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Research article

Initial boundary value problems for a multi-term time fractional diffusion equation with generalized fractional derivatives in time

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Abstract: For a multi-term time-fractional diffusion equation comprising Hilfer fractional derivatives in time variables of different orders between 0 and 1, we have studied two problems (direct problem and inverse source problem). The spectral problem under consideration is self-adjoint. The solution to the given direct and inverse source problems is formulated utilizing the spectral problem. For the solution of the given direct problem, we proposed existence, uniqueness, and stability results. The existence, uniqueness, and consistency effects for the solution of the given inverse problem were addressed, as well as an inverse source for recovering space-dependent source term at certain T. For the solution of the challenges, we proposed certain relevant cases.

Keywords: direct problem; inverse problem; fractional derivative; multinomial Mittag-Leffler function
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1. Introduction

Recently, fractional calculus has gained consideration in pure and applied analysis due to its advancement in bifurcation, chaos, random walks, image encryption, chaotic maps, visco-elastic materials, electrodynamics, hydrodynamics, physics, biology, and control theory, see [1–16]. As a result, mathematicians are becoming intensely interested in searching for the applications of fractional derivative/integral operators in partial differential equations [17–33] and integral equations/inequalities [34–44]. Accordingly, several definitions of fractional derivatives and their

integral transforms are widely discussed in [45–49].
In this paper, we will deal with the following multi-term time-fractional diffusion equation (TFDE)

$$ {}^{H}D_{0_{+},t}^{\nu_1,\omega_1} u(t,x) + \sum_{j=2}^{n} m_j {}^{H}D_{0_{+},t}^{\nu_j,\omega_j} u(t,x) = u_{xx}(t,x) + H(t,x), \quad (t,x) \in (0,T) \times (0,\pi), \tag{1.1} $$

subject to the boundary conditions

$$ u(t,0) = 0 = u(t,\pi), \qquad t \in (0,T) \tag{1.2} $$

and initial conditions

$$ \lim_{t\to 0} J_{0_{+},t}^{(1-\nu_j)(1-\omega_j)} u(t,x) = {}_{j}\rho(x), \qquad j = 1,2,...,n,\ n \in \mathbb{N}, \qquad x \in (0,\pi),, \tag{1.3} $$

where ${}^{H}D_{0_{+},t}^{\nu_j,\omega_j}$ stands for the Hilfer fractional derivatives (HFDs) in time variable of orders ν_j, $0 < \nu_n < ... < \nu_2 < \nu_1 < 1$ and type ω_j, $0 \le \omega_n \le ... \le \omega_2 \le \omega_1 \le 0$, $m > 0$, $\Omega := (0,T) \times (0,\pi)$ and $J_{0_{+},t}^{(1-\nu_j)(1-\omega_j)}$ denotes the Riemann-Liouville fractional (RLF) integrals in time variable of orders $(1-\nu_j)(1-\omega_j)$, $j = 1,2,...,n$, $n \in \mathbb{N}$ respectively.

We are going to discuss both problems, that are a direct problem where $H(t,x) = f(t,x)$ is a known function of time and space, and an inverse source term where $H(t,x) = f(x)$ is an unknown function of space only.

Let us provide a brief overview of the importance of considering direct problems (DPs) and inverse problems (IPs) for fractional diffusion equations (FDEs). Al-Musalhi et al. [51] studied DP and ISP of a fractional diffusion equation with regularized Caputo-like counterpart of a hyper-Bessel differential operator. Kirane et al. [52] examined TFDE's DP and IP in two dimensions and explained possible results of existence and uniqueness. Liu et al [53] calculated direct and inverse Cauchy problems by applying multiple-scale radial basis function method. Hu et al. [54] considered DP and IP for electromagnetic scattering by a doubly periodic structure with a partially coated dielectric. The authors [55] of the paper investigated at the DP and IPs of solving well-posed solvability for an abstract differential equation with Hadamard fractional derivatives. Turner et al. [56] studied DP and IP to estimating those parameters through computational techniques for a multi-term TFDE. For a space-time FDE, Ali et al. [57] reported two IPs in which the capturing of space-dependent and time-dependent source term. Tarar et al. [58] studied the IP of the determination of an uncertain source term for a space-time fractional differential equation (FDE).

