)$ when $B < 0$; two coexisting equilibria $E_{\pm} = (\bar{u}_{\pm}, \bar{v}_{\pm})$ when $B > 0,\ A < 0$ and $A^2 - 4B > 0$.

Let $E = (u^*, v^*)$ be arbitrary equilibrium. We use linearization technique to analyze the local stability of system (1.2). The Jacobian matrix of system (1.2) at $E = (u^*, v^*)$ is given by

$$
J(u^*, v^*) = \begin{pmatrix} 1 - 2u^* - \frac{sv(m+\alpha\beta)}{(m+\alpha\beta+u^*)^2} & -\frac{su}{m+\alpha\beta+u^*} \\ -\frac{\delta(\beta-v^*)v^*}{(\alpha\beta+u^*)^2} e^{-\lambda\tau} & \delta(n + \frac{\beta - v^*}{\alpha\beta+u^*}) - \frac{\delta v^*}{\alpha\beta+u^*} e^{-\lambda\tau} \end{pmatrix}.
$$

It is easy to confirm that $E_0 = (0,0)$ is an unstable node and $E_1 = (1,0)$ is a saddle.

At Prey-extinction equilibrium $E_2 = (0, n\alpha\beta + \beta)$, the corresponding Jacobian matrix is

$$J(E_2) = \begin{pmatrix} 1 - \frac{s(\alpha\beta n+\beta)}{m+\alpha\beta} & 0 \\ \frac{n\delta(\alpha\beta n)}{\alpha\beta} e^{-\lambda\tau} & -\delta(n + \frac{1}{\alpha}) e^{-\lambda\tau} \end{pmatrix}$$

and the characteristic equation becomes

$$(\lambda - P_1)(\lambda + P_2 e^{-\lambda\tau}) = 0, \tag{3.2}$$

where $P_1 = 1 - \frac{s(\alpha\beta n+\beta)}{\alpha\beta}, P_2 = \delta(n + \frac{1}{\alpha})$.

Case 1. $\tau = 0$.

It follows from (3.2) that $E_2 = (0, n\alpha\beta + \beta)$ is locally asymptotically stable when $P_1 < 0$ (equivalent to $B > 0$) and unstable when $P_1 > 0$.

Case 2. $\tau > 0$.

Let $\lambda = i\omega^*(\omega^* > 0)$ be a root of the equation $\lambda + P_2 e^{-\lambda\tau} = 0$, then $i\omega + P_2(\cos\omega\tau - i\sin\omega\tau) = 0$. By a direct calculation, we get $\omega_0^* = P_2$ and $\tau_k^* = \frac{1}{P_2}(2k\pi + \frac{\pi}{2}), k = 0, 1, 2, \cdots$. Differentiating the both sides of $\lambda + P_2 e^{-\lambda\tau} = 0$ with respect to τ, we have

$$\frac{d\lambda}{d\tau} + P_2\Big(-\tau\frac{d\lambda}{d\tau} - \lambda\Big)e^{-\lambda\tau} = 0,$$

that is

$$\Big(\frac{d\lambda}{d\tau}\Big)^{-1} = \frac{1}{\lambda P_2 e^{-\lambda\tau}} - \frac{\tau}{\lambda} = \frac{1}{\omega^2} - \frac{\tau}{\lambda},$$

which implies that

$$\operatorname{sgn}\Big\{\frac{d\mathrm{Re}\lambda}{d\tau}\Big\}_{\lambda=i\omega_0^*} = \operatorname{sgn}\Big\{\mathrm{Re}\Big(\frac{d\lambda}{d\tau}\Big)^{-1}\Big\}_{\lambda=i\omega_0^*} = \frac{1}{\omega_0^{*2}} > 0.$$

Lemma 3.1 If $B > 0$, then all roots of the characteristic equation (3.2) have negative real part when $0 \le \tau < \frac{\pi}{2P_2}$ and at least one positive real part when $\tau > \frac{\pi}{2P_2}$.

Therefore, we have the following conclusions for the boundary equilibria.

Theorem 3.2 (i) The trivial equilibrium $E_0 = (0, 0)$ and predator-extinction equilibrium $E_1 = (1, 0)$ are always unstable for all $\tau \ge 0$.
(ii) When $B > 0$, the prey-extinction equilibrium $E_2 = (0, n\alpha\beta + \beta)$ is asymptotically stable for all $0 \le \tau < \frac{\pi}{2P_2}$ and unstable for all $\tau > \frac{\pi}{2P_2}$. The system (1.2) undergoes a Hopf bifurcation at E_2 for $\tau = \frac{\pi}{2P_2}$.

Now we investigate the stability of the coexisting equilibrium $E = (\bar{u}, \bar{v})$. The corresponding Jacobian matrix is

$$J(E) = \begin{pmatrix} -\bar{u} + C(1 - \bar{u}) & -sC \\ nDe^{-\lambda\tau} & -De^{-\lambda\tau} \end{pmatrix}$$

and the corresponding characteristic equation becomes

$$\lambda^2 + G\lambda + D\lambda e^{-\lambda\tau} + DFe^{-\lambda\tau} = 0. \tag{3.3}$$

Case 1. $\tau = 0$.

Equation (3.3) becomes

$$\lambda^2 + (G + D)\lambda + DF = 0.$$

We calculate F as follows:

$$\begin{aligned} F &= \bar{u} - C(1-\bar{u}) + snC \\ &= C(m + \alpha\beta + sn + 2\bar{u} - 1) \\ &= C(A + 2\bar{u}), \end{aligned}$$

which implies that $F < 0$ when $\bar{u} = \bar{u}_-$ and $F > 0$ when $\bar{u} = \bar{u}_+$. Thus, the following conclusions are obvious.

Lemma 3.3 If the coexisting equilibria exist and $\tau = 0$, then $E_+ = (\bar{u}_+, \bar{v}_+)$ is locally asymptotically stable when $G + D > 0$ and $E_- = (\bar{u}_-, \bar{v}_-)$ is always unstable.

Case 2. $\tau > 0$.

Let $\lambda = i\omega(\omega > 0)$ be a root of the equation (3.3), then

$$-\omega^2 + i\omega G + i\omega D(\cos\omega\tau - i\sin\omega\tau) + DF(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{3.4}$$

We obtain

$$\begin{aligned} -\omega^2 + \omega D\sin\omega\tau + DF\cos\omega\tau = 0, \\ \omega G + \omega D\cos\omega\tau - DF\sin\omega\tau = 0, \end{aligned} \tag{3.5}$$

that is

$$\begin{aligned} \cos\omega\tau &= \frac{\omega^2(F-G)}{\omega^2 D + DF^2}, \\ \sin\omega\tau &= \frac{\omega^3 + \omega GF}{\omega^2 D + DF^2}. \end{aligned} \tag{3.6}$$

From (3.6), we have

$$\omega^4 + (G^2 - D^2)\omega^2 - D^2F^2 = 0. \tag{3.7}$$

Let $z = \omega^2$, (3.7) turns to

$$z^2 + (G^2 - D^2)z - D^2F^2 = 0. \tag{3.8}$$

Clearly, (3.8) has a unique positive root $z_0 = \frac{-(G^2-D^2)+\sqrt{(G^2-D^2)^2+4D^2F^2}}{2}$. So (3.8) has a pair of purely imaginary roots $\pm i\omega_0(\omega_0 = \sqrt{z_0})$. From the first equation of (3.6), we have

$$\tau_k = \frac{1}{\omega_0}\left(2k\pi + \arccos\frac{\omega^2(F-G)}{\omega^2 D + DF^2}\right), k = 0, 1, 2, 3, \cdots.$$

Differentiating both sides of (3.3) with respect to τ, we have

$$2\lambda\frac{d\lambda}{d\tau} + G\frac{d\lambda}{d\tau} + D\frac{d\lambda}{d\tau}e^{-\lambda\tau} + D\lambda e^{-\lambda\tau}\left(-\tau\frac{d\lambda}{d\tau} - \lambda\right) + DFe^{-\lambda\tau}\left(-\tau\frac{d\lambda}{d\tau} - \lambda\right) = 0,$$

namely,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + G)e^{\lambda\tau}}{D\lambda(\lambda + F)} + \frac{1}{\lambda(\lambda + F)} - \frac{\tau}{\lambda}.$$

Then

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0} = \frac{\omega_0 G\cos\omega_0\tau_0 - 2\omega_0\sin\omega_0\tau_0 - 2\omega_0 F\cos\omega_0\tau_0 + G\sin\omega_0\tau_0 + D\omega_0}{-\omega_0 D(\omega_0^2+F^2)}$$
$$= \frac{2\omega_0^2 + (G^2-D^2)}{D^2(\omega_0^2+F^2)}$$
$$= \frac{\sqrt{(G^2-D^2)^2+4D^2F^2}}{D^2(\omega_0^2+F^2)},$$

which implies that

$$\mathrm{sgn}\left\{\frac{dRe\lambda}{d\tau}\right\}_{\lambda=i\omega_0} = \mathrm{sgn}\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_0}\right\} > 0.$$

Lemma 3.4 If condition $G + D > 0$ holds, then all roots of the characteristic equation (3.3) at $E_+ = (\bar{u}_+, \bar{v}_+)$ have negative real part when $0 < \tau < \tau_0$ and at least one positive real part when $\tau > \tau_0$.

