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Research article

Complete integration convergence for arrays of rowwise extended negatively dependent random variables under the sub-linear expectations

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Abstract: Limit theorems of sub-linear expectations are challenging field that has attracted widespread attention in recent years. In this paper, we establish some results on complete integration convergence for weighted sums of arrays of rowwise extended negatively dependent random variables under sub-linear expectations. Our results generalize the complete moment convergence of the probability space to the sub-linear expectation space.

Keywords: complete integration convergence; sub-linear expectation; extended negative dependence random variables; complete convergence; identical distribution
Mathematics Subject Classification: 60F15

1. Introduction

The study of sub-linear expectation is a new direction of probability limit theory and has attracted great attention in the field in recent years. Probability limit theory is mainly good at dealing with those situations where the corresponding probability model can be determined by mathematical statistical methods and data analysis. However, most random variables in the real world have different degrees of uncertainty. How to analyze and calculate the financial and economic problems under uncertainty has become an important concern at present. Compared with the classical linear model, sub-linear expectation theory is more complex and more challenging. By relaxing the linear property of classical expectation to sub-additivity and positive homogeneity, we can obtain many interesting properties. Peng [1, 2] first introduced the concept and framework of sub-linear expectation, and provided the corresponding basic properties. Under the sub-linear expectation, Peng [3,4] developed the law of weak large numbers and the central limit theorem for independent identically distributed random variables. Subsequently, Zhang [5–7] did a series of studies on the basis of the framework established by Peng, proved the exponential inequality and Rosenthal's inequality, and obtained Kolmogorov's strong law of larger numbers and Hartman-Wintner's law of iterated logarithm. Zhang's work provides us with a powerful tool for studying theorems under sub-linear expectations.

Complete convergence and complete moment convergence are two very important ideas in probability limit theory. Chow [8] first proposed the concept of complete convergence for sequences of independent random variables. Complete moment convergence is a more accurate convergence than complete convergence. A large number of scholars have studied the theorems of complete moment convergence for random variable sequences, and obtained many related results in the classical probability space. Wang [9, 10] studied the complete moment convergence of difference sequences and random variables satisfying the Rosenthal-type inequality. Qiu [11] and Yi [12] extended the sequence suitable for the complete moment convergence again, and both obtained the complete moment convergence of END sequence. In recent years, Some scholars have also begun to study the complete integral convergence, but the results are relatively few. Lu [13] proved the complete integral convergence of wise widely negative dependent random variables under sub-linear expectation. Zhong and Wu [14] studied the complete integral convergence of the weighted sum of the END variable under sub-linear expectation space context. Liang and Wu [15] extend complete integral convergence theorems for END sequences of random variables. Li and Wu [16] discuss the property of complete integral convergence and obtain the result of q-th integral convergence of arrays of rowwise extended negatively dependent random variables under sub-linear expectation. Based on the above results, the study of complete integral convergence needs to be improved.

The content of this article is as follows. In Section 2, we introduce some basic notation, concepts and related properties. In Section 3, complete integral convergence theorems for weighted sums of arrays of rowwise extended negatively dependent random variables are established. Our results generalize Ge's [17] conclusions from probability space to sub-linear expectation. In Section 4, we give lemmas that are useful for proving the main results, and use these lemmas to prove the main results of this paper.

2. Preliminaries

We use the framework and notions of Peng [1, 2]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \ \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

Definition 2.1. A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have
(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$;
(b) Constant preserving: $\hat{\mathbb{E}}c = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$; whenever $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda\hat{\mathbb{E}}X, \lambda \geq 0$.

Here $\bar{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space.

Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}(X) := -\hat{\mathbb{E}}(-X),\ \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$,

$$\hat{\varepsilon}X \leq \hat{\mathbb{E}}X,\ \hat{\mathbb{E}}(X+c) = \hat{\mathbb{E}}X + c,$$

$$|\hat{\mathbb{E}}(X-Y)| \leq \hat{\mathbb{E}}|X-Y| \text{ and } \hat{\mathbb{E}}(X-Y) \geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y.$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, then $\hat{\mathbb{E}}(X+aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0,1]$ is called a capacity if

$$V(\emptyset) = 0,\ V(\Omega) = 1 \text{ and } V(A) \leq V(B) \text{ for } \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}\xi; I(A) \leq \xi, \xi \in \mathcal{H}\},\ \mathcal{V}(A) := 1 - \mathbb{V}(A^c),\ \forall A \in \mathcal{F},$$

where $\mathbb{V}(A^c)$ is the complement set of A. By definition of $\mathbb{V}$ and $\mathcal{V}$, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\mathcal{V} \leq \mathbb{V},\ \forall A \in \mathcal{F};\ \mathbb{V}(A) = \hat{\mathbb{E}}(I(A)),\ \mathcal{V}(A) = \hat{\varepsilon}(I(A)),\ \text{if } I(A) \in \mathcal{H},$$

$$\hat{\mathbb{E}}f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g,\ \hat{\varepsilon}f \leq \mathcal{V}(A) \leq \hat{\varepsilon}g,\ \text{if } f \leq I(A) \leq g,\ f, g \in \mathcal{H}.$$

Definition 2.2. We define the Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ by

$$C_V(X) := \int_0^\infty V(X > x)\mathrm{d}x + \int_{-\infty}^0 (V(X > x) - 1)\mathrm{d}x,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$ respectively.

Definition 2.3. (i) (Identical distribution) Let X_1 and X_2 be two random variables defined severally in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1(\varphi(X_1)) = \hat{\mathbb{E}}_2(\varphi(X_2)),\ \forall \varphi \in C_{l,Lip}(\mathbb{R}),$$

whenever the sub-expectations are finite. A sequence $\{X_n; n \geq 1\}$ of random variables is said to be identically distributed if X_i and X_1 are identically distributed for each $i \geq 1$.

(ii) (Extended negatively dependence) A sequence of random variables $\{X_n; n \geq 1\}$ is said to be upper (resp.lower) extended negatively dependent if there is some dominating constant $K \geq 1$ such that

$$\hat{\mathbb{E}}\left(\prod_{i=1}^n \varphi_i(X_i)\right) \leq K \prod_{i=1}^n \hat{\mathbb{E}}(\varphi_i(X_i)),\ \forall n \geq 2,$$

whenever the non-negative functions $\varphi_i(x) \in C_{l,Lip}(\mathbb{R}), i = 1, 2, \ldots,$ are all non-decreasing (resp. all non-increasing). They are said to be extended negatively dependent (END) if they are both upper extended negatively dependent and lower extended negatively dependent.

