Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 12182–12224.
DOI: 10.3934/math.2021707
Received: 13 June 2021
Accepted: 19 August 2021
Published: 24 August 2021

Research article

Stability properties of Radon measure-valued solutions for a class of nonlinear parabolic equations under Neumann boundary conditions

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Abstract: In this paper, we address the existence, uniqueness, decay estimates, and the large-time behavior of the Radon measure-valued solutions for a class of nonlinear strongly degenerate parabolic equations involving a source term under Neumann boundary conditions with bounded Radon measure as initial data.

$$\begin{cases} u_t = \Delta\psi(u) + h(t)f(x,t) & \text{in } \Omega\times(0,T), \\ \frac{\partial\psi(u)}{\partial\eta} = g(u) & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $T > 0$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is an open bounded domain with smooth boundary $\partial\Omega$, η is an outward normal vector on $\partial\Omega$. The initial value data u_0 is a nonnegative bounded Radon measure on Ω, the function f is a solution of the linear inhomogeneous heat equation under Neumann boundary conditions with measure data, and the functions ψ, g and h satisfy the suitable assumptions.

Keywords: Radon measure-valued solutions; nonlinear strongly degenerate parabolic equations; linear inhomogeneous heat equation; Neumann boundary conditions; large-time behavior
Mathematics Subject Classification: 35K65, 35K61, 35B40, 28A33, 35R06, 28A50

1. Introduction

In this paper, we study the existence, uniqueness, decay estimates, and the large-time behavior of the solutions for a class of the nonlinear strongly degenerate parabolic equations involving the linear inhomogeneous heat equation solution as a source term under Neumann boundary conditions with bounded Radon measure as initial data. This problem is described as follows:

$$\begin{cases} u_t = \Delta\psi(u) + h(t)f(x,t) & \text{in } Q := \Omega\times(0,T), \\ \frac{\partial\psi(u)}{\partial\eta} = g(u) & \text{on } S := \partial\Omega\times(0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{P}$$

where $T > 0$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is an open bounded domain with smooth boundary $\partial\Omega$, η is an unit outward normal vector. The initial value data u_0 is a nonnegative bounded Radon measure on Ω. The functions ψ and g fulfill the following assumptions

$$\begin{cases} (i) \quad \psi \in L^\infty(\mathbb{R}_+)\cap C^2(\mathbb{R}_+),\ \ \psi(0) = 0,\ \ \psi' > 0 \ \text{ in } \ \mathbb{R}_+, \\ (ii) \quad \psi'\ ,\ \psi'' \in L^\infty(\mathbb{R}_+) \ \text{ and } \ \psi'(s) \to 0 \text{ as } s \to +\infty, \\ (iii) \quad \psi(s) \to \gamma \text{ as } s \to +\infty, \\ (iv) \quad \frac{|\psi''|}{\psi'} \leq \kappa \quad \text{in} \quad \mathbb{R}^*_+, \quad \text{for some } \ \kappa \in \mathbb{R}^*_+, \end{cases} \tag{I}$$

and

$$\begin{cases} (i) \quad g \in L^\infty(\mathbb{R}_+)\cap C^1(\mathbb{R}_+),\ \ g' < 0 \ \text{ in } \ \mathbb{R}_+ \ \text{ and } \ g > 0 \ \text{ in } \ \mathbb{R}_+, \\ (ii) \quad g' \in L^\infty(\mathbb{R}_+) \quad \text{and} \quad g(s) \to 0 \text{ as } s \to +\infty, \end{cases} \tag{A}$$

where $\mathbb{R}_+ \equiv [0,+\infty)$, $\mathbb{R}^*_+ \equiv (0,+\infty)$ and $\gamma \in \mathbb{R}^*_+$. By ψ' and ψ'' we denote the first and second derivatives of the function ψ. The assumption (I)-(iii) stem from (I)-(i), hence we extend the function ψ in $[0,+\infty]$ defining $\psi(+\infty) = \gamma$.

The typical example of the functions ψ and g are given

$$\psi(s) = \gamma\left[1 - e^{1-(1+s)^m}\right] \quad \text{and} \quad g(s) = e^{1-(1+s)^m}. \tag{1.1}$$

where $0 < m \leq 1$.

The function h satisfies the following hypothesis

$$h \in C^1(\mathbb{R}_+)\cap L^1(\mathbb{R}_+)\ ,\ h(0) = 0\ ,\ h' > 0 \ \text{ in } \ \mathbb{R}_+. \tag{J}$$

The function f is a solution of the linear inhomogeneous heat equation under Neumann boundary conditions with measure data

$$\begin{cases} f_t = \Delta f + \mu & \text{in } Q := \Omega\times(0,T), \\ \frac{\partial f}{\partial\eta} = g(f) & \text{on } S := \partial\Omega\times(0,T), \\ f(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{H}$$

where μ is a nonnegative bounded Radon measure on Q and g fulfills the assumption (A).

Throughout this paper, we consider solutions of the problem (P) as maps from (0,T) to the cone of nonnegative finite Radon measure on Ω, which satisfy (P) in the following sense: For a suitable class of test functions ξ there holds

$$\int_0^T \langle u_r(\cdot,t), \xi_t(\cdot,t)\rangle_\Omega\, dt + \int_0^T h(t)\,\langle f(\cdot,t), \xi(\cdot,t)\rangle_\Omega\, dt + \langle u_0, \xi(\cdot,0)\rangle +$$

$$+\int_0^T \langle g(u_r(\cdot,t)),\xi\rangle_{\partial\Omega}\,dt = \int_0^T \langle \nabla\psi(u_r)(\cdot,t),\nabla\xi(\cdot,t)\rangle_{\Omega}\,dt \tag{1.2}$$

(see Definition 2.1). Here the measure $u(\cdot,t)$ is defined for almost every $t \in (0,T)$, $u_r \in L^1(Q)$.

The type of the problem (P) has been intensively studied by many authors for instance (see [5, 18–20, 27, 28, 30]) few to mention. For the general form of the problem (P), we consider the following problem studied in [18],

$$\begin{cases} u_t = \operatorname{div}(\nabla\phi(x,t,u) + h(x,t,u)) + F(x,t,u) & \text{in } \Omega_T, \\ (\nabla\phi(x,t,u) + h(x,t,u))\cdot\eta = r(x,t,u) & \text{on } \sum_T, \\ u(x,0) = u_0 \ \text{ in } (\partial\Omega\setminus\sum)_T \cup \overline{\Omega}\times\{0\}, \end{cases} \tag{A.1}$$

where $\Omega_T = \Omega\times(0,T)$, $\sum_T = \sum\times(0,T)$, $(\partial\Omega\setminus\sum)_T = (\partial\Omega\setminus\sum)\times(0,T)$ with $\sum$ is a relative open subset of $\partial\Omega$, $\overline{\sum}$ and $\partial\Omega\setminus\sum$ are C^2 surface with boundary which meet in C^2 manifold dimension $N-2$ and $0 \le u_0 \in L^\infty((\partial\Omega\setminus\sum)_T \cup \overline{\Omega}\times\{0\})$. The author in [18], proved the local existence, uniqueness and the blow-up at the finite time of the degenerate parabolic equations (A.1). Furthermore, the existence and regularity of the solutions to the quasilinear parabolic systems under nonlinear boundary conditions is discussed in detail by the studies [28, 29]

$$\begin{cases} u_t + \mathcal{A}(t,u)u = F(t,u) & \text{in } \Omega\times(s,T), \\ \beta(t,u)u = r(t,u) & \text{in } \partial\Omega\times(s,T], \\ u(s) = u_0 & \text{on } \Omega, \end{cases} \tag{A.2}$$

where $s < t \le T$ and $u_0 \in W^{\tau,p}(\Omega,\mathbb{R}^N)(\tau \in [0,\infty))$ and the definition of the operators $\mathcal{A}(t,u)u$ and $\beta(t,u)u$ are in [28]. Similarly, studies in [19, 20] showed the existence and regularity of the degenerate parabolic equations with nonlinear boundary conditions and $u_0 \in L^2(\Omega)$ as an initial datum. Thus, we point out that the difference between the previous works (A.1), (A.2) and our work is on the following points; firstly, the initial value $u_0 \in \mathcal{M}^+(\Omega)$ (the nonnegative bounded Radon measure on Ω), secondly, the assumptions of the functions ψ, g given by (I) and (A). Finally, the source term f is a solution to the linear inhomogeneous heat equation under Neumann boundary conditions with measure data.

Furthermore, the study of the degenerate parabolic problem with forcing term has been intensively investigated by many authors (see [31–33]). In particular, [31] deals with existence solutions in the sense distributions of the nonlinear inhomogeneous porous medium type equations

$$u_t - \operatorname{div}A(x,t,u,Du) = \mu \qquad \text{in} \quad Q := \Omega\times(0,T) \tag{A.3}$$

where μ is a nonnegative Radon measure on Q with $\mu(Q) < \infty$ and $\mu|_{\mathcal{R}^{N+1}\setminus Q} = 0$. In last decade, some authors studied the existence, uniqueness and qualitative properties of the Radon measure-valued solutions to the nonlinear parabolic equations under zero Dirichlet or zero Neumann boundary conditions with bounded Radon measure as initial data (e.g. [1, 6, 7, 9–13, 15, 25] and references therein). Specially, [6] discuss the existence, uniqueness and the regularity of the Radon measure-valued solutions for a class of nonlinear degenerate parabolic equations

$$\begin{cases} u_t = \Delta\theta(u) & \text{in } Q, \\ \theta(u) = 0 & \text{on } S, \\ u_0(x,0) = u_0 & \text{on } \Omega, \end{cases} \tag{$A.4$}$$

where $u_0 \in \mathcal{M}^+(\Omega)$ and the function θ fulfills the assumptions expressed in [6]. The difference between the abovementioned studies and the problem (P) is the presence of the nonzero-Neumann boundary conditions and the source term which is a solution of the linear inhomogeneous heat equations under Neumann boundary conditions with measure data.

In general, the study of the partial differential equations through numerical methods is investigated by several authors (e.g. [47–50]). In particular, there are some authors who deal with the computation of the measure-valued solutions of the incompressible or compressible Euler equations (see [47, 48]). Mostly, the authors employ the numerical experiment corresponding to initial data of the partial differential equations and prove that the resulting approximation converge to a weak solution. For instance, in [50], the authors study numerical experiment to prove that the convergence of the solution to the nonlinear degenerate parabolic equations is measure-valued. Similarly, [49] employs the numerical method to show that the resulting approximation of a non-coercive elliptic equations with measure data converges to a weak solution. Hence, the numerical experiments represent the straightforward application of the theoretical study of the type of the problem (P).

To address the large-time behavior of the Radon measure-valued solutions of the problem (P), we construct the steady-state problem as a nonlinear strongly degenerate elliptic equations given as follows

$$\begin{cases} -\Delta\psi(U) + U = u_0 & \text{in } \Omega, \\ \frac{\partial\psi(U)}{\partial\eta} = g(U) & \text{on } \partial\Omega, \end{cases} \tag{E}$$

where $u_0 \in \mathcal{M}^+(\Omega)$ and the function ψ and g satisfy the hypotheses (I) and (A) respectively.

We consider solutions of the problem (E) as maps from Ω to the cone of nonnegative bounded Radon measure on Ω which satisfies (E) in the following sense: For a suitable class of test function φ, there holds

$$\int_\Omega \nabla\psi(U_r)\nabla\varphi dx + \int_\Omega U\varphi dx = \int_\Omega \varphi du_0(x) + \int_{\partial\Omega} g(U_r)\varphi d\mathcal{H}(x)$$

(see Definition 2.6), where $U_r \in L^1(\Omega)$ denotes the density of the absolutely continuous part of U with respect to the Lebesgue measure.

The nonlinear elliptic equations under Neumann boundary conditions with absorption term and a source term has been intensively studied by several authors [26, 34, 38–40]. In these studies, the authors dealt with the existence, uniqueness and regularity of the solutions. Furthermore, in [34], the following problem is considered

$$\begin{cases} LU + B(U) = f_2 & \text{in } \Omega, \\ \frac{\partial U}{\partial\eta} + C(U) = g_2 & \text{on } \partial\Omega, \end{cases} \tag{$A.5$}$$

where $B(U) \in L^1(\Omega)$, $C(U) \in L^1(\partial\Omega)$, $f_2 \in L^1(\Omega)$, $g_2 \in L^1(\partial\Omega)$ and the expression of the differential operator L in [34, Section 2]. The authors proved the existence, uniqueness and regularity of the solutions $U \in W^{1,1}(\Omega)$ to the problem $(A.5)$ (see [34, Section 4, Theorem 22 and Corollary 21). The difference between the previous studies mentioned above and (A.5) is that we study the nonlinear strongly degenerate elliptic equations and the solutions obtained are Radon measure-valued. However, the existence, uniqueness, and regularity of the Radon measure-valued solutions of the quasilinear degenerate elliptic equations under zero Dirichlet boundary conditions are discussed in detail [13] by considering the following problem

$$\begin{cases}-\mathrm{div}(A(x,U)\nabla U) + U(x) = \overline{\mu} & \text{in } \Omega,\\ U(x) = 0 & \text{on } \partial\Omega,\end{cases} \tag{A.6}$$

where $\overline{\mu} \in \mathcal{M}(\Omega)$ and $A(x, U)$ satisfies the hypothesis in [13]. In this case, the difference between the problem (E) and (A.4) is a boundary conditions with the assumptions on ψ.

In this paper, we study a class of nonlinear parabolic problems involving a forcing term and initial data is a nonnegative Radon measure. In the recent years, there are different papers that investigate these kind of problems in the setting in which the solution is a Radon measure for positive time. This type of study was done for parabolic and hyperbolic equations. One of the main tool is to search a solution by an approximation of the initial data and then try to pass to the limit in a very weak topology. The innovative part of this work is mainly the study of the large time behavior of the solutions. In my opinion, it is essential to highlight that the explicit examples of equations study in this work have not already been dealt with in literature and the novelties of the techniques that they introduced in the work. Finally, the study of the asymptotic behavior is a novelty.

The main difficulty to study the problem (P) is due to the presence of the forcing term which depends on the property solutions of the inhomogeneous heat equation (H).

The main motivation of this study comes from the desire to deal with parabolic equations in which the forcing term can be either Radon measure or $L^p(Q)$ $(1 \leq p < \infty)$ functions. Whence, the idea to consider the linear inhomogeneous heat equation solution with measure data as a forcing term.

To deal with the existence and the uniqueness of the weak solutions to the problem (P), we use the definition of the Radon measure-valued solutions of the parabolic equations and the natural approximation method. In particular, to show the uniqueness of the problem (P), we will distinguish two cases for the forcing term f, either the function is in $L^2((0,T), H^1(\Omega))$ or the Radon measure on Q. Notice that when the linear inhomogeneous heat equation (H) does not admit an unique solution, the problem (P) has no unique solution as well.

Furthermore, we prove the necessary and sufficient condition between measure data and capacity in order to deal with the existence of the weak solutions to the problem (P).

To establish the decay estimates of the Radon measure-valued solutions to the problem (P), we construct the suitable function and we use it as a test function in the approximation of the problem (P). Then we easily infer the decay estimates after the use of some measure properties.

To address the large-time behavior of the Radon measure-valued solutions of the problem (P), we first show that the problem (E) has a Radon measure-valued solutions in Ω.

To the best of our knowledge no existing result of decay estimates and large-time behavior of Radon measure-valued solutions obtained as limit of the approximation of the problem (P) are known in the literature. Hence, this interesting case will be discussed in this paper. This paper is organized as follows: In the next section, we state the main results, while in Section 3, we present important preliminaries. In Section 4, we study the existence and uniqueness of the heat equation (H). Finally, we prove the main results in the Sections 5–8.

2. Statement of the main results

To study the weak solution of the problem (P), we refer to the following definition.

Definition 2.1. For any $u_0 \in \mathcal{M}^+(\Omega)$ and $\mu \in \mathcal{M}^+(Q)$, a measure u is called *a weak solution* of problem (P), if $u \in \mathcal{M}^+(Q)$ such that
(i) $u \in L^\infty((0,T), \mathcal{M}^+(\Omega))$,
(ii) $\psi(u_r) \in L^2((0,T), H^1(\Omega))$,
(iii) $g(u_r) \in L^1(S)$,
(iv) for every $\xi \in C^1((0,T), C^1(\Omega))$, $\xi(\cdot,T) = 0$ in Ω, u satisfies the identity

$$\int_0^T \langle u(\cdot,t), \xi_t(\cdot,t)\rangle_\Omega dt + \int_0^T h(t) \langle f(\cdot,t), \xi(\cdot,t)\rangle_\Omega \, dt + \langle u_0, \xi(\cdot,0)\rangle_\Omega +$$

$$+ \int_0^T \langle g(u_r(\cdot,t)), \xi\rangle_{\partial\Omega} \, dt = \int_0^T \langle \nabla\psi(u_r), \nabla\xi\rangle_\Omega \, dxdt \tag{2.1}$$

where u_r is the nonnegative density of the absolutely continuous part of Radon-measure with respect to the Lebesgue measure such that $u_r \in L^\infty((0,T), L^1(\Omega))$ and the function f is the solution of the problem (H).

Throughout this paper, we assume that Ω is a strong $C^{1,1}$ open subset of $\mathbb{R}^N$. Also, we assume that there exists a finite open cover (B_j) such that the set $\Omega \cap B_j$ epigraph of a $C^{1,1}$ function $\zeta : \mathbb{R}^{N-1} \to \mathbb{R}$ that is

$$\Omega \cap B_j = \left\{x \in B_j / \ x_N > \zeta(\overline{x})\right\} \quad \text{and} \quad \partial\Omega \cap B_j = \left\{x \in B_j / \ x_N = \zeta(\overline{x})\right\}$$

where $x = (\overline{x}, x_N)$, the local coordinates with $\overline{x} = (x_1, x_2, \ldots, x_{N-1})$. We denote $\vartheta = \{\overline{x}, \ x \in \Omega \cap B_j\} \subseteq \mathbb{R}^{N-1}$, the projection of $\Omega \cap B_j$ onto the $(N-1)$ first components, and $\vartheta_\varsigma = \{\overline{x}, \ x \in \text{supp}(\varsigma) \cap \Omega\}$.
If a function ϕ is defined on S, we denote ϕ_S the function defined on $(B_j \cap Q) \times [0,T]$ by $\xi_S(x,t) = \xi(\overline{x}, \zeta(\overline{x}), t)$. Notice that the restriction of ξ_S to $[0,T) \times \vartheta$.

The next definition of the trace is corresponding to the problem (P) adapts to the context of [36, Theorem 2.1].

Definition 2.2 Let $\mathcal{F} \in [L^2(Q)]^{N+1}$ be such that div$\mathcal{F}$ is a bounded Radon measure on Q. Then there exists a linear functional $\mathcal{T}_\eta$ on $W^{\frac{1}{2},2}(S) \cap C(S)$ which represents the normal traces $\mathcal{F} \cdot \nu$ on S in the sense that the following Gauss-green formula holds:

(i) For all $\xi \in C_c^\infty(\overline{Q})$,

$$\langle \mathcal{T}_\nu, \xi\rangle = \int_Q \xi \text{div}\mathcal{F} + \int_Q \nabla\xi \cdot \mathcal{F}$$

where $\langle \mathcal{T}_\nu, \xi\rangle$ depends only on ξ_S.
(ii) If (B_j, ς, f) is an above subsequence localization near boundary, then for all $\xi \in C_c^\infty([0,T) \times \overline{\Omega})$ there holds

$$\langle \mathcal{T}_\nu, \xi\rangle = -\lim_{s\to 0} \frac{1}{s} \left\{ \int_s^T \int_\vartheta \int_{\zeta(\overline{x})}^{\zeta(\overline{x})+s} \mathcal{F} \cdot \begin{pmatrix} -\nabla\zeta(\overline{x}) \\ 1 \\ 0 \end{pmatrix} \xi\sigma dx_N d\overline{x} dt \right\} - \lim_{s\to 0} \frac{1}{s} \int_0^s \int_\Omega \mathcal{F} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xi\sigma dxdt \tag{2.2}$$

where the divergence of the fields,

$$\mathcal{F}(x,t) = \begin{pmatrix} u(x,t) \\ \nabla\psi(u_r(x,t)) \end{pmatrix}$$

is a bounded Radon measure on Q.

