



*Research article*

## Lie symmetries of Generalized Equal Width wave equations

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**Abstract:** Lie symmetry analysis of differential equations proves to be a powerful tool to solve or atleast to reduce the order and non-linearity of the equation. The present article focuses on the solution of Generalized Equal Width wave (GEW) equation using Lie group theory. Over the years, different solution methods have been tried for GEW but Lie symmetry analysis has not been done yet. At first, we obtain the infinitesimal generators, commutation table and adjoint table of Generalized Equal Width wave (GEW) equation. After this, we find the one dimensional optimal system. Then we reduce GEW equation into non-linear ordinary differential equation (ODE) by using the Lie symmetry method. This transformed equation can take us to the solution of GEW equation by different methods. After this, we get the travelling wave solution of GEW equation by using the Sine-cosine method. We also give graphs of some solutions of this equation.

**Keywords:** Generalized Equal Width wave equation; Lie symmetry; optimal system; reduction; travelling wave solution; Sine-cosine method

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### 1. Introduction

Symmetry is a wonderful tool in explicating the laws of nature. One of the great stories of Lie's success was the initiation of a program to solve or at least to simplify the differential equation by using the Lie group theory on the analogy of *Évariste* Galois's work to solve the algebraic equations having degrees two, three and four. But he was unable to give closed forms of roots of equations whose

degrees are five or greater, by using only the arithmetic operation which are  $(+, -, \div, \times, \sqrt{\quad})$  [1]. Lie's original ideas was to establish a general theory of integration of ordinary differential equation on the similar basis as Galois and Abel did for algebraic equations.

At first the method could not attract much attention until last few decades when a revival of the interests in Lie's theory was observed which resulted in a significant progress. The main hinderance was the complicated system of many differential equations which we obtain in symmetry analysis. Today we have strong computer algebra systems like Maple and Mathematica to handle this problem. Physical laws of nature are best governed by the exploitation of symmetries involved in the system. Real world phenomenons can be transformed into mathematical language by using non-linear PDE's. The non-linear PDE's have attracted much attention and have wide range of applications in science and engineering like plasma waves, fluid mechanics, optics, biological systems, chemical physics and financial systems. One such immensely popular class of equation is Non-linear Evolution Equation (NLEE's). Complex systems can be characterized by NLDE's. One particularly famous NLDE is the KdV equation derived by Korteweg and de Vries [2]. It can be given in the following form,

$$u_t + uu_x + u_{xxx} = 0. \quad (1.1)$$

It can be observed that this equation is a non-linear PDE in one dimension and describes the time-dependent motion of shallow water waves. Other equation is the Regularised Long wave equation (RLW) [3]. It is more common than KdV equation to depict the behaviour of non-linear dispersive waves. This equation can be written as,

$$u_t + u_x + \varepsilon uu_x - \mu u_{xxt} = 0. \quad (1.2)$$

Morrison et al. [4] introduced one dimensional non-linear Evolution Equation, in the given form,

$$u_t + uu_x - \mu u_{xxt} = 0. \quad (1.3)$$

It is also known as Equal Width wave (EW) equation [5–10] derived by using both KdV and RLW equation. Authors in [5] discussed solitary waves related to EW equation. Different methods have been adopted to find the solution of EW equations both numerically and analytically, see for instance [5–10]. Because of soliton solution with permanent speed and form, the wave has equal width for all wave amplitude. That's why it is called Equal Width wave equation. Here  $\mu$  is a positive parameter,  $x, t$  denote space and time coordinate respectively, and  $u(x, t)$  represents wave amplitude with boundary condition  $u \rightarrow 0$  as  $x \rightarrow \pm\infty$ . In plasma physics  $u$  represents  $-ve$  of electrostatic potential and in fluid problems,  $u$  represents wave vertical displacement of water surface [11]. Solitary wave solutions can also be found in EW equation. Soliton is unique type of solitary waves that retain its shape after colliding with other objects.

The Modified Equal Width wave (MEW) equation is derived from EW equation and it has cubic non-linearity with dispersive wave form

$$u_t + u^2 u_x - \mu u_{xxt} = 0. \quad (1.4)$$

This equation has also been discussed on large scale recently [12–18]. Authors solved this equation numerically in [12, 18]. Some B-spline methods have been adopted to solve this equation in [13–17]. A

generalization of EW equation also known as Generalized Equal Width wave (GEW) equation [19–23] can be derived from EW equation in following form

$$u_t + u^n u_x - \mu u_{xxt} = 0, \quad (1.5)$$

with  $n + 1$  non linearity and dispersion wave having solitary solution, where  $n \in \mathbb{Z}^+$ . This equation can be related with Generalized KdV equation [24] and Generalized RLW equation [25, 26]. Solitary waves of GEW has been discussed in [19]. Raslan used collocation method to solve GEW in [20]. Some other methods also have been employed on this equation, see [21–23].

With the passage of time, application of DEs and it uses are increasing continuously. Many numerical as well as analytical method keep on emerging to solve PDE's. Some famous numerical method are Finite difference method [27], Multigrid method [28], Methods of lines [29], Domain decomposition method [30], Gradient discretization method [31], Mesh free method [32], Spectral method [33] and Quadratic B-spline method [34] etc. Some popular analytical methods are Daraboux transformation method [35, 36], Inverse scattering method [37, 38], Kudryashov method [39, 40], Simplest equation method [41], Homogenous balance method [42], Tanh-coth method [43], Hirota bilinear method [44], Jacobi elliptic function expansion method [45, 46], Sine-cosine method [47] and Lie symmetric method [48–53]. These methods give exact solutions of the PDE's. Many numerical and analytical method have been used to find the solution of EW equation, MEW equation and GEW equation. Zaki used the least square finite element method to find the numerical solution of EW equation [54]. Elein Yusufoglu et al. used He's variational iteration method to find the numerical solution of EW equation [55]. G. A Gardner et al. used Galerkin method to obtain numerical solution of EW equation [56]. B. G. Karakoc et al. used cubic B-spline lumped Galerkin method to find the numerical solution of MEW equation [57]. Khalique et al. used Jacobi elliptic expansion method to find the travelling wave solution of EW equation [58]. Evan et al. computed the solitary wave solution of GEW equation by using quadratic B-spline method [59]. S. Hamdi et al. discussed the exact solution of GEW equation [60]. Halil Zeybek obtained the numerical solution of GEW equation by using cubic B-spline Galerkin method [61]. S. B. G. Karakoc et al. used the septic B-spline collocation method to find the solution of GEW equation in [62] and derived numerical solution of GEW equation by using sextic B-spline finite element method in [63]. B. G. Karakoc et al. used the finite element method to find the numerical solution of GEW equation [64].