It is obvious that, if $\{X_n; n \geq 1\}$ is a sequence of END random variables and $f_1(x), f_2(x), \ldots \in C_{l,\mathrm{Lip}}(\mathbb{R})$ are non-descreasing (resp. non-increasing) functions, then $\{f_n(X_n); n \geqq 1\}$ is also a sequence of END random variables.

For $0 < \mu < 1$, let $g_\mu(x) \in C_{l,Lip}(\mathbb{R})$ be an even function and it is decreasing in $x > 0$ such that $0 \leq g_\mu(x) \leq 1$ for all x , $g_\mu(x) = 1$ if $|x| \leq \mu$, $g_\mu(x) = 0$ if $|x| > 1$. Then

$$I(|x| \leq \mu) \leq g_\mu(x) \leq I(|x| \leq 1), I(|x| > 1) \leq 1 - g_\mu(x) \leq I(|x| > \mu). \tag{2.1}$$

Let $A_n \ll B_n$ ($A_n \gg B_n$) denote that there exists a constant $c > 0$ such that $A_n \leq cB_n$ ($A_n \geq cB_n$) for sufficiently large n, and the symbol $I(\cdot)$ will be used to signify the indicator function. $\{a\}_+$ means $\max\{a, 0\}$.

3. Main results

Theorem 3.1. *Let $\left\{X_{nj}; 1 \leq j \leq b_n, n \geq 1\right\}$ be an array of rowwise END random variables, $\left\{a_{nj}, j \geq 1, n \geq 1\right\}$ be an array of positive numbers, $\{b_n, n \geq 1\}$ be a non-decreasing sequence of positive integers and $\{c_n, n \geq 1\}$ be a non-decreasing sequence of positive numbers. Suppose that for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} \mathbb{V}\left(\left|a_{nj}X_{nj}\right| \geq \varepsilon b_n^{1/t}\right) < \infty, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}}\left(X_{nj}\right)^2 g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right) < \infty. \tag{3.2}$$

Then

$$\sum_{n=1}^{\infty} c_n \mathbb{V}\left\{\sum_{j=1}^{b_n} a_{nj}\left(X_{nj} - \hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) \geq \varepsilon b_n^{1/t}\right\} < \infty, \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} c_n \mathbb{V}\left\{\sum_{j=1}^{b_n} a_{nj}\left(X_{nj} - \hat{\varepsilon}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) < -\varepsilon b_n^{1/t}\right\} < \infty. \tag{3.4}$$

Particularly, if $\hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right) = \hat{\varepsilon}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)$, then

$$\sum_{n=1}^{\infty} c_n \mathbb{V}\left\{\left|\sum_{j=1}^{b_n} a_{nj}\left(X_{nj} - \hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\right| \geq \varepsilon b_n^{1/t}\right\} < \infty. \tag{3.5}$$

Further, if the following condition also hold:

$$\sum_{n=1}^{\infty} c_n b_n^{1/t} \sum_{j=1}^{b_n} \left|a_{nj}\right| C_{\mathbb{V}}\left(\left|X_{nj}\right|\left(1 - g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\right) < \infty, \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}}\left(X_{nj}\right)^2 \left(1 - g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) < \infty. \tag{3.7}$$

Then

$$\sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left\{ b_n^{-1/t} \left| \sum_{j=1}^{b_n} a_{nj} \left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\right| - \varepsilon \right\}_+ < \infty. \tag{3.8}$$

Remark 3.1. *The results obtained from Theorem 3.1 is very extensive, we can obtain different forms of complete integration convergence theorems by taking different forms of c_n and b_n.*

Taking $c_n = n^{2(\alpha-1)}$, $b_n = n$, $t = 1/\alpha$ or $c_n = n^{\alpha-2}h(n)$, $b_n = n$, $a_{nj} = 1$ in Theorem 3.1, $h(n)$ be a slowly varying function here, then we obtain two complete integration convergence theorems such as Theorems 3.2 and 3.3.

Theorem 3.2. *Let $\{X_{nj}; j \geq 1, n \geq 1\}$ be an array of rowwise END random variables, $\{a_{nj}; j \geq 1, n \geq 1\}$ be an array of positive numbers satisfying $\hat{\mathbb{E}} X_{nj} g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon n^\alpha}\right) = \hat{\varepsilon} X_{nj} g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon n^\alpha}\right)$, and*

$$\sum_{j=1}^{n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 = O\left(n^\delta\right) \quad as \quad n \to \infty, \ for\ some\ \ 0 < \delta < 1. \tag{3.9}$$

Then, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{2(\alpha-1)} C_{\mathbb{V}} \left\{ n^{-\alpha} \left| \sum_{j=1}^{n} a_{nj} \left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon n^\alpha}\right)\right)\right| - \varepsilon \right\}_+ < \infty. \tag{3.10}$$

Particularly, if $\hat{\mathbb{E}} X_{nj} = 0$ and $\alpha > \delta/2$, then

$$\sum_{n=1}^{\infty} n^{2(\alpha-1)} C_{\mathbb{V}} \left\{ n^{-\alpha} \left| \sum_{j=1}^{n} a_{nj} X_{nj} \right| - \varepsilon \right\}_+ < \infty. \tag{3.11}$$

Theorem 3.3. *Let $\{X, X_{nj}; j \geq 1, n \geq 1\}$ be an array of rowwise identically distributed END random variables with $\hat{\mathbb{E}} X = 0$, and $h(x) > 0$ be a slowly varying function as $x \longrightarrow \infty$. If $\hat{\mathbb{E}} X_{nj} g_\mu\left(\frac{X_{nj}}{\varepsilon n^{1/t}}\right) = \hat{\varepsilon} X_{nj} g_\mu\left(\frac{X_{nj}}{\varepsilon n^{1/t}}\right)$, $\hat{\mathbb{E}}\left(|X|^{\alpha t} h\left(|X|^t\right)\right) \leq C_{\mathbb{V}}\left(|X|^{\alpha t} h\left(|X|^t\right)\right) < \infty$ for $\alpha > 1$, $1 < \alpha t < 2$, then*

$$\sum_{n=1}^{\infty} n^{\alpha-2} h(n) \sum_{j=1}^{n} C_{\mathbb{V}} \left\{ n^{-\frac{1}{t}} \left| \sum_{j=1}^{n} X_{nj} \right| - \varepsilon \right\}_+ < \infty. \tag{3.12}$$

Remark 3.2. *Theorem 3.1 extends Theorem 2.1 of Ge [16] from the conventional probability space to sub-linear expectation space. Theorems 3.2 and 3.3 generalize Corollary 2.6 and Theorem 2.4 of Ge [16].*

Remark 3.3. *Under sub-linear expectation, Li and Wu [16] study q-th integral convergence of arrays of rowwise extended negatively dependent random variables. Wu and Jiang [18] study complete convergence and complete integral convergence for negatively dependent random variables. Wu [19] establish precise asymptotics for complete integral convergence. In this paper, we study complete integral convergence of weighted sums, and the negatively dependent random variable expands to arrays of rowwise extended negatively dependent random variables.*

4. Proof of main results

To prove our results, we need the following lemmas.