From lemma 3.3 and 3.4, we confirm the following conclusions about the interior equilibrium.

Theorem 3.5 (i) If $E_- = (\bar{u}_-, \bar{v}_-)$ exists, then it is unstable for all $\tau \geq 0$.

(ii) If $E_+ = (\bar{u}_+, \bar{v}_+)$ exists and the condition $G + D > 0$ holds, then E_+ is locally asymptotically stable for all $0 \leq \tau < \tau_0$ and unstable for all $\tau > \tau_0$. The system (1.2) undergoes a Hopf bifurcation at E_+ for $\tau = \tau_0$.

Remark 3.6 If the prey is pest, we just need to control the quantity of additional food satisfy the following conditions:

$$\beta > \frac{1-m-sn}{\alpha}(i.e.\ A > 0) \ \text{ and } \ \beta > \frac{m}{sn\alpha + s - \alpha}(i.e.\ B > 0),$$

then additional food can induce pest eradication. Meanwhile, the density of predators eventually goes to $n\alpha\beta + \beta$ when the maturation time of predator species is less than $\frac{\pi}{2P_2}$.

Remark 3.7 If the conditions $B > 0$, $A < 0$, $A^2 - 4B > 0$ and $G + D > 0$ hold simultaneously, the system (1.2) is bistable when $\tau < \min\{\tau_0^*, \tau_0\}$.

4. Direction and stability of the Hopf bifurcation

We know from the literature [35] that the properties of Hopf bifurcation are determined by the following three quantities, namely

$$\begin{cases} \mu_2 = -\frac{\mathrm{Re}(c_1(0))}{\lambda'(\tilde{\tau})}, \\ \beta_2 = 2\mathrm{Re}(c_1(0)), \\ T_2 = -\frac{\mathrm{Im}(c_1(0))+\mu_2\mathrm{Im}\lambda'(\tilde{\tau})}{\tilde{\omega}}, \end{cases} \tag{4.1}$$

where $c_1(0) = \frac{i}{2\tilde{\omega}\tilde{\tau}}\left[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2}$, $\tilde{\tau}$ is the critical value. μ_2, β_2 and T_2 determine direction, stability and the period of the bifurcating periodic solutions, respectively. So we need to figure out the value of g_{ij} in $c_1(0)$. We will use the normal form theory and the center manifold theorem to obtain the expression of g_{ij} in this section.

It has known that system (1.2) undergoes Hopf bifurcation at coexisting equilibrium E_+ when $\tau = \tau_0$. We denote the critical values τ_k and $E_+ = (\bar{u}_+, \bar{v}_+)$ as $\tilde{\tau}$ and $E^* = (u^*, v^*)$, respectively. Let $x(t) = u(\tau t) - u^*$ and $y(t) = v(\tau t) - v^*$, using Taylor expansion, (1.2) can be rewritten as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \tau A_1 \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \tau B_1 \begin{pmatrix} x(t-1) \\ y(t-1) \end{pmatrix} + F(x_t, y_t, \tau), \tag{4.2}$$

where

$$A_1 = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ a_3 & a_4 \end{pmatrix},$$

$$F(x_t, y_t, \tau) = \tau \begin{pmatrix} a_5 x^2(t) + a_6 x(t)y(t) + \cdots \\ a_7 x^2(t-1) + a_8 x(t-1)y(t) + a_9 x(t-1)y(t-1) + a_{10} y(t)y(t-1) + \cdots \end{pmatrix},$$

$a_1 = 1 - 2u^* - \frac{s(m+\alpha\beta)v^*}{(m+\alpha\beta+u^*)^2}, \quad a_2 = -\frac{su^*}{m+\alpha\beta+u^*}, \quad a_3 = -\frac{\delta v^*(\beta - v^*)}{(\alpha\beta+u^*)^2} = \frac{\delta n v^*}{\alpha\beta+u^*},$
$a_4 = -\frac{\delta v^*}{\alpha\beta+u^*}, \quad a_5 = -2 + \frac{2s(m+\alpha\beta)v^*}{(m+\alpha\beta+u^*)^3}, \quad a_6 = -\frac{2s(m+\alpha\beta)}{(m+\alpha\beta+u^*)^2},$
$a_7 = \frac{2\delta v^*(\beta - v^*)}{(\alpha\beta+u^*)^3}, \quad a_8 = -\frac{2\delta(\beta - v^*)}{(\alpha\beta+u^*)^2}, \quad a_9 = \frac{2\delta v^*}{(\alpha\beta+u^*)^2}, \quad a_{10} = -\frac{2\delta}{\alpha\beta+u^*}.$

Let $\tau = \bar{\tau} + h$, then $h = 0$ is a Hopf bifurcation value of the system (1.2). Choose the phase space $C = C([-1, 0], \mathbb{R}^2)$, $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C, \theta \in [-1, 0]$, define $L(h)$

$$L(h)\phi = (\tilde{\tau} + h)A_1\phi(0) + (\tilde{\tau} + h)B_1\phi(-1).$$

By the Riesz representation theorem, we choose the bounded variation function

$$\eta(h, \theta) = (\tilde{\tau} + h)A_1\delta(\theta) - (\tilde{\tau} + h)B_1\delta(\theta + 1)$$

such that

$$L(h)\phi = \int_{-1}^{0} d\eta(h, \theta)\phi(\theta),$$

where $\delta(\theta)$ is delta function.

For $\phi \in \mathbb{C}^1([-1, 0], \mathbb{R}^2)$, define

$$A(h)\phi(\theta) = \dot{\phi}(\theta) + T_0(\theta)[L(h)(\phi) - \dot{\phi}(0)],$$

and

$$R(h)\phi(\theta) = T_0(\theta)F(\phi, \tau + h),$$

where $T_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & \theta \in [-1, 0). \end{cases}$

Then (4.2) is written as

$$\dot{u}_t = A(h)u_t + R(h)u_t, \tag{4.3}$$

where $u_t = u(t + \theta)$ and $u_t = (x_t, y_t)^{\mathrm{T}}$.

For $\phi \in \mathbb{C}^1([-1, 0], \mathbb{C}^2)$ and $\psi \in \mathbb{C}^1([0, 1], (\mathbb{C}^2)^*)$, define a adjoint operator of $A(0)$

$$A^*\psi(s) = -\dot{\psi}(s) + T_0(-s)[\int_{-1}^{0} \mathrm{d}\eta(0, t)\psi(-t) + \dot{\psi}(0)]$$

and the bilinear form

$$\langle\psi,\phi\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta}\bar{\psi}(\xi-\theta)d\eta(\theta,0)\phi(\xi)d\xi.$$