In this article, we find Lie symmetries and travelling wave solution of GEW equation by using Sine-cosine method. Lie symmetry approach has not been used for GEW equation. We use this method to reduce the complexity of GEW equation. It is worth mentioning that Lie symmetry method is the most important approach for constructing analytical solutions of nonlinear PDEs. We prefer to use Lie symmetry analysis because it studies the invariance of differential equations (DEs) under a one-parameter group of transformations which transforms a solution to another new solution and is used to reduce the order such as the number of variables of DEs; moreover, the conservation laws can be constructed by using the symmetries of the DEs. Kumar et al. effectively used this method for (3+1)-dimensional generalized KP equation in [48], (3+1)-dimensional KdV-type equation in [49], (2+1)-dimensional BK equation with variable coefficient in [50] and CHKP equation in [51]. Liu et al. used this approach to find the invariant solutions of SP equation in [52]. Chauhan et al. used this method to find traveling wave solutions of EW equation in [53].

## 2. General method to find the Lie symmetry

We describe a method to find the symmetry of non-linear PDEs in this section. For this, we consider the system of  $n$ -th order non-linear PDEs having  $p$  independent variable  $X = (x^1, x^2, \dots, x^p) \in R^p$  and  $q$  dependent variable  $U = (u^1, u^2, \dots, u^q) \in R^q$  has form

$$\Delta_b(X, U^{(n)}) = 0, \quad b = 1, 2, \dots, l, \quad (2.1)$$

where  $U^{(n)} = \frac{\partial u^n}{\partial x^n}$ .

To construct the Lie symmetry, firstly we introduce a Lie group of transformation acting on both dependent and independent variable such as

$$\begin{aligned} \tilde{x}^c &= x^c + \varepsilon \xi^c(X, U) + O(\varepsilon^2), & c &= 1, 2, \dots, p, \\ \tilde{u}^d &= u^d + \varepsilon \eta^d(X, U) + O(\varepsilon^2), & d &= 1, 2, \dots, q, \end{aligned} \quad (2.2)$$

where  $\varepsilon$  taken as a very small parameter, usually taken as  $\varepsilon \ll 1$  and  $\xi^c$  and  $\eta^d$  are infinitesimal generators with respect to the independent and dependent variable respectively.

The vector field of above transformation is

$$V = \sum_{c=1}^p \xi^c(X, U) \frac{\partial}{\partial x^c} + \sum_{d=1}^q \eta^d(X, U) \frac{\partial}{\partial u^d}. \quad (2.3)$$

Thus the  $n$ -th prolongation of vector field  $v$  is also the vector field, which is

$$pr^{(n)}V = V + \sum_{r=1}^q \sum_S \eta_r^S(X, U^{(n)}) \frac{\partial}{\partial u_r^S}, \quad (2.4)$$

defined on the space  $M^{(n)} \subset X \times U^{(n)}$ , where  $S = (s_1, s_2, \dots, s_k)$ , with  $1 \leq s_k \leq p$ ,  $1 \leq k \leq n$ , here

$$\eta_r^S(x, u^{(n)}) = D_S(\eta_r - \sum_{c=1}^p \xi^c u_c^r) + \sum_{c=1}^p \xi^c u_{S,c}^r, \quad (2.5)$$

where  $u_c^r = \frac{\partial u^r}{\partial x^c}$ , and  $u_{s,c}^r = \frac{\partial u_s^r}{\partial x^c}$ , where the total derivative  $D_x$  and  $D_y$  is defined as

$$\begin{aligned} D_x H &= \frac{\partial H}{\partial x} + u_x \frac{\partial H}{\partial u} + u_{xx} \frac{\partial H}{\partial u_x} + u_{xy} \frac{\partial H}{\partial u_y} + \dots, \\ D_y H &= \frac{\partial H}{\partial y} + u_y \frac{\partial H}{\partial u} + u_{xy} \frac{\partial H}{\partial u_x} + u_{yy} \frac{\partial H}{\partial u_y} + \dots, \end{aligned} \quad (2.6)$$

with condition

$$Pr^{(n)}v[\Delta_b(X, U^{(n)})] = 0, \quad b = 1, 2, \dots, l, \quad \text{whenever} \quad \Delta_b(X, U^{(n)}) = 0. \quad (2.7)$$

### 3. Lie symmetry of Generalized Equal Width wave equation

Taking one parameter local Lie group of transformation having variable  $x$ ,  $t$  and  $u$  are as follows:

$$\begin{aligned}x^* &= x + \varepsilon \zeta(t, x, u) + O(\varepsilon^2), \\t^* &= t + \varepsilon \tau(t, x, u) + O(\varepsilon^2), \\u^* &= u + \varepsilon \varphi(t, x, u) + O(\varepsilon^2),\end{aligned}\tag{3.1}$$

where  $\varepsilon \in R$  is the group parameter. The vector field  $v$  of equation is defined as

$$V = \zeta(t, x, u) \frac{\partial}{\partial x} + \tau(t, x, u) \frac{\partial}{\partial t} + \varphi(t, x, u) \frac{\partial}{\partial u}.\tag{3.2}$$

The third order prolongation is defined as:

$$Pr^{(3)}V = V + \varphi^x \frac{\partial}{\partial u_x} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}} + \varphi^{xxx} \frac{\partial}{\partial u_{xxx}} + \varphi^{xxt} \frac{\partial}{\partial u_{xxt}} + \varphi^{xtt} \frac{\partial}{\partial u_{xtt}} + \varphi^{ttt} \frac{\partial}{\partial u_{ttt}}.\tag{3.3}$$