An IP of identifying time-dependent source term for a parabolic TFDE has been proposed by Slodicka [59] whenever over-specified data is provided at the boundary of the spatial domain. Malik et al. [60] considered two IPs of recovering space-dependent and time-dependent source-term temporal variable for a fourth-order time (FDE). Karimov at al. [61] is discussed IPs in which identification of time-dependent source-term for higher order multi-term time fractional partial differential equation (PDE) involving Caputo Fabrizo derivative. Karimov [62] calculated an IP of investigating a source term for a semi-linear time-fractional telegraph equation. The total energy source has a temporal dimension for an IP of recovering of the space-time FDE established by Ali et al. [63]. Rundell et al. [64] considered an IP of determination of non-linear boundary conditions for a FDE. In [65], Sun et al. presented an IP for collectively extracting diffusion intensity and source- term for a multi-term TFDE.

In the next section we presented preliminaries, multinomial Mittag-Leffler function (in short $MMLF$) and spectral problem. We discussed the formal solution, existence, uniqueness and stability result for the direct problem in the next section. In the next section we presented the formal solution, existence, uniqueness and stability result for the inverse source problem. In the next section, we presented some particular cases. We also concluded our paper in the last section.

2. Preliminaries and spectral problem

For the given framework (1.1)-(1.2), we will present some basic definitions from FC, properties, and Lemma's related to $MMLF$ and spectral problem.

Definition 2.1. [66] Let $h \in L^1_{loc}([a,b]), -\infty < a < z < b < \infty$ be a locally integrable real-valued function. The left and right sided RLF integral of order $\xi > 0$ are defined as

$$J^{\xi}_{a_+,z} h(z) := \frac{1}{\Gamma(\xi)} \int_a^z (z-\tau)^{\xi-1} h(\tau)\, d\tau,$$

and

$$J^{\xi}_{b_-,z} h(z) := \frac{1}{\Gamma(\xi)} \int_z^b (\tau-z)^{\xi-1} h(\tau)\, d\tau,$$

respectively.

Definition 2.2. [67] Let $h \in L^1([a,b]), -\infty < a < z < b < \infty$ and $h * g_{(1-\xi)(1-\eta)} \in AC([a,b])$. The HFD of order ξ, $0 < \xi < 1$ and type η, $0 \leq \eta \leq 1$ is defined as

$${}^{H}D^{\xi,\,\eta}_{a_+,z} h(z) := \left(J^{\eta(1-\xi)}_{a_+,z} \frac{d}{dz} J^{(1-\xi)(1-\eta)}_{a_+,z} h \right)(z),$$

where $g_\xi = z^{\xi-1}/\Gamma(\xi)$ and " $*$ " indicates integral convolution stated as

$$(h * g)(z) = \int_0^z h(\tau) g(z-\tau) d\tau.$$

The HFD interpolates both the RLF and the Caputo fractional (CF) derivatives.
• For $\eta = 0$, the HFD becomes the RLFD, i.e.,

$$D^{\xi,0}_{a_+,z} h(z) = \frac{d}{dz} J^{(1-\xi)}_{a_+,z} h(z) := {}^{RL}D^{\xi}_{a_+,z} h(z).$$

In this case the initial conditions (1.3) reduce to the following condition

$$\lim_{t\to 0} J^{(1-\gamma_1)}_{0_+,t} u(t,x) = {}_1\rho(x), \quad x \in (0,\pi).$$

• For $\eta = 1$, the HFD becomes the Caputo fractional derivative, i.e.,

$$D^{\xi,1}_{a_+,z} h(z) = J^{1-\xi}_{a_+,z} \frac{d}{dz} h(z) := {}^{C}D^{\xi}_{a_+,z} h(z).$$

In this case the initial conditions (1.3) reduce to one condition, i.e., $u(x,0) = {}_1\rho(x)$, $x \in (0,\pi)$.

Lemma 2.3. *[66] Let h_i be a sequence of functions defined on $(a, b]$ for each $i \in \mathbb{N}$, such that*
(1) ${}^{H}D^{\xi,\eta}_{a+,z}h_i(z)$ exists $\forall i \in \mathbb{N}$, $z \in (a, b]$,
(2) both series $\sum_{i=1}^{\infty} h_i(z)$ and $\sum_{i=1}^{\infty} {}^{H}D^{\xi,\eta}_{a+,z}h_i(z)$ are uniformly convergent on the interval $[a + \epsilon, b]$ for any $\epsilon > 0$.
Then,

$$ {}^{H}D^{\xi,\eta}_{a+,z} \sum_{i=1}^{\infty} h_i(z) = \sum_{i=1}^{\infty} {}^{H}D^{\xi,\eta}_{a+,z}h_i(z), \quad 0 < \xi \leq \eta < 1,\ a < z < b. $$

2.1. Mittag-Leffler type functions

In this subsection, we will define a *MMLF* and some of its important estimates.