Lemma 4.1. *Zhong and Wu [14] suppose $X \in \mathcal{H}$, $\alpha > 0$, $t > 0$, and $h(x)$ is a slow varying function. (i) Then*

$$C_{\mathbb{V}}\left(|X|^{\alpha t} h\left(|X|^{t}\right)\right) < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha-1} h(n)\, \mathbb{V}\left(|X| > cn^{1/t}\right) < \infty, for\ \forall c > 0. \tag{4.1}$$

(ii) If $C_{\mathbb{V}}\left(|X|^{\alpha t} h\left(|X|^{t}\right)\right) < \infty$, then for $\forall c > 0$ and $\theta > 1$,

$$\sum_{k=1}^{\infty} \theta^{k\alpha} h\left(\theta^{k}\right) \mathbb{V}\left(|X| > c\theta^{k/t}\right) < \infty. \tag{4.2}$$

Lemma 4.2. *Zhang [5] Let $X_1, X_2, \ldots, X_n$ be a sequence of upper extended negatively dependent random variables in with $\hat{\mathbb{E}}[X_k] \leq 0$. Set $S_n = \sum_{k=1}^{n} X_k$, then*

$$\mathbb{V}(S_n \geq x) \leq (1 + Ke) \frac{\sum_{k=1}^{n} \hat{\mathbb{E}} X_k^2}{x^2}, \forall x > 0. \tag{4.3}$$

Proof of Theorem 3.1. For array of rowwise END random variables $\left\{X_{nj}; 1 \leq j \leq b_n,\ \ n \geq 1\right\}$, we need that truncated function belong to $C_{l,Lip}(\mathbb{R})$ and is non-decreasing to make the truncated random variables are also END. For any $c > 0, 1 \leq j \leq b_n, n \geq 1$,

$$Y_{nj} = \frac{-\varepsilon b_n^{1/t}}{a_{nj}} I\left(a_{nj}X_{nj} < -\varepsilon b_n^{1/t}\right) + X_{nj} I\left(\left|a_{nj}X_{nj}\right| \leq \varepsilon b_n^{1/t}\right) + \frac{\varepsilon b_n^{1/t}}{a_{nj}} I\left(a_{nj}X_{nj} > \varepsilon b_n^{1/t}\right),$$

$$Z_{nj} = \left(X_{nj} + \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right) I\left(a_{nj}X_{nj} < -\varepsilon b_n^{1/t}\right) + \left(X_{nj} - \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right) I\left(a_{nj}X_{nj} > \varepsilon b_n^{1/t}\right).$$

There are $X_{nj} = Y_{nj} + Z_{nj}$. Note that

$$\begin{aligned}
&\sum_{j=1}^{b_n} a_{nj}\left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\\
&= \sum_{j=1}^{b_n} a_{nj}Z_{nj} + \sum_{j=1}^{b_n} a_{nj}\left(Y_{nj} - \hat{\mathbb{E}} Y_{nj}\right) + \sum_{j=1}^{b_n} a_{nj}\left(\hat{\mathbb{E}} Y_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Hence, to prove (3.3), it suffices to verify that

$$\sum_{n=1}^{\infty} c_n \mathbb{V}\left(I_i \geq \varepsilon b_n^{1/t}\right) < \infty, i = 1, 2, \text{ and } b_n^{-1/t} |I_3| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

By (3.1), we have

$$\begin{aligned}\sum_{n=1}^{\infty} c_n \mathbb{V}\left\{I_1 > \varepsilon b_n^{1/t}\right\} &= \sum_{n=1}^{\infty} c_n \mathbb{V}\left\{\sum_{j=1}^{b_n} a_{nj} Z_{nj} > \varepsilon b_n^{1/t}\right\} \\ &\leq \sum_{n=1}^{\infty} c_n \mathbb{V}\left\{\exists j : 1 \leq j \leq b_n, \text{ such that } \left|a_{nj} X_{nj}\right| > \varepsilon b_n^{1/t}\right\} \\ &\leq \sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} \mathbb{V}\left\{\left|a_{nj} X_{nj}\right| > \varepsilon b_n^{1/t}\right\} < \infty.\end{aligned} \tag{4.4}$$

For any $r > 0$, combining the Cr inequality and (2.1),

$$\begin{aligned}\left|Y_{nj}\right|^r &\ll \left|X_{nj}\right|^r I\left(\left|X_{nj}\right| \leq \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right) + \left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)^r I\left(\left|X_{nj}\right| > \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right) \\ &\leq \left|X_{nj}\right|^r g_\mu\left(\frac{\mu a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right) + \left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)^r \left(1 - g_\mu\left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right)\right),\end{aligned}$$

thus,

$$\begin{aligned}\hat{\mathbb{E}}\left(\left|Y_{nj}\right|^r\right) &\ll \hat{\mathbb{E}}\left(\left|X_{nj}\right|^r g_\mu\left(\frac{\mu a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) + \left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)^r \hat{\mathbb{E}}\left(1 - g_\mu\left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) \\ &\leq \hat{\mathbb{E}}\left(\left|X_{nj}\right|^r g_\mu\left(\frac{\mu a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) + \left(\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)^r \mathbb{V}\left(\left|a_{nj} X_{nj}\right| \geq \mu \varepsilon b_n^{1/t}\right).\end{aligned} \tag{4.5}$$