Now by applying the third order prolongation to the Eq (1.5), we get invariance condition

$$\varphi^t + nu^{n-1} \varphi u_x + u^n \varphi^x - \mu \varphi^{xxt} = 0,\tag{3.4}$$

by putting the value of coefficients  $[\varphi^t]$ ,  $[\varphi^x]$  and  $[\varphi^{xxt}]$  in above equation

$$\begin{aligned}\varphi^t &= D_t(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xt} + \tau u_{tt}, \\ \varphi^x &= D_x(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xt} + \tau u_{tt}, \\ \varphi^{xxt} &= D_x D_x D_t(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xxx} + \tau u_{xxt},\end{aligned}\tag{3.5}$$

where  $D_t$  and  $D_x$  are total derivative. By applying the third order prolongation and use the value of  $\varphi^t$ ,  $\varphi^x$  and  $\varphi^{xxt}$ , we get

$$\begin{aligned} & nu^{n-1} \varphi u_x - u^n \zeta u_x^2 - u^n \tau u_x u_t + u^n \varphi u_x - u^n \zeta_x u_x - u^n \tau_x u_t \\ & + u^n \varphi_x - \zeta u_x u_t - \zeta_t u_x - \tau u_t^2 - \tau_t u_t + \varphi u_t + \varphi_t + 3\mu \zeta_{uu} u_x u_{xx} u_t \\ & + 4\mu \tau_{uu} u_x u_t u_{xt} - \mu \varphi_{xxt} + 2\mu \tau_{ux} u_x u_{tt} + \mu \zeta_{uuu} u_x^3 u_t + \mu \tau_{uuu} u_x^2 u_t^2 \\ & + \mu \tau_{uuu} u_x^2 u_t - \mu \varphi_{uuu} u_x^2 u_t + 3\mu \zeta_{uu} u_x^2 u_{xt} + \mu \tau_{uu} u_x^2 u_{tt} + 3\mu \zeta_{uu} u_x u_{xx} \\ & + 2\mu \tau_{ut} u_x u_{xt} - 2\mu \varphi_{uu} u_x u_{xt} + \mu \tau_{uu} u_{xx} u_t^2 + \mu \tau_{ut} u_{xx} u_t - \mu \varphi_{uu} u_{xx} u_t \\ & + 3\mu \zeta_{uu} u_{xx} u_{xt} + \mu \tau_{uu} u_{xx} u_{tt} + \mu \zeta_{uu} u_{xxx} u_t + 3\mu \zeta_{uu} u_x u_{xxt} + 2\mu \tau_{uu} u_{xxt} u_t \\ & + 2\mu \tau_{uu} u_x u_{xt} + 4\mu \tau_{ux} u_t u_{xt} + 2\mu \zeta_{uuu} u_x^2 u_t + 2\mu \tau_{uuu} u_x u_t^2 + 2\mu \tau_{uxt} u_x u_t \\ & + \mu \zeta_{xxu} u_x u_t - 2\mu \varphi_{uuu} u_x u_t + 4\mu \zeta_{ux} u_x u_{xt} + 2\mu \zeta_{ux} u_{xx} u_t + \mu \zeta_{uuu} u_x^3 \\ & - \mu \varphi_{uuu} u_x^2 - \mu \varphi_{uu} u_{xx} + 2\mu \tau_{uu} u_{xt}^2 + \mu \zeta_t u_{xxx} + \mu \tau_t u_{xxt} - \mu \varphi_{uu} u_{xxt} \\ & + 2\mu \zeta_x u_{xxt} + 2\mu \tau_x u_{xt} + \mu \zeta_{xxt} u_x + \mu \tau_{xxt} u_t - 2\mu \varphi_{uxt} u_x - \mu \varphi_{u_{xx}} u_t \\ & + 2\mu \zeta_{xt} u_{xx} + 2\mu \tau_{xt} u_{xt} - 2\mu \varphi_{ux} u_{xt} + \mu \zeta_{xx} u_{xt} + \mu \tau_{xx} u_{tt} + 2\mu \zeta_{uxt} u_x^2 \\ & + \mu \tau_{u_{xx}} u_t^2 = 0.\end{aligned}\tag{3.6}$$

By putting the value of  $u_t$  and put the coefficients of various monomial equal to zero, we get the system of equations in term of partial derivatives, which is

$$\begin{aligned}
 u_x : \quad & nu^{n-1}\varphi - u^n\zeta_x + u^n\tau_t - 2\mu\varphi_{uxt} + \mu u^n\varphi_{uux} = 0, \\
 u_{xxt} : \quad & 2\mu\zeta_x - \mu^2\varphi_{uux} = 0, \\
 u_{xt} : \quad & -2\mu\varphi_{ux} + \mu\zeta_{xx} = 0, \\
 u_x u_{xt} : \quad & \varphi_{uu} = 0, \\
 u_{xx} : \quad & \varphi_{ut} = 0, \\
 \text{constant} : \quad & \varphi_t + u^n\varphi_x - \mu\varphi_{xxt} = 0.
 \end{aligned} \tag{3.7}$$

We can solve this system manually or by using software Mathematica or Maple, so by solving the system we get infinitesimal

$$\zeta(t, x, u) = C_3, \quad \tau(t, x, u) = C_1t + C_2, \quad \varphi(t, x, u) = -\frac{1}{n}C_1u, \tag{3.8}$$

where  $C_1$ ,  $C_2$  and  $C_3$  are arbitrary constants. As a result, three vector fields spanned the Lie algebra of infinitesimal generators of Eq (1.5) .

$$\begin{aligned}
 H_1 &= t\frac{\partial}{\partial t} - \frac{1}{n}u\frac{\partial}{\partial u}, \\
 H_2 &= \frac{\partial}{\partial t}, \\
 H_3 &= \frac{\partial}{\partial x}.
 \end{aligned} \tag{3.9}$$

Thus the one-parameter Lie group  $G_i$ , ( $i = 1, 2, 3$ ) generated by the three vector fields  $H_1$ ,  $H_2$  and  $H_3$  which are

$$\begin{aligned}
 G_1 &= (t, x, u) \rightarrow (te^\epsilon, x, ue^{-\frac{1}{n}\epsilon}), \\
 G_2 &= (t, x, u) \rightarrow (t + \epsilon, x, u), \\
 G_3 &= (t, x, u) \rightarrow (t, x + \epsilon, u),
 \end{aligned} \tag{3.10}$$

where  $\epsilon \in R$  is the group parameter.

Since each group  $G_i$  is a symmetric group, so if  $u = h(t, x)$  is a solution of (1.5), so are the functions

$$\begin{aligned}
 u^{(1)} &= e^{-\frac{1}{n}\epsilon}h(te^{-\epsilon}, x), \\
 u^{(2)} &= h(t - \epsilon, x), \\
 u^{(3)} &= h(t, x - \epsilon).
 \end{aligned} \tag{3.11}$$

### 3.1. Optimal system of one-dimensional subalgebras

An optimal system of one parameter Lie group is the collection of all inequivalent one parameter Lie group that any other subgroup is conjugate to one of group in the collection. To compute the optimal system, first we compute the commutator table.