The following result states the existence of the trace of the boundary condition to the problem (P).

Lemma 2.1 Let Ω is a strong $C^{1,1}$ open subset of $\mathbb{R}^N$. Then there exists an unique trace $\mathcal{T}_\eta : W^{1,1}(\Omega) \to L^1(\partial\Omega)$ such that

$$\langle \mathcal{T}_\eta, \xi \rangle = \int_S g(u_r)\xi d\mathcal{H}(x)dt \tag{2.3}$$

where the function $g(u_r) \in L^1(S)$ and $\xi \in C_c^\infty([0,T) \times \overline{\Omega})$.

To prove the uniqueness of the solution to the problem (P), we define the notion *very weak solution* of the problem (P) as follows.

Definition 2.3. For any $\mu \in \mathcal{M}^+(Q)$ and $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$, a measure u is called a very weak solution to problem (P) if $u \in L^\infty((0,T), \mathcal{M}^+(\Omega))$ such that

$$\int_0^T \langle u(\cdot,t), \xi_t(\cdot,t)\rangle_\Omega dt + \int_Q \psi(u_r)\Delta\xi dxdt + \int_Q h(t)f(x,t)\xi dxdt + \int_S g(u)\xi d\mathcal{H}dt + \langle u_0, \xi(0)\rangle_\Omega = 0 \tag{2.4}$$

for every $\xi \in C^{2,1}(\overline{Q})$, which vanishes on $\partial\Omega \times [0,T]$, for $t = T$.

To prove the uniqueness of the problem (P) when f lies in $\mathcal{M}^+(Q)$, we consider the following every weak solution gives below:

Definition 2.4 Let $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$ and $\mu \in \mathcal{M}^+(Q)$ such that

$$u_0 = f_0 - \mathrm{div} G_0 , \quad f_0 \in L^1(\Omega) \text{ and } G_0 \in \left[L^2(\Omega)\right]^N .$$

A function u is called a *very weak solutions obtained as limit of approximation*, if

$$u_n \overset{*}{\rightharpoonup} u \text{ in } \mathcal{M}^+(Q) \tag{2.5}$$

where $\{u_n\} \subseteq L^\infty(Q) \cap L^2((0,T), H^1(\Omega))$ is the sequences of weak solutions to problem (P_n) satisfies

$$\begin{cases} u_{0n} = f_{0n} - F_{0n} \in C_c^\infty(\Omega), \\ F_{0n} \to \mathrm{div} G_0 \quad \text{in } (H^1(\Omega))^*, \\ f_{0n} \to f_0 \quad \text{in } L^1(\Omega). \end{cases} \tag{2.6}$$

We denote $(H^1(\Omega))^*$ the dual space of $H^1(\Omega)$ and the embedding $H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))^*$ holds.

Definition 2.5 Let $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$ and $\mu \in \mathcal{M}^+_{d,2}(Q)$ such that

$$u_0 = f_0 - \mathrm{div} G_0 , \quad f_0 \in L^1(\Omega) \text{ and } G_0 \in \left[L^2(\Omega)\right]^N .$$

$$\mu = f_1 - \mathrm{div} G + \varphi_t , \ f_1 \in L^1(Q) , \ G \in \left[L^2(Q)\right]^N \quad \text{and} \quad \varphi \in L^2((0,T), H^1(\Omega)).$$

A measure f is called a *very weak solutions obtained as limit of approximation*, if

$$f_n \overset{*}{\rightharpoonup} f \ \text{ in } \mathcal{M}^+(Q) \tag{2.7}$$

where $\{u_n\}$ and $\{f_n\} \subseteq L^\infty(Q) \cap L^2((0,T), H^1(\Omega))$ are the sequences of weak solutions to problem (P_n) and (H_n) respectively satisfy

$$\begin{cases} \mu_n = f_{1n} - F_n + g_{nt} \in C_c^\infty(Q), \\ u_{0n} = f_{0n} - F_{0n} \in C_c^\infty(\Omega), \\ f_{1n} \to f_1 \quad \text{in } \ L^1(Q), \\ F_n \to \text{div}G \quad \text{in } \ L^2((0,T), (H^1(\Omega))^*), \\ \varphi_n \to \varphi \quad \text{in } \ L^2((0,T), H^1(\Omega)), \\ F_{0n} \to \text{div}G_0 \quad \text{in } \ (H^1(\Omega))^*, \\ f_{0n} \to f_0 \quad \text{in } \ L^1(\Omega). \end{cases} \tag{2.8}$$

Then, the function u is *very weak solutions of the problem* (P) *obtained as limit of approximation* if the function f is a *very weak solutions of the problem* (H) *obtained as limit of approximation*.

Notice that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } \ \mathcal{M}^+(Q)\,, \ \mu_n \overset{*}{\rightharpoonup} \mu \quad \text{in } \ \mathcal{M}^+(Q) \quad \text{and} \quad u_{0n} \overset{*}{\rightharpoonup} u_0 \quad \text{in } \ \mathcal{M}^+(\Omega).$$

$\mathcal{M}^+_{d,2}(\Omega)$ denotes the set of nonnegative measures on Ω which are *diffuse with respect to the Newtonian capacity* and the definition of the diffuse measure with respect to the parabolic capacity $\mathcal{M}^+_{d,2}(Q)$ will be recalled in the Section 3.

Before dealing with the existence of the problem (P), we first prove the existence and uniqueness of the solutions to the problem (H) given by the following result.

Theorem 2.1. Assume that $u_0 \in \mathcal{M}^+(\Omega)$ and $\mu \in \mathcal{M}^+(Q)$ hold.

(i) Then, there exists a nonnegative Radon measure-valued solution to the problem (H) in the space $L^\infty((0,T), \mathcal{M}^+(\Omega))$ such that

$$f(x,t) = \int_\Omega G_N(x-y,t)du_0(y) + \int_0^t \int_\Omega G_N(x-y,t-\sigma)d\mu(y,\sigma) + \int_0^t \int_{\partial\Omega} G_N(x-y,t-\sigma)g(f(y,\sigma))d\mathcal{H}(y)d\sigma \tag{2.9}$$

for almost every $t \in (0,T)$. Furthermore, the Radon measure-valued solution f satisfies the following estimate

$$\| f(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \leq e^{Ct} (\| \mu \|_{\mathcal{M}^+(Q)} + \| u_0 \|_{\mathcal{M}^+(\Omega)}) \tag{2.10}$$

for any $C = C(T)$ a positive constant.

(ii) Suppose that $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$, $\mu \in \mathcal{M}^+_{d,2}(Q)$ and $g(f) = \overline{K}$ almost everywhere on S ($\overline{K}$ is a positive constant) are satisfied. Then, the nonnegative weak Radon measure-valued solution to the problem (H) obtained as limit of the approximation is unique in $L^\infty((0,T), \mathcal{M}^+(\Omega))$.

We denote by $G_N(x-y, t-s)$ as the Green function of the heat equation under homogeneous Neumann boundary conditions. By [4], the Green function satisfies the following properties

$$G_N(x-y, t-s) \geq 0\,, \quad x, y \in \Omega\,, \ 0 \leq s < t < T, \tag{2.11}$$

$$\int_{\Omega} G_N(x-y,t-s)dx = 1\,, \quad y \in \Omega\,,\ 0 \le s < t < T. \tag{2.12}$$

There exist two positive constants τ_1 and τ_2 such that

$$\left| G_N(x-y,t-s) - \frac{1}{|\,\Omega\,|} \right| \le \tau_1 e^{-\tau_2(t-s)}, \quad x,y \in \Omega\,,\ 1+s < t. \tag{2.13}$$

$$\lim_{t\to s} \int_{\Omega} G_N(x-y,t-s)\phi(y)dy = \phi(x) \tag{2.14}$$

for any $\phi \in C_c(\Omega)$ and $|\,\Omega\,|$ is a Lebesgue measure of the set Ω.

Remark 2.1 (i) For any test function $\xi \in C^1((0,T),C^1(\Omega))$ such that $\xi(\cdot,T) = 0$ in Ω and $\frac{\partial\xi}{\partial\eta} = 0$ on S, the inner product $\langle f(\cdot,t),\xi(\cdot,t)\rangle_{\Omega}$ in (2.1) is given by the following expression

$$\langle f(\cdot,t),\xi(\cdot,t)\rangle_{\Omega} = \int_{\Omega}\int_{\Omega} G(x-y,t)\xi(y,0)du_0(y)dx +$$

$$+ \int_{\Omega}\int_0^t \int_{\Omega} G_N(x-y,t-\sigma)\left(f\xi_\sigma - 2\nabla f\nabla\xi - f\Delta\xi\right) dyd\sigma dx +$$

$$+ \int_{\Omega}\int_0^t \int_{\Omega} G_N(x-y,t-\sigma)\xi(y,\sigma)d\mu(y,\sigma)dx +$$

$$+ \int_{\Omega}\int_0^t \int_{\partial\Omega} G_N(x-y,t-\sigma)\xi(y,\sigma)g(f(y,\sigma))d\mathcal{H}(y)d\sigma dx \tag{2.15}$$

where ξ_σ is a first derivative order of ξ with respect σ.

(ii) By the regularity properties of the Green function $G_N(x-y,t-\sigma)$ in [42], the solution of the problem (H) given by (2.9), $f \in L^2((0,T),H^1(\Omega))$.

(iii) By virtue of the assumptions (J), (2.11) and (2.12), the term $h(t)f(x,t)$ is well-defined at $t = 0$. Indeed, the function $t \mapsto \int_{\Omega} G_N(x-y,t-\sigma)h(\sigma)d\mu(y,\sigma)$, $t \mapsto \int_{\partial\Omega} G_N(x-y,t-\sigma)h(\sigma)g(f(y,\sigma))d\mathcal{H}(y)$ and $t \mapsto \int_{\Omega} G_N(x-y,t-\sigma)f(y,\sigma)h'(\sigma)dy$ are continuous in $\mathbb{R}_+$. Then there holds

$$\lim_{t\to 0^+} h(t)f(x,t) = \lim_{t\to 0^+} \int_0^t \int_{\Omega} G_N(x-y,t-\sigma)f(y,\sigma)h'(\sigma)dyd\sigma +$$

$$+ \lim_{t\to 0^+} \int_0^t \int_{\partial\Omega} G_N(x-y,t-\sigma)h(\sigma)g(f)d\mathcal{H}(y)d\sigma + \lim_{t\to 0^+} \int_0^t \int_{\Omega} G_N(x-y,t-\sigma)h(\sigma)d\mu(y,\sigma) = 0.$$

Hence we extend the function $h(t)f(x,t)$ in [0,T] defining $h(0)f(x,0) = 0$. Furthermore, the presence of the function h is to well-defined the forcing term of the nonlinear parabolic problem (P).

In order to study the existence and uniqueness of the solutions to the problem (P), we give the necessary and sufficient condition on the measures μ and u_0 for the existence of the weak solutions to the problem (P) with respect to the parabolic and Newtonian capacity respectively. This result is given by the following theorem.

Theorem 2.2. Suppose that the hypotheses (I), (A), $\mu \in \mathcal{M}^+(Q)$ and $u_0 \in \mathcal{M}^+(\Omega)$ hold. For any function h satisfying (J), there exists $t \in (0,T)$ such that $\int_0^t h(\sigma)d\sigma = 1$ and u is a weak solution to the problem (P). Then μ and u_0 are absolutely continuous measures with respect to the parabolic capacity.

Notice that Newtonian and parabolic capacity are equivalent, then μ and u_0 are absolutely continuous measures with respect to C_2-capacity as well.

In the next theorem, we present the result of the existence Radon measure-valued solutions to the problem (P).

Theorem 2.3 Suppose that the assumptions (I), (J), (A) $\mu \in \mathcal{M}^+(Q)$ and $u_0 \in \mathcal{M}^+(\Omega)$ are satisfied. Then there exists a weak solution u to problem (P) obtained as a limiting point of the sequence $\{u_n\}$ of solutions to problems (P_n) such that for every $t \in (0,T) \setminus H^*$, there holds

$$\| u(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \leq C \left(\| \mu \|_{\mathcal{M}^+(Q)} + \| u_0 \|_{\mathcal{M}^+(\Omega)}\right). \tag{2.16}$$

The result of the uniqueness of the problem (P) is given by the following theorem:

Theorem 2.4 Assume that the hypotheses (I), (J) and (A), $\mu \in \mathcal{M}^+_{d,2}(Q)$ and $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$ hold. Then there exists a unique *very weak solution obtained as the limit of approximation u* of the problem (P), if $g(u_r) = L$ almost everywhere in S, whenever L is a positive constant.

To establish the decay estimate of the solution to the problem (P), we recall two particular problems of the problem (P). Now we consider the following problem.

$$\begin{cases} v_t = \Delta\vartheta(v) & \text{in } Q, \\ \frac{\partial\vartheta(v)}{\partial\eta} = g_1(v) & \text{on } S, \\ v(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{P_0}$$

The functions ψ and g satisfy the assumption (I) and (A) respectively and have the same properties with the functions ϑ and g_1 given as follows

$$\vartheta(s) = \gamma\left[1 - \frac{1}{(1+s)^m}\right] \quad (m > 0) \quad \text{and} \quad g_1(s) = \frac{1}{(1+s)^m} \tag{2.17}$$

where $m > 0$ and $s > 0$. Therefore, by Theorem 2.3, the problem (P_0) possesses a solution in the space $L^\infty((0,T), \mathcal{M}^+(\Omega))$, such that

$$\| v(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \leq C \| u_0 \|_{\mathcal{M}^+(\Omega)}$$

for almost every $t \in (0,T)$.

Similarly, we consider the following problem

$$\begin{cases} w_t = \Delta\psi(w) + h(t)f(x,t) & \text{in } Q, \\ \frac{\partial\psi(w)}{\partial\eta} = g(w) & \text{on } S, \\ w(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{P_1}$$

By Theorem 2.3, the problem (P_1) admits a solution in $L^\infty((0,T), \mathcal{M}^+(\Omega))$, such that

$$\| w(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \leq C \| \mu \|_{\mathcal{M}^+(\Omega)}$$

for almost every $t \in (0,T)$.

Now we state the decay estimates in the next theorem:

Theorem 2.5 Suppose that (I), (J), (A), $\mu \in \mathcal{M}^+(Q)$ and $u_0 \in \mathcal{M}^+(\Omega)$ are satisfied. The measure u is the weak solution to the problem (P). According to Theorem 2.3, v is the weak solution to the problem

(P_0) and w is the weak solution to the problem (P_1). Then for every $t \in (0, T) \setminus H^*$ with $| H^* |= 0$, there holds

$$\| u(\cdot, t) - v(\cdot, t) \|_{\mathcal{M}^+(\Omega)} \leq \frac{C}{(T-t)^\alpha} \left(\| \mu \|_{\mathcal{M}^+(Q)} + \| u_0 \|_{\mathcal{M}^+(\Omega)} \right), \tag{2.18}$$

$$\| u(\cdot, t) - w(\cdot, t) \|_{\mathcal{M}^+(\Omega)} \leq C \frac{\| u_0 \|_{\mathcal{M}^+(\Omega)}}{(T-t)^\alpha}, \tag{2.19}$$

and

$$\| u(\cdot, t) \|_{\mathcal{M}^+(\Omega)} \leq \frac{C}{t^\alpha} (\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}) \tag{2.20}$$

for any positive constant C and $\alpha > 1$.

To deal with the large-time behavior of the Radon measure-valued solutions to the problem (P), we first extend $(0, T)$ to $(0, +\infty)$, then we assume that the hypothesis

$$\limsup_{t \to +\infty} \| u(\cdot, t) \|_{\mathcal{M}^+(\Omega)} \leq C \tag{2.21}$$

where C is a positive constant.

To analyze the large-time behavior of the Radon measure-valued solutions, we first study the existence of the Radon measure-valued solutions corresponding to the steady state problem (E) by considering the following definition.

Definition 2.6 Assume that the hypotheses (I), (A) and $u_0 \in \mathcal{M}^+(\Omega)$ are satisfied. A measure U is a solution of the problem (E), if $U \in \mathcal{M}^+(\Omega)$ such that
(i) $\psi(U_r) \in W^{1,1}(\Omega)$,
(ii) $g(U_r) \in L^1(\partial\Omega)$,
(iii) for every $\varphi \in C^1(\Omega)$, the following assertion

$$\int_\Omega \nabla\psi(U_r(x))\nabla\varphi(x)dx + \int_\Omega U(x)\varphi(x)dx = \int_\Omega \varphi(x)du_0(x) + \int_{\partial\Omega} g(U(x))\varphi(x)d\mathcal{H}(x) \tag{2.22}$$

holds true.

The existence result of the problem (E) is given by the following theorem:

Theorem 2.6 Suppose that the hypotheses (I), (A) and $u_0 \in \mathcal{M}^+(\Omega)$ are satisfied. Then there exists a weak solution $U \in \mathcal{M}^+(\Omega)$ of the problem (E) obtained as a limiting point of the sequence $\{U_n\}$ of solutions to the approximation problem (E_n) such that

$$\| U \|_{\mathcal{M}^+(\Omega)} \leq C \| u_0 \|_{\mathcal{M}^+(\Omega)} \tag{2.23}$$

where $C > 0$ is a constant.

The result of the large-time behavior of the Radon measure-valued solutions of the problem (P) is given by the following theorem

Theorem 2.7. Suppose that the assumption (I), (A), (J), $u_0 \in \mathcal{M}^+(\Omega)$ and $\mu \in \mathcal{M}^+(Q)$. U is a Radon measure-valued solutions of the steady-state problem (E) in sense of Theorem 2.6 and u is a Radon measure-valued solutions in the sense of Theorem 2.3 such that (2.21) holds. Then there holds

$$u(\cdot, t) \to U \quad \text{in} \quad \mathcal{M}^+(\Omega) \quad \text{as} \quad t \to \infty \tag{2.24}$$

3. Preliminaries

In the following section, we define the truncation function for $k > 0$ and $s \in \mathbb{R}$,

$$T_k(s) = \min\{| s |, k\}\mathrm{sign}(s).$$

To prove the main results from the previous section, we need to recall the preliminaries about capacity and measure collected in [9–16]. Likewise, we recall some important notations as follows:

For any Borel set $E \subset \Omega$, the C_2-capacity of E in Ω is defined as

$$C_2(E) = \inf\left\{\int_\Omega (| u |^2 + | \nabla u |^2) dx / u \in \mathbb{Z}_\Omega^E\right\}$$

where $\mathbb{Z}_\Omega^E$ denotes the set of u which belongs to $H^1(\Omega)$ such that $0 \leq u \leq 1$ almost everywhere in Ω, and $u = 1$ almost everywhere in a neighborhood E.