Through definition 2.3 (ii), we know that $\left\{a_{nj}\left(Y_{nj} - \hat{\mathbb{E}} Y_{nj}\right), 1 \leq j \leq b_n, n \geq 1\right\}$ is still END with $\hat{\mathbb{E}} a_{nj}\left(Y_{nj} - \hat{\mathbb{E}} Y_{nj}\right) = 0$, by (4.3), (3.2) and (4.5), it is easy to prove that

$$\begin{aligned}\sum_{n=1}^{\infty} c_n \mathbb{V}\left\{I_2 > \varepsilon b_n^{1/t}\right\} &= \sum_{n=1}^{\infty} c_n \mathbb{V}\left(\sum_{j=1}^{b_n} a_{nj}\left(Y_{nj} - \hat{\mathbb{E}} Y_{nj}\right) > \varepsilon b_n^{1/t}\right) \\ &\ll \sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} \hat{\mathbb{E}} a_{nj}^2 Y_{nj}^2 \\ &\leq C \sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 g_\mu\left(\frac{\mu a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right) \\ &\quad + C \sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} \mathbb{V}\left(\left|a_{nj} X_{nj}\right| > \mu \varepsilon b_n^{1/t}\right) \\ &\ll \sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 g_\mu\left(\frac{\mu a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right) < \infty.\end{aligned} \tag{4.6}$$

Next, we verify $b_n^{-1/t}|I_3| \longrightarrow 0$ as $n \longrightarrow \infty$. We can obtain $\sum_{j=1}^{b_n} \mathbb{V}\left(\left|a_{nj} X_{nj}\right| \geq \varepsilon b_n^{1/t}\right) \longrightarrow 0$ as $n \longrightarrow \infty$

from (3.1) and $\{c_n, n \geq 1\}$ is a non-decreasing sequence of positive numbers, then, by $g_\mu(x) \downarrow$ in $x > 0$,

$$\begin{aligned}
b_n^{-1/t}|I_3| &= b_n^{-1/t}\left|\sum_{j=1}^{b_n} a_{nj}\left(\hat{\mathbb{E}}Y_{nj} - \hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\right| \\
&\leq b_n^{-1/t}\sum_{j=1}^{b_n} a_{nj}\left|\hat{\mathbb{E}}Y_{nj} - \hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right| \\
&\leq b_n^{-1/t}\sum_{j=1}^{b_n} a_{nj}\hat{\mathbb{E}}\left|Y_{nj} - X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right| \\
&\leq b_n^{-1/t}\sum_{j=1}^{b_n} a_{nj}\hat{\mathbb{E}}\left|-\frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(X_{nj} < -\frac{\varepsilon b_n^{1/t}}{a_{nj}}\right) + \left(X_{nj} - X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\right. \\
&\quad \left.\times I\left(\frac{\mu\varepsilon b_n^{1/t}}{a_{nj}} < X_{nj} \leq \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right) + \frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(X_{nj} > \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)\right| \\
&\ll b_n^{-1/t}\sum_{j=1}^{b_n} a_{nj}\hat{\mathbb{E}}\left[\frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(\left|X_{nj}\right| > \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right) + \left|X_{nj}\right| I\left(\frac{\mu\varepsilon b_n^{1/t}}{a_{nj}} < X_{nj} \leq \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right)\right] \\
&\leq b_n^{-1/t}\sum_{j=1}^{b_n} a_{nj}\hat{\mathbb{E}}\left[\frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(\left|X_{nj}\right| > \frac{\varepsilon b_n^{1/t}}{a_{nj}}\right) + \frac{\varepsilon b_n^{1/t}}{a_{nj}}I\left(\left|X_{nj}\right| > \frac{\mu\varepsilon b_n^{1/t}}{a_{nj}}\right)\right] \\
&\ll \sum_{j=1}^{b_n}\mathbb{V}\left(\left|a_{nj}X_{nj}\right| \geq \mu^2\varepsilon b_n^{1/t}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \tag{4.7}$$

Combining (4.4), (4.6) and (4.7), we obtain (3.3). Obviously, $\left\{-X_{nj}; 1 \leq j \leq b_n, n \geq 1\right\}$ also satisfies the conditions of Theorem 3.1, replacing $\left\{X_{nj}; 1 \leq j \leq b_n, n \geq 1\right\}$ by $\left\{-X_{nj}; 1 \leq j \leq b_n, n \geq 1\right\}$ in (3.3), by $g_\mu(x)$ is an even function, we obtain (3.4). According to the $\hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right) = \hat{\varepsilon}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)$, (3.3) and (3.4), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n\mathbb{V}\left\{\left|\sum_{j=1}^{b_n} a_{nj}\left(X_{nj} - \hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\right| \geq \varepsilon b_n^{1/t}\right\} \\
&\leq \sum_{n=1}^{\infty} c_n\mathbb{V}\left\{\sum_{j=1}^{b_n} a_{nj}\left(X_{nj} - \hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) \geq \varepsilon b_n^{1/t}\right\} \\
&\quad + \sum_{n=1}^{\infty} c_n\mathbb{V}\left\{\sum_{j=1}^{b_n} a_{nj}\left(X_{nj} - \hat{\varepsilon}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) < -\varepsilon b_n^{1/t}\right\} < \infty.
\end{aligned} \tag{4.8}$$

Therefore (3.5) holds.

Next, we prove (3.8). For $\forall\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n C_{\mathbb{V}}\left\{b_n^{-1/t}\left|\sum_{j=1}^{b_n} a_{nj}\left(X_{nj} - \hat{\mathbb{E}}X_{nj}g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right)\right| - \varepsilon\right\}_+$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V}\left\{ b_n^{-1/t} \left| \sum_{j=1}^{b_n} a_{nj} \left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right) \right) \right| - \varepsilon > u \right\} \mathrm{d}u \\
&= \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon} \mathbb{V}\left\{ b_n^{-1/t} \left| \sum_{j=1}^{b_n} a_{nj} \left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right) \right) \right| - \varepsilon > u \right\} \mathrm{d}u \\
&\quad + \sum_{n=1}^{\infty} c_n \int_\varepsilon^{\infty} \mathbb{V}\left\{ b_n^{-1/t} \left| \sum_{j=1}^{b_n} a_{nj} \left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right) \right) \right| - \varepsilon > u \right\} \mathrm{d}u \\
&\le \sum_{n=1}^{\infty} c_n \mathbb{V}\left\{ \left| \sum_{j=1}^{b_n} a_{nj} \left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right) \right) \right| > \varepsilon b_n^{1/t} \right\} \\
&\quad + \sum_{n=1}^{\infty} c_n \int_\varepsilon^{\infty} \mathbb{V}\left\{ \left| \sum_{j=1}^{b_n} a_{nj} \left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right) \right) \right| > u b_n^{1/t} \right\} \mathrm{d}u \\
&=: H_1 + H_2.
\end{aligned}
$$