The commutator table of  $H_i$ , ( $i = 1, 2, 3$ ) is

**Table 1.** Commutator table.

*	$H_1$	$H_2$	$H_3$
$H_1$	0	$-H_2$	0
$H_2$	$H_2$	0	0
$H_3$	0	0	0

Now we proceed to compute the adjoint table.

The adjoint table for  $H_i$ , ( $i = 1, 2, 3$ ) is

**Table 2.** Adjoint table.

[adj]	$H_1$	$H_2$	$H_3$
$H_1$	$H_1$	$e^\epsilon H_2$	$H_3$
$H_2$	$H_1 - \epsilon H_2$	$H_2$	$H_3$
$H_3$	$H_1$	$H_2$	$H_3$

Consider a generator of the form

$$H = a_1 H_1 + a_2 H_2 + a_3 H_3, \quad (3.12)$$

where  $a_1$ ,  $a_2$  and  $a_3$  are arbitrary constant. We solve it by using different cases.

**Case 1:**

Let  $a_1 \neq 0$  and  $a_1 = 1$ , then

$$H = H_1 + a_2 H_2 + a_3 H_3. \quad (3.13)$$

By acting  $Adj_{e^{a_2} H_2}$  on  $H$ , the coefficient  $a_2$  disappear.

$$\dot{H} = H_1 + a_3 H_3. \quad (3.14)$$

**Subcase 1:** If  $a_3 < 0$ , then

$$\dot{H} = H_1 - H_3. \quad (3.15)$$

**Subcase 2:** If  $a_3 > 0$ , then

$$\dot{H} = H_1 + H_3. \quad (3.16)$$

**Case 2:**

Let  $a_1 = 0$  and  $a_2 = 1$ , then

$$H = H_2 + a_3 H_3. \quad (3.17)$$

**Subcase 1:** If  $a_3 < 0$ , then

$$H = H_2 - H_3. \quad (3.18)$$

**Subcase 2:** If  $a_3 > 0$ , then

$$H = H_2 + H_3. \quad (3.19)$$

**Case 3:**

Let  $a_1 = 0$ ,  $a_2 = 0$  and  $a_3 = 1$ , then

$$H = H_3. \quad (3.20)$$

**Case 4:**

Let  $a_3 = 0$ , then

$$H = a_1 H_1 + a_2 H_2. \quad (3.21)$$

By acting  $Adj_{e^{\frac{a_2}{a_1} H_2}} H$  on  $H$ , we get

$$\dot{H} = a_1 H_1. \quad (3.22)$$

**Case 5:**

Let  $a_1 = a_3 = 0$  and  $a_2 \neq 0$ , then

$$H = a_2 H_2. \quad (3.23)$$

So by using the optimal system method, we get the optimal system of Eq (1.5), which is  $\bullet H_1$

- $\bullet H_2$
- $\bullet H_3$
- $\bullet H_1 \pm H_3$
- $\bullet H_2 \pm H_3.$

### 3.2. Lie symmetry reduction of Generalized Equal Width wave equation

In this section, we use the Lie symmetry method to find the exact solution of GEW equation. First by using the Lie symmetry, we convert PDEs into ODEs, then by using any appropriate method, we get the exact solution of ODEs. To obtain reduction form, we use subalgebra  $H_1$ ,  $H_2$ ,  $H_3$ ,  $H_1 + H_3$  and  $H_2 + H_3$ .

**Case 1:**

The characteristic equation for  $H_1 = -\frac{1}{n}u \frac{\partial}{\partial u} + t \frac{\partial}{\partial t}$  is

$$\frac{dx}{0} = \frac{dt}{t} = \frac{du}{-\frac{1}{n}u}. \quad (3.24)$$

From this, we have

$$x = r, \quad s = ut^{\frac{1}{n}}, \quad (3.25)$$



where  $r$  and  $s$  are constant of integration. So

$$u = \frac{F(r)}{t^{\frac{1}{n}}}, \quad \text{and} \quad (3.26)$$

$$u_t = -\frac{1}{n} \frac{F(r)}{t^{\frac{1}{n}+1}}, \quad u_x = \frac{F'(r)}{t^{\frac{1}{n}}}, \quad u_{xxt} = -\frac{1}{n} \frac{F''(r)}{t^{\frac{1}{n}+1}}.$$

So by putting the value of  $u_t$ ,  $u_x$  and  $u_{xxt}$  in Eq (1.5), we get

$$-F(r) + nF^n(r)F'(r) + \mu F''(r) = 0, \quad (3.27)$$

which is a 2nd order non-linear ODE, which we can solved numerically.

**Case 2:**

The characteristics equation for  $H_2 = \frac{\partial}{\partial t}$  is

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}. \quad (3.28)$$

From this, we have

$$x = r, \quad u = s, \quad (3.29)$$

where  $r$  and  $s$  are constant of integration. So

$$u = F(r), \quad \text{and} \quad (3.30)$$

$$u_t = 0, \quad u_x = F'(r), \quad u_{xxt} = 0. \quad (3.31)$$

So by putting the value of  $u_t$ ,  $u_x$  and  $u_{xxt}$  in Eq (1.5), we get

$$F'(r) = 0, \quad \text{and} \quad (3.32)$$

$$F(r) = c_1, \quad (3.33)$$

where  $c_1$  is the constant of integration.

**Case 3:**

The characteristics equation for  $H_3 = \frac{\partial}{\partial x}$  is

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}. \quad (3.34)$$

So, we have

$$t = r, \quad u = s, \quad (3.35)$$

where  $r$  and  $s$  are constant of integration. Thus we have

$$u = F(r), \quad \text{and} \quad (3.36)$$

$$u_t = F'(r), \quad u_x = 0, \quad u_{xxt} = 0. \quad (3.37)$$

By putting the value of  $u_x, u_t$  and  $u_{xxt}$  in Eq (1.5), we have

$$F'(r) = 0, \quad (3.38)$$

which implies that

$$F(r) = c_2, \quad (3.39)$$

where  $c_2$  is the constant of integration.