Let $W = \left\{u \in L^2((0,T), H^1(\Omega)) \ \text{ and } \ u_t \in L^2((0,T), (H^1(\Omega))^*)\right\}$ endowed with its natural norm $\| u \|_W = \| u \|_{L^2((0,T),H^1(\Omega))} + \| u_t \|_{L^2((0,T),(H^1(\Omega))^*)}$ a Banach space. For any open set $U \subset Q$, we define the parabolic capacity as

$$\mathrm{Cap}(U) = \inf\left\{\| u \|_W / u \in \mathbb{V}_Q^U\right\}$$

where $\mathbb{V}_Q^U$ denotes the set of u belongs to W such that $0 \leq u \leq 1$ almost everywhere in Q, and $u = 1$ almost everywhere in a neighborhood U.

Let $\mathcal{M}(B)$ be the space of bounded Radon measures on B, and $\mathcal{M}^+(B) \subset \mathcal{M}(B)$ the cone of nonnegative bounded Radon measures on B. For any $\mu \in \mathcal{M}(B)$ a bounded Radon measure on B, we set

$$\| \mu \|_{\mathcal{M}(\Omega)} := | \mu | (B)$$

where $| \mu |$ stands for the total variation of μ.

The duality map $\langle \cdot, \cdot \rangle_B$ between the space $\mathcal{M}(B)$ and $C_c(B)$ is defined by

$$\langle \mu, \varphi \rangle_B = \int_B \varphi d\mu.$$

$\mathcal{M}_s^+(B)$ denotes the set of nonnegative measures *singular with respect to the Lebesgue measure*, namely

$$\mathcal{M}_s^+(B) := \{\mu \in \mathcal{M}^+(\Omega) / \exists \text{ a Borel set } F \subseteq B \text{ such that } | F | = 0 , \mu = \mu \llcorner B\}$$

we will consider either $| \ . \ |$ the Lebesgue measure on $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$. Similarly, $\mathcal{M}_{ac}^+(B)$ the set of nonnegative measures *absolutely continuous with respect to the Lebesgue measure*, namely

$$\mathcal{M}_{ac}^+(B) := \{\mu \in \mathcal{M}^+(\Omega) / \mu(F) = 0, \text{ for every Borel set } F \subseteq B \text{ such that } | F | = 0\}$$

Let $\mathcal{M}_{c,2}^+(B)$ be the set of nonnegative measures on B which are *concentrated with respect to the Newtonian capacity*

$$\mathcal{M}_{c,2}^+(B) := \{\mu \in \mathcal{M}^+(B) / \exists \text{ a Borel set } F \subseteq B, \text{ such that } \mu = \mu \llcorner F \text{ and } C(F) = 0\}$$

$\mathcal{M}^+_{d,2}(B)$ denotes the set of nonnegative measures on B which are *diffuse with respect to the Newtonian capacity*

$$\mathcal{M}^+_{d,2}(B) := \{\mu \in \mathcal{M}^+(B)/\mu(F) = 0, \text{ for every Borel set } F \subseteq B \text{ such that } C(F) = 0\}.$$

It is known that a measure $\overline{\mu}_{d,2} \in \mathcal{M}^+_{d,2}(\Omega)$ (resp. $\mu_{d,2} \in \mathcal{M}^+_{d,2}(Q)$) if there exist $f_0 \in L^1(\Omega)$ and $G_0 \in \left[L^2(B)\right]^N$ (resp. if $\mu_{d,2} \in \mathcal{M}^+_{d,2}(Q)$, there exist $f \in L^1(Q)$, $g \in L^2((0,T), H^1(\Omega))$ and $G \in \left[L^2(Q)\right]^N$) such that

$$\overline{\mu}_{d,2} = f_0 - \text{div} G_0 \quad \text{in} \quad D'(\Omega) \quad (\text{resp. } \mu_{d,2} = f - \text{div} G + g_t \quad \text{in} \quad D'(Q)). \tag{3.1}$$

For any $\lambda \in \mathcal{M}^+(B)$, if there exists a unique couple $\lambda_{d,2} \in \mathcal{M}^+_{d,2}(B)$, $\lambda_{c,2} \in \mathcal{M}^+_{c,2}(B)$ such that

$$\lambda = \lambda_{d,2} + \lambda_{c,2}. \tag{3.2}$$

On the other hand, there exists a unique couple $\lambda_{ac} \in \mathcal{M}^+_{ac}(B)$, $\lambda_s \in \mathcal{M}^+_s(B)$ such that

$$\lambda = \lambda_{ac} + \lambda_s \tag{3.3}$$

where either $B = \Omega$, $C(F) = C_2(E)$ or $B = Q$, $C(F) = Cap(U)$.

By $L^\infty((0,T), \mathcal{M}^+(\Omega))$, the set of nonnegative Radon measures $u \in \mathcal{M}^+(\overline{Q})$ such that for every $t \in (0,T)$, there exists a measure $u(\cdot,t) \in \mathcal{M}^+(\Omega)$ such that

(i) for every $\xi \in C(\overline{Q})$ the map

$$t \mapsto \langle u(\cdot,t), \xi(\cdot,t)\rangle_\Omega \text{ is Lebesgue measurable}$$

and

$$\langle u(\cdot,t), \xi(\cdot,t)\rangle_\Omega = \int_0^T \langle u(\cdot,t), \xi(\cdot,t)\rangle_\Omega \, dt$$

(ii) there exists a constant $C > 0$ such that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \| u(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \leq C$$

with the norm denotes

$$\| u \|_{L^\infty((0,T),\mathcal{M}^+(\Omega))} = \operatorname*{ess\,sup}_{t\in(0,T)} \| u(\cdot,t) \|_{\mathcal{M}^+(\Omega)}. \tag{3.4}$$

4. Existence and Uniqueness results of the problem (H)

In the literature, many authors dealt with the existence, uniqueness, blow-up at finite and infinite time, decay estimates, stability properties and asymptotic behavior of the solutions to the heat equation under Neumann boundary conditions with a source term and initial data, such as (see [2–5, 42, 43] and references therein). Moreover, most of the authors employed the maximum principle theorem through the monotonicity technique and semi-group method to show the existence, blow-up, stability properties and asymptotic behavior of these solutions. Meanwhile, in this section we prove the existence and uniqueness of the linear inhomogeneous heat equation (H) by using the fundamental solution of the heat equation (see [2–4, 42]). Also, we use the definition of the Radon measure-valued solutions in

[9] and some properties of the Radon measure provided in [24, 44]. Moreover, we consider for every $n \in \mathbb{N}$, the approximation of problem (H) such that

$$\begin{cases} f_{nt} = \Delta f_n + \mu_n & \text{in } Q, \\ \frac{\partial f_n}{\partial \eta} = g(f_n) & \text{on } S, \\ f_n(x,0) = u_{0n}(x) & \text{in } \Omega, \end{cases} \tag{H_n}$$

Since $u_0 \in \mathcal{M}^+(\Omega)$, the approximation of the Radon measure u_0 is given by [9, Lemma 4.1] such that $\{u_{0n}\} \subseteq C_c^\infty(\Omega)$ satisfies the following assumptions

$$\begin{cases} u_{0n} \overset{*}{\rightharpoonup} u_0 \quad \text{in} \quad \mathcal{M}^+\Omega), \\ u_{0n} \to u_{0r} \quad \text{a.e in} \quad \Omega, \\ \| u_{0n} \|_{L^1(\Omega)} \leq \| u_0 \|_{\mathcal{M}^+(\Omega)} . \end{cases} \tag{4.1}$$

Moreover $\mu \in \mathcal{M}^+(Q)$, the approximation of the Radon measure μ is given by [15] such that $\{\mu_n\} \subseteq C_c^\infty(Q)$ fulfills the following hypotheses

$$\begin{cases} \mu_n \overset{*}{\rightharpoonup} \mu \quad \text{in} \quad \mathcal{M}^+(Q), \\ \mu_n \to \mu_r \quad \text{a.e in} \quad Q, \\ \| \mu_n \|_{L^1(Q)} \leq \| \mu \|_{\mathcal{M}^+(Q)}, \end{cases} \tag{4.2}$$

for every $n \in \mathbb{N}$. By [21, 22, 43], the approximation problem (H_n) has a unique solution f_n in $C^1((0,T), L^2(\Omega)) \cap L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$.

In the next proposition, we establish the relationship between the approximation solution f_n and any test function in (P_n).

Proposition 4.1. Suppose that $\xi \in C_c^1(Q)$ such that $\frac{\partial \xi}{\partial \eta} = 0$ on S, the test function in (H_n) and f_n the approximation solution of the problem (H_n). Then, the following expression holds

$$f_n(x,t)\xi(x,t) = \int_\Omega G_N(x-y,t)\,\xi(y,0)u_{0n}(y)dy + \int_0^t \int_\Omega G_N(x-y,t-\sigma)\,\{f_n\xi_\sigma - 2\nabla f_n \nabla \xi - f_n \Delta \xi\}\,dyd\sigma +$$

$$+ \int_0^t \int_\Omega G_N(x-y,t-\sigma)\xi(y,\sigma)\mu_n(y,\sigma)dyd\sigma + \int_0^t \int_{\partial\Omega} G_N(x-y,t-\sigma)\xi(y,\sigma)g(f(y,\sigma))d\mathcal{H}(y)d\sigma \tag{4.3}$$

where ξ_σ is first-order derivative order of ξ with respect to σ.

Remark 4.2. Assume that the test function $\xi = \rho \in C_c^2(\Omega)$, then we obtain

$$f_n(x,t)\rho(x) = \int_\Omega G_N(x-y,t)\,\rho(y)u_{0n}(y)dy - \int_0^t \int_\Omega G_N(x-y,t-\sigma)\,\{2\nabla f_n \nabla \rho + f_n \Delta \rho\}\,dyd\sigma +$$

$$+ \int_0^t \int_\Omega G_N(x-y,t-\sigma)\rho(y)\mu_n(y,\sigma)dyd\sigma + \int_0^t \int_\Omega G_N(x-y,t-\sigma)\rho(y)g(f(y,\sigma))d\mathcal{H}(y)d\sigma. \tag{4.4}$$

On the other hand, we suppose that the test function $\xi = \theta(t) \in C^1(0,T)$ then (4.3) reads

$$f_n(x,t)\theta(t) = \int_\Omega G_N\,(x-y,t)\,\theta(0)u_{0n}(y)dy + \int_0^t \int_\Omega G_N(x-y,t-\sigma)f_n(y,\sigma)\theta'(\sigma)dyd\sigma +$$

$$+\int_0^t \int_\Omega G_N(x-y,t-\sigma)\theta(\sigma)\mu_n(y,\sigma)dyd\sigma + \int_0^t \int_\Omega G_N(x-y,t-\sigma)\theta(\sigma)g(f(y,\sigma))d\mathcal{H}(y)d\sigma. \quad (4.5)$$

Proof of Proposition 4.1. Assume that $\xi \in C^1((0,T), C_c^2(\Omega))$ such that $\frac{\partial \xi}{\partial \eta} = 0$ on S, a test function in (H_n), then the following equation

$$\begin{cases} (f_n\xi)_t = \Delta(f_n\xi) + f_n\xi_t - 2\nabla f_n \nabla\xi - f_n\Delta\xi + \mu_n\xi & \text{in } \Omega\times(0,T),\\ \frac{\partial(f_n\xi)}{\partial\eta} = g(f_n)\xi & \text{on } \partial\Omega\times(0,T),\\ f_n(x,0)\xi(x,0) = u_{0n}\xi(x,0) & \text{in } \Omega, \end{cases} \quad (H_\xi)$$

is well-defined. By [35, Chapter 20, Section 20.2], the problem (H_ξ) admits a unique solution $f_n\xi$ expressed in (4.3). □

Proof of Theorem 2.1 (i) We argue this proof into two steps:

Step 1. We show that $\{f_n(\cdot,t)\}$ is a Cauchy sequence in $L^1(\Omega)$ a.e in $(0,T)$. To attain this, we use the expression (4.3) to prove the Cauchy sequence. Indeed, for any $m,n \in \mathbb{N}$ there holds

$$f_n(x,t) - f_m(x,t) = \int_\Omega G_N(x-y,t)[u_{0n}(y) - u_{0m}(y)]dy +$$

$$+\int_0^t \int_\Omega G_N(x-\xi,t-\sigma)[\mu_n(\xi,\sigma) - \mu_m(y,\sigma))]dyds +$$

$$+\int_0^t \int_{\partial\Omega} G_N(x-y,t-\sigma)[g(f_n(\xi,\sigma)) - g(f_m(\xi,s))]d\mathcal{H}(y)ds. \quad (4.6)$$

From the assumption (2.12), the Eq (4.6) yields

$$\int_\Omega |\, f_n(x,t) - f_m(x,t)\,|\, dx \le \int_\Omega |\, u_{0n}(y) - u_{0m}(y)\,|\, dy + \int_0^t \int_\Omega |\, \mu_n(y) - \mu_m(y)\,|\, dyd\sigma$$

$$+\int_0^t \int_{\partial\Omega} G_N(x-\xi,t-s)d\mathcal{H}(\xi)\left(\int_\Omega |\, g(f_n(x,s)) - g(f_m(x,s))\,|\, dx\right)ds. \quad (4.7)$$

Furthermore, by using the mean value theorem, we find that there exists a function $\theta(x,s)$ which is continuous in $\overline{Q}_{T_1}$ such that $\alpha_1 < \theta(x,s) < \alpha_2$, $g(f_n(x,s)) - g(f_m(x,s)) = g'(\theta(x,s))(f_n(x,s) - f_m(x,s))$, where $g'(\theta(x,s)) \in L^\infty(\mathbb{R}_+)$(see assumption (I)-(i)) and $0 < \alpha_1 < \alpha_2$ are constants, therefore we obtain

$$\int_\Omega |\, f_n(x,t) - f_m(x,t)\,|\, dx \le \int_\Omega |\, u_{0n}(y) - u_{0m}(y)\,|\, dy + \int_0^T \int_\Omega |\, \mu_n(y) - \mu_m(y)\,|\, dy +$$

$$+C(T_1)\int_0^t \int_\Omega |\, f_n(x,\sigma) - f_m(x,\sigma)\,|\, dxd\sigma, \quad (4.8)$$

whenever $C(T_1) = \sup_{(\xi,\sigma)\in\overline{Q}_{T_1}} \int_{\partial\Omega} G_N(x-\xi,T_1)g'(\theta(x,s))d\mathcal{H}(\xi) > 0$. By the property (2.13) of the Green function $G_N(x-\xi,t-\sigma)$ of the heat equation with nonhomogeneous Neumann boundary and the fact that $g'(\theta(x,s)) \in L^{\infty}(\mathbb{R}_+)$, then $C(T_1)$ is a constant depending on T_1. From the Gronwall's inquality, the inequality (4.8) yields

$$\int_{\Omega} | f_n(x,t) - f_m(x,t) | \, dx \leq C(T,T_1) \int_{\Omega} | u_{0n}(y) - u_{0m}(y) | \, dy+$$
$$+C(T,T_1) \int_{\Omega} | \mu_n(y,\sigma) - \mu_m(y,\sigma) | \, dyd\sigma \tag{4.9}$$

for a.e $0 \leq t < T_1 < T$ and $C(T,T_1) = 1 + C(T_1)Te^{C(T_1)T} > 0$ a constant.

Since the sequences $\{u_{0n}\}$ and $\{u_{0m}\}$ are satisfying the assumption (4.1) and $\{\mu_n\}$ and $\{\mu_m\}$ are verifying the assumption (4.2), then by passing to the limit as n and m go to infinity, there holds

$$\lim_{n,m\to+\infty} \int_{\Omega} | f_n(x,t) - f_m(x,t) | \, dx \leq C(T,T_1) \limsup_{n\to+\infty} \int_{\Omega} | u_{0n}(y) - u_{0r}(y) | \, dy+$$
$$+C(T,T_1) \limsup_{m\to+\infty} \int_{\Omega} | u_{0m}(y) - u_{0r}(y) | \, dy+$$
$$+C(T,T_1) \limsup_{n\to+\infty} \int_{\Omega} | \mu_n(y,\sigma) - \mu_r(y,\sigma) | \, dyd\sigma+$$
$$+C(T,T_1) \limsup_{m\to+\infty} \int_{\Omega} | \mu_m(y,\sigma) - \mu_r(y,\sigma) | \, dyd\sigma \leq 0. \tag{4.10}$$

Hence the sequence $\{f_n(\cdot,t)\}$ is Cauchy in $L^1(\Omega)$ for almost every $t \in (0,T)$.

Step 2. We show that $f_n(\cdot,t) \overset{*}{\rightharpoonup} f(\cdot,t)$ in $\mathcal{M}^+(\Omega)$ a.e in $(0,T)$.

Since the function $f_n(x,t)$ is a solution of the approximation problem (H_n) and $\mu_n(x) \geq 0$ in Q, $u_{0n}(x) \geq 0$ in Ω, $g > 0$ in $\mathbb{R}_+$, then we apply the maximum principal theorem in [22, 43] and then the solution of the approximation problem (H_n) is nonnegative in $\overline{Q}$. Likewise, we assume that $\xi(x,t) \equiv 1$, then we obtain

$$f_n(x,t) = \int_{\Omega} G_N(x-y,t-\sigma)u_{0n}(y)dy + \int_0^t \int_{\Omega} G_N(x-y,t-\sigma)\mu_n(y,\sigma)dyd\sigma+$$
$$+ \int_0^t \int_{\partial\Omega} G_N(x-y,t-\sigma)g(f(y,\sigma))d\mathcal{H}(y)d\sigma. \tag{4.11}$$

By the assumptions (A), (2.12), (2.13), (4.1) and (4.2), we infer that

$$\int_{\Omega} f_n(x,t)dx \leq \| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)} + C \int_0^t \int_{\Omega} f_n(x,\sigma)dxd\sigma. \tag{4.12}$$

By Gronwall's inequality, we deduce that

$$\| f_n(\cdot,t) \|_{L^1(\Omega)} \leq e^{Ct} \left(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)} \right), \tag{4.13}$$

for almost every $t \in (0,T)$.

By *Step 1*, the sequence $\{f_n(\cdot,t)\}$ is Cauchy in $L^1(\Omega)$, then we infer that $f_n(\cdot,t) \to f(\cdot,t)$ a.e in $(0,T)$. Hereby we argue as in [9, Proposition 5.3], one proves that $f(\cdot,t) \in \mathcal{M}^+(\Omega)$ and the following convergence

$$f_n(\cdot,t) \overset{*}{\rightharpoonup} f(\cdot,t) \quad \text{in} \quad \mathcal{M}^+(\Omega) \tag{4.14}$$

for almost every $t \in (0,T)$ holds.

From [44, Chapter 5, Section 5.2.1, Theorem 1], the estimate (4.13) yields

$$\| f(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \leq \liminf_{n\to+\infty} \int_\Omega f_n(\cdot,t)dx \leq e^{Ct}\left(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}\right).$$

The estimate (2.10) is achieved.

(ii) Now we show the uniqueness solutions to the problem (H).

To attain this, we consider f_1 and f_2 two every weak solutions of the problem (H) in sense of Definition 2.5 with initial data u_{01} and u_{02} respectively.