By (3.5), we can get $H_1 < \infty$. Furthermore, the proof of $H_2 < \infty$ is similar to the considerations in (4.8), by $\hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right) = \hat{\varepsilon} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right)$, we only need to prove

$$
H_3 =: \sum_{n=1}^{\infty} c_n \int_\varepsilon^{\infty} \mathbb{V}\left\{ \sum_{j=1}^{b_n} a_{nj} \left(X_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right) \right) > u b_n^{1/t} \right\} \mathrm{d}u < \infty.
$$

For any $1 \le j \le b_n$, $u \ge \varepsilon$, $\delta > 0$, let

$$
Y'_{nj} = \frac{-u b_n^{1/t}}{a_{nj}} I\left(a_{nj} X_{nj} < -u b_n^{1/t} \right) + X_{nj} I\left(\left| a_{nj} X_{nj} \right| \le u b_n^{1/t} \right) + \frac{u b_n^{1/t}}{a_{nj}} I\left(a_{nj} X_{nj} > u b_n^{1/t} \right),
$$

$$
Z'_{nj} = \left(X_{nj} + \frac{u b_n^{1/t}}{a_{nj}} \right) I\left(a_{nj} X_{nj} < -u b_n^{1/t} \right) + \left(X_{nj} - \frac{u b_n^{1/t}}{a_{nj}} \right) I\left(a_{nj} X_{nj} > u b_n^{1/t} \right).
$$

There are $X_{nj} = Y'_{nj} + Z'_{nj}$. Noting that

$$
\begin{aligned}
H_3 &\le \sum_{n=1}^{\infty} c_n \int_\varepsilon^{\infty} \mathbb{V}\left\{ \sum_{j=1}^{b_n} a_{nj} Z'_{nj} > \frac{u b_n^{1/t}}{3} \right\} \mathrm{d}u \\
&\quad + \sum_{n=1}^{\infty} c_n \int_\varepsilon^{\infty} \mathbb{V}\left\{ \sum_{j=1}^{b_n} a_{nj} \left(Y'_{nj} - \hat{\mathbb{E}} Y'_{nj} \right) > \frac{u b_n^{1/t}}{3} \right\} \mathrm{d}u \\
&\quad + \sum_{n=1}^{\infty} c_n \int_\varepsilon^{\infty} \mathbb{V}\left\{ \sum_{j=1}^{b_n} a_{nj} \left(\hat{\mathbb{E}} Y'_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}} \right) \right) > \frac{u b_n^{1/t}}{3} \right\} \mathrm{d}u \\
&=: H_{31} + H_{32} + H_{33}.
\end{aligned}
$$

By (3.6), we have

$$
H_{31} = \sum_{n=1}^{\infty} c_n \int_\varepsilon^{\infty} \mathbb{V}\left\{ \sum_{j=1}^{b_n} a_{nj} Z'_{nj} > \frac{u b_n^{1/t}}{3} \right\} \mathrm{d}u
$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{\infty} c_n \int_{\varepsilon}^{\infty} \mathbb{V}\left\{\exists j: 1 \leq j \leq b_n, \text{ such that } \left|a_{nj}X_{nj}\right| > \delta u b_n^{1/t}\right\} \mathrm{d}u \\
&\leq \sum_{n=1}^{\infty} c_n \int_{\varepsilon}^{\infty} \sum_{j=1}^{b_n} \mathbb{V}\left\{\left|a_{nj}X_{nj}\right| > \delta u b_n^{1/t}\right\} \mathrm{d}u \\
&\leq \sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} C_{\mathbb{V}}\left(\frac{\left|a_{nj}X_{nj}\right|}{\delta b_n^{1/t}} I\left(\frac{\left|a_{nj}X_{nj}\right|}{\delta b_n^{1/t}} > \varepsilon\right)\right) \\
&\ll \sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} \sum_{j=1}^{b_n} \left|a_{nj}\right| C_{\mathbb{V}}\left(\left|X_{nj}\right|\left(1 - g_\mu\left(\frac{\left|a_{nj}X_{nj}\right|}{\varepsilon\delta b_n^{1/t}}\right)\right)\right) < \infty.
\end{aligned}
$$

The proof of is $H_{32} < \infty$ similar to considerations in (4.6), by Lemma 4.2 we have

$$
\begin{aligned}
H_{32} &\leq \sum_{n=1}^{\infty} c_n b_n^{-2/t} \int_{\varepsilon}^{\infty} u^{-2} \sum_{j=1}^{b_n} \hat{\mathbb{E}} a_{nj}^2 Y_{nj}'^2 \mathrm{d}u \\
&\leq C \sum_{n=1}^{\infty} c_n b_n^{-2/t} \int_{\varepsilon}^{\infty} u^{-2} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 g_\mu\left(\frac{\mu a_{nj}X_{nj}}{u b_n^{1/t}}\right) \mathrm{d}u \\
&\quad + C \sum_{n=1}^{\infty} c_n \int_{\varepsilon}^{\infty} \sum_{j=1}^{b_n} \mathbb{V}\left(\left|a_{nj}X_{nj}\right| > \mu u b_n^{1/t}\right) \mathrm{d}u \\
&\leq C \sum_{n=1}^{\infty} c_n b_n^{-2/t} \int_{\varepsilon}^{\infty} u^{-2} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right) \mathrm{d}u \\
&\quad + C \sum_{n=1}^{\infty} c_n b_n^{-2/t} \int_{\varepsilon}^{\infty} u^{-2} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 \left(g_\mu\left(\frac{\mu a_{nj}X_{nj}}{u b_n^{1/t}}\right) - g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) \mathrm{d}u \\
&\quad + C \sum_{n=1}^{\infty} c_n \int_{\varepsilon}^{\infty} \sum_{j=1}^{b_n} \mathbb{V}\left(\left|a_{nj}X_{nj}\right| > \mu u b_n^{1/t}\right) \mathrm{d}u \\
&=: H_{321} + H_{322} + H_{323}.
\end{aligned}
$$

By (3.2), we have

$$
H_{321} \ll \sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right) \int_{\varepsilon}^{\infty} u^{-2} \mathrm{d}u < \infty.
$$