Definition 2.4. [68] For $\eta > 0,\ \xi_i > 0,\ z_i \in \mathbb{C}, i = 1, 2, ..., n,\ n \in \mathbb{N}$, the *MMLF* is stated as

$$ E_{(\xi_1,\xi_2,...,\xi_n),\eta}(z_1, z_2, ..., z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+l_2+...+l_n=k \\ l_1 \geq 0,...,l_m \geq 0}} (k; l_1, ..., l_n) \frac{\Pi_{i=1}^{n} z_i^{l_i}}{\Gamma\left(\eta + \sum_{i=1}^{n} \xi_i l_i\right)}, $$

where $(k; l_1, ..., l_n) = \dfrac{k!}{l_1! \times ... \times l_n!}$.

Remark 1. For $n = 2$ *MMLF* reduces to

$$ \begin{aligned} E_{(\xi_1,\xi_2),\eta}(z_1; z_2) &= \sum_{k=0}^{\infty} \sum_{\substack{l_1+l_2=k \\ l_1 \geq 0, l_2 \geq 0}} \frac{k!}{l_1! l_2!} \frac{z_1^{l_1} z_2^{l_2}}{\Gamma(\eta + \xi_1 l_1 + \xi_2 l_2)}, \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \frac{z_1^{i} z_2^{k-i}}{\Gamma(\eta + \xi_1 l_1 + \xi_2 l_2)}. \end{aligned} $$

Moreover, note that

$$ E_{(\xi_1,\xi_2,...,\xi_n),\eta}(z_1; z_2; ...; z_n) = E_{(\xi_n,...,\xi_2,\xi_1),\eta}(z_n; ...; z_2; z_1). $$

Remark 2. For $z_1 \neq 0$ and $z_2 = 0$, the *MMLF* takes the following form

$$ E_{(\xi_1,\xi_2),\eta}(z_1; 0) = \sum_{k=0}^{\infty} \frac{z_1^k}{\Gamma(\eta + \xi_1 k)} := E_{\xi_1,\eta}(z_1). $$

Let us fix following notation

$$ \mathcal{E}_{(\xi_1,\xi_2,...,\xi_n),\eta}(\tau; q_1, q_2, ...q_n) := \tau^{\eta-1} E_{(\xi_1,\xi_2),\eta}(-q_1\tau^{\xi_1}, -q_2\tau^{\xi_2}, ..., -q_n\tau^{\xi_n}), $$

where $q_i > 0,\ i = 1, 2$.

Lemma 2.5. *[69] For $\boldsymbol{\xi}, \eta, \tau, \sigma_i > 0,\ i = 1, 2, ..., n,\ n \in \mathbb{N}$ the Laplace transform of the multinomial Mittag-Leffler function is given by*

$$ \mathcal{L}\{\mathcal{E}_{(\xi_1,\xi_2,...,\xi_n),\eta}(\tau; q_1, ..., q_n)\} = \frac{s^{-\eta}}{1 + \sum_{i=1}^{n} q_i s^{-\xi_i}}, \qquad \textit{if} \qquad \left| \sum_{i=1}^{n} q_i s^{-\xi_i} \right| < 1. $$

Lemma 2.6. *[70] For $0<\eta<1$ and $0<\xi_2<\xi_1<1$ be given. Assume that $\xi_1\pi/2<\mu<\xi_1\pi$, $\mu\leq|arg(q_2\tau^{\xi_1})|\leq\pi$ and $q_i\tau>0$. Then there exists a constant C_0 depending only on $\mu,\xi_i,\ i=1,2$ such that*

$$|E_{(\xi_1-\xi_n,\ldots,\xi_1-\xi_2,\xi_1),\xi_1+1}(-q_n\tau^{\xi_1-\xi_n},\ldots,-q_2\tau^{\xi_1-\xi_2},-q_1\tau^{\xi_1})|\leq\frac{C_0}{1+|q_1\tau^{\xi_1}|}.$$