Let $\{f_{1n}\}$, $\{f_{2n}\} \subseteq L^\infty(Q) \cap L^2((0,T), H^1(\Omega))$ be two weak solutions given by the proof (i) of Theorem 2.1. Assume that $\{u_{01n}\}$, $\{u_{02n}\}$ $\{\mu_{1n}\}$, $\{\mu_{2n}\}$ are approximating Radon measures in sense of Definition 2.4 and f_{1n}, f_{2n} in (4.11) hold. Since we have assumed that $g(f) = \overline{K}$ almost everywhere in S, thus $g(f_{1n}) = g(f_{2n}) = \overline{K}$ on S. For any $\xi \in C_c^1(Q)$ such that $\frac{\partial\xi}{\partial\eta} = 0$ on S, there holds

$$\int_0^T [f_{1n}(y,t) - f_{2n}(y,t)]\xi(y,t)dt = \int_Q [f_{1n}(x,t) - f_{2n}(x,t)]\delta_y(x)\xi(x,t)dxdt$$

$$= \int_0^T \int_\Omega G_N(0,t)\,(u_{01n}(y) - u_{02n}(y))\xi(y,t)dydt +$$

$$+ \int_0^T \int_0^t \int_\Omega G_N(0,t-\sigma)(\mu_{1n}(y,\sigma) - \mu_{2n}(y,\sigma))\xi(y,t)dyd\sigma =: I_1 + I_2. \tag{4.15}$$

Let us evaluate the limit of I_1 and I_2 when $n \to \infty$. To attain this, we begin with the expression I_1:

$$I_1 = \left[\int_\Omega (u_{01n}(y) - u_{01n}(y))\xi(y,t)dy\right]\left[\int_0^T G_N(0,t)\,dt\right].$$

Taking $\xi(y,t) = \rho(y)\widetilde{h}(t)$ with $\rho \in C^2(\Omega)$ such that $\frac{\partial\rho}{\partial\eta} = 0$ on $\partial\Omega$ and $\widetilde{h} \in C_c(0,T)$, then we have

$$I_1 = \int_0^T \widetilde{h}(t)G_N(0,t)\,dt \int_\Omega (u_{01n}(y) - u_{02n}(y))\rho(y)dy$$

$$= \int_0^T \widetilde{h}(t)G(0,t)\,dt\left[\int_\Omega (f_{01n}(y) - f_{02n}(y))\rho(y)dy - \int_\Omega (F_{01n}(y) - F_{02n}(y))\rho(y)dy\right].$$

Passing to the limit when $n \to \infty$, there holds

$$\lim_{n\to\infty} I_1 = 0. \tag{4.16}$$

Now we consider the expression I_2,

$$I_2 = \left[\int_0^T \int_\Omega (\mu_{1n}(y,t) - \mu_{2n}(y,t))\xi(y,t)dydt\right]\left[\int_0^t G(0,t-\sigma)d\sigma\right].$$

According to Definition 2.4, it is worth observing that

$$I_2 = \left[\int_0^t G(0, t-\sigma)d\sigma\right]\left[\int_Q (f_{11n}(y,t) - f_{12n}(y,t))\xi(y,t)dydt\right] -$$

$$-\left[\int_0^t G(0, t-\sigma)d\sigma\right]\left[\int_Q (F_{1n}(y,t) - F_{2n}(y,t))\xi(y,t)dydt\right] +$$

$$+\left[\int_0^t G(0, t-\sigma)d\sigma\right]\left[\int_Q (\varphi_{1n}(y,t) - \varphi_{2n}(y,t))\xi_t(y,t)dydt\right] +$$

$$+\left[\int_0^t G(0, t-\sigma)d\sigma\right]\left[\int_\Omega (\varphi_{1n}(y,0) - \varphi_{2n}(y,0))\xi(y,0)dy\right].$$

We pass to the limit when n goes to infinity, therefore

$$\lim_{n\to\infty} I_2 = 0. \tag{4.17}$$

By (4.16), (4.17) and Dominated Convergence theorem, we obtain

$$\int_Q (f_1(x,t) - f_2(x,t))\xi(x,t)dxdt = 0, \tag{4.18}$$

which leads to

$$f_{1n} \overset{*}{\rightharpoonup} f_1 \ \ \mathcal{M}^+(Q) \ \text{ and } \ f_{2n} \overset{*}{\rightharpoonup} f_2 \ \ \mathcal{M}^+(Q).$$

Hence $f_1 = f_2$ holds. □

Remark 4.1 (i) Since $f \in \mathcal{M}^+(Q)$, then it is worthy observing that f_n in (4.11) is a sequence of the approximation Radon measure f satisfying the following properties

$$\begin{cases} f_n \overset{*}{\rightharpoonup} f \quad \text{in} \quad \mathcal{M}^+(Q), \\ f_n \to f \quad \text{a.e in} \quad Q, \\ \| f_n \|_{L^1(Q)} \leq C(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}), \\ f \ \text{ is given in } \ (2.9), \end{cases} \tag{4.19}$$

for every $n \in \mathbb{N}$ and $C > 0$ is a constant.

(ii) By (2.11)–(2.13) and the assumption (A), we deduce from the compactness theorem in [23] the approximation problem (H_n) possesses a weak solution f in $L^2((0,T), H^1(\Omega))$ such that the properties

$$\begin{cases} f_n = T_n(f), \\ f_n \to f \quad \text{a.e in} \quad Q, \\ | f_n | \leq | f |, \end{cases} \tag{4.20}$$

hold true.

5. Existence results of the problem (P)

Proof of Lemma 2.1. To prove this result, we use Definition 2.2 and we recall the Gauss-green formula given by the functional

$$\langle \mathcal{T}_\nu, \xi \rangle = \int_Q \xi \operatorname{div}\mathcal{F} + \int_Q \nabla\xi \cdot \mathcal{F} \tag{5.1}$$

Since there exists a linear continuous functional $\mathcal{T}_\nu$ on $W^{\frac{1}{2},2}(S) \cap C(S)$ which stands for $\mathcal{F}\cdot\nu$, then we define a notion of the normal trace of the flux $\nabla\psi(u_r)\cdot\nu$ such that

$$\langle \mathcal{T}_\eta, \xi \rangle = \langle \mathcal{T}_\nu, \xi \rangle + \int_\Omega \xi(x,0)du_0(x) + \int_Q h(t)f(x,t)\xi dxdt. \tag{5.2}$$

The definition make sense because of the definition of the weak solution when we assume that the value of the initial data

$$\lim_{s\to 0}\frac{1}{s}\int_0^s \int_\Omega u(x,t)\xi(x,t)dxdt = \int_\Omega \xi(x,0)du_0 \tag{5.3}$$

holds, for any $\frac{s-t}{s}\chi_{(0,s)}(t)\phi$ as a test function in (2.1). In particular $\langle \mathcal{T}_\nu, \xi\sigma\rangle$ depends only on ξ_S and from (2.2), we infer the formula

$$\langle \mathcal{T}_\nu, \xi \rangle = -\lim_{s\to 0}\frac{1}{s}\int_s^T \int_\vartheta \int_{\zeta(\overline{x})}^{\zeta(\overline{x})+s} \nabla\psi(u_r)\begin{pmatrix} -\nabla\zeta(\overline{x}) \\ 1 \end{pmatrix}\xi\varsigma dx_N d\overline{x}dt \tag{5.4}$$

for any $\xi \in C_c^\infty([0,T)\times\overline{\Omega})$. We denote $\{v_\delta\}$ a boundary-layer sequence of $C^2(\Omega)\cap C(\overline{\Omega})$ such that

$$\lim_{\delta\to 0^+} v_\delta = 1 \ \text{ a.e in } \ \Omega, \quad 0 \le v_\delta \le 1, \ \ v_\delta = 0 \quad \text{on } \ \partial\Omega. \tag{5.5}$$

For more properties concerning the boundary-layer sequence $\{v_\delta\}$ (see [37, Lemma 5.5 and Lemma 5.7]). If $\xi \in \left(H^1(\Omega)\right)^N$, then

$$\lim_{\delta\to 0^+}\int_\Omega \varsigma\xi\nabla v_\delta = -\lim_{\delta\to 0^+}\int_\Omega \operatorname{div}(\varsigma\xi)v_\delta = -\int_\Omega \operatorname{div}(\varsigma\xi) = -\int_{\partial\Omega}\varsigma\xi\cdot\eta d\mathcal{H}(x) \tag{5.6}$$

The previous statement (5.6) explains that for any function-valued $F : \Omega \to \mathbb{R}^N$, then $-F\cdot\nabla v_\delta$ approaches the normal trace $F\cdot\eta$. Let $\xi \in C_c^\infty([0,T)\times\overline{\Omega})$ and $\xi = \xi(1-v_\delta)$ on S, it implies that $\langle \mathcal{T}_\nu, \xi\varsigma\rangle = \langle \mathcal{T}_\nu, \varsigma\xi(1-v_\delta)\rangle$. By Definition 2.2 and the equation (5.2), the Gauss-Green formula yields

$$\begin{aligned}\left\langle \mathcal{T}_\eta, \varsigma\xi \right\rangle &= \langle \mathcal{T}_\nu, \varsigma\xi(1-v_\delta)\rangle + \int_\Omega \xi(x,0)\varsigma(1-v_\delta)du_0 + \int_Q h(t)f(x,t)\xi(x,t)\varsigma(1-v_\delta)dxdt \\ &= \int_Q \xi\varsigma(1-v_\delta)\operatorname{div}\mathcal{F} + \int_Q \nabla(\xi\varsigma(1-v_\delta))\cdot\mathcal{F} + \int_\Omega \xi(x,0)\varsigma(1-v_\delta)du_0 + \\ &\quad + \int_Q h(t)f(x,t)\xi(x,t)\varsigma(1-v_\delta)dxdt.\end{aligned}$$

Since $0 \leq v_\delta \leq 1$ and $v_\delta \to 1$ a.e in Ω as $\delta \to 0^+$, then Dominated Convergence Theorem ensures that

$$\lim_{\delta\to 0^+}\int_Q \xi\varsigma(1-v_\delta)\mathrm{div}\mathcal{F} = 0 \quad \text{and} \quad \left\langle \mathcal{T}_\eta, \xi\varsigma\right\rangle = -\lim_{\delta\to 0^+}\int_Q \nabla\xi(u_r)\nabla v_\delta \xi\varsigma dxdt \tag{5.7}$$

On the other hand, we consider $\xi\varsigma(1-v_\delta)$ as a test function in the problem (P) then the following expression holds

$$-\int_\Omega \phi(x,0)\varsigma(1-v_\delta)du_0 + \int_\Omega u(x,T)\xi(x,T)\varsigma(1-v_\delta)dx - \int_Q u(x,t)\xi_t(x,t)\varsigma(1-v_\delta)dxdt$$

$$= -\int_Q \nabla\psi(u_r)\nabla(\xi\varsigma)(1-v_\delta)dxdt + \int_Q \nabla\psi(u_r)\nabla v_\delta\xi\varsigma dxdt + \int_S g(u_r)\xi\varsigma d\mathcal{H}(x)dt + .$$

$$+\int_Q h(t)f(x,t)\xi(x,t)\varsigma(1-v_\delta)dxdt.$$

Since $0 \leq v_\delta \leq 1$ and $v_\delta \to 1$ a.e in Ω as $\delta \to 0^+$, then Dominated Convergence Theorem yields

$$\int_S g(u_r)\xi\varsigma d\mathcal{H}(x)dt = -\lim_{\delta\to 0^+}\int_Q \nabla\psi(u_r)\nabla v_\delta\xi\varsigma dxdt. \tag{5.8}$$

By combining the assertions (5.7) with (5.8), the statement (2.3) is satisfied. □

Proof of Theorem 2.2. Assume that for any compact set $K = K_0 \times [0,T] \subset \Omega\times(0,T) \subset \mathbb{R}^N \times \mathbb{R}_+$ (resp. for any compact set $K_0 \subset \Omega \subset \mathbb{R}^N$) such that $\mu^-(K) = 0$ (resp. $u_0^-(K_0) = 0$) and $\mathrm{Cap}(K) = 0$ (resp. $C_2(K_0) = 0$). To show that μ and u_0 are absolutely continuous measures with respect to the parabolic capacity, it is enough to prove that $\mu^+(K) = 0$ (resp. $u_0^+(K_0) = 0$). To this purpose $\mathrm{Cap}(K) = 0$ (resp. $C_2(K_0) = 0$), there exists a sequence $\{\varphi_n(t)\} \subset C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+)$ (resp. $\{\varphi_n(0)\} \subset C_c^\infty(\mathbb{R}^N)$) such that $0 \leq \varphi_n(t) \leq 1$ in Q (resp. $0 \leq \varphi_n(0) \leq 1$ in Ω), $\varphi_n(t) \equiv 1$ in K (resp. $\varphi_n(0) \equiv 1$ in K_0) and $\varphi_n(t) \to 0$ in W as $n \to \infty$ (resp. $\varphi_n(0) \to 0$ in $H^1(\Omega)$ as $n \to \infty$). In particular $\| \Delta\varphi_n(t) \|_{L^1(Q)} \to 0$ as $n \to \infty$.

Let us consider the nonnegative function $\varphi_n(t) \in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+)$ such that $\varphi_n(T) = 0$ in Ω and $\frac{\partial\varphi_n(t)}{\partial\eta} = 0$ in S as a test function in the problem (P), then there holds

$$\int_\Omega \varphi_n(0)du_0 + \int_Q u\varphi_{nt}(t)dxdt = -\int_Q \psi(u_r)\Delta\varphi_n(t)dxdt - \int_Q h(t)f(x,t)\varphi_n(t)dxdt-$$

$$-\int_S g(u_r)\varphi_n(t)d\mathcal{H}(x)dt. \tag{5.9}$$

By (4.3)(Probably n is large enough), the following statement holds

$$\int_Q f(x,t)h(t)\varphi_n(t)dxdt = \int_Q\int_0^t\int_\Omega G_N(x-y,t-\sigma)f(y,\sigma)(h(\sigma)\varphi_n(\sigma))_\sigma dyd\sigma dxdt$$

$$-\int_Q\int_0^t\int_\Omega G_N(x-y,t-\sigma)[2\nabla f\nabla(h(\sigma)\varphi_n(\sigma)) + f\Delta(h(\sigma)\varphi_n(\sigma))]dyd\sigma dxdt+$$

$$+\int_Q\int_0^t\int_\Omega G_N(x-y,t-\sigma)h(\sigma)\varphi_n(\sigma)d\mu(y,\sigma)dxdt+$$

$$+\int_Q\int_0^t\int_{\partial\Omega} G_N(x-y,t-\sigma)h(\sigma)\varphi_n(\sigma)g(f(y,\sigma))d\mathcal{H}(y)d\sigma dxdt. \tag{5.10}$$

Combining the Eq (5.9) with (5.10), we obtain

$$\int_Q\int_\Omega\int_0^t G_N(x-y,t-\sigma)h(\sigma)\varphi_n(\sigma)d\mu(y,\sigma)dxdt+\int_\Omega \varphi_n(0)du_0 =$$

$$=\int_Q\int_0^t\int_\Omega h(\sigma)G_N(x-y,t-\sigma)[2\nabla f\nabla\varphi_n(\sigma)+f\Delta\varphi_n(\sigma)]dyd\sigma dxdt-$$

$$-\int_Q\int_0^t\int_\Omega G_N(x-y,t-\sigma)f(y,\sigma)(h(\sigma)\varphi_n(\sigma))_\sigma dyd\sigma dxdt-$$

$$-\int_Q\int_0^t\int_{\partial\Omega} G_N(x-y,t-\sigma)h(\sigma)\varphi_n(\sigma)g(f(y,\sigma))d\mathcal{H}(y)d\sigma dxdt-$$

$$-\int_S \varphi_n(t)g(u_r)d\mathcal{H}(x)dt-\int_Q \psi(u_r)\Delta\varphi_n(t)dxdt-\int_Q u\varphi_{nt}(t)dxdt. \tag{5.11}$$

By (2.14), $f\in L^2((0,T),H^1(\Omega))$ (see Remark 4.1-(ii)) and letting $\sigma\to t$, $\int_0^t h(\sigma)d\sigma=1$ and dropping down the nonnegative terms on the left hand-side of the previous equation. Therefore (5.11) yields

$$\int_Q \varphi_n(t)d\mu(x,t)+\int_\Omega \varphi_n(0)du_0(x)\le \| u\|_{L^2((0,T),H^1(\Omega))}\|\varphi_n(t)\|_W+$$

$$C(\gamma)\int_Q |\Delta\varphi_n(t)|\,dxdt+\| f\|_{L^2((0,T),H^1(\Omega))}\|\varphi_n(t)\|_W\,. \tag{5.12}$$

Since the following assertions are valid, then

$$\mu^+(K)\le\int_Q\varphi_n(t)d\mu(x,t)+\int_Q\varphi_n(t)d\mu^-(x,t) \tag{5.13}$$

where $\mu^+(K)=\mu(K)+\mu^-(K)$ and

$$u_0^+(K_0)\le\int_\Omega\varphi_n(0)du_0(x)+\int_\Omega\varphi_n(0)du_0^-(x) \tag{5.14}$$

with $u_0^+(K_0)=u_0(K_0)+u_0^-(K_0)$. In view of (5.13) and (5.14), the inequality (5.12) reads as

$$\mu^+(K)+u_0^+(K_0)\le\| u\|_{L^2((0,T),H^1(\Omega))}\|\varphi_n(t)\|_W+C(\gamma)\int_Q|\Delta\varphi_n(t)|\,dxdt+$$

$$+\| f\|_{L^2((0,T),H^1(\Omega))}\|\varphi_n(t)\|_W+\int_Q\varphi_n(t)d\mu^-(x,t)+\int_\Omega\varphi_n(0)du_0^-(x). \tag{5.15}$$

Since $\mu^-(K)=0$ (resp. $u_0^-(K_0)=0$), then for any $\epsilon>0$ one has

$$\int_Q\varphi_n(t)d\mu^-(x,t)<\frac{\epsilon}{2}\qquad\left(\text{resp.}\quad\int_\Omega\varphi_n(0)du_0^-(x)<\frac{\epsilon}{2}\right). \tag{5.16}$$

Then, the limit in (5.16) as $n \to \infty$, the following holds $\mu^+(K) + u_0^+(K_0) \leq \epsilon$. Therefore, $\mu^+(K) = 0$ for any compact set $K \subset Q$ (resp. $u_0^+(K_0) = 0$ for any compact set $K_0 \subset \Omega$). □

To prove the existence and decay estimates of the solutions, we consider the following problem

$$\begin{cases} u_{nt} = \Delta\psi_n(u_n) + h(t)f_n(x,t) & \text{in } Q, \\ \frac{\partial\psi_n(u_n)}{\partial\eta} = g(u_n) & \text{on } S, \\ u_n(x,0) = u_{0n} & \text{in } \Omega, \end{cases} \tag{P_n}$$

where the sequence $\{u_{0n}\} \subseteq C_c^\infty(\Omega)$ satisfies the assumption (4.1) and the sequence $\{f_n\} \subseteq C_c^\infty(\overline{Q})$ fulfills the hypothesis (4.19). We set

$$\psi_n(s) = \psi(s) + \frac{1}{n} \tag{5.17}$$

By [8, 18, 21, 22], the approximating problem (P_n) has a solution u_n in $C((0,T), L^1(\Omega)) \cap L^\infty(Q)$. Then, the definition of the weak solution $\{u_n\} \subseteq C^\infty(\overline{Q})$ of (P_n) satisfies the following expression

$$\int_0^T \langle u_n(\cdot,t), \xi_t(\cdot,t)\rangle_\Omega dt + \int_0^T h(t)\langle f_n(\cdot,t), \xi(\cdot,t)\rangle_\Omega\, dt + \langle u_{0n}, \xi(\cdot,0)\rangle_\Omega +$$

$$+ \int_0^T \langle g(u_n), \xi\rangle_{\partial\Omega}\, dt = \int_0^T \langle \nabla\psi_n(u_n), \nabla\xi\rangle_\Omega\, dxdt \tag{5.18}$$

for every ξ in $C^1(\overline{Q})$ such that $\xi(\cdot,T) = 0$ in Ω and $\frac{\partial\xi}{\partial\eta} = 0$ on S.