From the previous discussion, we know $\pm i\tilde{\omega}\tilde{\tau}$ are the eigenvalues of $A(0)$ and A^*. Let $q(\theta) = (1,\alpha_1)^T e^{i\tilde{\omega}\tilde{\tau}\theta}$ be the eigenvector of $A(0)$ corresponding to $i\tilde{\omega}\tilde{\tau}$ and $q^*(s) = M(1,\alpha_2)e^{i\tilde{\omega}\tilde{\tau}s}$ be the eigenvector of A^* corresponding to $-i\tilde{\omega}\tilde{\tau}$, and they satisfy the conditions $\langle q^*,q\rangle = 1$ and $\langle q^*,\bar{q}\rangle = 0$. Therefore, we have

$$\tilde{\tau}\begin{pmatrix} a_1 & a_2 \\ a_3e^{-i\tilde{\omega}\tilde{\tau}} & a_4e^{-i\tilde{\omega}\tilde{\tau}}\end{pmatrix}\begin{pmatrix}1\\ \alpha_1\end{pmatrix} = i\tilde{\omega}\tilde{\tau}\begin{pmatrix}1\\ \alpha_1\end{pmatrix}$$

and

$$\tilde{\tau}M(1\quad \alpha_2)\begin{pmatrix} a_1 & a_2 \\ a_3e^{i\tilde{\omega}\tilde{\tau}} & a_4e^{i\tilde{\omega}\tilde{\tau}}\end{pmatrix} = -i\tilde{\omega}\tilde{\tau}M(1\quad \alpha_2),$$

then $\alpha_1 = \frac{i\tilde{\omega}-a_1}{a_2}$ and $\alpha_2 = \frac{-i\tilde{\omega}-a_1}{a_3e^{i\tilde{\omega}\tilde{\tau}}}$.
By the bilinear form, we have

$$\begin{aligned}\langle q^*(s),q(\theta)\rangle &= \bar{M}(1,\bar{\alpha}_2)(1,\alpha_1)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{M}(1,\bar{\alpha}_2)e^{-i\tilde{\omega}\tilde{\tau}(\xi-\theta)}d\eta(\theta,0)(1,\alpha_1)^Te^{i\tilde{\omega}\tilde{\tau}\xi}d\xi,\\ &= \bar{M}(1+\alpha_1\bar{\alpha}_2) - \int_{-1}^{0}\bar{M}(1,\bar{\alpha}_2)\theta e^{i\tilde{\omega}\tilde{\tau}\theta}d\eta(\theta,0)(1,\alpha_1)^T,\\ &= \bar{M}((1+\alpha_1\bar{\alpha}_2) + \tilde{\tau}\bar{\alpha}_2(a_3+a_4\alpha_1)e^{-i\tilde{\omega}\tilde{\tau}}).\end{aligned} \tag{4.4}$$

Thus $M = \frac{1}{(1+\bar{\alpha}_1\alpha_2)+\tilde{\tau}\alpha_2(a_3+a_4\bar{\alpha}_1)e^{i\tilde{\omega}\tilde{\tau}}}$.

We need the coordinates to describe the center manifold C_0 near $h = 0$. Let z and $\bar{z}$ be local coordinates for C_0 in the directions of q^* and $\bar{q}^*$. Assume that u_t is a solution of (4.3) at $h = 0$, define

$$z(t) = \langle q^*,u_t\rangle,\qquad W(t,\theta) = u_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta).$$

On C_0, we have $W(t,\theta) = W(z(t),\bar{z}(t),\theta)$, where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\tfrac{z^2}{2} + W_{11}z\bar{z} + W_{02}(\theta)\tfrac{\bar{z}^2}{2} + W_{30}(\theta)\tfrac{z^3}{2} + \cdots. \tag{4.5}$$

The manifold of (4.2) on the center manifold is determined by the following equation

$$\dot{z}(t) = i\tilde{\tau}\tilde{\omega}z(t) + \bar{q}^*(0)F(zq(0)+\bar{z}\bar{q}(0)+W(z,\bar{z},0)) \triangleq i\tilde{\omega}\tilde{\tau}z(t) + \bar{q}^*(0)F_0$$

which is abbreviated to

$$\dot{z}(t) = i\tilde{\tau}\tilde{\omega}z(t) + g(z,\bar{z}), \tag{4.6}$$

where the power series form of $g(z,\bar{z})$ is

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots \tag{4.7}$$

and

$$F_0 = \tilde{\tau}\begin{pmatrix} a_5x^2(t) + a_6x(t)y(t) \\ a_7x^2(t-1) + a_8x(t-1)y(t) + a_9x(t-1)y(t-1) + a_{10}y(t)y(t-1) \end{pmatrix}.$$

By a direct calculation, we have

$$\begin{aligned} x(t) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + W_{30}^{(1)}(0)\frac{z^3}{2} + \cdots, \\ y(t) &= \alpha_1 z + \bar{\alpha}_1\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}, \\ x(t-1) &= e^{-i\tilde{\omega}\tilde{\tau}}z + e^{i\tilde{\omega}\tilde{\tau}}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + W_{30}^{(1)}(-1)\frac{z^3}{2} + \cdots, \\ y(t-1) &= \alpha_1 e^{-i\tilde{\omega}\tilde{\tau}}z + \bar{\alpha}_1 e^{i\tilde{\omega}\tilde{\tau}}\bar{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + W_{30}^{(2)}(-1)\frac{z^3}{2} + \cdots. \end{aligned} \tag{4.8}$$

Substituting (4.8) into F_0 and comparing with (4.7), we obtain

$$\begin{aligned} g_{20} &= 2\tilde{\tau}\bar{M}(a_5 + a_6\alpha_1 + \bar{\alpha}_2(a_7e^{-2i\tilde{\omega}\tilde{\tau}} + a_8\alpha_1e^{-i\tilde{\omega}\tilde{\tau}} + a_9\alpha_1e^{-2i\tilde{\omega}\tilde{\tau}} + a_{10}\alpha_1^2e^{-i\tilde{\omega}\tilde{\tau}})), \\ g_{02} &= 2\tilde{\tau}\bar{M}(a_5 + a_6\bar{\alpha}_1 + \bar{\alpha}_2(a_7e^{2i\tilde{\omega}\tilde{\tau}} + a_8\bar{\alpha}_1e^{i\tilde{\omega}\tilde{\tau}} + a_9\bar{\alpha}_1e^{2i\tilde{\omega}\tilde{\tau}} + a_{10}\bar{\alpha}_1^2e^{i\tilde{\omega}\tilde{\tau}})), \\ g_{11} &= 2\tilde{\tau}\bar{M}(a_5 + a_6Re\{\alpha_1\} + \bar{\alpha}_2(a_7 + a_8Re\{\alpha_1e^{i\tilde{\omega}\tilde{\tau}}\} + a_9Re\{\alpha_1\} + a_{10}Re\{\alpha_1\bar{\alpha}_1e^{i\tilde{\omega}\tilde{\tau}}\})), \\ g_{21} &= 2\tilde{\tau}\bar{M}(a_5k_5 + a_6k_6 + \bar{\alpha}_2(a_7k_7 + a_8k_8 + a_9k_9 + a_{10}k_{10})), \end{aligned}$$

where

$$\begin{aligned} k_5 &= W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0), \\ k_6 &= \tfrac{1}{2}\bar{\alpha}_1W_{20}^{(1)}(0) + \alpha_1W_{11}^{(1)}(0) + \tfrac{1}{2}W_{20}^{(2)}(0) + W_{11}^{(2)}(0), \\ k_7 &= e^{i\tilde{\omega}\tilde{\tau}}W_{20}^{(1)}(-1) + 2e^{-i\tilde{\omega}\tilde{\tau}}W_{11}^{(1)}(-1), \\ k_8 &= \tfrac{1}{2}\bar{\alpha}_1W_{20}^{(1)}(-1) + \alpha_1W_{11}^{(1)}(-1) + \tfrac{1}{2}e^{i\tilde{\omega}\tilde{\tau}}W_{20}^{(2)}(0) + e^{-i\tilde{\omega}\tilde{\tau}}W_{11}^{(2)}(0), \\ k_9 &= \tfrac{1}{2}e^{i\tilde{\omega}\tilde{\tau}}W_{20}^{(2)}(-1) + e^{-i\tilde{\omega}\tilde{\tau}}W_{11}^{(2)}(-1) + \tfrac{1}{2}\bar{\alpha}_1e^{i\tilde{\omega}\tilde{\tau}}W_{20}^{(1)}(-1) + \alpha_1e^{-i\tilde{\omega}\tilde{\tau}}W_{11}^{(1)}(-1), \\ k_{10} &= \tfrac{1}{2}\bar{\alpha}_1W_{20}^{(2)}(-1) + \alpha_1W_{11}^{(2)}(-1) + \tfrac{1}{2}\bar{\alpha}_1e^{i\tilde{\omega}\tilde{\tau}}W_{20}^{(2)}(0) + \alpha_1e^{-i\tilde{\omega}\tilde{\tau}}W_{11}^{(2)}(0). \end{aligned}$$

In order to obtain the normal form of (4.6) confined to the center manifold, we need compute $W_{20}(\theta)$ and $W_{11}(\theta)$.