**Case 4:**

The characteristics equation for  $H_1 + H_3 = \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{1}{n} u \frac{\partial}{\partial u}$  is

$$\frac{dx}{1} = \frac{dt}{t} = \frac{du}{-\frac{1}{n}u}. \quad (3.40)$$

So we have

$$r = \frac{e^x}{t}, \quad s = ut^{\frac{1}{n}}, \quad (3.41)$$

where  $r$  and  $s$  are constant of integration. Thus we have

$$u = \frac{F(r)}{t^{\frac{1}{n}}}, \quad \text{and} \quad (3.42)$$

$$u_t = -\left[\frac{F(r) + nrF'(r)}{nt^{\frac{1}{n}+1}}\right], \quad u_x = \frac{rF'(r)}{t^{\frac{1}{n}}}, \quad u_{xxt} = -\left[\frac{nr^3F'''(r) + (3n+1)r^2F''(r) + (n+1)rF'(r)}{nt^{\frac{1}{n}+1}}\right]. \quad (3.43)$$

By putting the value of  $u_x, u_t$  and  $u_{xxt}$  in Eq (1.5), we get

$$-F(r) - nrF'(r) + nrF^n(r)F'(r) + \mu(nr^3F'''(r) + (3n+1)r^2F''(r) + (n+1)rF'(r)) = 0, \quad (3.44)$$

which is a 3rd order non-linear ODE, which we can solve numerically.

**Case 5:**

The characteristic equation for  $H_2 + H_3 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$  is

$$\frac{dx}{1} = \frac{dt}{1} = \frac{du}{0}. \quad (3.45)$$

So, we have

$$r = x - t, \quad u = s, \quad (3.46)$$

where  $r$  and  $s$  are constant of integration. Thus we have

$$u = F(r), \quad \text{and} \quad (3.47)$$

$$u_t = -F'(r), \quad u_x = F'(r), \quad u_{xxt} = -F'''(r). \quad (3.48)$$

By putting the value  $u_t, u_x$  and  $u_{xxt}$  in Eq (1.5), we get a 3rd order non-linear ODE.

$$-F'(r) + F^n(r)F'(r) + \mu F'''(r) = 0. \quad (3.49)$$

#### 4. Travelling wave solution of Generalized Equal Width wave equation by using Sine-cosine method

**Travelling wave:** A wave in which medium move in the direction of propagation of wave is called travelling wave. We can find the travelling wave solution of those equation which propagation the wave property. The wave occur in the form of  $u(x, t) = f(x - ct)$ , where  $c$  is the wave speed which move in negative direction as  $c < 0$  and in positive direction as  $c > 0$ .

**Procedure:**

We consider the non-linear PDE's in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \quad (4.1)$$

where  $u(x, t)$  is the travelling wave solution of non-linear PDE's. The wave variable  $\xi = x - ct$  is used to obtain the travelling wave solution so that  $u(x, t) = u(\xi)$ , where  $\xi = x - ct$ .

This allow us to make the following modification:

$$\frac{\partial}{\partial t} = -c \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial t^2} = c^2 \frac{d^2}{d\xi^2}, \quad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}. \quad (4.2)$$

The above equation was used to transfer PDEs to ODEs.

$$Q(u, -cu', u', u'', c^2u'', \dots) = 0, \quad (4.3)$$

where  $u' = \frac{du}{d\xi}$ .

Equation (4.3) is then integrated till all terms contain derivative. For the sake of simplicity, the integration constant has been set to zero.

The solution of Sine-cosine method is in the forms

$$u(t, x) = \lambda \cos^\beta(\xi\mu), \quad |\xi| \leq \frac{\pi}{2\mu}, \quad (4.4)$$

and

$$u(t, x) = \lambda \sin^\beta(\xi\mu), \quad |\xi| \leq \frac{\pi}{\mu}, \quad (4.5)$$

where  $\lambda$  and  $\beta$  are parameters and  $\mu$  is the wave number and  $c$  is the wave speed. From Eqs (4.4) and (4.5), we have

$$(u^n)'(\xi\mu) = -n\beta\lambda^n\mu \cos^{n\beta-1}(\xi\mu) \sin(\xi\mu), \quad (4.6)$$

$$(u^n)''(\xi\mu) = -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\xi\mu) + n\mu^2\lambda^n\beta(n\beta - 1) \cos^{n\beta-2}(\xi\mu),$$

and

$$(u^n)'(\xi\mu) = n\beta\lambda^n\mu \sin^{n\beta-1}(\xi\mu) \cos(\xi\mu), \quad (4.7)$$

$$(u^n)''(\xi\mu) = -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\xi\mu) + n\mu^2\lambda^n\beta(n\beta - 1) \sin^{n\beta-2}(\xi\mu).$$

By substitute Eqs (4.6) and (4.7) into (4.3), we get equation in the form of  $\sin^R(\xi\mu)$  or  $\cos^R(\xi\mu)$ . Then to compute the parameter, we compare the exponent of each pair and then coefficient of equal power of  $\cos^k(\xi\mu)$  or  $\sin^k(\xi\mu)$ . Then we get the system of equation in  $\mu, \beta$  and  $\lambda$  that will be determined.

The Sine-cosine method reduce the size of computational work than any other method which we have mention before.

#### 4.1. Implementation of Sine-cosine method on Generalized Equal Width wave equation

The GEW equation is

$$u_t + u^n u_x - bu_{xxx} = 0, \quad (4.8)$$

here  $b$  is a parameter.

By using  $u(x, t) = u(\xi)$ ,  $\xi = x - ct$ . Equation (4.8) is transferred to non-linear ODE

$$-cu' + u^n u' + cbu''' = 0. \quad (4.9)$$

Integrating (4.9) one time and use the constant of integration to be zero for the sake of simplicity, we find that

$$-cu + \frac{u^{n+1}}{n+1} + cbu'' = 0. \quad (4.10)$$

By using  $u(x, t) = \lambda \cos^\beta(\xi\mu)$  into (4.10), we get

$$-c\lambda \cos^\beta(\xi\mu) + \frac{1}{n+1} \lambda^{n+1} \cos^{(n+1)\beta}(\xi\mu) + cb\mu^2 \lambda \beta(\beta-1) \cos^{(\beta-2)}(\xi\mu) - cb\beta^2 \mu^2 \lambda \cos^\beta(\xi\mu) = 0. \quad (4.11)$$

By comparing the exponent of each pair and coefficient of equal power of  $\cos^k(\xi\mu)$ , we get system of algebraic equation in  $\beta$ ,  $\mu$  and  $\xi$ , which is

$$\begin{aligned} \beta - 1 &\neq 0, \\ \beta(n+1) &= \beta - 2, \\ -c\lambda - cb\beta^2 \mu^2 \lambda &= 0, \\ \frac{1}{n+1} \lambda^{n+1} &= -cb\mu^2 \lambda \beta(\beta-1). \end{aligned} \quad (4.12)$$