Now we establish some technical estimates which will be used in the proof of the existing solution.

Lemma 5.2 Assume that (I), (J), (A), $\mu \in \mathcal{M}^+(Q)$ and $u_0 \in \mathcal{M}^+(\Omega)$ are satisfied. Let u_n be the solution of the approximation problem (P_n), then

$$\| u_n(\cdot,t) \|_{L^1(\Omega)} \leq C\left(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}\right). \tag{5.19}$$

$$\| \nabla\psi_n(u_n) \|_{L^2(Q)} + \| \psi_n(u_n) \|_{L^2(Q)} \leq C\left(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}\right). \tag{5.20}$$

for almost every $t \in (0,T)$ and C is a positive constant.

The sequence $\{[\psi_n(u_n)]_t\}$ is bounded in $L^2((0,T),(H^1(\Omega))^*) + L^1(\overline{Q})$.

Proof of Lemma 5.2. To prove the estimate (5.19), we consider the approximation problem (P_n) such that

$$\begin{cases} u_{ns} = \Delta\psi_n(u_n) + h(s)f_n(x,s) & \text{in } \Omega\times(\tau,\tau+t), \\ \frac{\partial\psi_n(u_n)}{\partial\eta} = g(u_n) & \text{on } \partial\Omega\times(\tau,\tau+t), \\ u_n(x,\tau) = u_{0n}(x) & \text{in } \Omega\times\{\tau\}, \end{cases} \tag{5.21}$$

where $\tau + t \leq T$ and $\tau, t \in (0,T)$.

Let us consider $\xi \in C^{1,2}(\overline{\Omega}\times[\tau,\tau+t])$ such that $\frac{\partial\xi}{\partial\eta} = 0$ on $\partial\Omega\times(\tau,\tau+t)$ and $\xi(\cdot,\tau+t) = 0$ in Ω as a test function in the above approximation problem (5.21), then we have

$$\int_{\Omega\times(\tau,\tau+t)} u_n\xi_s dxds + \int_{\Omega\times(\tau,\tau+t)} \psi_n(u_n)\Delta\xi dxds + \int_{\partial\Omega\times(\tau,\tau+t)} g(u_n)\xi d\mathcal{H}(x)ds +$$

$$+ \int_{\Omega\times(\tau,\tau+t)} h(s)f_n(x,s)\xi(x,s)dxds + \int_\Omega \mu_n(x)\xi(x,\tau)dx = 0. \tag{5.22}$$

By the mean value theorem and the assumption (I), the Eq (5.22) yields

$$\int_{\Omega\times(\tau,\tau+t)} u_n(\xi_s+\theta_n\Delta\xi)dxds+\int_{\partial\Omega\times(\tau,\tau+t)} g(u_n)\xi d\mathcal{H}(x)ds+$$
$$+\int_{\Omega\times(\tau,\tau+t)} h(s)f_n(x,s)dxds+\int_{\Omega}\mu_n(x)\xi(x,\tau)dx=0, \tag{5.23}$$

where $\theta_n(x,t)=\int_0^1 \psi'_{u_n}(\alpha u_n)d\alpha$.

On the other hand, we consider the following backward parabolic equations

$$\begin{cases} -\phi_s-\theta_\epsilon\Delta\phi=\frac{1}{\tau} & \text{in } Q_\tau=\Omega\times(\tau,\tau+t),\\ \frac{\partial\phi}{\partial\eta}=0 & \text{on } S_\tau=\partial\Omega\times(\tau,\tau+t),\\ \phi(\cdot,\tau+t)=0 & \text{in } \Omega\times\{\tau+t\}, \end{cases} \tag{5.24}$$

has an unique solution ϕ in $C^{1,2}(\overline{Q}_\tau)\cap C(Q_\tau)$ and $0<\phi\le C$ for any $\tau,t\in(0,T)$ (see [18, Lemma 4.2]). Then for $\xi=\phi$, there holds

$$\frac{1}{\tau}\int_\tau^{\tau+t}\int_\Omega u_n(x,s)dxds=\int_\Omega \mu_n(x)\phi(x,\tau)dx+\int_\tau^{\tau+t}\int_{\partial\Omega} g(u_n)\phi d\mathcal{H}(x)ds+$$
$$+\int_\tau^{\tau+t}\int_\Omega h(s)f_n(x,s)\phi dxds \tag{5.25}$$

By the assumptions (A), (J), (4.19) and (4.1), there exists a positive constant C such that the expression below is satisfied

$$\frac{1}{\tau}\int_\tau^{\tau+t}\int_\Omega u_n(x,s)dxds\le C(\| u_0\|_{\mathcal{M}^+(\Omega)}+\|\mu\|_{\mathcal{M}^+(Q)}). \tag{5.26}$$

By letting $\tau\to 0^+$, we obtain the estimate (5.19). Where $C=C(h(T),\| g(u_n)\|_{L^\infty(\mathbb{R}_+)},| S |)>0$. To prove the estimate (5.20), we consider $T_{\gamma+1}(\psi_n(u_n))$ as a test function in the approximation problem (P_n), then we have

$$\int_{\{(x,t)\in Q_T/\psi_n(u_n)\le\gamma+1\}} |\nabla\psi_n(u_n)|^2\, dxdt=\int_\Omega\left(\int_0^{u_{0n}(x)} T_{\gamma+1}(\psi_n(s))ds\right)dx-$$
$$-\int_\Omega\left(\int_0^{u_n(x,T)} T_{\gamma+1}(\psi_n(s))ds\right)dx+\int_0^T\int_{\partial\Omega} g(u_n)T_{\gamma+1}(\psi_n(u_n))d\mathcal{H}(x)dt+$$
$$+\int_0^T\int_\Omega T_{\gamma+1}(\psi_n(u_n))h(t)f_n(x,t)dxdt \tag{5.27}$$

where $T_\lambda(s)=\min\{\lambda,s\}$. It follows that there exists a positive constant C such that

$$\int_{\{(x,t)\in Q_T/\psi_n(u_n)\le\gamma+1\}} |\nabla\psi_n(u_n)|^2\, dxdt$$
$$\le(\gamma+1)\int_\Omega\mu_n(x)dx+C(\gamma+1)\| g(u_n)\|_{L^\infty(\mathbb{R}_+)}+h(T)\int_Q f_n(x,t)dxdt.$$

For the suitable positive constant $C = C(h(T), \gamma, \| g(u_n) \|_{L^\infty(\mathbb{R}_+)}, \| \mu \|_{\mathcal{M}^+(\Omega)}, \| u_0 \|_{\mathcal{M}^+(\Omega)}, | S |) > 0$, the following estimate holds

$$\int_{\{(x,t)\in Q_T/\psi_n(u_n)\leq\gamma+1\}} | \nabla\psi_n(u_n) |^2 \, dxdt \leq C. \tag{5.28}$$

On the other hand, we assume that $\mathcal{G}_\lambda(s) = \max\{\lambda, s\}$ and we choose $\mathcal{G}_{\gamma+1}(\psi_n(u_n))$ as a test function in the approximation problem (P_n), then we have

$$\begin{aligned}\int_{\{(x,t)\in Q_T/\psi_n(u_n)>\gamma+1\}} | \nabla\psi_n(u_n) |^2 \, dxdt &= \int_\Omega \left(\int_0^{u_{0n}(x)} \mathcal{G}_{\gamma+1}(\psi_n(s)))ds \right) dx - \\ - \int_\Omega \left(\int_0^{u_n(x,T)} \mathcal{G}_{\gamma+1}(\psi_n(s))ds \right) dx &+ \int_0^T \int_{\partial\Omega} g(u_n)\mathcal{G}_{\gamma+1}(\psi_n(u_n))d\mathcal{H}(x)dt + \\ &+ \int_0^T \int_\Omega h(t) f_n(x,t)\mathcal{G}_{\gamma+1}(\psi_n(u_n))d\mathcal{H}(x)dt.\end{aligned} \tag{5.29}$$

It implies that

$$\begin{aligned}&\int_{\{(x,t)\in Q/\psi_n(u_n)>\gamma+1\}} | \nabla\psi_n(u_n) |^2 \, dxdt \\ &\leq (\gamma+1) \int_\Omega \mu_n(x)dx + (\gamma+1)M \| g(u_n) \|_{L^\infty(\mathbb{R}_+)} + h(T) \int_Q f_n(x,t)dxdt\end{aligned}$$

It follows that

$$\int_{\{(x,t)\in Q/\psi_n(u_n)>\gamma+1\}} | \nabla\psi_n(u_n) |^2 \, dxdt \leq C. \tag{5.30}$$

Combining the inequality (5.28) with (5.30), we deduce that

$$\int_Q | \nabla\psi_n(u_n) |^2 \, dxdt \leq C \tag{5.31}$$

By the assumption (I), then $\psi_n(u_n) \in L^2(Q)$, whence the estimate (5.20) holds.

To end the proof of this Lemma, we consider that for every $\xi \in C_c^1(Q)$ such that if we choose $\phi = \psi_n'(u_n)\xi$ arbitrary as a test function in problem (P_n), then the following stands true

$$\int_Q \xi_t[\psi_n(u_n)]dxdt = -\int_Q \xi\psi_n'(u_n)\mathrm{div}\,(\nabla\psi_n(u_n))\,dxdt - \int_Q h(t)f_n(x,t)\psi_n'(u_n)\xi dxdt \tag{5.32}$$

It follows that

$$\begin{aligned}\int_Q \xi_t[\psi_n(u_n)]dxdt &= \int_Q \psi_n'(u_n)\nabla\psi_n(u_n)\nabla\xi dxdt - \int_Q h(t)f_n(x,t)\phi dxdt - \\ &- \int_S g(u_n)\psi_n'(u_n)\xi d\mathcal{H}(x)dt + \int_Q \frac{\psi_n''(u_n)}{\psi_n'(u_n)} | \nabla\psi_n(u_n) |^2 \xi dxdt.\end{aligned} \tag{5.33}$$

Now we estimate each term in the right hand side of (5.33), we obtain

$$\left| \int_Q \psi_n'(u_n)\nabla\psi_n(u_n)\nabla\xi dxdt \right| \leq \| \psi_n' \|_{L^\infty(\mathbb{R}_+)} \int_Q | \nabla\xi \| \nabla\psi_n(u_n) | \, dxdt. \tag{5.34}$$

From Hölder's inequality and (5.31), the inequality (5.34) reads as

$$\left|\int_Q \psi_n'(u_n)\nabla\psi_n(u_n)\nabla\xi dxdt\right| \leq C \parallel \nabla\xi \parallel_{L^2(Q)} . \tag{5.35}$$

By the assumption (J) and (4.19), we deduce the estimate

$$\left|\int_Q h(t)f_n(x,t)\xi dxdt\right| \leq C \parallel \xi \parallel_{L^\infty(Q)} \tag{5.36}$$

where $C = C(h(T), \parallel \mu \parallel_{\mathcal{M}^+(\Omega)}, \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}) > 0$ is a constant.

By the assumptions (A) and (I), there exists a positive constant $C = C(\parallel g(u_n) \parallel_{L^\infty(\mathbb{R}_+)}, \parallel \psi_n'(u_n) \parallel_{L^\infty(\mathbb{R}_+)}) > 0$ such that

$$\int_S g(u_n)\psi_n'(u_n)\xi d\mathcal{H}(x)dt \leq C \parallel \xi \parallel_{L^\infty(S)} . \tag{5.37}$$

Furthermore, one has

$$\left|\int_Q \frac{\psi_n''(u_n)}{\psi_n'(u_n)} \mid \nabla\psi_n(u_n) \mid^2 \xi dxdt\right| \leq \kappa \parallel \xi \parallel_{L^\infty(Q)} \int_Q \mid \nabla\psi_n(u_n) \mid^2 dxdt.$$

In view of (5.28), the expression below holds true

$$\left|\int_Q \frac{\psi_n''(u_n)}{\psi_n'(u_n)} \mid \nabla\psi_n(u_n) \mid^2 \xi dxdt\right| \leq C \parallel \xi \parallel_{L^\infty(Q)} \tag{5.38}$$

where $C = C(\kappa, \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}, \parallel \mu \parallel_{\mathcal{M}^+(Q)}) > 0$. By (5.35)–(5.38) and (5.33), we infer that the sequence $\{[\psi_n(u_n)]_t\}$ is bounded in $L^2((0,T),(H^1(\Omega))^*) + L^1(\overline{Q})$. □

Now we study the limit points of the sequences $\{u_n\}$ and $\psi_n(u_n)$ as $n \to \infty$.

Proposition 5.1 Suppose that the assumptions (I), (A) and (J) are satisfied. Let u_n be the solution of the approximation problem (P_n). Then there exists a subsequence $\{u_{n_k}\} \subseteq \{u_n\}$ and $v \in L^2((0,T), H^1(\Omega)) \cap L^\infty(Q)$ such that

$$\psi_{n_k}(u_{n_k}) \overset{*}{\rightharpoonup} v \quad \text{in} \quad L^\infty(Q). \tag{5.39}$$

$$\psi_{n_k}(u_{n_k}) \rightharpoonup v \quad \text{in} \quad L^2((0,T), H^1(\Omega)). \tag{5.40}$$

$$[\psi_{n_k}(u_{n_k})]_t \rightharpoonup v_t \quad \text{in} \quad L^2((0,T),(H^1(\Omega))^*). \tag{5.41}$$

$$\psi_{n_k}(u_{n_k}) \to v \quad \text{a.e in} \quad Q, \tag{5.42}$$

where $v_t \in L^1(Q)$ and $v \leq \gamma$.

Proof of Proposition 5.1. The convergences (5.39) and (5.40) are the consequence of assumption (I)-(i) and estimate (5.20) respectively. By Lemma 5.1, the sequence $\{[\psi_n(u_n)]_t\}$ is bounded in $L^2((0,T),(H^1(\Omega))^*) + L^1(\overline{Q})$. By [45], there exists a subsequence $\{u_{n_k}\} \subseteq \{u_n\}$ and $v^* \in L^2((0,T), H^1(\Omega)) \cap L^\infty(\overline{Q})$ such that

$$\psi_{n_k}(u_{n_k}) \to v^* \quad \text{a.e in} \quad Q.$$

Furthermore, by [9, Proposition 5.1] and (5.41) holds true and we have

$$\psi_{n_k}(u_{n_k}) \to v \quad \text{a.e in} \quad Q$$

with $v = v^*$ which leads to (5.42) be satisfied. In view of the assumptions (I)-(i) and (5.17), we get

$$\| \psi_{n_k}(u_{n_k}) - \psi(u_{n_k}) \|_{L^\infty(Q)} = \frac{1}{n_k}.$$

Therefore the following convergence $\psi(u_{n_k}) \overset{*}{\rightharpoonup} v \quad \text{in} \quad L^\infty(Q)$ holds true. □

Remark 5.1 For any subsequence $\{u_{n_k}\} \subseteq \{u_n\}$ and v the function given in Proposition 5.1, the following assertions
$\psi^{-1}(v) \in L^\infty((0,T), L^1(\Omega))$, $u_{n_k} \to \psi^{-1}(v) \quad \text{a.e in} \quad Q$ and $u_{n_k} \to g^{-1}(v) \quad \text{a.e in} \quad S$ hold.

Proposition 5.2 Assume that the hypotheses (I), (J), (A), $\mu \in \mathcal{M}^+(Q)$ and $u_0 \in \mathcal{M}^+(\Omega)$ are satisfied. Let $\{u_{n_k}\}$ be the subsequence and v the function mentioned in Proposition 5.1. Then there exist a subsequence $\{u_{n_k}(\cdot, t)\} \subseteq \{u_n(\cdot, t)\}$ and u_a , $u(\cdot, t)$, $u_b(\cdot, t) \in \mathcal{M}^+(\Omega)$ such that

$$u_{n_k}(\cdot, t) \overset{*}{\rightharpoonup} u(\cdot, t) := u_a(\cdot, t) + u_b(\cdot, t) \quad \text{in} \quad \mathcal{M}^+(\Omega), \tag{5.43}$$

$$\psi_{n_k}(u_{n_k})(\cdot, t) \overset{*}{\rightharpoonup} \psi(u_b)(\cdot, t) \quad \text{in} \quad \mathcal{M}^+(\Omega), \tag{5.44}$$

$$g(u_{n_k})(\cdot, t) \overset{*}{\rightharpoonup} g(u_b)(\cdot, t) \quad \text{in} \quad L^\infty(\partial\Omega). \tag{5.45}$$

Moreover, there hold

$$u_b(\cdot, t) = u_r(\cdot, t) \ \text{ a.e in } \ \Omega \ \text{ and } \ u_a(\cdot, t) = u_s(\cdot, t) \ \text{ in } \quad \mathcal{M}^+(\Omega) \tag{5.46}$$

for almost every $t \in (0, T)$. Furthermore $u \in L^\infty((0,T), \mathcal{M}^+(\Omega))$ and for almost every $t \in (0, T)$, there holds

$$\| u(\cdot, t) \|_{\mathcal{M}^+(\Omega)} \leq C(\| \mu \|_{\mathcal{M}^+(Q)} + \| u_0 \|_{\mathcal{M}^+(\Omega)}). \tag{5.47}$$

Proof. By the assumption (I)-(i), $\psi_{n_k}(u_{n_k}) \in L^\infty(Q)$ and using Hölder's inequality, we have

$$\int_Q | \nabla\psi_{n_k}(u_{n_k}) | \, dxdt \leq \left[\int_Q \frac{| \nabla\psi_{n_k}(u_{n_k}) |^2}{(1 + \psi_{n_k}(u_{n_k}))^2} dxdt \right]^{\frac{1}{2}} \left[\int_Q (1 + \psi_{n_k}(u_{n_k}))^2 dxdt \right]^{\frac{1}{2}}$$

$$\leq C \left[\int_Q | \nabla\psi_{n_k}(u_{n_k}) |^2 \, dxdt \right]^{\frac{1}{2}}.$$

From the estimate (5.20), there exists a positive constant $C = C(\| \psi_{n_k}(u_{n_k}) \|_{L^\infty(\mathbb{R}_+)}, \| \mu \|_{\mathcal{M}^+(Q)}, \| u_0 \|_{\mathcal{M}^+(\Omega)}) > 0$ such that

$$\int_Q | \nabla\psi_{n_k}(u_{n_k}) | \, dxdt \leq C. \tag{5.48}$$

According to Lemma 5.1, the assumption (I) and (5.48), we infer that

$$\| \psi_{n_k}(u_{n_k}) \|_{BV(Q)} = \| \psi_{n_k}(u_{n_k}) \|_{L^1(Q)} + \| \nabla\psi_{n_k}(u_{n_k}) \|_{L^1(Q)} + \| [\psi_{n_k}(u_{n_k})]_t \|_{L^1(Q)} \leq C. \tag{5.49}$$