Using $\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q}$ which combines with (4.3) and (4.6), we obtain

$$\dot{W} = \begin{cases} AW - gq(\theta) - \bar{g}\bar{q}(\theta), & \theta \in [-1, 0), \\ AW - gq(\theta) - \bar{g}\bar{q}(\theta) + F_0, & \theta = 0. \end{cases} \tag{4.9}$$

On the other hand, from (4.5) and (4.6), we have

$$\begin{aligned} \dot{W} &= W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}} \\ &= [W_{20}(\theta)z + W_{11}(\theta)\bar{z}](i\tilde{\tau}\tilde{\omega}z(t) + g(z,\bar{z})) + [W_{11}(\theta)z + W_{02}(\theta)\bar{z}](i\tilde{\tau}\tilde{\omega}\bar{z}(t) + \bar{g}(z,\bar{z})) + \cdots. \end{aligned} \tag{4.10}$$

We substitute (4.6) into (4.9) and compare the coefficients of $\frac{z^2}{2}$ and $z\bar{z}$ with (4.10), respectively. It gives

$$(2\tilde{\omega}\tilde{\tau}I - A)W_{20}(\theta) = \begin{cases} -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), & \theta \in [-1, 0), \\ -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) + F_{z^2}, & \theta = 0. \end{cases} \tag{4.11}$$

and

$$-AW_{11}(\theta) = \begin{cases} -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), & \theta \in [-1, 0), \\ -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) + F_{z\bar{z}}, & \theta = 0. \end{cases} \tag{4.12}$$

According to (4.11) and (4.12), by a direct calculation for $\theta \in [-1, 0)$, we obatin

$$W_{20}(\theta) = \frac{ig_{20}}{\tilde{\omega}\tilde{\tau}} q(\theta) + \frac{i\bar{g}_{02}}{3\tilde{\omega}\tilde{\tau}} \bar{q}(\theta) + E_1 e^{2i\omega_0\theta}$$

and

$$W_{11}(\theta) = -\frac{ig_{11}}{\tilde{\omega}\tilde{\tau}} q(\theta) + \frac{i\bar{g}_{11}}{\tilde{\omega}\tilde{\tau}} \bar{q}(\theta) + E_2,$$

where E_1 and E_2 hold the following equations

$$\begin{pmatrix} 2i\tilde{\omega}\tilde{\tau} - a_1 & -a_2 \\ -a_3 e^{-2i\tilde{\omega}\tilde{\tau}} & 2i\tilde{\omega}\tilde{\tau} - a_4 e^{-2i\tilde{\omega}\tilde{\tau}} \end{pmatrix} E_1 = 2 \begin{pmatrix} a_5 + a_6\alpha_1 \\ a_7 e^{-2i\tilde{\omega}\tilde{\tau}} + a_8\alpha_1 e^{-i\tilde{\omega}\tilde{\tau}} + a_9\alpha_1 e^{-2i\tilde{\omega}\tilde{\tau}} + a_{10}\alpha_1^2 e^{-2i\tilde{\omega}\tilde{\tau}} \end{pmatrix},$$

$$\begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix} E_2 = 2 \begin{pmatrix} a_5 + a_6 \mathrm{Re}\{\alpha_1\} \\ a_7 + a_8 \mathrm{Re}\{\alpha_1 e^{i\tilde{\omega}\tilde{\tau}}\} + a_9 \mathrm{Re}\{\alpha_1\} + a_{10}\alpha_1\bar{\alpha}_1 \mathrm{Re}\{e^{i\tilde{\omega}\tilde{\tau}}\} \end{pmatrix}.$$

From the above discussions, we have the following conclusions on the center manifold.

Theorem 4.1 (i) If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical).
(ii) If $\beta_2 < 0$ ($\beta_2 > 0$), the bifurcating periodic solution is stable (unstable).
(iii) If $T_2 > 0$ ($T_2 < 0$), the period increases (decreases).

5. Global existence of periodic solutions

Next, we investigate the global continuation of bifurcating periodic solutions from the positive equilibrium (E_+, τ_k). We employ the global Hopf bifurcation theorem and follow closely the notations in Wu [28].

Denote the conditions for the existence of E_+ as

(H) Either $B < 0$ or $B > 0, A < 0$ and $A^2 - 4B > 0$.

Let $\mathbb{R}_+ = \{(u, v) \in \mathbb{R}^2, u > 0, v > 0\}$, $X = C([-\tau, 0], \mathbb{R}^2)$, $z_t = (u_t, v_t)$, the system (1.2) is rewritten as

$$\dot{z}(t) = F(z_t, \tau, p), \tag{5.1}$$

where $z_t(\theta) = z(t + \theta) \in X$ and $(\tau, p) \in \mathbb{R}_+ \times \mathbb{R}_+$. Clearly, the mapping $F : X \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}^2$ is completely continuous. If we take $\mathbb{R}^2$ for the subspace of constant functions of X, we obtain $\hat{F} \mid_{\mathbb{R}^2\times\mathbb{R}_+\times\mathbb{R}_+}: \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}^2$. It is easy to know (E_0, τ, p), (E_1, τ, p), (E_2, τ, p), (E_-, τ, p) and (E_+, τ, p) are all stationary solutions of (5.1). Now, we verify that (E_+, τ, p) holds the conditions (A_1), (A_2), (A_3) and (A_4) in [28].

From (1.2), We know easily that $\hat{F} \in C^2(\mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+^2)$ and $F(\phi, \tau, p)$ is differential with respect to ϕ. That is to say, the conditions (A_1) and (A_3) are satisfied. We also obtain

$$D\hat{F}(E_+, \tau, p) = \begin{pmatrix} -\bar{u}_+ + C(1 - \bar{u}_+) & -sC \\ nD & -D \end{pmatrix}.$$

Directly calculate, we have

$$\mathrm{Det} D_z \hat{F}(E_+, \tau, p) = -D(\bar{u}_+ - C(1 - \bar{u}_+) + snD) = -DF < 0.$$

Then $D_z\hat{F}(E_+, \tau, p)$ is a homeomorphism on $\mathbb{R}^2$ at E_+, which satisfies the condition (A_2).

The characteristic matrix of (5.1) at (E_+, τ, p) is taken as:

$$\Delta_{(E_+,\tau,p)}(\lambda) = \lambda Id - DF(E_+, \tau, p)(e^{\lambda \cdot} Id). \tag{5.2}$$

A stationary solution $(E_+, \tau, p)(\lambda)$ of (5.1) is called a center if $\det\Delta_{(E_+,\tau,p)}(\lambda) = 0$ has purely imaginary characteristic roots of the form $\mathrm{i}m\frac{2\pi}{p_0}$ for some positive integer m. It follows from (5.2) that

$$\det\Delta_{(E_+,\tau,p)}(\lambda) = \lambda^2 + E\lambda + D\lambda e^{-\lambda\tau} + DFe^{-\lambda\tau} = 0, \tag{5.3}$$

which is the same as (3.3). Taking $p_0 = \frac{2\pi}{\omega_0}$, we know $\mathrm{i}\frac{2\pi}{p_0}$ is a root of (5.3), namely, $\mathrm{i}\frac{2\pi}{p_0}$ is an eigenvalue of $(E_+, \tau_k, \frac{2\pi}{\omega_0})$. Thus, $(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is a center, where τ_k and ω_0 are defined in Section 3. Furthermore, the center $(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is an isolated center, because it satisfies the following two conclusions:
(i) $J(E_+, \tau_k, \frac{2\pi}{\omega_0}) = 1$, where $J(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is a positive integer set with respect to m such that $\mathrm{i}m\frac{2\pi}{p_0}$ are eigenvalues of $(E_+, \tau_k, \frac{2\pi}{\omega_0})$.
(ii) For arbitrary $k \geq 0$, there exist $\varepsilon_k > 0, \delta_k > 0$ and a smooth function $\lambda : (\tau_k - \sigma_k, \tau_k + \sigma_k \to C)$(C is complex field) such that $\det\Delta_{(E_+,\tau_k,\frac{2\pi}{\omega_0})}(\lambda(\tau)) = 0$, $|\lambda(\tau) - \mathrm{i}\omega_0| < \varepsilon_k$ for arbitrary $\tau \in (\tau_k - \sigma_k, \tau_k + \sigma_k)$ and $\lambda(\tau_k) = \mathrm{i}\omega_0$, $\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}|_{\tau=\tau_k} > 0$. That is, $(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is the only center in certain neighborhood of $(E_+, \tau_k, \frac{2\pi}{\omega_0})$.