By (3.7), we obtain

$$
\begin{aligned}
H_{322} &\ll \sum_{n=1}^{\infty} c_n b_n^{-2/t} \int_{\varepsilon}^{\infty} u^{-2} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 \left(1 - g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) \mathrm{d}u \\
&\leq \sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 \left(1 - g_\mu\left(\frac{a_{nj}X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) \int_{\varepsilon}^{\infty} u^{-2} \mathrm{d}u
\end{aligned}
$$

$$\ll \sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 \left(1 - g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) < \infty.$$

The proof of $H_{323} < \infty$ is similar to the proof of the convergence of H_{31}, by (3.6), we have

$$\begin{aligned} H_{323} &\ll \sum_{n=1}^{\infty} c_n \int_{\varepsilon}^{\infty} \sum_{j=1}^{b_n} \mathbb{V}\left(\left|a_{nj} X_{nj}\right| > \mu u b_n^{1/t}\right) \mathrm{d}u \\ &\leq \sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} C_{\mathbb{V}} \left(\frac{\left|a_{nj} X_{nj}\right|}{\mu b_n^{1/t}} I\left(\frac{\left|a_{nj} X_{nj}\right|}{\mu b_n^{1/t}} > \varepsilon\right)\right) \\ &\ll \sum_{n=1}^{\infty} c_n b_n^{-1/t} \sum_{j=1}^{b_n} \left|a_{nj}\right| C_{\mathbb{V}} \left(\left|X_{nj}\right| \left(1 - g_\mu \left(\frac{\left|a_{nj} X_{nj}\right|}{\varepsilon \mu b_n^{1/t}}\right)\right)\right) < \infty. \end{aligned}$$

Therefore, we get $H_{32} < \infty$. Next, we consider $H_{33} < \infty$. For $u > \varepsilon$, $1 - g_\mu\left(\frac{a_{nj}X_{nj}}{\mu u b_n^{1/t}}\right) < 1 - g_\mu\left(\frac{a_{nj}X_{nj}}{\mu \varepsilon b_n^{1/t}}\right)$, the way is similar to the proof of (4.7), we have

$$\sup_{u > \varepsilon} \left(u b_n^{1/t}\right)^{-1} \left| \sum_{j=1}^{b_n} a_{nj} \left(\hat{\mathbb{E}} Y_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) \right| \longrightarrow 0.$$

Thereby, for sufficiently large n, we can get

$$\mathbb{V}\left\{\left(u b_n^{1/t}\right)^{-1} \sum_{j=1}^{b_n} a_{nj} \left(\hat{\mathbb{E}} Y_{nj} - \hat{\mathbb{E}} X_{nj} g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) > \frac{1}{3}\right\} = 0. \tag{4.9}$$

By (4.9), we get $H_{33} < \infty$. Then $H_3 < \infty$. The proof of Theorem 3.1 is completed.

Proof of Theorem 3.2. In Theorem 3.1, let $c_n = n^{2(\alpha-1)}$, $b_n = n$, $t = 1/\alpha$, by (3.9) and $0 < \delta < 1$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} \left(X_{nj}\right)^2 g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right) \\ &\leq \sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} \left(X_{nj}\right)^2 \\ &\leq C \sum_{n=1}^{\infty} n^{2(\alpha-1)} n^{-2\alpha} n^{\delta} \\ &\ll \sum_{n=1}^{\infty} n^{-2+\delta} < \infty. \end{aligned} \tag{4.10}$$

Similarly, we can obtain

$$\sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} a_{nj}^2 \hat{\mathbb{E}} \left(X_{nj}\right)^2 \left(1 - g_\mu \left(\frac{a_{nj} X_{nj}}{\varepsilon b_n^{1/t}}\right)\right) \ll \sum_{n=1}^{\infty} n^{-2+\delta} < \infty. \tag{4.11}$$

And for $\forall c > 0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n \int_{\varepsilon}^{\infty} \sum_{j=1}^{b_n} \mathbb{V}\left(\left|a_{nj}X_{nj}\right| > cub_n^{1/t}\right) \mathrm{d}u \\
&\ll \sum_{n=1}^{\infty} c_n \int_{\varepsilon}^{\infty} \sum_{j=1}^{b_n} \frac{\hat{\mathbb{E}}\left|a_{nj}X_{nj}\right|^2}{u^2 b_n^{2/t}} \mathrm{d}u \\
&\leq \sum_{n=1}^{\infty} c_n b_n^{-2/t} \sum_{j=1}^{b_n} \hat{\mathbb{E}} a_{nj}^2 X_{nj}^2 \int_{\varepsilon}^{\infty} u^{-2} \mathrm{d}u \\
&\ll \sum_{n=1}^{\infty} n^{2(\alpha-1)} n^{-2\alpha} n^{\delta} = \sum_{n=1}^{\infty} n^{-2+\delta} < \infty.
\end{aligned} \tag{4.12}$$

Hence, (4.10) and (4.11) satisfy conditions (3.2) and (3.7) of Theorem 3.1. By (4.12), we can get $H_{31} < \infty$, then we obtain (3.10). Finally, we show that (3.11). We only need to verify that

$$I =: \left| n^{-\alpha} \sum_{j=1}^{n} a_{nj} \hat{\mathbb{E}} X_{nj} g_{\mu}\left(\frac{a_{nj}X_{nj}}{\varepsilon n^{\alpha}}\right) \right| \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.$$

For $\alpha > \delta/2$, by $\hat{\mathbb{E}} X_{nj} = 0$, we have

$$\begin{aligned}
I &= \left| n^{-\alpha} \sum_{j=1}^{n} a_{nj} \left(\hat{\mathbb{E}} X_{nj} - \hat{\mathbb{E}} X_{nj} g_{\mu}\left(\frac{a_{nj}X_{nj}}{\varepsilon n^{\alpha}}\right) \right) \right| \\
&\leq n^{-\alpha} \sum_{j=1}^{n} a_{nj} \hat{\mathbb{E}} \left| X_{nj} - X_{nj} g_{\mu}\left(\frac{a_{nj}X_{nj}}{\varepsilon n^{\alpha}}\right) \right| \\
&= n^{-\alpha} \sum_{j=1}^{n} a_{nj} \hat{\mathbb{E}} \left| X_{nj} \left(1 - g_{\mu}\left(\frac{a_{nj}X_{nj}}{\varepsilon n^{\alpha}}\right) \right) \right| \\
&\leq n^{-\alpha} \sum_{j=1}^{n} a_{nj} \hat{\mathbb{E}} \left| \frac{\mu a_{nj} X_{nj}^2}{\varepsilon n^{\alpha}} \right| \\
&\ll n^{-2\alpha} \sum_{j=1}^{n} a_{nj}^2 \hat{\mathbb{E}} X_{nj}^2 \\
&\ll n^{-2\alpha+\delta} \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.
\end{aligned}$$

Therefore the proof of Theorem 3.2 is completed.