By Fatou's Lebesgue Lemma, we obtain

$$\int_0^T \liminf_{k\to\infty} \int_\Omega \{|\ \psi_{n_k}(u_{n_k})\ | + |\ \nabla\psi_{n_k}(u_{n_k})\ | + |\ [\psi_{n_k}(u_{n_k})]_t\ |\} \le C. \tag{5.50}$$

Then there exists zero Lebesgue measure set $\mathcal{N}_1 \subset (0,T)$ such that

$$\liminf_{k\to\infty} \int_\Omega \{|\ \psi_{n_k}(u_{n_k})\ | + |\ \nabla\psi_{n_k}(u_{n_k})\ | + |\ [\psi_{n_k}(u_{n_k})]_t\ |\}(x,t) \le C \tag{5.51}$$

for every $t \in (0,T)\backslash\mathcal{N}_1$. In view of (5.51), the sequence $\{\psi_{n_k}(u_k)(\cdot,t)\} \subseteq BV(\Omega)$ for every $t \in (0,T)\backslash\mathcal{N}_1$. By [44, Chapter IV, Section 1.1, Proposition 5], there exists a subsequence $\{\psi_{n_k}(u_{n_k})(\cdot,t)\}$ and $v(\cdot,t) \in \mathcal{M}^+(\Omega)$ a.e in $(0,T)$ such that the convergence

$$\psi_{n_k}(u_{n_k})(\cdot,t) \overset{*}{\rightharpoonup} v(\cdot,t) \quad \text{in} \quad \mathcal{M}^+(\Omega) \tag{5.52}$$

holds true. Furthermore, from the assertions (5.19), (5.52) and the Prohorov Theorem (see [44, Chapter II, Section 2.6, Theorem 1] or [25, Proposition A.2] or [17, Proposition 1]), there exists a sequence $\{\widetilde{\tau}^{n_k}\}$ of the Young measures associated with the sequence $\{u_{n_k}\} \subseteq \{u_n\}$ converges narrowly over $\overline{Q}\times\mathbb{R}$ to a Young measure $\widetilde{\tau}$ which the disintegration $\overline{\mu}_{(\cdot,t)}$ is the Dirac mass concentrated at the point $\psi^{-1}(v(\cdot,t))$ for a.e in Ω (see [17]). By [25, Proposition A.4], there exist sequences of measure sets $A_k \subseteq \Omega$, $A_k \subseteq A_{k+1}$ and $|\ A_k\ | \to 0$, such that

$$u_{k_j}(\cdot,t)\chi_{\Omega\backslash A_k} \rightharpoonup u_b(\cdot,t) := \int_{[0,+\infty)} \lambda d\overline{\mu}_{(\cdot,t)}(\lambda) \quad \text{in} \quad L^1(\Omega), \tag{5.53}$$

where $u_b \in L^\infty((0,T),L^1(\Omega))$, $u_b \ge 0$ is a barycenter of the limiting Young measure $\overline{\mu}_{(\cdot,t)}$ associated with the subsequence $\{u_{n_k}(\cdot,t)\}$ and $\text{supp}\overline{\mu}_{(\cdot,t)} \subseteq [0,+\infty)$ for almost every $t \in (0,T)$.

By (5.19) and the compactness result, the sequence $\{u_{n_k}\chi_{\Omega\backslash A_j}\}$ is uniformly bounded in $L^1(\Omega)$. Therefore, there exists a Radon measure $u_a(\cdot,t) \in \mathcal{M}^+(\Omega)$ such that $u_{n_k}(\cdot,t) \overset{*}{\rightharpoonup} u(\cdot,t) \quad \text{in} \quad \mathcal{M}^+(\Omega)$. Finally, the sequence u_{n_k} is of $u_{n_k}(\cdot,t) = u_{n_k}(\cdot,t)\chi_{A_k} + u_{n_k}(\cdot,t)\chi_{\Omega\backslash A_k} \overset{*}{\rightharpoonup} u_a(\cdot,t) + u_b(\cdot,t) \quad \text{in} \quad \mathcal{M}^+(\Omega)$. Hence $u(\cdot,t) := u_a(\cdot,t) + u_b(\cdot,t)$ in $\mathcal{M}^+(\Omega)$ and the statement (5.43) is completed. By the assumption (I)-(*iii*), there holds

$$\lim_{s\to+\infty} \frac{\psi_{n_k}(s)}{s} = 0. \tag{5.54}$$

By the assertion (5.54) and [45, Proposition 5.2] or [25], we obtain

$$\psi_{n_k}(u_k)(\cdot,t) \overset{*}{\rightharpoonup} \psi^*(\cdot,t) \quad \text{in} \quad \mathcal{M}^+(\Omega) \tag{5.55}$$

where $\psi^*(\cdot,t) \in L^1(\Omega)$ and

$$\psi^*(\cdot,t) = \int_{[0,+\infty)} \psi(\lambda) d\overline{\mu}_{(\cdot,t)}(\lambda). \tag{5.56}$$

Furthermore, we also obtain the next result via (5.55)

$$\psi^*(\cdot,t) = \int_{[0,+\infty)} \psi(\lambda) d\overline{\mu}_{(\cdot,t)}(\lambda) = \psi\left(\int_{[0,+\infty)} \lambda d\overline{\mu}_{(\cdot,t)}(\lambda)\right) = \psi(u_b)(\cdot,t).$$

By combining the assertion (5.53) and the previous equality, we conclude that $\psi(u_b)(\cdot,t) = v(\cdot,t)$ a.e in $(0,T)$, when the convergence (5.44) is satisfied.
By virtue of the convergence (5.53), the next convergence result

$$g(u_{n_k}) \to g(\psi^{-1}(v)) := g(u_b) \quad \text{a.e in} \quad S \tag{5.57}$$

holds true. Since the function $g(u_{n_k}) \in L^\infty(\mathbb{R}_+)$ (see assumption (H)-(i)) and from Fatou's Lebesgue Lemma, then there exists a positive constant C such that

$$\int_0^T \liminf_{k\to+\infty} \int_{\partial\Omega} g(u_{n_k}) dxdt \leq C. \tag{5.58}$$

Therefore, there exists a zero Lebesgue measure set $\mathcal{N}_2 \subseteq (0,T)$ such that

$$\liminf_{k\to+\infty} \int_{\partial\Omega} g(u_{n_k})(x,t) dx \leq C \tag{5.59}$$

for every $t \in (0,T)\backslash\mathcal{N}_2$. In view of (5.59) and (5.57), there exists a function $z(\cdot,t) := g(u_b)(\cdot,t) \in L^\infty(\partial\Omega)$ such that the convergence (5.45) is achieved.

To show (5.46), we consider the functions F , G : $\mathbb{R}_+ \to \mathbb{R}_+$ defined by setting

$$\mathrm{F}_\epsilon(s) = \begin{cases} 0 & \text{if} \quad s \leq \frac{1}{\epsilon}, \\ \frac{(\epsilon s-1)^2}{2\epsilon^2} & \text{if} \ \frac{1}{\epsilon} \leq s \leq \frac{1}{\epsilon}+1, \\ s - \frac{1}{\epsilon} - \frac{1}{2} & \text{if} \ s \geq \frac{1}{\epsilon}+1, \end{cases}$$

and $\mathrm{G}_\epsilon(s) = s - \mathrm{F}_\epsilon(s)$ for every $\epsilon > 0$. It is worthy observing that $\mathrm{F}'_\epsilon(s) \geq 0$ in $\mathbb{R}_+$ and $0 \leq \mathrm{F}''_\epsilon(s) \leq \chi_{\{s\geq\frac{1}{\epsilon}\}}(s)$. According to the above results, there exists a subsequence $\{u_{n_k}\}$ in Lemma 5.1 and Proposition 5.1. For any nonnegative function $\rho \in C^2(\overline{\Omega})$, we choose $\mathrm{F}'_\epsilon(u_{n_k})\rho(x)$ as a test function in the approximation problem (P_n), then we obtain the following identity

$$\int_\Omega \mathrm{F}_\epsilon(u_{n_k})(\cdot,\tau)\rho(x)dx \leq \int_\Omega \mathrm{F}_\epsilon(u_{0n_k})\rho(x)dx - \int_0^\tau \int_\Omega \mathrm{F}'_\epsilon(u_{n_k})\nabla\psi_{n_k}(u_{n_k})\nabla\rho(x)dxdt +$$

$$+ \int_0^\tau \int_{\partial\Omega} g(u_{n_k})\mathrm{F}'_\epsilon(u_{n_k})\rho(x)d\mathcal{H}(x)dt + \int_0^\tau \int_\Omega h(t)f_{n_k}(x,t)\mathrm{F}'_\epsilon(u_{n_k})\rho(x)dxdt \tag{5.60}$$

where $\tau \in (0,T)$. Since the sequence $\{\mathrm{F}'_\epsilon(u_{n_k})\}$ is uniformly bounded in $L^\infty(Q)$, then $\mathrm{F}'_\epsilon(u_{n_k}) \to 0$ as $\epsilon \to 0^+$ and $\mathrm{F}_\epsilon(u_{n_k}) \to 0$ as $\epsilon \to 0^+$. By Lemma 5.1 and Proposition 5.1, and by applying the Dominated Convergence Theorem, results to

$$\lim_{k\to+\infty} \int_0^\tau \int_\Omega \mathrm{F}'_\epsilon(u_{n_k})\nabla\psi_{n_k}(u_{n_k})\nabla\rho(x)dxdt = \int_0^\tau \int_\Omega \mathrm{F}'_\epsilon(\psi^{-1}(v))\nabla v\nabla\rho(x)dxdt. \tag{5.61}$$

Similary, we get

$$\lim_{k\to+\infty} \int_0^\tau \int_{\partial\Omega} g(u_{n_k})\mathrm{F}'_\epsilon(u_{n_k})\rho(x)d\mathcal{H}(x)dydt = \int_0^\tau \int_{\partial\Omega} g(\psi^{-1}(v))\mathrm{F}'_\epsilon(\psi^{-1}(v))\rho(x)d\mathcal{H}(x)dt, \tag{5.62}$$

By the statement (4.19) and Proposition 5.1, we have

$$\lim_{k\to+\infty}\int_0^\tau\int_\Omega h(t)f_{n_k}(x,t)\mathrm{F}'_\epsilon(u_{n_k})\rho(x)dxdt=\int_0^\tau\int_\Omega h(t)f(x,t)\mathrm{F}'_\epsilon(\psi^{-1}(v))\rho(x)d\mathcal{H}(x)dt. \tag{5.63}$$

Given the properties of the sequence $\{\mathrm{F}_\epsilon(u_{n_k})\}$ and passing to limit in (5.61), (5.62) and (5.63) when $\epsilon\to 0^+$, then the following holds

$$\lim_{\epsilon\to0^+}\lim_{k\to+\infty}\int_0^\tau\int_\Omega \mathrm{F}'_\epsilon(u_{n_k})\nabla\psi_{n_k}(u_{n_k})\nabla\rho(x)dxdt=0. \tag{5.64}$$

Similarly we obtain

$$\lim_{\epsilon\to0^+}\lim_{k\to+\infty}\int_0^\tau\int_{\partial\Omega} g(u_{n_k})\mathrm{F}'_\epsilon(u_{n_k})\rho(x)d\mathcal{H}(x)dt=0. \tag{5.65}$$

And

$$\lim_{\epsilon\to0^+}\lim_{k\to+\infty}\int_0^\tau\int_\Omega h(t)f_{n_k}(x,t)\mathrm{F}'_\epsilon(u_{n_k})\rho(x)dxdt=0. \tag{5.66}$$

On the other hand, we have

$$\mathrm{F}_\epsilon(u_{0n_k})=u_{0n_k}-\mathrm{G}_\epsilon(u_{0n_k})=u_{0rn_k}+u_{0sn_k}-\mathrm{G}_\epsilon(u_{0n_k}).$$

Since $u_{0rn_k}\to u_{0r}$ in $L^1(\Omega)$, $u_{0sn_k}\overset{*}{\rightharpoonup}u_{0s}$ in $\mathcal{M}^+(\Omega)$ and the sequence $\{\mathrm{G}_\epsilon(u_{0n_k})\}$ is uniformly bounded in $L^\infty(\Omega)$, then we deduce that

$$u_{0rn_k}-\mathrm{G}_\epsilon(u_{0n_k})\to u_{0r}-\mathrm{G}_\epsilon(u_{0r}):=\mathrm{F}_\epsilon(u_{0r})\quad\text{in}\quad L^1(\Omega). \tag{5.67}$$

According to the convergence statement (5.43), we have

$$\mathrm{F}_\epsilon(u_{n_k})(\cdot,t)=u_{n_k}(\cdot,t)-\mathrm{G}_\epsilon(u_{n_k})(\cdot,t)\overset{*}{\rightharpoonup}u_a(\cdot,t)+\psi^{-1}(v)-\mathrm{G}_\epsilon(\psi^{-1}(v))(\cdot,t)\ \text{ in }\quad\mathcal{M}^+(\Omega) \tag{5.68}$$

where $\mathrm{F}_\epsilon(\psi^{-1}(v))(\cdot,t):=\psi^{-1}(v)(\cdot,t)-\mathrm{G}_\epsilon(\psi^{-1}(v))(\cdot,t)$.
Furthermore, from the Eqs (5.43) and (5.66) we obtain the following

$$\lim_{\epsilon\to0^+}\lim_{k\to+\infty}\int_\Omega \mathrm{F}_\epsilon(u_{n_k})(\cdot,t)\rho(x)dx=\langle u_a(\cdot,t),\rho\rangle_\Omega+\lim_{\epsilon\to0^+}\int_\Omega \mathrm{F}_\epsilon(\psi^{-1}(v))(\cdot,t)\rho(x)dx. \tag{5.69}$$

It follows that

$$\lim_{\epsilon\to0^+}\lim_{k\to+\infty}\int_\Omega \mathrm{F}_\epsilon(u_{n_k})(\cdot,t)\rho(x)dx=\langle u_a(\cdot,t),\rho\rangle_\Omega\,. \tag{5.70}$$

Likewise, from (5.67) one has

$$\lim_{\epsilon\to0^+}\lim_{k\to+\infty}\int_\Omega \mathrm{F}_\epsilon(u_{0n_k})(\cdot,t)\rho(x)dx=\langle u_{0s},\rho\rangle_\Omega+\lim_{\epsilon\to0^+}\int_\Omega \mathrm{F}_\epsilon(u_{0r})\rho(x)dx.$$

It implies that

$$\lim_{\epsilon\to0^+}\lim_{k\to+\infty}\int_\Omega \mathrm{F}_\epsilon(u_{0n_k})\rho(x)dx=\langle u_{0s},\rho\rangle_\Omega\,. \tag{5.71}$$

Combining the statements (5.64)–(5.66), (5.70), (5,71) with (5.60) yields

$$\langle u_a(\cdot,t),\rho\rangle_\Omega \le \langle u_{0s},\rho\rangle_\Omega \,.$$

Since $u_a(\cdot,t)$ is a singular measure with respect to the Lebesgue measure $u_a(\cdot,t) = [u_a(\cdot,t)]_s = u_s(\cdot,\bar{t})$ for a suitable $\bar{t} \in (0,T)\setminus H^*$, where H^* is zero Lebesgue measure in $(0,T)$. Hence the assertion (5.46) is obtained.

From [44, Chapter 5, Section 5.2.1, Theorem 1], the estimate (5.19) yields

$$\| u(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \le \liminf_{k\to+\infty} \int_\Omega u_{n_k}(\cdot,t)dx \le C(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}). \tag{5.72}$$

The estimate (5.47) is completed. □

Proof of Theorem 2.2. By Proposition 5.1 and Proposition 5.2, we have $\psi(u_r) = v$ a.e in Q. Hence the problem (P) has a weak Radon measure-valued solution u in $L^\infty((0,T),\mathcal{M}^+(\Omega))$. □

Remark 5.1 By Theorem 2.2, the result holds

$$[u(\cdot,t)]_s \le u_{0s} \quad \text{in} \quad \mathcal{M}^+(\Omega) \tag{5.73}$$

for almost every $t \in (0,T)$. By (5.73), there exists zero Lebesgue measure set $\mathcal{N}_3 \subset (0,T)$ such that

$$[u(\cdot,t)]_{c,2}(E) \le [u_0]_{c,2}(E) \quad \text{in} \quad \mathcal{M}^+(\Omega) \tag{5.74}$$

for all Borel sets $E \subset \Omega$, with $C_2(E) = 0$ and $t \in (0,T)\setminus\mathcal{N}_3$.

Proposition 5.3. Suppose that the assumptions (I) and (A) are fulfilled. Let $\{u_{n_k}\}$ be the subsequence and v the function given in Proposition 5.1. Then the following sets

$$\mathcal{S} = \left\{(x,t) \in \overline{Q} \setminus \psi(u_r)(x,t) = \gamma\right\} \text{ and } \mathcal{N} = \left\{(x,t) \in \overline{Q} \setminus g(u_r)(x,t) = 0\right\}$$

have zero Lebesgue measure. Moreover $\mathcal{S} \subseteq \mathcal{N}$ and $\mathcal{B} = \mathcal{S} \cup \mathcal{N}$ has zero Lebesgue measure.

Proof of Proposition 5.3. By [9, Proposition 5.2], the set $\mathcal{S}$ has zero Lebesgue measure. Assume that

$$A_j = \left\{(x,t) \in \overline{Q} \setminus v(x,t) \le \frac{1}{j}\right\}.$$

Then, it is worth observing that

$$A_{j+1} \supseteq A_j \,,\ \mathcal{N} = \bigcup_{j=1}^{\infty} A_j \,,\ |\, \mathcal{N} \,| = \lim_{j\to+\infty} |\, A_j \,| \tag{5.75}$$

To prove that $|\, \mathcal{N} \,| = 0$, it is enough to show that $|\, A_j \,| \to 0$ as $j \to +\infty$.