Let

$$\Omega_{\varepsilon_k,p_0} = \{(r, p)|0 < r < \varepsilon_k, p_0 - \varepsilon_k < p < p_0 + \varepsilon_k\}.$$

Clearly, for $\tau \in (\tau_k - \sigma_k, \tau_k + \sigma_k)$ and $(r, P) \in \partial\Omega_{\varepsilon_k,p_0}$ such that $\det\Delta_{(E_+,\tau,p)}\Big(r + \mathrm{i}\frac{2\pi}{p}\Big) = 0$ if and only if $\tau = \tau_k, p = p_0, r = 0$. This is the condition (A_4).

So far, we have verified the conditions $(A_1) - (A_4)$ in [28]. Define

$$H^{\pm}_m(E_+, \tau_k, \frac{2\pi}{\omega_0})(r, p) = \det \triangle_(E_+, \tau_k \pm \delta_k, p)\Big(r + \mathrm{i}m\frac{2\pi}{p}\Big).$$

The condition (A_4) and $J(E_+, \tau_k, \frac{2\pi}{\omega_0}) = 1$ imply $H^{\pm}_1(E_+, \tau_k, \frac{2\pi}{\omega_0})(r, p) \neq 0$ for $(r, p) \in \partial\Omega_{\varepsilon_k,p_0}$. Therefore, the first crossing number $\gamma_1(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is calculated as

$$\gamma_1(E_+, \tau_k, \frac{2\pi}{\omega_0}) = \deg(H^-_1)(E_+, \tau_k, \frac{2\pi}{\omega_0})(r, p), \Omega_{\varepsilon_k,p_0}) - \deg(H^+_1)(E_+, \tau_k, \frac{2\pi}{\omega_0})(r, p), \Omega_{\varepsilon_k,p_0}) = -1. \tag{5.4}$$

By the similar arguments, we can know (E_0, τ, p) and (E_1, τ, p) are not the centers, but $(E_2, \tau^*_k, \frac{2\pi}{\omega^*_0})$ and $(E_-, \tau_k, \frac{2\pi}{\omega_0})$ are isolated centers. we may also obtain $\gamma_1(E_2, \tau^*_k, \frac{2\pi}{\omega^*_0}) = -1$ and $\gamma_1(E_-, \tau_k, \frac{2\pi}{\omega_0}) = -1$.

In what follows we define

$\Sigma = Cl\{(z, \tau, p)|z\ is\ p-periodic\ solution\ of\ (5.1)\}$,

$N = \{(\bar{z}, \tau, p)|F(\bar{z}, \tau, p) = 0\}$.

Let $C\Big(E_+, \tau_k, \frac{2\pi}{\omega_0}\Big)$ denote the connected component through $(E_+, \tau_k, \frac{2\pi}{\omega_0})$ in Σ.

By Theorem 3.2 in [28], there exists an isolated center $(E_+, \tau_k, \frac{2\pi}{\omega_0})$ which satisfies $J(E_+, \tau_k, \frac{2\pi}{\omega_0}) = 1$ and $\gamma_1(E_+, \tau_k, \frac{2\pi}{\omega_0}) \neq 0$ such that $C\Big(E_+, \tau_k, \frac{2\pi}{\omega_0}\Big)$ through $(E_+, \tau_k, \frac{2\pi}{\omega_0})$ in Σ is nonempty. In addition, all the centers of (5.1) are isolated centers and satisfy

$$\sum_{(\bar{z},\tau,p)\in(E_+,\tau_k,\frac{2\pi}{\omega_0})\cap N(F)} \gamma_m(\bar{z}, \tau, p) < 0.$$

By Theorem 3.3 in [28], we obtain the following Lemma.
Lemma 5.1 $C\Big(E_+, \tau_k, \frac{2\pi}{\omega_0}\Big)$ is unbounded.
On the other hand, from Theorem 2.2 in Section 2, it is easy to obtain the following lemma.
Lemma 5.2 If τ is bounded, then any nontrivial periodic solution of system (1.2) is uniformly bounded.
Lemma 5.3 When the conditions (H) and $s < \delta$ hold, (1.2) has no any nontrivial $\tau-$periodic solutions.
Proof. If $(u^*(t), v^*(t))$ is a nontrivial $\tau-$periodic solution of (1.2), then it is also a nontrivial periodic solution of the following (5.5).

$$\begin{cases} \dfrac{du}{dt} &= u(1-u) - \frac{suv}{m+\alpha\beta+u}, \\ \dfrac{dv}{dt} &= \delta v\Big(n + \frac{\beta - v}{\alpha\beta+u}\Big). \end{cases} \tag{5.5}$$

When the condition (H) hold, (5.5) has at most five equilibria, namely, $E_0 = (0,0)$, $E_1 = (1,0)$, $E_2 = (0, n\alpha\beta + \beta)$ and $E_\pm = (\bar{u}_\pm, \bar{v}_\pm)$. Note that E_0, E_1, and E_2 are located in u-axis and v-axis, they can not produce any nontrivial periodic solution. On the other hand, $E_\pm$ also can not produce any nontrivial periodic solution due to $s < \delta$. Thus, there is no any nontrivial periodic solution in (5.5). The proof is complete.
Theorem 5.4 Suppose the conditions (H), $G+D > 0$ and $s < \delta$ hold, then for each $\tau > \tau_k, k = 1, 2, 3\cdots$, system (1.2) has at least $k-1$ periodic solutions.
Proof. Let $\mathrm{Proj}_\Theta C(E_+, \tau_k, \frac{2\pi}{\omega_0})$ be the projection of $C(E_+, \tau_k, \frac{2\pi}{\omega_0})$ onto Θ-space. From Lemma 5.2, it is easy to know that $\mathrm{Proj}_z C(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is bounded. It follows from the proof of Lemma 5.3 that (1.2) with $\tau = 0$ has no nontrivial periodic solution. Thus, $\mathrm{Proj}_\tau C(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is away from zero.

We suppose that $\mathrm{Proj}_\tau C(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is bounded. Then there exist τ^* such that $\mathrm{Proj}_\tau C(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is a subset of interval $(0, \tau^*)$. From the definition of τ_k and ω_0 in section 3, we have $\frac{2\pi}{\omega_0} < \tau_k, k = 1, 2, 3\cdots$. Applying Lemma 5.3, we have $p \in (0, \tau^*)$ when $(z, \tau, p) \in C(E_+, \tau_k, \frac{2\pi}{\omega_0})$.

The above discussion shows that $C(E_+, \tau_k, \frac{2\pi}{\omega_0})$ is bounded, which contradicts Lemma 5.1. Therefore, $\mathrm{Proj}_\tau C(E_+, \tau_k, \frac{2\pi}{\omega_0})$ contains at least an interval $(\tau_k, +\infty)$. The proof is complete.

6. Numerical simulations

Based on the previous discussion, three numerical results of system (1.2) are presented.
Case 1. We consider system (1.2) with $s = 0.8$, $m = 0.2$, $\alpha = 1, \beta = 0.6$, $\delta = 0.4$, $n = 0.8$, that is

$$\begin{cases} \dfrac{du}{dt} &= u(t)(1-u(t)) - \frac{0.8u(t)v(t)}{0.2+0.6+u(t)}, \\ \dfrac{dv}{dt} &= 0.4v(t)\Big(0.8 + \frac{0.6 - v(t-\tau)}{0.6+u(t-\tau)}\Big). \end{cases} \tag{6.1}$$

We have $A = 0.4400$, $B = 0.0640$ and $\sqrt{A^2 - 4B} = -0.624$. It implies system (6.1) has no positive equilibrium and satisfies the condition (ii) in Theorem 3.2. Therefore, the prey-extinction equilibrium $E_2 = (0, 1.08)$ is stable when $\tau < \tau_0^*$, where $\tau_0^* = 2.1817$. When τ pass through the critical value τ_0^*, E_2 loses its stability(see Figure 1).