By figuring out the system, we get

$$\beta = \frac{-2}{n}, \quad \mu = \sqrt{\frac{1}{-b} \left(\frac{n}{2}\right)}, \quad \lambda = \left[\frac{c}{2}(n+1)(n+2)\right]^{\frac{1}{n}}, \quad (4.13)$$

by putting the value of  $\beta$ ,  $\mu$  and  $\lambda$  in (4.4), we get the following periodic solution for  $b < 0$  as shown in Figure 1.

$$u_1(x, t) = \left[\frac{c}{2}(n+1)(n+2) \sec^2\left(\sqrt{\frac{1}{-b} \left(\frac{n}{2}\right)}(x-ct)\right)\right]^{\frac{1}{n}}, \quad \left|\sqrt{\frac{1}{-b} \left(\frac{n}{2}\right)}(x-ct)\right| < \frac{\pi}{2}. \quad (4.14)$$

When we take  $b > 0$ , we get the following soliton solution as shown in Figure 2.

$$u_2(x, t) = \left[\frac{c}{2}(n+2)(n+1) \operatorname{sech}^2\left(\sqrt{\frac{1}{b} \left(\frac{n}{2}\right)}(x-ct)\right)\right]^{\frac{1}{n}}, \quad 0 < \sqrt{\frac{1}{b} \left(\frac{n}{2}\right)}(x-ct) < \pi. \quad (4.15)$$

If we put ansatz  $u(x, t) = \lambda \sin^\beta(\mu\xi)$ , then for  $b < 0$ , we get periodic solution as shown in Figure 3.

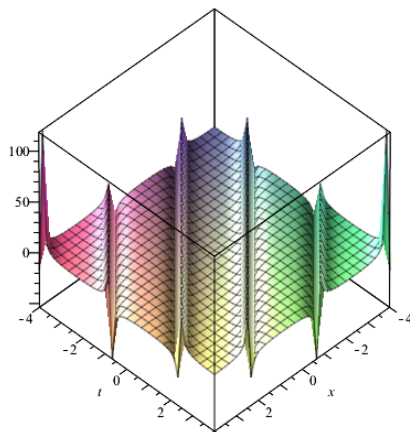
$$u_3(x, t) = \left[\frac{c}{2}(n+2)(n+1) \csc^2\left(\sqrt{\frac{1}{-b} \left(\frac{n}{2}\right)}(x-ct)\right)\right]^{\frac{1}{n}}. \quad (4.16)$$

And for  $b > 0$ , we have soliton solution as shown in Figure 4.

$$u_4(x, t) = \left[\frac{c}{2}(n+2)(n+1) \operatorname{csch}^2\left(\sqrt{\frac{1}{b} \left(\frac{n}{2}\right)}(x-ct)\right)\right]^{\frac{1}{n}}. \quad (4.17)$$

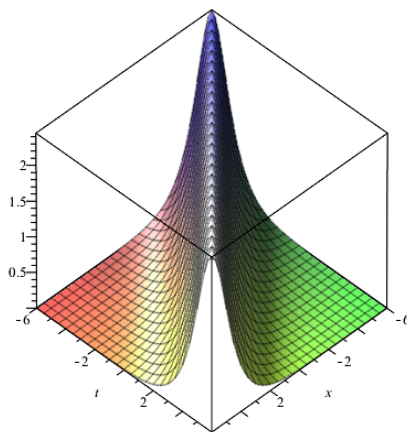
## 5. Graphical representation of some solutions

In this section we present the graphs of solution of the GEW equation. In Figure 1, we obtain periodic solution of GEW which is given in (4.14). It has wave speed value  $c = 1$ .



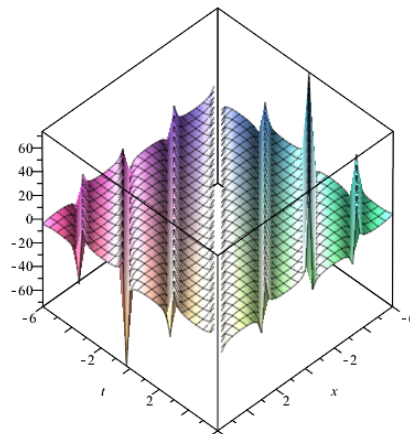
**Figure 1.** Periodic solution,  $u_1(x, t)$  of GEW with wave speed  $c = 1$ ,  $b = -1$ ,  $n = 2$  and  $x = -4..4$ ,  $t = -4..4$ .

In the Figure 2, we give a soliton solution of GEW equation for given values of parameters  $c = 1$ ,  $b = 1$ ,  $n = 2$  and  $x = -6..6$ ,  $t = -6..6$ .



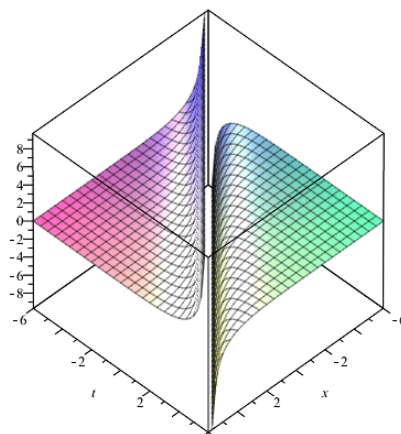
**Figure 2.** Soliton solution  $u_2(x, t)$  of GEW with wave speed  $c = 1$ ,  $b = 1$ ,  $n = 2$  and  $x = -6..6$ ,  $t = -6..6$ .

In the Figure 3, we give a periodic solution of GEW for given values of parameters  $c = 1$ ,  $b = -1$ ,  $n = 2$  and  $x = -6..6$ ,  $t = -6..6$ .



**Figure 3.** Periodic solution  $u_3(x, t)$  of GEW with wave speed  $c = 1$ ,  $b = -1$ ,  $n = 2$  and  $x = -6..6$ ,  $t = -6..6$ .

In the Figure 4, we give a soliton solution of GEW equation for given values of parameters  $c = 1$ ,  $b = 1$ ,  $n = 2$  and  $x = -6..6$ ,  $t = -6..6$ .



**Figure 4.** Soliton solution  $u_4(x, t)$  of GEW with wave speed  $c = 1$ ,  $b = 1$ ,  $n = 2$  and  $x = -6..6$ ,  $t = -6..6$ .