Proof of Theorem 3.3. For $j \geq 1$, let $g_j(x) \in C_{l,Lip}(\mathbb{R})$ be an even function, such that $0 \leq g_j(x) \leq 1$ for all x; $g_j\left(\frac{x}{2^{j/t}}\right) = 1$ if $2^{(j-1)/t} < |x| \leq 2^{j/t}$ and $g_j\left(\frac{x}{2^{j/t}}\right) = 0$ if $|x| \leq \mu 2^{(j-1)/t}$ or $|x| > (1+\mu) 2^{j/t}$. Then

$$g_j\left(\frac{X}{2^{j/t}}\right) \leq I\left(\mu \left(2^{j-1}\right)^{\frac{1}{t}} < |X| \leq (1+\mu) 2^{\frac{j}{t}}\right), \tag{4.13}$$

$$|X|^q g_{\mu}\left(\frac{X}{2^{k/t}}\right) \leq 1 + \sum_{j=1}^{k} |X|^q g_j\left(\frac{X}{2^{j/t}}\right), \tag{4.14}$$

$$1-g_{\mu}\left(\frac{X}{2^{k-1/t}}\right)\leq\sum_{j=k-1}^{\infty}g_{j}\left(\frac{X}{2^{j/t}}\right). \tag{4.15}$$

Let $c_n = n^{\alpha-2}h(n)$, $b_n = n$, $a_{nj} = 1$. By Theorem 3.1, we can obtain

$$\sum_{n=1}^{\infty}n^{\alpha-2}h(n)\,C_{\mathbb{V}}\left\{n^{-1/t}\left|\sum_{j=1}^{n}\left(X_{nj}-\hat{\mathbb{E}}X_{nj}g_{\mu}\left(\frac{X_{nj}}{\varepsilon n^{1/t}}\right)\right)\right|-\varepsilon\right\}_{+}<\infty. \tag{4.16}$$

By (4.2), (4.13), (4.14) and $\alpha t < 2$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha-2-\frac{2}{t}}h(n)\sum_{j=1}^{n}\hat{\mathbb{E}}\left(X_{nj}\right)^{2}g_{\mu}\left(\frac{X_{nj}}{\varepsilon n^{1/t}}\right)\\
&=\sum_{n=1}^{\infty}n^{\alpha-1-\frac{2}{t}}h(n)\,\hat{\mathbb{E}}X_{11}^{2}g_{\mu}\left(\frac{X}{\varepsilon n^{1/t}}\right)\\
&=\sum_{k=1}^{\infty}\sum_{2^{k-1}\leq n<2^{k}}n^{\alpha-1-\frac{2}{t}}h(n)\,\hat{\mathbb{E}}X^{2}g_{\mu}\left(\frac{X_{11}}{\varepsilon n^{1/t}}\right)\\
&\ll\sum_{k=1}^{\infty}\left(2^{k}\right)^{\alpha-\frac{2}{t}}h\left(2^{k}\right)\hat{\mathbb{E}}X^{2}g_{\mu}\left(\frac{X}{\varepsilon\left(2^{k}\right)^{1/t}}\right)\\
&\leq\sum_{k=1}^{\infty}\left(2^{k}\right)^{\alpha-\frac{2}{t}}h\left(2^{k}\right)\hat{\mathbb{E}}\left(1+\sum_{j=1}^{k}X^{2}g_{j}\left(\frac{X}{\varepsilon 2^{j/t}}\right)\right)\\
&\leq\sum_{k=1}^{\infty}\left(2^{k}\right)^{\alpha-\frac{2}{t}}h\left(2^{k}\right)+\sum_{k=1}^{\infty}\left(2^{k}\right)^{\alpha-\frac{2}{t}}h\left(2^{k}\right)\sum_{j=1}^{k}\hat{\mathbb{E}}X^{2}g_{j}\left(\frac{X}{\varepsilon 2^{j/t}}\right)\\
&\ll\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\left(2^{k}\right)^{\alpha-\frac{2}{t}}h\left(2^{k}\right)\hat{\mathbb{E}}X^{2}g_{j}\left(\frac{X}{\varepsilon 2^{j/t}}\right)\\
&\leq\sum_{j=1}^{\infty}\left(2^{j}\right)^{\alpha-\frac{2}{t}}h\left(2^{j}\right)\left(2^{j}\right)^{\frac{2}{t}}\mathbb{V}\left(|X|>\left(2^{j-1}\right)^{1/t}\right)\\
&=\sum_{j=1}^{\infty}2^{j\alpha}h\left(2^{j}\right)\mathbb{V}\left(|X|>c2^{j/t}\right)<\infty.
\end{aligned}$$