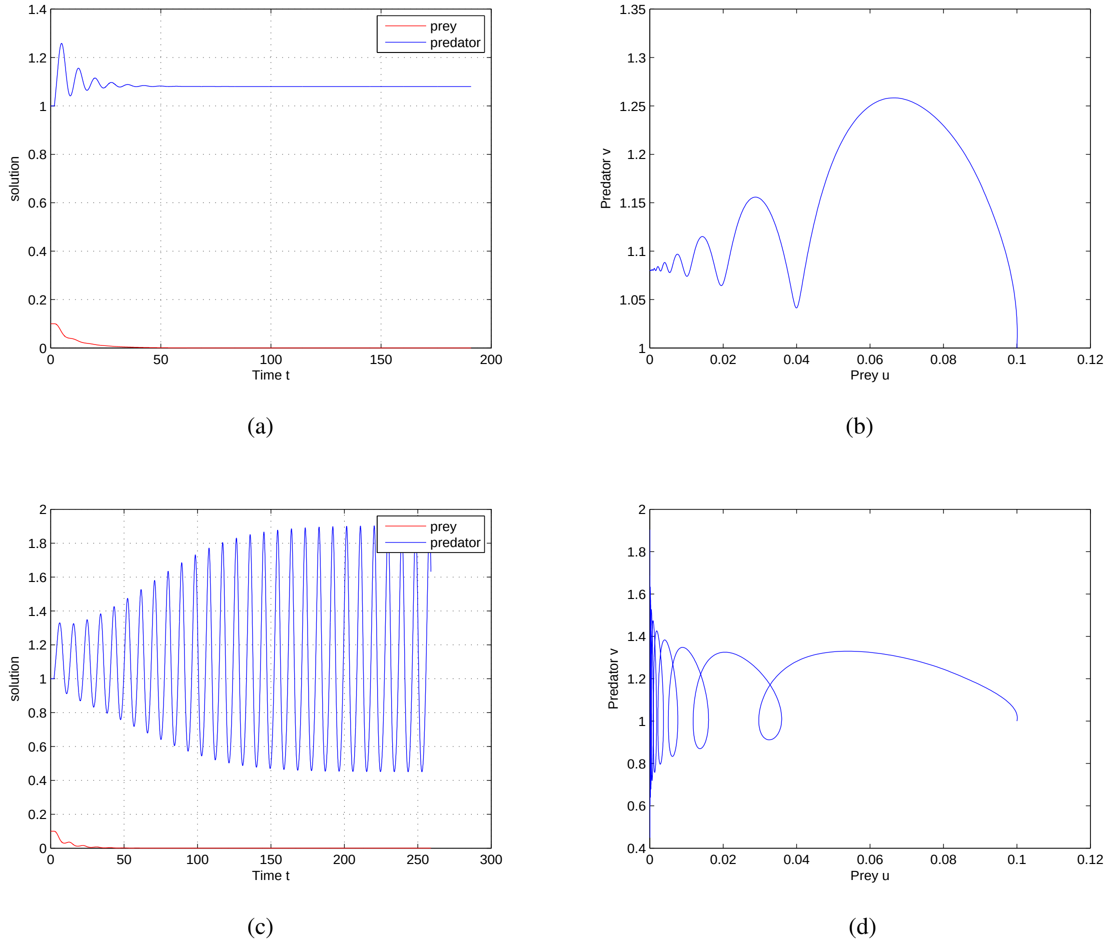


Figure 1. (a)&(b) When $\tau = 1.6817 < \tau_0^*$, the prey-extinction equilibrium $E_2 = (0, 1.08)$ is stable; (c)&(d) When $\tau = 2.2817 > \tau_0^*$, the prey-extinction equilibrium $E_2 = (0, 1.08)$ is unstable.

Case 2. we discuss the following system

$$\begin{cases} \dfrac{du}{dt} &= u(t)(1 - u(t)) - \frac{0.2u(t)v(t)}{0.1+0.6+u(t)}, \\ \dfrac{dv}{dt} &= 0.3v(t)\Big(0.4 + \frac{0.6-v(t-\tau)}{0.6+u(t-\tau)}\Big), \end{cases} \tag{6.2}$$

where $s = 0.2$, $m = 0.1$, $\alpha = 1$, $\beta = 0.6$, $\delta = 0.3$, $n = 0.4$. Calculate directly, we have $B = -0.5320$, $G = 0.7641$, $D = 0.2444$, then there is a unique coexisting equilibrium $E_+ = (0.8476, 1.1791)$. From the condition (ii) in Theorem 3.5, there exists the critical value $\tau_0 = 6.0476$ such that E_+ is stable when $\tau < \tau_0$ and unstable when $\tau > \tau_0$(see Figure 2).

On the other hand, we obtain the following values by using Matlab

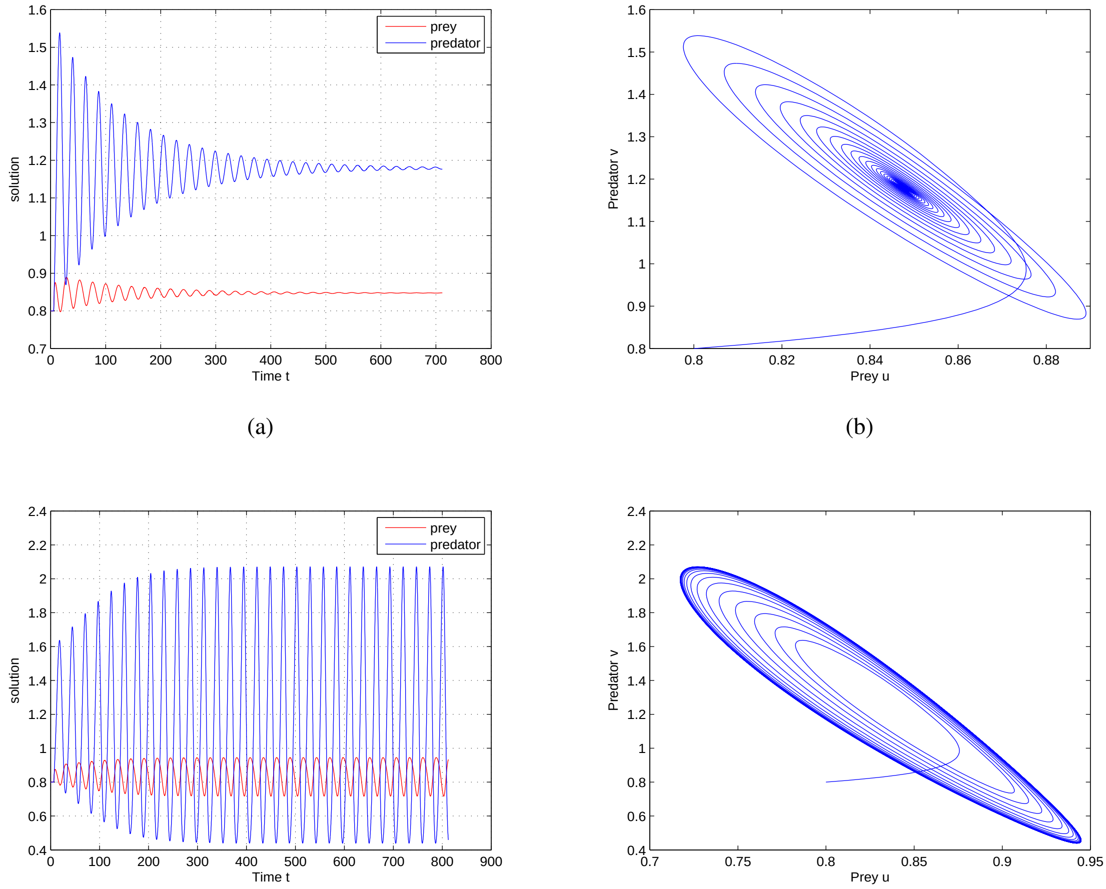
$\alpha_1 = -6.9759 - 2.3462i$, $\alpha_2 = 2.7585 + 7.7731i$, $M = -0.0090 - 0.0008i$,

$g_{20} = -14.4293 - 14.6239i$, $g_{02} = 19.8351 - 5.9477i$, $g_{11} = 0.6641 - 2.1798i$,

$$g_{21} = -41.2805 + 50.5284i, \quad c_1(0) = -27.6348 - 37.3972i, \quad \lambda'(\tau_0) = 0.0192 - 0.0298i.$$

It follows that $\mu_2 = 1440.4 > 0$ and $\beta_2 = -55.2696 < 0$ and $T_2 = 312.4076$, which, together with Theorem 4.1, implies that the bifurcating periodic solution exists when $\tau > \tau_0$ and the bifurcating periodic solution is stable on the center manifold and the period increases.