## 6. Conclusions

In the present article, we discussed the solution of Generalized Equal Width wave equation. We analysed the solution of GEW using two methods. At the first step transformed this PDE into ODE using Lie symmetry analysis. We used Lie symmetry analysis to reduce the complexity of the equations actually. It is worth-mentioning that GEW equation has not been discussed from the point of view of Lie symmetries. In the second step, we also used method of Sine-cosine to evaluate the exact solutions of this equation. We also analyzed the graphs of the solutions and found how they behave depending upon the parameters involved.

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## Conflict of interest

The authors declare no conflict of interest.

## References

1. R. Gilmore, *Lie Groups, Physics and Geometry: An Introduction for Physicists, Engineers and Chemists*, Cambridge U. Press, New York, 2008.
2. T. Xiang, A summary of the Korteweg-de Vries equation, 2015.
3. H. N. Hassan, H. K. Saleh, The solution of the regularized long wave equation using the fourier Leap-frog method, *Z. Naturforsch. A*, **65** (2010), 268–276.
4. P. J. Morrison, J. D. Meiss, J. R. Cary, Scattering of regularized-long-wave solitary waves, *Physica D*, **11** (1984), 324–336.
5. S. Dhawan, Turgut Ak, G. Apaydin, Algorithms for numerical solution of the Equal Width wave equation using multi-quadric quasi-interpolation method, *Int. J. Mod. Phys. C*, **30** (2019), 17.
6. L. R. T. Gardner, G. A. Gardner, F. A. Ayoub, N. K. Amein, Simulations of the EW undular bore, *Commun. Numer. Meth. En.*, **13** (1997), 583–592.
7. M. G. Rani, S. Padmasekaran, T. Shanmugapriya, Symmetry reductions of (2+1)-dimensional Equal Width wave equation, *Int. J. Appl. Comput. Math.*, **12** (2017).
8. A. Esen, A numerical solution of the Equal Width wave equation by a Lumped Galerkin method, *Appl. Math. Comput.*, **168** (2005), 270–282.
9. B. Saka, A finite element method for Equal Width wave equation, *Appl. Math. Comput.*, **175** (2006), 730–747.
10. I. Dag, B. Saka, A cubic B-spline collocation method for the EW equation, *Math. Comput. Appl.*, **9** (2004), 381–392.
11. S. K. Bhowmik, S. B. G. Karakoc, Numerical solutions of the Generalized Equal Width wave equation using Petrov-Galerkin method, *Appl. Anal.*, **100** (2019), 714–734.
12. R. Arora, M. J. Siddiqui, V. P. Singh, Solution of the Modified Equal Width wave equation, its variant and non-homogeneous Burger's equation by RDT Method, *Am. J. Comput. Appl. Math.*, **1** (2011), 53–56.
13. S. T. Mohyud-Din, A. Yildirim, M. E. Berberler, M. M. Hosseini, Numerical solution of Modified Equal Width wave equation, *World Appl. Sci. J.*, **8** (2010), 792–798.
14. T. Geyikli, S. B. G. Karakoc, Septic B-spline collocation method for the numerical solution of the Modified Equal Width wave equation, *Appl. Math.*, **2** (2011), 739–749.

15. T. Geyikli, S. B. G. Karakoc, Subdomain finite element method with quartic B-splines for the Modified Equal Width wave equation, *Comput. Math. Math. Phys.*, **55** (2015), 410–421.
16. M. Merdan, A. Yildirim, A. Gokdogan, Numerical solution of time-fraction Modified Equal Width wave equation, *Eng. Comput.*, **29** (2012), 766–777.
17. A. Esen, A Lumped Galerkin method for the numerical solution of the Modified Equal Width wave equation using quadratic B-splines, *Int. J. Comput. Math.*, **83** (2006), 449–459.
18. B. Saka, Algorithms for numerical solution of the Modified Equal Width wave equation using collocation method, *Math. comput. model.*, **45** (2007), 1096–1117.
19. M. Yaseer, A. Essa, Multigrid method for solving the Generalized Equal Width wave equation, *Int. J. Math. Arch.*, **8** (2017).
20. K. R. Raslan, Collocation method using cubic B-spline for the Generalized Equal Width wave equation, *Int. J. Simul. Process Model.*, **2** (2006), 37–44.
21. N. Taghizadeh, M. Mirzazadeh, M. Akbari, M. Rahimian, Exact soliton solutions for Generalized Equal Width wave equation, *Math. Sci. Lett.*, **2** (2013), 99–106.
22. H. Zeybek, B. G. Karakoc, Application of the collocation method with B-spline to the GEW equation, *Electron. Trans. Numer. Anal.*, **46** (2017), 71–88.
23. T. Roshan, A Petrov-Galerkin method for solving the Generalized Equal Width wave (GEW) equation, *J. Comput. Appl. Math.*, **235** (2011), 1641–1652.
24. L. R. T. Gardner, G. A. Gardner, T. Geyikli, The boundary forced MKdV equation, *J. Comput. Phys.*, **113** (1994), 5–12.
25. D. Kaya, A numerical simulation of solitary-wave solutions of the generalized regularized long wave equation, *Appl. Math. Comput.*, **149** (2004), 833–841.
26. D. Kaya, S. M. El-Sayed, An application of the decomposition method for the Generalized KdV and RLW equations, *Chaos Soliton. Fract.*, **17** (2003), 869–877.
27. S. Kumar, Finite difference method: A brief study, *SSRN Elect. J.*, **6** (2014).
28. S. C. Shiralashetti, M. H. Kantli, A. B. Deshi, A new wavelet multigrid method for the numerical solution of elliptic type differential equations, *Alex. Eng. J.*, **57** (2018), 203–209.
29. M. N. O. Sadiku, C. N. Obiozor, A simple introduction to the method of lines, *Int. J. Elec. Eng. Edu.*, **37** (2000), 282–296.
30. V. Dolean, P. Jolivet, F. Nataf, *An Introduction to Domain Decomposition Methods: Algorithms, Theory and Parallel Implementation*, Siam Press Manag. Co. Ltd., 2016.
31. J. Droniou, R. Eymard, T. Gallouet, R. Herbin, The gradient discretisation method for linear advection problems, *Comput. Methods Appl. Math.*, **20** (2020), 437–458.
32. Z. Jiang, Lingde Su, T. Jiang, A Meshfree method for numerical solution of non-homogeneous time-dependent problems, *Abstr. Appl. Anal.*, **11** (2014).
33. N. Rai, S. Mondal, Spectral methods to solve non-linear problems: A review, *Part. Diff. Equ. Appl. Math.*, **4** (2021).
34. F. Cheng, X. Wang, B. A. Barsky, Quadratic B-spline curve interpolation, *Comput. Math.*, **41** (2001), 39–50.