Since the function $g' < 0$ in $\mathbb{R}_+$ (see the assumption (A)-(i)), then we have

$$g(u_{n_k}) \le \frac{2}{j} \Leftrightarrow u_{n_k} \ge g^{-1}\left(\frac{2}{j}\right) \ ((x,t) \in \overline{Q}). \tag{5.76}$$

It follows that

$$g^{-1}\left(\frac{2}{j}\right) \int_{\left\{(x,t)\in\overline{Q}\setminus v(x,t)\le\frac{1}{j}\right\}} \chi_{\left\{g(u_{n_k})\le\frac{2}{j}\right\}} dxdt \le \int_Q u_{n_k}(x,t)dxdt. \tag{5.77}$$

By the estimate (5.19), we have

$$g^{-1}\left(\frac{2}{j}\right) \mid A_j \mid \leq CT \parallel \mu \parallel_{\mathcal{M}^{+}(\Omega)} . \tag{5.78}$$

Since $g^{-1}\left(\frac{2}{j}\right) \to +\infty$ as $j \to +\infty$, then (3.62) yields $\mid A_j \mid \to 0$ as $j \to +\infty$.
Assume that $(x_0, t_0) \in \mathcal{S}$, then $\psi(u_r(x_0, t_0)) = \gamma$ for every $\gamma \in (0, +\infty)$. Since $g(u_r(x_0, t_0)) = \frac{\partial \psi(u_r)}{\partial \eta}(x_0, t_0) = \frac{\partial}{\partial \eta}(\gamma) = 0$. Therefore, $(x_0, t_0)) \in \mathcal{N}$, that is $\mathcal{S} \subseteq \mathcal{N}$ holds true. The fact that $\mathcal{S} \subseteq \mathcal{N}$, then $\mathcal{B} = \mathcal{N}$. Consequently, $\mathcal{B}$ is zero Lebesgue measure set. □

6. Uniqueness results of the problem (*P*)

Proposition 6.1. Under assumptions (*I*), (*A*) and (*J*). Let u be a very weak Radon measure-valued solution to the problem (*P*) and for every $\rho \in C^2(\overline{\Omega})$ such that $\frac{\partial \rho}{\partial \eta} = 0$ on $\partial\Omega$, there holds

$$\operatorname{ess}\lim_{t \to 0^+} \langle u(\cdot, t), \rho \rangle_\Omega = \langle u_0, \rho \rangle_\Omega \tag{6.1}$$

Proof of Proposition 6.1 Let us consider that for every $\tau > 0$, the smooth function $\eta_\tau \in C_c^1(0, T)$, $0 \leq \eta_\tau \leq 1$ such that

$$\eta_\tau(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_1 - \tau, \\ \frac{1}{\tau}(t + \tau - t_1) & \text{if } t_1 - \tau \leq t \leq t_1, \\ 1 & \text{if } t_1 \leq t \leq t_2, \\ \frac{1}{\tau}(-t + \tau + t_2) & \text{if } t_2 \leq t \leq t_2 + \tau, \\ 0 & \text{if } t_2 + \tau \leq t \leq T. \end{cases} \tag{6.2}$$

Let us choose $\rho_j(x)\eta_\tau(t)$ as a test function in (*P*), there holds

$$\int_0^T \int_\Omega \left\{-u\rho_j(x)\eta_\tau'(t) - \psi(u_r)\eta_\tau(t)\Delta\rho_j(x)\right\} dxdt =$$

$$\int_0^T \int_{\partial\Omega} g(u_r)\rho_j(x)\eta_\tau(t)d\mathcal{H}(x)dt + \int_0^T \int_\Omega h(t)f\eta_\tau(t)\rho_j(x)dxdt. \tag{6.3}$$

It is worth observing that the first term on the left hand side of the equality (6.3) gives

$$\int_0^T \int_\Omega -u\rho_j(x)\eta_\tau'(t)dxdt = -\frac{1}{\tau}\int_{t_1-\tau}^{t_1} \int_\Omega u(x,t)\rho_j(x)dxdt +$$

$$+\frac{1}{\tau}\int_{t_2}^{t_2+\tau} \int_\Omega u(x,t)\rho_j(x)dxdt. \tag{6.4}$$

Let us consider a zero Lebesgue measure set D_j in $(0, T)$ such that for any $t_1,\ t_2 \in (0, T) \setminus D_j$, one has

$$\lim_{\tau \to 0} \int_0^T \int_\Omega -u\rho_j(x)\eta_\tau'(t)dxdt = -\int_\Omega u(x, t_1)\rho_j(x)dx + \int_\Omega u(x, t_2)\rho_j(x)dx. \tag{6.5}$$

We assume that a sequence $\{\rho_j(x)\}$ of test functions in Ω such that

$$\rho \in C^2(\Omega) \quad , \quad \rho_j(x) \to \rho(x) \quad \text{with} \quad \rho(x) \in C^2(\Omega)$$

and

$$\Delta\rho_j(x) \to \Delta\rho(x) \quad \text{uniformly in} \quad \Omega.$$

Then for every $t \in (0,T) \setminus D_j$, there holds

$$\int_\Omega u(x,t)\rho_j(x)dx - \int_\Omega u_0\rho_j(x)dx = \int_0^t \int_\Omega \psi(u_r)\Delta\rho_j(x)dxds+$$
$$+\int_0^t \int_{\partial\Omega} g(u_r)\rho_j(x)d\mathcal{H}(x)dt + \int_0^t \int_\Omega h(t)f\rho_j(x)dxdt. \tag{6.6}$$

By Dominated Convergence Theorem, we obtain

$$\int_\Omega u(x,t)\rho(x)dx - \int_\Omega u_0\rho(x)dx = \int_0^t \int_\Omega \psi(u_r)\Delta\rho(x)dxds+$$
$$+\int_0^t \int_{\partial\Omega} g(u_r)\rho(x)d\mathcal{H}(x)dt + \int_0^t \int_\Omega h(t)f\rho(x)dxdt \tag{6.7}$$

for every $t \in (0,T) \setminus D$ with $D = \bigcup_{j\geq 0} D_j$
Since $\psi(u_r) \in L^\infty(Q)$, for every $\rho \in C^2(\Omega)$ and for every sequence $\{t_j\} \subseteq (0,T) \setminus D$, $t_j \to 0^+$ as $j \to \infty$ such that

$$\int_\Omega u(x,t_j)\rho(x)dx - \int_\Omega u_0\rho(x)dx = \int_0^{t_j} \int_\Omega \psi(u_r)\Delta\rho(x)dxds+$$
$$+\int_0^{t_j} \int_{\partial\Omega} g(u_r)\rho(x)d\mathcal{H}(x)dt + \int_0^{t_j} \int_\Omega h(t)f\rho(x)dxdt \tag{6.8}$$

holds true.
Since $u \in L^\infty((0,T), \mathcal{M}^+(\Omega))$, then we have

$$\sup_j \| u(\cdot,t_j) \|_{\mathcal{M}^+(\Omega)} \leq C. \tag{6.9}$$

So that there exists a subsequence $\{t_{j_m}\} \subseteq \{t_j\}$ and a Radon measure $\mu_0 \in \mathcal{M}^+(\Omega)$ such that

$$u(\cdot,t_{j_m}) \overset{*}{\rightharpoonup} \mu_0 \quad \text{in} \quad \mathcal{M}^+(\Omega) \quad \text{as} \quad j_m \to \infty. \tag{6.10}$$

By the standard density arguments, one has

$$\text{ess}\lim_{j_m\to\infty} \left\langle u(\cdot,t_{j_m}),\rho\right\rangle_\Omega = \langle u_0,\rho\rangle_\Omega \tag{6.11}$$

where $\mu_0 = u_0$, hence (6.1) is obtained. □
Proof of Theorem 2.4 Let u_1 , u_2 be two *very weak solutions obtained as limit of approximation* of (P) with initial data u_{01n} and u_{02n} respectively . Let $\{u_{1n}\}$, $\{u_{2n}\} \subseteq L^\infty(Q) \cap L^2((0,T), H^1(\Omega))$ be two approximating sequence solutions to the problem (P_n). We consider a test function $\xi \in C^{2,1}(Q)$ such

that $\xi(\cdot,T)=0$ in Ω and $\frac{\partial\xi}{\partial\eta}=0$ on $\partial\Omega\times(0,T)$ in the approximation problem (P_n) in the sense of the Definition 2.3, then there holds

$$\int_Q (u_{1n}-u_{2n})\,\xi_t dxdt = -\int_Q (\psi_n(u_{1n})-\psi_n(u_{2n}))\,\Delta\xi dxdt-$$
$$-\int_Q h(t)\,(f_{1n}-f_{2n})\,\xi dxdt - \int_S (g_{1n}-g_{2n})\,\xi d\mathcal{H}(x)dt-$$
$$\int_\Omega (u_{01n}-u_{02n})\,\xi(x,0)dx, \tag{6.12}$$

where $\{f_{1n}\}$, $\{f_{2n}\}$, $\{u_{01n}\}$, and $\{u_{02n}\}$ are two approximating functions.

By the assumption $g(u_r)=L$ a.e in S, then for any sequences $\{u_{1n}\}$, $\{u_{2n}\}$ one has $g(u_{1n}=g(u_{2n})=L$ on S. Consequently the third term on the right hand-side of the equation (6.12) vanishes.

For almost every $(x,t)\in Q$, we consider the function $a_n(x,t)$ defined as

$$a_n(x,t)=\begin{cases}\frac{\psi_n(u_{1n}(x,t))-\psi_n(u_{2n}(x,t))}{u_{1n}(x,t)-u_{2n}(x,t)} & \text{if } u_{1n}(x,t)\neq u_{2n}(x,t),\\ \psi_n'(u_{1n}(x,t)) & \text{if } u_{1n}(x,t)=u_{2n}(x,t).\end{cases} \tag{6.13}$$

Obviously $a_n\in L^\infty(Q)$ and for every $n\in\mathbb{N}$ there exists a positive constant C_n such that

$$\operatorname{ess}\inf_{(x,t)\in Q} a_n(x,t)\geq C_n>0. \tag{6.14}$$

This ensures that for every $z\in C_c^2(Q)$, the problem

$$\begin{cases}\xi_{nt}+a_n\Delta\xi_n+z=0 & \text{in } Q,\\ \frac{\partial\xi_n}{\partial\eta}=0 & \text{on } S,\\ \xi_n(\cdot,T)=0 & \text{in } \Omega,\end{cases} \tag{6.15}$$

has a unique solution $\xi_n\in L^\infty((0,T),H^2(\Omega))\cap L^2((0,T),H^1(\Omega))$ with $\xi_{nt}\in L^2(Q)$ (see [18, 21]). Moreover, it can be seen that

$$\mid \xi_n(x,t)\mid\leq (T-t)\parallel z\parallel_{L^\infty(Q)}. \tag{6.16}$$

Let us consider the function β such that for any $t_1+1<t_2$ and t_1 , $t_2\in(0,T)$

$$\beta(t)=\begin{cases}0 & \text{if } 0\leq t\leq t_1,\\ t-t_1 & \text{if } t_1<t<t_2,\\ t_2-t_1 & \text{if } t\geq t_2.\end{cases} \tag{6.17}$$

Choosing $\beta\Delta\xi_n$ as a test function in (6.15), then we obtain

$$\int_Q \xi_{nt}\beta(t)\Delta\xi_n dxdt+\int_Q \beta(t)a_n(x,t)[\Delta\xi_n]^2dxdt+\int_Q z\beta(t)\Delta\xi_n dxdt=0. \tag{6.18}$$

It follows that

$$\frac{1}{2}\int_Q \mid\nabla\xi_n\mid^2 dxdt+\int_Q a_n(x,t)[\Delta\xi_n]^2dxdt\leq C_0(T,z) \tag{6.19}$$

holds, for some constant $C_0(T,z)$ independent on n.
From (6.16) and (6.19), there exists a constant $C_1(T,z)$ such that

$$\| \xi_n \|_{L^2((0,T),H^1(\Omega))} + \| \sqrt{a_n}\Delta\xi_n \|_{L^2(Q)} \leq C_1(T,z). \tag{6.20}$$

On the other hand, multiplying (6.15) by $\Delta\xi_n$ and we obtain

$$-\int_Q \nabla\xi_n\nabla\xi_{nt} + \int_Q a_n[\Delta\xi_n]^2 dxdt = -\int_Q \xi_n\Delta z dxdt$$

which leads to

$$\frac{1}{2}\int_\Omega |\nabla\xi_n|^2 (x,0)dx + \int_Q a_n[\Delta\xi_n]^2 dxdt \leq C_2(T,z), \tag{6.21}$$

where $C_2(T,z) = C\left(\| \xi_n \|_{L^\infty(Q)}, \| z \|_{C^2(\overline{Q})}\right) > 0$. Therefore, we get

$$\| \xi_n(\cdot,0) \|_{H^1(\Omega)} + \| \sqrt{a_n}\Delta\xi_n \|_{L^2(Q)} \leq C_2(T,z). \tag{6.22}$$

By standard density arguments, we can choose $\xi = \xi_n$ as a test function in (6.15). It implies that (6.12) yields

$$\int_Q (u_{1n} - u_{2n})\, z dxdt = \int_Q h(t)\,(f_{1n} - f_{2n})\,\xi_n(x,t)dxdt +$$

$$+\int_\Omega (u_{01n} - u_{02n})\,\xi_n(x,0)dx. \tag{6.23}$$

Letting n to infinity in (6.23). Then it enough to observe from (6.20), there exists $\xi_n \in L^\infty((0,T),H^2(\Omega)) \cap L^2((0,T),H^1(\Omega))$ which is obtained by extracting the subsequence of the $\{\xi_n\}$ such that

$$\xi_n(x,t) \overset{*}{\rightharpoonup} \xi(x,t) \text{ in } L^\infty(Q). \tag{6.24}$$

$$\xi_n(x,t) \rightharpoonup \xi(x,t) \text{ in } L^2((0,T),H^1(\Omega)). \tag{6.25}$$

Since $\xi_{nt} \in L^2(Q)$ and the compactness theorem states in [21], we deduce that

$$\xi_{nt}(x,t) \to \xi_t(x,t) \text{ in } L^2((0,T),(H^1(\Omega))^*), \tag{6.26}$$

$$\xi_n(x,t) \to \xi(x,t) \text{ a.e in } Q. \tag{6.27}$$

By (6.16) and (6.22), there exists $\xi(\cdot,0) \in L^\infty(\Omega) \cap H^1(\Omega)$ such that the following statements

$$\xi_n(x,0) \overset{*}{\rightharpoonup} \xi(x,0) \text{ in } L^\infty(\Omega), \tag{6.28}$$

$$\xi_n(x,0) \rightharpoonup \xi(x,0) \text{ in } H^1(\Omega) \tag{6.29}$$

holds true. By Theorem 2.1, the solutions of the problem (H) are unique in $\mathcal{M}^+(Q)$. Therefore $f_{1n} \overset{*}{\rightharpoonup} f$ in $\mathcal{M}^+(Q)$ and $f_{2n} \overset{*}{\rightharpoonup} f$ in $\mathcal{M}^+(Q)$. Furthermore, the sequences $\{u_{01n}\}$ and $\{u_{02n}\}$ satisfy the assumption (2.6). By combining the above assumptions and Dominated Convergence Theorem, the Eq (6.23) reads

$$\int_Q (u_1 - u_2)z(x,t)dxdt = \lim_{n\to+\infty}\int_Q [h(t)(f_{1n} - f_{2n})]\,\xi(x,t)dxdt +$$

$$+\lim_{n\to\infty}\int_\Omega (f_{01n} - f_{02n})\,\xi(x,0)dxdt - \lim_{n\to\infty}\int_\Omega (F_{01n} - F_{02n})\,\xi(x,0)dx = 0$$

It follows that $u_1 = u_2$ in $\mathcal{M}^+(Q)$. □

7. Decay estimates solutions of the problem (P)

In this section, we prove the result of decay estimate solutions.

Proof of Theorem 2.5. We consider u_n and v_n two solutions of the approximation problems (P_n) and (P_{0n}) respectively. For any $\xi \in C^1((0,T), C^1(\Omega))$ such that $\xi(\cdot, T) = 0$ in Ω and $\frac{\partial \xi}{\partial \eta} = 0$ on S as a test function of the approximation problem $(P_n) - (P_{0n})$, then there holds

$$\int_Q (u_n - v_n)\xi_t(x,t)dxdt = \int_Q \nabla[\psi(u_n) - \vartheta(v_n)]\nabla\xi dxdt - \int_Q h(t) f_n(x,t)\xi dxdt -$$

$$- \int_S (g(u_n) - g_1(v_n))\xi d\mathcal{H}(x)dt. \tag{7.1}$$

For every $\epsilon > 0$, we consider $\{z_\epsilon\}$ be a sequence of smooth functions such that $\| z_\epsilon \|_{L^1(0,T)} \leq C$ and $z_\epsilon(t) \overset{*}{\rightharpoonup} \delta_t$ in $\mathcal{M}^+(0,T)$. Let us choose $\xi(x,t) = \text{sign}(u_n(x,t) - v_n(x,t)) \int_t^T z_\epsilon(s)(T-s)^\alpha ds (\alpha > 1)$ into the Eq (7.1), then (7.1) reads

$$\left[\int_0^T z_\epsilon(t)(T-t)^\alpha dt\right]\left[\int_\Omega | u_n(\cdot,t) - v_n(\cdot,t) | dx\right]$$

$$= -\left[\int_t^T h(t)\left(\int_0^T z_\epsilon(s)(T-s)^\alpha ds\right)dt\right]\left[\int_\Omega f_n \text{sign}(u_n(x,t) - v_n(x,t))dx\right] -$$

$$-\left[\int_t^T \left(\int_0^T z_\epsilon(s)(T-s)^\alpha ds\right)dt\right]\left[\int_{\partial\Omega} (g(u_n) - g_1(u_n))\text{sign}(u_n(x,t) - v_n(x,t))d\mathcal{H}(x)\right] \tag{7.2}$$

Letting $\epsilon \to 0^+$ in the previous equation and using the properties of the Dirac mass at t, then we have the following expression

$$(T-t)^\alpha \int_\Omega | u_n(\cdot,t) - v_n(\cdot,t) | dx \leq C \int_Q f_n(x,t)(T-t)^\alpha dxdt \tag{7.3}$$

for any $t \in (0,T)\backslash H^*$ with $| H^* |= 0$ and $C = C(| S |, \| g(u_n) \|_{L^\infty(\mathbb{R}_+)}, \| g_1(v_n) \|_{L^\infty(\mathbb{R}_+)}, T^\alpha) > 0$ is a constant. On the other hand, by (4.5) we have

$$f_n(x,t)(T-t)^\alpha = T^\alpha \int_\Omega G_N(x-y,t)u_{0n}(y)dy + \int_0^t \int_{\partial\Omega} G_N(x-y,t-\sigma)g(f_n)(T-\sigma)^\alpha dyd\sigma +$$

$$+ \int_0^t \int_\Omega G_N(x-y,t-\sigma)\left\{-\alpha f_n(T-\sigma)^{\alpha-1} + \mu_n(T-\sigma)^\alpha\right\} dyd\sigma.$$

By (2.11)–(2.13) and the properties of the Green function G_N , we get the following result

$$\int_Q f_n(x,t)(T-t)^\alpha dxdt \leq T^{\alpha+1} \int_\Omega u_{0n}(y)dy + \alpha \int_0^t \int_Q f_n(y,\sigma)(T-\sigma)^\alpha dyd\sigma dt +$$

$$+ \int_0^t \int_Q \mu_n(T-\sigma)^\alpha dyd\sigma dt + \int_0^T \int_S g(f_n)(T-\sigma)^\alpha dyd\sigma dt.$$

By the assumptions (A), (4.1) and (4.2), there exists a positive constant $C = C(T^{\alpha+1}, \| g(f_n) \|_{L^\infty(\mathbb{R}_+)}, | S |) > 0$ such that

$$\int_Q f_n(x,t)(T-t)^\alpha dxdt \leq C(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(\Omega)}) + \alpha \int_0^t \left(\int_Q f_n(x,\sigma)(T-\sigma)^\alpha dxdt \right) d\sigma. \quad (7.4)$$

By Gronwall's inequality, (7.4) yields

$$\int_Q f_n(x,t)(T-t)^\alpha dxdt \leq Ce^{\alpha T}(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(\Omega)}) \quad (7.5)$$

where $C = C(T^{\alpha+1}, \| g(f_n) \|_{L^\infty(\mathbb{R}_+)}, | S |, e^{\alpha T}) > 0$ is a constant. Combining (7.3) with (7.5), we deduce that

$$(T-t)^\alpha \int_\Omega | u_n(\cdot,t) - v_n(\cdot,t) | dx \leq C(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}). \quad (7.6)$$

By [24, Chapter V, Section 5.2.1, Theorem 1], the semi-continuity of the total variation yields,

$$\begin{aligned}(T-t)^\alpha \| u(\cdot,t) - v(\cdot,t) \|_{\mathcal{M}^+(\Omega)} &\leq (T-t)^\alpha \liminf_{n\to\infty} \int_\Omega | u_n(\cdot,t) - v_n(\cdot,t) | dx \\ &\leq C(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}). \quad (7.7)\end{aligned}$$

Hence (2.18) holds.