By (4.1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha-2-\frac{1}{t}}h(n)\sum_{j=1}^{n}\left|a_{nj}\right|C_{\mathbb{V}}\left[\left|X_{nj}\right|\left(1-g_{\mu}\left(\frac{a_{nj}X_{nj}}{\varepsilon n^{1/t}}\right)\right)\right]\\
&=\sum_{n=1}^{\infty}n^{\alpha-1-\frac{1}{t}}h(n)\,C_{\mathbb{V}}\left[|X|\left(1-g_{\mu}\left(\frac{X}{\varepsilon n^{1/t}}\right)\right)\right]\\
&\leq\sum_{n=1}^{\infty}n^{\alpha-1-\frac{1}{t}}h(n)\int_{0}^{\infty}\mathbb{V}\left\{|X|\,I\left(|X|>\mu\varepsilon n^{1/t}\right)>x\right\}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\alpha-1-\frac{1}{t}} h(n) \int_0^{\mu\varepsilon n^{1/t}} \mathbb{V}\left\{|X| > \mu\varepsilon n^{1/t}\right\} \mathrm{d}x + \sum_{n=1}^{\infty} n^{\alpha-1-\frac{1}{t}} h(n) \int_{\mu\varepsilon n^{1/t}}^{\infty} \mathbb{V}\{|X| > x\} \mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha-1} h(n) \mathbb{V}\left(|X| > \mu\varepsilon n^{1/t}\right) + \sum_{n=1}^{\infty} n^{\alpha-1-\frac{1}{t}} h(n) \sum_{j=n}^{\infty} \int_{j^{1/t}}^{(j+1)^{1/t}} \mathbb{V}\left(|X| > j^{1/t}\right) \mathrm{d}x \\
&\ll \sum_{n=1}^{\infty} n^{\alpha-1-\frac{1}{t}} h(n) \sum_{j=n}^{\infty} j^{\frac{1}{t}-1} \mathbb{V}\left(|X| > j^{1/t}\right) \\
&\leq \sum_{j=1}^{\infty} j^{\frac{1}{t}-1} h(j) \mathbb{V}\left(|X| > j^{1/t}\right) \sum_{n \leq j} n^{\alpha-1-\frac{1}{t}} \\
&\leq \sum_{j=1}^{\infty} j^{\alpha-1} h(j) \mathbb{V}\left(|X| > j^{1/t}\right) < \infty.
\end{aligned}
$$

And by (4.2), (4.13) and (4.15), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha-2-\frac{2}{t}} h(n) \sum_{j=1}^{n} \hat{\mathbb{E}}\left(X_{nj}\right)^2 \left(1 - g_\mu\left(\frac{X_{nj}}{\varepsilon n^{1/t}}\right)\right) \\
&= \sum_{n=1}^{\infty} n^{\alpha-1-\frac{2}{t}} h(n) \hat{\mathbb{E}} X^2 \left(1 - g_\mu\left(\frac{X}{\varepsilon n^{1/t}}\right)\right) \\
&= \sum_{k=1}^{\infty} \sum_{2^{k-1} \leq n < 2^k} n^{\alpha-1-\frac{2}{t}} h(n) \hat{\mathbb{E}} X^2 \left(1 - g_\mu\left(\frac{X}{\varepsilon n^{1/t}}\right)\right) \\
&\ll \sum_{k=1}^{\infty} \left(2^k\right)^{\alpha-\frac{2}{t}} h\left(2^k\right) \hat{\mathbb{E}} X^2 \left(1 - g_\mu\left(\frac{X}{\varepsilon \left(2^{k-1}\right)^{1/t}}\right)\right) \\
&\leq \sum_{k=1}^{\infty} \left(2^k\right)^{\alpha-\frac{2}{t}} h\left(2^k\right) \sum_{j=k-1}^{\infty} \hat{\mathbb{E}} X^2 g_j\left(\frac{X}{\varepsilon 2^{j/t}}\right) \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{j} \left(2^k\right)^{\alpha-\frac{2}{t}} h\left(2^k\right) \hat{\mathbb{E}} X^2 g_j\left(\frac{X}{\varepsilon 2^{j/t}}\right) \\
&\leq \sum_{j=1}^{\infty} 2^{j\alpha} h\left(2^j\right) \mathbb{V}\left(|X| > c 2^{j/t}\right) < \infty.
\end{aligned}
$$

thus (3.2), (3.6), (3.7) are satisfied. In order to prove (3.12), it remain to show that $n^{-1/t}\left|\sum_{j=1}^{n} \hat{\mathbb{E}} X_{nj} g_\mu\left(\frac{X_{nj}}{\varepsilon n^{1/t}}\right)\right| \longrightarrow 0$ as $n \longrightarrow \infty$. First of all, by the properties of the slowly varying function, we obtain $|X|^{\alpha t} h\left(|X|^t\right) \uparrow$ when $\alpha t > 1$. By (2.1), $\alpha t > 1$, $\alpha > 1$ and $|X|^{\alpha t} h\left(|X|^t\right) \uparrow$, we can get

$$
\begin{aligned}
|X|\left(1 - g_\mu\left(\frac{x}{\varepsilon n^{1/t}}\right)\right) &\leq |X| I\left(|X| > \mu\varepsilon n^{1/t}\right) \leq |X| \frac{|X|^{\alpha t-1} h\left(|X|^t\right)}{\left(\mu\varepsilon n^{1/t}\right)^{\alpha t-1} h\left(\mu^t \varepsilon^t n\right)} \\
&\ll n^{-\alpha+1/t} h^{-1}(cn) |X|^{\alpha t} h\left(|X|^t\right)
\end{aligned}
\tag{4.17}
$$

By $\hat{\mathbb{E}}X = 0$, (4.17), $\alpha t > 1$, $\alpha > 1$ and $\hat{\mathbb{E}}\left(|X|^{\alpha t} h\left(|X|^t\right)\right) \leq C_{\mathbb{V}}\left(|X|^{\alpha t} h\left(|X|^t\right)\right) < \infty$, we have

$$\begin{aligned} n^{-1/t}\left|\sum_{j=1}^{n} \hat{\mathbb{E}}X_{nj}g_{\mu}\left(\frac{X_{nj}}{\varepsilon n^{1/t}}\right)\right| &= n^{1-1/t}\left|\hat{\mathbb{E}}Xg_{\mu}\left(\frac{X}{\varepsilon n^{1/t}}\right)\right| \\ &= n^{1-1/t}\left|\hat{\mathbb{E}}X - \hat{\mathbb{E}}Xg_{\mu}\left(\frac{X}{\varepsilon n^{1/t}}\right)\right| \\ &\leq n^{1-1/t}\hat{\mathbb{E}}\left|X - Xg_{\mu}\left(\frac{X}{\varepsilon n^{1/t}}\right)\right| \\ &= n^{1-1/t}\hat{\mathbb{E}}\,|X|\left(1 - g_{\mu}\left(\frac{X}{\varepsilon n^{1/t}}\right)\right) \\ &\leq n^{1-1/t}\hat{\mathbb{E}}\,|X|\,\frac{|X|^{\alpha t-1} h\left(|X|^t\right)}{\left(\mu\varepsilon n^{1/t}\right)^{\alpha t-1} h\left(\mu^t\varepsilon^t n\right)} \\ &\ll n^{1-\alpha}h^{-1}(cn)\,\hat{\mathbb{E}}\left(|X|^{\alpha t} h\left(|X|^t\right)\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{4.18}$$

Combining (4.16) and (4.18), we can get (3.12). Then Theorem 3.3 holds.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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