35. Q. Zhao, L. Wu, Darboux transformation and explicit solutions to the generalized TD equation, *Appl. Math. Lett.*, **67** (2017), 1–6.
36. R. Li, X. Geng, B. Xue, Darboux transformations for a matrix long-wave-short-wave equation and higher-order rational rogue-wave solutions, *Appl. Math. Lett.*, **43** (2020), 948–967.
37. M. A. Ablowitz, P. A. Clarkson, *Solitons, Non-linear Evolution Equations and Inverse Scattering*, Cambridge Uni. Press, 1991.
38. W. X. Ma, Inverse scattering for non-local reverse-time non-linear schrödinger equations, *Appl. Math. Lett.*, **102** (2020).
39. F. Mahmud, Md Samsuzzoha, M. A. Akbar, The generalized Kudryashov method to obtain exact traveling wave solutions of the Phi-four equation and the fisher equation, *Results Phys.*, **7** (2017), 4296–4302.
40. M. S. Islam, K. Khan, A. H. Arnous, Generalized Kudryashov method for solving some (3+1)-dimensional non-linear evolution equations, *New Trend Math. Sci.*, **3** (2015), 46–57.
41. T. Motsepa, C. M. Khalique, Conservation laws and solutions of a generalized coupled (2+1)-dimensional Burger’s system, *Comput. Math.*, **74** (2017), 1333–1339.
42. M. L. Wang, Y. B. Zhou, Z. B. Li, Application of a homogeneous balance method to exact solutions of non-linear equations in mathematical physics, *Phys. Lett. A*, **216** (1996), 67–75.
43. A. M. Wazwaz, The tanh-coth method for solitons and kink solutions for non-linear parabolic equations, *Appl. Math. Comput.*, **188** (2007), 1467–1475.
44. R. Hirota, *The Direct Method in Soliton Theory*, Cambridge Uni. Press, 2004.
45. Z. Y. Zhang, Jacobi elliptic function expansion method for the Modified Korteweg-de Vries-Zakharov-Kuznetsov and the Hirota equations, *Phys. Lett. A*, **60** (2001), 1384–1394.
46. I. Simbanefayi, C. M. Khalique, Travelling wave solutions and conservation laws for the Korteweg-de Vries-Bejamin-Bona-Mahony equation, *Results Phys.*, **8** (2018), 57–63.
47. A. M. Wazwaz, Exact solutions for the ZK-MEW equation by using the tanh and sine–cosine methods, *Int. J. Comput. Math.*, **82** (2005), 699–708.
48. S. Kumar, W. X. Ma, A. Kumar, Lie symmetries, optimal system and group-invariant solutions of the (3+1)-dimensional generalized KP equation, *Chin. J. Phys.*, **69** (2021), 1–23.
49. S. Kumar, D. Kumar, A. M. Wazwaz, Lie symmetries, optimal system, group-invariant solutions and dynamical behaviors of solitary wave solutions for a (3+1)-dimensional KdV-type equation, *Eur. Phys. J. Plus*, **136** (2021).
50. S. Kumar, L. Kaur, M. Niwas, Some exact invariant solutions and dynamical structures of multiple solitons for the (2+1)-dimensional Bogoyavlensky-Konopelchenko equation with variable coefficients using Lie symmetry analysis, *Chin. J. Phys.*, **71** (2021), 518–538.
51. S. Kumar, M. Niwas, I. Hamid, Lie symmetry analysis for obtaining exact soliton solutions of generalized Camassa-Holm-Kadomtsev-Petviashvili equation, *Int. J. Mod. Phys. B*, **35** (2021).
52. H. Liu, J. Li, Lie symmetry analysis and exact solutions for the short pulse equation, non-linear analysis: Theory, methods and applications, *Chin. J. of Phys.*, **71** (2009), 2126–2133.

53. A. Chauhan, R. Arora, A. Tomar, Lie symmetry analysis and traveling wave solutions of Equal Width wave equation, *Proyecciones*, **39** (2020), 179–198.
54. S. I. Zaki, A least-squares finite element scheme for the EW equation, *Comput. Methods Appl. Mech. Eng.*, **189** (2000), 587–594.
55. E. Yusufoglu, A. Bekir, Numerical simulation of Equal Width wave equation, *Comput. Math.*, **54** (2007), 1147–1153.
56. L. R. T Gardner, G. A Gardner, Solitary waves of the Equal Width wave equation, *J. Comput. Phys.*, **101** (1992), 218–223.
57. S. B. G. Karakoc, T. Geyikli, Numerical solution of the Modified Equal Width wave equation, *Int. J. Diff. Equ.*, **15** (1992).
58. C. M. Khalique, K. Plaattjie, I. Simbanefayi, Exact solutions of Equal Width equation and its conservation laws, *Open Phys.*, **17** (2019), 505–511.
59. D. J. Evans, K. R. Raslan, Solitary waves for the Generalized Equal Width (GEW) equation, *Int. J. Comput. Math.*, **82** (2005), 445–455.
60. S. Hamdi, W. H. Enright, W. E. Schiesser, J. J. Gottlieb, Exact solutions of the Generalized Equal Width wave equation, *Int. Conf. Comput. Sci. Appl.*, **2668** (2003), 725–734.
61. S. B. G. Karakoc, H. Zeybek, A cubic B-spline Galerkin approach for the numerical simulation of the GEW equation, *Stat. Optim. Inf. Comput.*, **4** (2016), 30–41.
62. S. B. G. Karakoc, H. Zeybek, A septic B-spline collocation method for solving the Generalized Equal Width wave equation, *Kuwait J. Sci.*, **43** (2016), 20–31.
63. S. B. G. Karakoc, K. Omrani, D. Sucu, Numerical investigations of shallow water waves via Generalized Equal Width (GEW) equation, *Appl. Numer. Math.*, **162** (2021), 249–264.
64. S. B. G. Karakoc, A Numerical analysing of the GEW equation using finite element method, *J. Sci. Arts*, **2** (2019), 339–348.



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