We consider u_n and w_n two solutions of the approximation problems (P_n) and (P_{1n}) respectively. For any $\xi \in C^1((0,T), C^1(\Omega))$ such that $\xi(\cdot,T) = 0$ in Ω and $\frac{\partial \xi}{\partial \eta} = 0$ on S as a test function of the approximation problem $(P_n) - (P_{1n})$. Therefore, we have the following equation

$$\int_Q (u_n - w_n)\xi_t(x,t)dxdt = \int_Q \nabla[\psi(u_n) - \psi(w_n)]\nabla\xi dxdt - \int_\Omega u_{0n}\xi(x,0)dxdt \quad (7.8)$$

Taking $\xi(x,t) = \text{sign}(u_n(x,t) - w_n(x,t)) \int_t^T z_\epsilon(s)(T-s)^\alpha ds (\alpha > 1)$ into the equality (7.8), then we obtain

$$\begin{aligned}&\left[\int_0^T z_\epsilon(t)(T-t)^\alpha dt\right]\left[\int_\Omega | u_n(\cdot,t) - w_n(\cdot,t) | dx\right] \\ &= -\left[\int_0^T z_\epsilon(s)(T-s)^\alpha dt\right]\left[\int_\Omega u_{0n}\text{sign}(u_n(x,0) - w_n(x,0))\right].\end{aligned}$$

Letting $\epsilon \to 0^+$ in the previous equation and using the properties of the Dirac mass at t, then we have

$$(T-t)^\alpha \int_\Omega | u_n(\cdot,t) - w_n(\cdot,t) | dx \leq T^\alpha \int_\Omega u_{0n} dx. \quad (7.9)$$

By (4.1), the above inequality (7.9) yields

$$(T-t)^\alpha \int_\Omega | u_n(\cdot,t) - w_n(\cdot,t) | dx \leq C \| u_0 \|_{\mathcal{M}^+(\Omega)} . \quad (7.10)$$

By [24, Chapter V, Section 5.2.1, Theorem 1], the semi-continuity of the total variation yields,

$$(T-t)^\alpha \| u(\cdot,t) - w(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \leq (T-t)^\alpha \liminf_{n\to\infty} \int_\Omega | u_n(\cdot,t) - w_n(\cdot,t) | dx$$

$$\leq C \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}$$

where $C = C(T, \alpha) > 0$ a constant. Hence (2.19) is achieved. Now we consider the auxiliary function $\mathcal{W}_n$ such that

$$\mathcal{W}_n(x,t) = t^\alpha u_n(x,t)\text{sign}(\mathcal{W}_n) \tag{7.11}$$

for every $\alpha > 1$. The derivation of the expression $\mathcal{W}_n$ with respect to the variable t gives

$$\mathcal{W}_{nt}(x,t) = \alpha t^{\alpha-1} u_n(x,t)\text{sign}(\mathcal{W}_n) + t^\alpha u_{nt}(x,t)\text{sign}(\mathcal{W}_n). \tag{7.12}$$

Since $u_{nt} = \Delta\psi(u_n) + h(t) f_n(x,t)$ and we multiply the Eq (7.12) by the function sign$(\mathcal{W}_n)$ and then we integrate the result over $\Omega \times (0,t)$(for any $t \in (0,T)$), then we obtain

$$\int_\Omega | \mathcal{W}_n | (x,t)dx = \alpha \int_0^t s^{\alpha-1} \int_\Omega u_n(x,s)dxds + \int_0^t \int_{\partial\Omega} g(u_n)s^\alpha d\mathcal{H}(x)ds + \int_0^t \int_\Omega s^\alpha h(s) f_n(x,s)dxds. \tag{7.13}$$

By replacing the expression of $\mathcal{W}_n$ in (7.13), we deduce that

$$t^\alpha \int_\Omega u_n(x,t)dx \leq \alpha T^\alpha \int_Q u_n(x,t)dxdt + T^\alpha \int_S g(u_n)d\mathcal{H}(x)dt + h(T)T^\alpha \int_Q f_n(x,t)dxdt. \tag{7.14}$$

By assumptions (A), (J), (2.16) and (4.19), there exists a constant $C = C(\alpha T^{\alpha+1}, h(T)T^\alpha, \parallel g(u_n) \parallel_{L^\infty(\mathbb{R}_+)}) > 0$ such that

$$t^\alpha \int_\Omega u_n(x,t)dx \leq C(\parallel u_0 \parallel_{\mathcal{M}^+(\Omega)} + \parallel \mu \parallel_{\mathcal{M}^+(Q)}). \tag{7.15}$$

According to [24, Chapter V, Section 5.2.1, Theorem 1], we conclude from the estimate (7.15), the following estimate

$$t^\alpha \parallel u(\cdot,t) \parallel_{\mathcal{M}^+(\Omega)} \leq t^\alpha \liminf_{n\to\infty} \int_\Omega u_n(\cdot,t)dx \leq C(\parallel u_0 \parallel_{\mathcal{M}^+(\Omega)} + \parallel \mu \parallel_{\mathcal{M}^+(Q)}).$$

Hence the estimate (2.21) is completed. □

8. Asymptotic behavior solutions of the problem (P)

To show the existence of the problem (E), we employ the natural approximation method. Therefore, the solution of the problem (P) is constructed by limiting point of a family $\{u_n\}$ of solutions to the approximation problem. To this purpose, we consider the function $\phi \in C_c^\infty(\Omega)$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ in K_0 (for any compact set $K_0 \subset \Omega \subset \mathbb{R}^N$), then we get

$$-\Delta(\phi\psi(U)) + \phi U = \phi u_0 + \varepsilon(\phi) \quad \text{in} \quad \mathcal{D}'(\Omega)$$

where $\varepsilon(\phi) = -\psi(U)\Delta\phi - 2\nabla\phi\nabla\psi(U)$ and $\varepsilon(\phi) = 0$ in K_0 with $\varepsilon(\phi) \in L^1(\Omega)$.

Now we consider the approximation of problem (E)

$$\begin{cases} -\Delta\psi(U_n) + U_n = u_{0n} & \text{in } \Omega, \\ \frac{\partial\psi(U_n)}{\partial\eta} = g(U_n) & \text{on } \partial\Omega, \end{cases} \tag{E_n}$$

where $u_{0n} = (\phi u_0 + \varepsilon(\phi)) * \rho_n$ and $\{\rho_n\}$ a sequence of standard mollifiers. Furthermore, the sequence $\{u_{0n}\} \subseteq C^\infty(\overline{\Omega})$ satisfies the assumption (4.1).

Then for every $n \in \mathbb{N}$, there exists $U_n \in H^1(\Omega) \cap L^\infty(\Omega)$ solution of the approximation problem (E_n).

In the next Lemma, we state the technical estimates important for the proof of the existing solutions.

Lemma 8.1 Assume that (I), (A) and $u_0 \in \mathcal{M}^+(\Omega)$ are satisfied. The sequence $\{U_n\}$ be a weak solution of the approximation problem (E_n). Then, there holds

$$\| U_n \|_{L^1(\Omega)} \leq C \| u_0 \|_{\mathcal{M}^+(\Omega)}, \tag{8.1}$$

$$\| \nabla\psi(U_n) \|_{L^2(\Omega)} + \| \psi(U_n) \|_{L^2(\Omega)} \leq C, \tag{8.2}$$

where $C > 0$ is a constant. Moreover, for every $1 \leq p < \frac{N}{N-1}$ there holds

$$\| \nabla\psi(U_n) \|_{L^p(\Omega)} + \| \psi(U_n) \|_{L^p(\Omega)} \leq C, \tag{8.3}$$

where $C = C(p) > 0$ is a constant.

Proof of Lemma 8.1 We consider $\varphi \in C^1(\overline{\Omega})$ as a test function in the approximation problem (E_n), then we have

$$\int_\Omega \nabla\psi(U_n)\nabla\varphi dx + \int_\Omega U_n\varphi dx = \int_\Omega u_{0n}\varphi dx + \int_{\partial\Omega} g(U_n)\varphi d\mathcal{H}(x) \tag{8.4}$$

Assume that $\Omega_- = \left\{x \in \Omega / \ U_n(x) \leq 0 \ \text{ in the sense of } L^1(\Omega)\right\}$ and $\varphi(x) = \inf_{x\in\Omega}\{U_n(x), 0\}$. It is worth observing that $\varphi(x) \in L^1(\Omega)$. To show that $U_n \geq 0$ in Ω, it is enough to prove that $\varphi(x) = 0$ in Ω. Indeed, we choose $\varphi(x) = \text{sign}(U_n(x))$, then we get

$$\int_{\Omega_-} | U_n(x) | \, dx = \int_{\Omega_-} u_{0n}(x)\text{sign}(U_n(x))dx + \int_{\partial\Omega_-} g(U_n)\text{sign}(U_n(x))d\mathcal{H}(x) \leq 0 \tag{8.5}$$

where $u_{0n} \geq 0$ in Ω and $g > 0$ in $\mathbb{R}_+$ (see the assumption (A)). Therefore $\varphi(x) = 0$ a.e in Ω. Hence the solution of the approximation problem (E_n), $U_n(x) \geq 0$ a.e $x \in \Omega$.

Now we consider the regularizing sequence $\{\mathcal{T}_\epsilon\} \subseteq C^1(\mathbb{R}_+)$ for every $\epsilon > 0$ such that

(i) $0 \leq \mathcal{T}_\epsilon(s) \leq 1$ in $\mathbb{R}_+$, $\mathcal{T}_\epsilon(s) = 0$, $\mathcal{T}'_\epsilon \geq 0$ in $\mathbb{R}_+$,

(ii) $\mathcal{T}_\epsilon(s) \to 1$ as $\epsilon \to 0^+$ for every $s \neq 0$.

We choose $\mathcal{T}_\epsilon(U_n) \in H^1(\Omega) \cap L^\infty(\Omega)$ as a test function in the approximation problem (E_n) and by employing the assumptions (A) and (I), then we get

$$\int_\Omega \mathcal{T}'_\epsilon(U_n)\psi'(U_n(x)) \mid \nabla U_n(x) \mid^2 dx + \int_\Omega U_n(x)\mathcal{T}_\epsilon(U_n)dx \leq C \| u_0 \|_{\mathcal{M}^+(\Omega)} \tag{8.6}$$

where $C = C(\| g(U_n) \|_{L^\infty(\mathbb{R}_+)}, | \partial\Omega |) > 0$. Since $\mathcal{T}'_\epsilon(U_k)\psi'(U_n(x)) \geq 0$ in $\mathbb{R}_+$ (see the hypothesis (I)), then (8.6) reads

$$\int_\Omega U_n(x)\mathcal{T}_\epsilon(U_n)dx \leq C \| u_0 \|_{\mathcal{M}^+(\Omega)} \tag{8.7}$$

Again, by considering the limit when $\epsilon \to 0^+$, the estimate (8.1) holds true. Now we consider another regularizing sequence $\{\mathrm{T}_\epsilon\} \subseteq C^1(\mathbb{R}_+)$ for every $\epsilon > 0$ such that $\mathrm{T}_\epsilon(s) = 1$ if $0 \leq s \leq \frac{1}{\epsilon}$, $\mathrm{T}_\epsilon(s) = \epsilon s$ if

$\frac{1}{\epsilon} \leq s \leq \frac{2}{\epsilon}$, $\mathrm{T}_\epsilon(s) = 2$ if $s \geq \frac{2}{\epsilon}$. It is obvious to see that $1 \leq \mathrm{T}_\epsilon(s) \leq 2$ in $\mathbb{R}_+$. We take the function $\varphi(s) = \int_0^s \mathrm{T}_\epsilon(\sigma)d\sigma$ and we choose $\varphi(\psi(U_n))$ as a test function in (E_n), then we obtain

$$\int_\Omega |\nabla\psi(U_n)|^2\, \mathrm{T}_\epsilon(\psi(U_n))dx + \int_\Omega U_n\varphi(\psi(U_n))dx = \int_\Omega \varphi(\psi(U_n))u_{0n}dx + \int_{\partial\Omega} g(U_n)\varphi(\psi(U_n))d\mathcal{H}(x). \tag{8.8}$$

Since $1 \leq \mathrm{T}_\epsilon(\psi(U_n)) \leq 2$ and $\psi(U_n) \leq \varphi(\psi(U_n)) \leq 2\psi(U_n)$, therefore there exists a positive constant C such that

$$\int_\Omega |\nabla\psi(U_n)|^2\, dx \leq C \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)} \tag{8.9}$$

where $C = C(\parallel \psi(U_n) \parallel_{L^\infty(\mathbb{R}_+)}, \parallel g(U_n) \parallel_{L^\infty(\mathbb{R}_+)}, |\,\partial\Omega\,|) > 0$. By the assumption (I), the statement $\psi(U_n) \in L^2(\Omega)$ holds. Whence the estimate (8.2) is achieved.

Again, recalling the Hölder's inequality, we get

$$\int_\Omega |\nabla\psi(U_n)|^p\, dx \leq \left[\int_\Omega \frac{|\nabla\psi(U_n)|^2}{(1+\psi(U_n))^2}dx\right]^{\frac{1}{q}} \left[\int_\Omega (1+\psi(U_n))^{q'}dx\right]^{\frac{1}{q'}}$$

where $q := \frac{2}{p}$ and $q' := \frac{2}{2-p}$. Therefore, there exists a positive constant $C = C(p, \parallel \psi(U_n) \parallel_{L^\infty(\mathbb{R}_+)}) > 0$ such that

$$\int_\Omega |\nabla\psi(U_n)|^p\, dx \leq C \tag{8.10}$$

By the assumption (I), the statement $\psi(U_n) \in L^p(\Omega)$ holds. Hence the estimate (8.3) is achieved. □

Proof of Theorem 2.6. From the estimate (8.2) and assumption (A), we can extract from $\{\psi(U_n)\}$ a subsequence $\{\psi(U_{n_k})\}$ such that

$$\psi(U_{n_k}) \to V \quad \text{in} \quad H^1(\Omega) \quad \text{and} \quad \psi(U_{n_k}) \to V \quad \text{a.e in} \quad \Omega \tag{8.11}$$

$$g(U_{n_k}) \overset{*}{\rightharpoonup} V \quad L^\infty(\partial\Omega) \quad \text{and} \quad g(U_{n_k}) \to V \quad \text{a.e in} \quad \partial\Omega \tag{8.12}$$

By (8.3), the sequence $\{\psi(U_{n_k})\} \subseteq BV(\Omega)$ and applying [44, Chapter IV, Section 1.1, Proposition 5], there exists a subsequence $\{\psi(U_{n_k})\}$ and $V_1 \in \mathcal{M}^+(\Omega)$ such that the convergence

$$\psi(U_{n_k}) \overset{*}{\rightharpoonup} V_1 \quad \text{in} \ \ \mathcal{M}^+(\Omega). \tag{8.13}$$

By repeating the same method as in the Proposition 5.2, we deduce that

$$U_{n_k} \overset{*}{\rightharpoonup} U := \psi^{-1}(V) + \lambda_1 \ \ \text{in} \ \ \mathcal{M}^+(\Omega) \tag{8.14}$$

where $U_r = \psi^{-1}(V)$ a.e in Ω, $U_s = \lambda_1$ in $\mathcal{M}^+(\Omega)$ and $U_r = g^{-1}(V)$ a.e in $\partial\Omega$.

By [24, Chapter V, Section 5.2.1, Theorem 1], the estimate (8.1) yields

$$\parallel U \parallel_{\mathcal{M}^+(\Omega)} \leq \liminf_{n\to\infty} \int_\Omega U_n(x)dx \leq C \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)} .$$

Hence the estimate (2.23) is completed. □

Remark 8.1 The sets

$$\mathcal{S}_0 = \left\{x \in \overline{\Omega} \setminus \psi(U_r)(x) = \gamma\right\} \text{ and } \mathcal{N}_0 = \left\{x \in \overline{\Omega} \setminus g(U_r)(x) = 0\right\}$$

have zero Lebesgue measure. Moreover $\mathcal{S}_0 \subseteq \mathcal{N}_0$ and $\text{supp}(U_s) \subseteq \mathcal{S}_0$.
Proof of Theorem 2.7. We choose $\xi(x,t) = \text{sign}(u_n(x,t) - U_n(x)) \int_t^T z_\epsilon(s) s^\alpha ds (\alpha > 1)$ as a test function in the approximation problem $(P_n) - (E_n)$, then we have

$$\int_\Omega \int_0^T | u_n(x,t) - U_n(x) | z_\epsilon(t) t^\alpha dt = \int_\Omega \int_0^T | u_{0n}(x) - U_n(x) | z_\epsilon(t) t^\alpha dt dx +$$
$$+ \int_\Omega \int_0^T [g(u_n) - g(U_n)] \text{sign}(u_n(x,t) - U_n(x)) \int_t^T z_\epsilon(s) s^\alpha ds dt d\mathcal{H}(x) +$$
$$+ \int_\Omega \int_0^T u_n(x,t) - U_n(x)] \text{sign}(u_n(x,t) - U_n(x)) \int_t^T z_\epsilon(s) s^\alpha ds dt dx +$$
$$+ \int_\Omega \int_0^T h(t) f_n(x,t) \text{sign}(u_n(x,t) - U_n(x)) \int_t^T z_\epsilon(s) s^\alpha ds dt dx.$$

By the previous proof mentioned above, we deduce that

$$\int_\Omega \int_0^T | u_n(x,t) - U_n(x) | z_\epsilon(t) t^\alpha dt \leq C(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}) \tag{8.15}$$

where $C = C(\alpha T^{\alpha+1}, h(T), T^\alpha, \| g(u_n) \|_{L^\infty(\mathbb{R}_+)}, | S |) > 0$ is a constant. By letting $\epsilon \to 0^+$, then (8.15) reads

$$t^\alpha \int_\Omega | u_n(x,t) - U_n(x) | dx \leq C(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}). \tag{8.16}$$

By virtue of [24, Chapter V, Section 5.2.1, Theorem 1], then the semi-continuity of the total variation yields

$$t^\alpha \| u(\cdot,t) - U(\cdot) \|_{\mathcal{M}^+(\Omega)} \leq \liminf_{n \to +\infty} t^\alpha \int_\Omega | u_n(x,t) - U_n(x) | dx \leq C(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}) \tag{8.17}$$

for almost every $t \in (0,T)$ and $\alpha > 1$. By considering to the limit as $t \to +\infty$ in the following inequality

$$\| u(\cdot,t) - U(\cdot) \|_{\mathcal{M}^+(\Omega)} \leq \frac{C}{t^\alpha}(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)}).$$

Hence the statement (2.24) follows. □

9. Conclusions

In this paper, we study the existence, uniqueness, decay estimates, and the asymptotic behavior of the Radon measure-valued solutions for a class of nonlinear parabolic equations with a source term and nonzero Neumann boundary conditions. To attain this, we use the natural approximation method, the definition of the weak solutions, and the properties of the Radon measure. Concerning the study of the existence and uniqueness of the solutions to the problem (P), we first show that the source term corresponding to the solution of the linear inhomogeneous heat equation with measure data is a unique Radon measure-valued. Moreover, we establish the decay estimates of these solutions by using the suitable test functions and the auxiliary functions. Finally, we analyze the asymptotic behavior of these solutions by establishing the decay estimate of the difference between the solution to the problem (P) and the solution to the steady state problem (E).

Acknowledgment

This work was partially supported by National Natural Sciences Foundation of China, grant No: 11571057

Conflict of interest

The authors declare no conflict of interest.

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