Lemma 2.7. *[70] For $h\in C^1([a,b])$ and $\xi_i,q_i>0$, for $i=1,2,\ldots,n,\quad n\in\mathbb{N}$, we have*

$$|h(\tau)*\mathcal{E}_{(\xi_1-\xi_n,\ldots,\xi_1-\xi_2,\xi_1),\xi_1}(\tau;q_n,\ldots,q_2,q_1)|\leq\frac{C_0}{q_1}\parallel h\parallel_{C^1([0,T])},$$

where $\|h\|_{C^1([0,T])}=\sup\limits_{t\in[0,T]}|h(t)|+\sup\limits_{t\in[0,T]}|h'(t)|$.

2.2. Auxiliary result

The spectral analysis pertaining to system (1.1)-(1.2) will be discussed in this subsection. The spectral problem analogous to (1.1)-(1.2) is described as

$$X''(x)=\lambda X(x),\qquad X(0)=0,\qquad X(\pi)=0.$$

The spectral problem is self-adjoint. It has the following eigenvalues and eigenfunctions:

$$\lambda_n=n^2,\qquad\{X_n(x)\}_{n=1}^{\infty}=\{\sin(nx)\}_{n=1}^{\infty}.$$

The set of eigenfunctions $\{X_n(x)\}_{n=1}^{\infty}$ forms an orthogonal basis in $L^2((0,\pi))$, see [71].

Lemma 2.8. *For $h(t,.)\in C^2([0,\pi])$ satisfying $h(t,0)=0=h(t,\pi)$, we have*

$$|h_n(t)|\leq\frac{D_0}{|\lambda_n|^2}\|h''(t,x)\|_{C^{2,0}(\bar{\Omega})},$$

where

$$h_n(t)=\langle h(t,x),X_n(x)\rangle.\tag{2.1}$$

Proof. From the expression of $h_n(t)$ given by (2.1) and integration by parts, we obtain

$$h_n=\frac{1}{|\lambda_n|^2}\Big\langle h''(x,t),X_n(x)\Big\rangle,$$

Using Cauchy Schwarz inequality, we have

$$|h_n|\leq\frac{1}{|\lambda_n|}\|h''(t,x)\|_{C^{2,0}(\bar{\Omega})}\|X_n(x)\|_{C^2([0,\pi])},$$

which implies

$$|h_n|\leq\frac{D_0}{|\lambda_n|}\|h''(t,x)\|_{C^{2,0}(\bar{\Omega})},$$

where $\|X_n(x)\|_{C^2([0,\pi])}\leq D_0$. □

3. Direct problem (DP)

We'll develop the series solution of the DP for the given system (1.1)-(1.3) in this portion. It would therefore also be exhibited that $u(t,x)$, $tu_{xx}(t,x)$, $t\,{}^{H}D^{\nu_j,\omega_j}_{0+,t}u(t,x)$ $j = 1,2,...n$ denotes a continuous function with results for uniqueness and stability.

3.1. Series form solution

Applying Fourier's process, the solution of the DP (1.1)-(1.3) can be expressed

$$u(t,x) = \sum_{n=1}^{\infty} X_n(x)U_n(t), \tag{3.1}$$

where $U_n(t)$ is unknowns and satisfy the following fractional equation

$${}^{H}D^{\nu_1,\omega_1}_{0+,t}U_n(t) + \sum_{j=2}^{n} m_j\,{}^{H}D^{\nu_j,\omega_j}_{0+,t}U_n(t) = \lambda_n U_n(t) + f_n(t),$$

$f_n(t) = \langle f(t,x), X_n(x)\rangle$.

Employing Laplace transform method and the initial conditions (1.3), see detail [69], one gets

$$\mathcal{L}\{U_n(t);s\} = \frac{s^{\omega_1(\nu_1-1)}\,{}_1\rho_n}{s^{\nu_1} + \sum\limits_{j=2}^{n} m_j s^{\nu_j} - \lambda_n} + \frac{\sum\limits_{j=2}^{n} m_j s^{\omega_j(\nu_j-1)}\,{}_j\rho_n}{s^{\nu_1} + \sum\limits_{j=2}^{n} m_j s^{\nu_j} - \lambda_n} + \frac{\mathcal{L}\{f_n(t);s\}}{s^{\nu_1} + \sum\limits_{j=2}^{n} m_j s^{\nu_j} - \lambda_n}$$

where ${}_j\rho_n = \langle {}_j\rho(x), X_n(x)\rangle$, $j = 1,2,...,n$, $n \in \mathbb{N}$.