When $\tau = \tau_0$, all roots of characteristic equation (3.3) have negative real parts except $\pm i\omega_0$. Since the periodic solution on the center manifold is stabile, the periodic solution in the whole phase space is stable(see Figure 2 (c) and (d)).



(a) (b)

(c) (d)

Figure 2. (a)&(b) When $\tau = 5.6475 < \tau_0$, $E_+ = (0.8476, 1.1791)$ is stable; (c)&(d) When $\tau = 6.4475 > \tau_0$, $E_+ = (0.8476, 1.1791)$ is unstable and a stable bifurcating periodic solution occurs.

Case 3. the simulation of the system (1.2) with $s = 0.6$, $m = 0.2$, $\alpha = 0.3$, $\beta = 0.5$, $\delta = 0.65$, $n = 0.6$ is given by

$$\begin{cases} \dfrac{du}{dt} &= u(t)(1 - u(t)) - \frac{0.6u(t)v(t)}{0.2+0.15+u(t)}, \\ \dfrac{dv}{dt} &= 0.65v(t)\Big(0.6 + \frac{0.5-v(t-\tau)}{0.15+u(t-\tau)}\Big), \end{cases} \tag{6.3}$$

which has two positive equilibriums $E_- = (0.0145, 0.5987)$ and $E_+ = (0.2755, 0.7553)$ due to $A = -0.2900$, $B = 0.0040$ and $\sqrt{A^2 - 4B} = 0.2610$. By the condition (i) and (ii) in Theorem 3.5, we can know that $E_- = (0.0145, 0.5987)$ is always unstable for any $\tau > 0$ and there exists the critical value $\tau_0 = 1.2379$ such that E_+ is stable for any $\tau \in [0, \tau_0)$. When τ crosses τ_0, E_+ is unstable and a Hop bifurcation occurs. We obtain $c_1(0) = -23.7642 + 34.8292i$ and $\lambda'(\tau_0) = 0.5009 - 0.6450i$ by using Matlab, then $\mu_2 = 47.4457 > 0$ and $\beta_2 = -47.5284 < 0$ and $T_2 = -3.6496$, which implies that the bifurcating periodic solution is stable on the center manifold and the period increases. As discussed in system (6.2), for $\tau > \tau_0$, the periodic solution of system (6.3) in the whole phase space is stable. The corresponding simulation results are shown in Figure 3.

In addition, system (6.3) also has a prey-extinction equilibrium $E_2 = (0, 0.59)$ corresponding to the critical value $\tau_0^* = 0.6144$. Thus, $E_2 = (0, 0.59)$ and $E_+ = (0.2755, 0.7553)$ are the stability equilibria of system (6.3) when $\tau < 0.6144$, which is depicted in Figure 4.

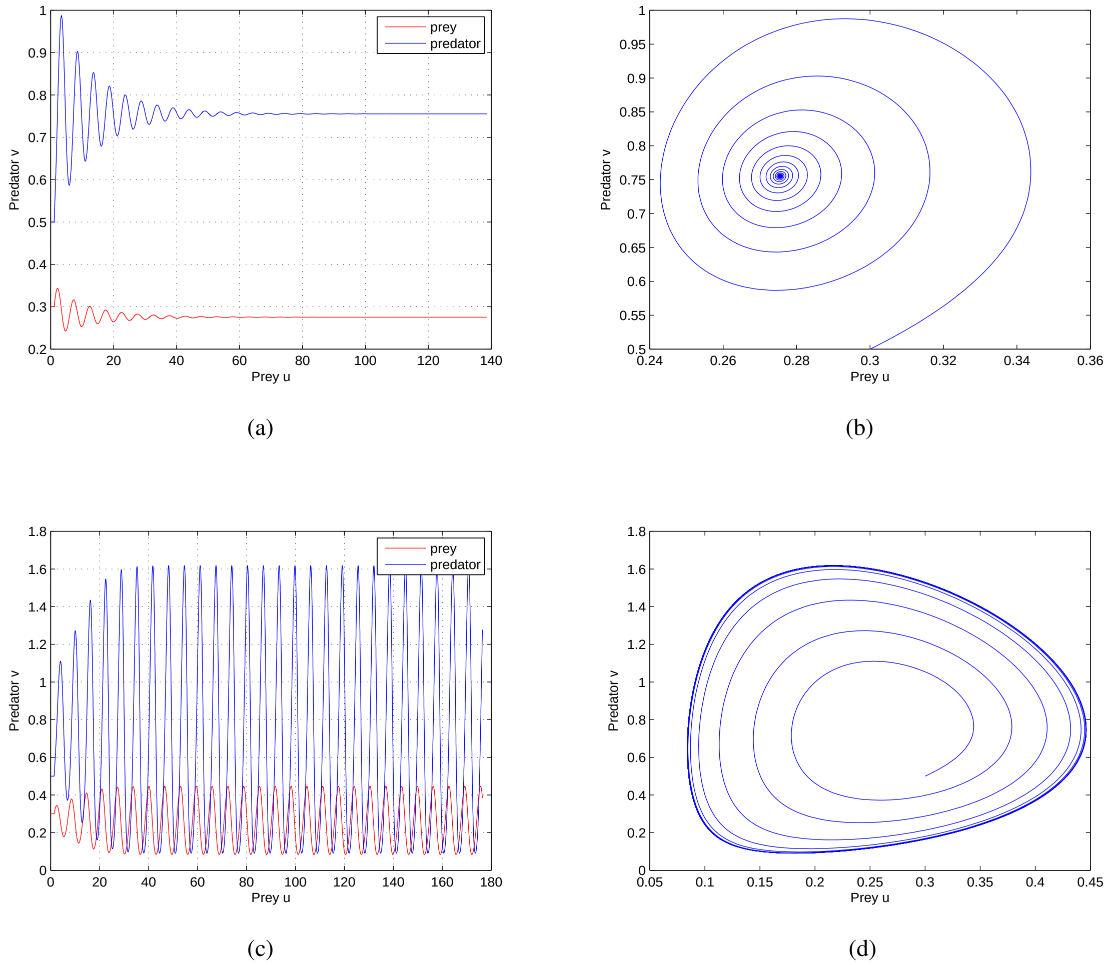
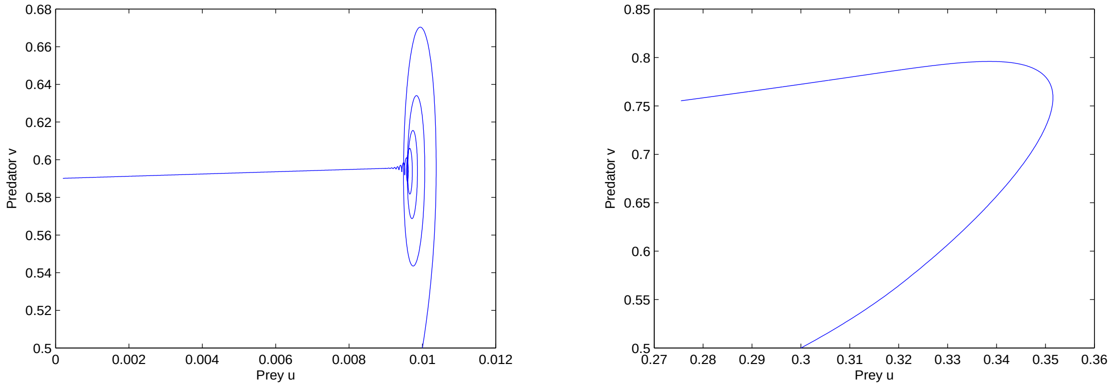


Figure 3. (a)&(b) When $\tau = 1.1 < \tau_0$, $E_+ = (0.2755, 0.7553)$ is stable;, (c)&(d) When $\tau = 1.4 > \tau_0$, $E_+ = (0.2755, 0.7553)$ is unstable and a stable bifurcating periodic solution occurs.



(a) (b)

Figure 4. When $\tau = 0.5144 < 0.6144$, (a) $E_2 = (0, 0.59)$ is stable with initial date $(0.01, 0.5)$ and (b) $E_+ = (0.2755, 0.7553)$ is also stable with initial date $(0.3, 0.5)$.

7. Conclusions

In the paper, we investigate a delayed predator-prey system with additional food and asymmetric functional response. The local stability of all possible equilibria are studied. It shows that we can exterminate the prey by adjusting the quality and quantity of additional food when the prey density and time delay are relatively small. We know that the number of positive equilibria is determined by the value of B. For $B < 0$, there is only one conditionally stable or unstable coexisting equilibrium E_+, which depends on the delay. However, there exist an absolutely unstable coexisting equilibriumE_- and a conditionally stable or unstable coexisting equilibrium E_+ for $B > 0$ in system (1.2). We also find that the model is bistable when $B > 0,\ A < 0$ and $A^2 - 4B > 0$ (see Figure 4).

Our investigation shows that coexisting equilibrium E_+ is always unstable after τ passes through the first critical value τ_0. That is to say, there does not exist any stability switching. However, stability switching can occurs in some predator-prey systems (see e.g. [26, 29, 33]). Moreover, the formulas determining the direction (μ_2) and stability (β_2) of Hopf bifurcation are given. We also show that the local Hopf bifurcation implies the global Hopf bifurcation of positive equilibrium after the second critical value of delay. Finally, we give three examples to illustrate the stability of the system (1.2) near the first critical value, and the simulation results are consistent with our conclusions.

This paper mainly considers the effects of providing additional food and designing a delayed feedback on Holling-Tanner model theoretically. We cannot claim that our method always holds for Holling-Tanner model due to the variety of ways to provide additional food and design delayed feedback. However, our investigation has potential significance for biological control. Therefore, the future works may consider how does the different ways of additional food and delayed feedback affect a predator-prey system, and cover the effects of additional food on an ecoepidemic model and so forth.

Acknowledgments

The authors thank the editor and referees for their valuable suggestions and comments, which improved the presentation of this manuscript.

Conflict of interests

For the publication of this article, no conflict of interests among the authors is disclosed.

References

1. S. Tang , L. Chen, Global attractivity in a "food-limited" population model with impulsive effects, *J. Math. Anal. Appl.,* **292** (2004), 211–221.
2. G. Saunders, B. Cooke, K. McColl, R. Shine, T. Peacock, Modern approaches for the biological control of vertebrate pests: an Australian perspective, *Biol. Control,* **52** (2010), 288–295.
3. B. Ghosh, F. Grognard, L. Mailleret, Natural enemies deployment in patchy environments for augmentative biological control, *Appl. Math. Comput.,* **266** (2015), 982–999.
4. A. Tena, A. Pekas, D. Cano, F. Wackers, A. Urbaneja, Sugar provisioning maximizes the biocontrol service of parasitoids, *J. Appl. Ecol.,* **52** (2015), 795–804.
5. P. Srinivasu, B. Prasad, M. Venkatesulu, Biological control through provision of additional food to predators: a theoretical study, *Theor. Popul. Biol.,* **72** (2007), 111–120.
6. B. Sahoo, S. Poria, The chaos and control of a food chain model supplying additional food to top-predator, *Chaos Soliton. Fract.,* **58** (2014), 52–64.
7. T. Kar, B. Ghosh, Sustainability and optimal control of an exploited prey predator system through provision of alternative food to predator, *Biosystems,* **109** (2012), 220–232.
8. P. Srinivasu, B. Prasad, Role of quantity of additional food to predators as a control in predator prey systems with relevance to pest management and biological conservation, *Bull Math. Biol.,* **73** (2011), 2249–2276.
9. J. Song, M. Hu, Y. Bai, Y. Xia, Dynamic analysis of a non-autonomous ratio-dependent predator-prey model with additional food, *J. Appl. Anal. Comput.,* **8** (2018), 1893–1909.
10. B. Sahoo, S. Poria, Effects of additional food on an ecoepidemic model with time delay on infection, *Appl. Math. Comput.,* **245** (2014), 17–35.
11. B. Sahoo, Role of additional food in eco-epidemiological system with disease in the prey, *Appl. Math. Comput.,* **259** (2015), 61–79.
12. A. Basheer, E. Quansah, R. Parshad, The effect of additional food in Holling Tanner type models, *Int. J. Dyn. Control,* **7** (2019), 1195–1212.
13. Y. Xia, Global analysis of an impulsive delayed Lotka-Volterra competition system, *Commun. Nonlinear Sci. Numer. Simul.,* **16** (2011), 1597–1616.
14. X. Meng, L. Chen, Permanence and global stability in an impulsive lotka-volterra n-species competitive system with both discrete delays and continuous delays, *Int. J. Biomath.,* **1** (2012), 179–196.

15. X. Meng, L. Chen, Almost periodic solution of non-autonomous Lotka-Volterra predator-prey dispersal system with delays, *J. Theoret. Biol.,* **243** (2006), 562–574.

16. Z. Wei, Y. Xia, T. Zhang, Stability and bifurcation analysis of a amensalism model with weak Allee effect, *Qual. Theory Dyn. Syst.,* **19** (2020), 23.

17. K. Kou, Y. Lou, Y. Xia, Zeros of a Class of Transcendental Equation with Application to Bifurcation of DDE, *Int. J. Bifurcat. Chaos,* **26** (2016), 1650062.

18. R. Xu, Z. Ma, Stability and Hopf bifurcation in a ratio-dependent predator-prey system with stage structure, *Chaos Soliton. Fract.,* **38** (2008), 669–684.

19. A. Maiti, B. Dubey, A. Chakraborty, Global analysis of a delayed stage structure prey-predator model with Crowley-Martin type functional response, *Math. Comput. Simulat.,* **162** (2019), 58–84.

20. Y. Chen , F. Zhang, Dynamics of a delayed predator-prey model with predator migration, *Appl. Math. Model.,* **37** (2013), 1400–1412.

21. T. Kar, U. Das, Bifurcation analysis of a delayed predator-prey model with Holling type III functional response and predator harvesting, *J. Nonlinear Dyn.,* **2014** (2014), 1–10.

22. J. Jiao, Y. Song, Stability and bifurcation analysis of a delayed Leslie-Gower predator-prey system with nonmonotonic functional response, *Abstr. Appl. Anal.,* **2013** (2013), 152459.

23. Y. Song, Y. Tao, H. Shu, Dynamics of a ratio-dependent stage-structured predator-prey model with delay, *Math. Methods Appl. Sci.,* **40** (2017), 6451–6467.

24. X. Jiang, X. Chen, T. Huang, H. Yan, Bifurcation and Control for a Predator-Prey System With Two Delays, *IEEE T. Circuits-II,* **68** (2021), 376–380.

25. X. Jiang, X. Chen, M. Chi, J. Chen, On Hopf bifurcation and control for a delay systems, *Appl. Math. Comput.,* **370** (2019), 124906.

26. B. Barman, B. Ghosh, Explicit impacts of harvesting in delayed predator-prey models, *Chaos Soliton. Fract.,* **122** (2019), 213–228.

27. L. Erbe, K. Geba, W. Krawcewicz, J. Wu, S^1-degree and global Hopf bifurcations, *J. Differ. Equations,* **98** (1992), 277–298.

28. J. Wu, Symmetric functional differential equations and neural networks with memory, *Trans. Amer. Math. Soc.,* **350** (1998), 4799–4838.

29. Y. Song, J. Wei, Local Hopf bifurcation and global periodic solutions in a delayed predator-prey system, *J. Math. Anal. Appl.,* **301** (2005), 1–21.

30. S. Yuan, F. Zhang, Stability and global Hopf bifurcation in a delayed predator-prey system, *Nonlinear Anal. Real World Appl.,* **11** (2010), 959–977.

31. S. Yuan, Y. Song, Stability and Hopf bifurcations in a delayed Leslie-Gower predator-prey system, *J. Math. Anal. Appl.,* **355** (2009), 82–100.

32. P. Hao, X. Wang, J. Wei, Global Hopf bifurcation of a population model with stage structure and strong Allee effect, *Discrete Contin. Dyn. Syst. Ser. S,* **10** (2017), 973–993.

33. X. Yan, W. Li, Hopf bifurcation and global periodic solutions in a delayed predator-prey system, *Appl. Math. Comput.,* **177** (2006), 427–445.

34. X. Liu, T. Zhang, X. Meng, T. Zhang, Turing-Hopf bifurcations in a predator-prey model with herd behavior, quadratic mortality and prey-taxis, *Phys. A,* **496** (2018), 446–460.

35. B. Hassard, N. Kazarinoff, Y. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge: Cambridge Univ. Press, 1981.

AIMS Press

©2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)