By Lemma 2.5, we obtain

$$\begin{aligned} U_n(t) =& {}_1\rho_n \mathcal{E}_{\boldsymbol{\nu},\nu_1+\omega_1(1-\nu_1)}(t;m_n,...,m_3,m_2,\lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\boldsymbol{\nu},\nu_1+\omega_j(1-\nu_j)}(t;m_n,...,m_3,m_2,\lambda_n) \\ &+ f_n(t) * \mathcal{E}_{\boldsymbol{\nu},\nu_1}(t;m_n,...,m_3,m_2,\lambda_n), \end{aligned} \tag{3.2}$$

where

$$\boldsymbol{\nu} := (\nu_1-\nu_n,...,\nu_1-\nu_2,\nu_1).$$

As a result, the DP solution, $u(x,t)$, is presented by

$$\begin{aligned} u(t,x) = \sum_{n=1}^{\infty} \Big(& {}_1\rho_n \mathcal{E}_{\boldsymbol{\nu},\nu_1+\omega_1(1-\nu_1)}(t;m_n,...,m_3,m_2,\lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\boldsymbol{\nu},\nu_1+\omega_j(1-\nu_j)}(t;m_n,...,m_3,m_2,\lambda_n) \\ &+ f_n(t) * \mathcal{E}_{\boldsymbol{\nu},\nu_1}(t;m_n,...,m_3,m_2,\lambda_n)\Big)\sin(nx). \end{aligned} \tag{3.3}$$

3.2. Existence of DP's solutions

In this subsection, we will investigate existence of the series representation of the solution of the DP given by (3.3).

Theorem 3.1. *Assume that* ${}_1\rho(x)$ *and* ${}_j\rho(x)$ *to fulfill the subsequent assumptions*

1. ${}_j\rho \in C^2([0,\pi])$, *such that* ${}_j\rho(0) = 0 = {}_j\rho(\pi)$, $\quad j = 1,2,...,n, \quad n \in \mathbb{N}$,
2. $f(t,\cdot) \in C^2([0,\pi])$, *such that* $f(t,0) = 0 = f(t,\pi)$.

Then, there exists a classic solution of the DP.

Proof. To demonstrate that the DP solution is classic, we will illustrate that

$$tu(t,x) \in C(\bar{\Omega}),\ \bar{\Omega} = [0,\pi]\times[0,T],\ tu_{xx}(t,x) \in C(\bar{\Omega}),\ t^{\nu_1+1}\ {}^{H}D^{\nu_j,\omega_j}_{0_+,t}u(t,x) \in C(\bar{\Omega})\ j = 1,2,...,n.$$

Due to Lemma 2.6, Lemma 2.7 and Eq (3.3), we have

$$|u(t,x)| \leq \sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|}\Big(|{}_1\rho_n|t^{\omega_1(1-\nu_1)-1} + \sum_{j=2}^{n} m_j|{}_j\rho_n|t^{\omega_j(1-\nu_j)-1} + \|f\|_{C^1([0,T])}\Big),$$

which implies

$$t|u(t,x)| \leq \sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|}\Big(|{}_1\rho_n|t^{\omega_1(1-\nu_1)} + \sum_{j=2}^{n} m_j|{}_j\rho_n|t^{\omega_j(1-\nu_j)} + t\|f\|_{C^1([0,T])}\Big). \tag{3.4}$$

Since $\lambda_n = n^2$, we conclude that the series involved in (3.4) is convergent uniformly. Hence, by virtue of Weierstrass M-test (WSMT), $t|u(t,x)|$ shows a continuous function.

Further, we will derive that $tu_{xx}(t,x)$ represent a continuous function. For the convergence of $tu_{xx}(t,x)$, we take term by term differentiation of Eq (3.1), we have

$$u_{xx}(t,x) = \sum_{n=1}^{\infty} X_n''(x)U_n(t), \tag{3.5}$$

where

$$X_n''(x) = n^2\sin(nx), \qquad \Rightarrow \qquad |X_n''(x)| \leq |n^2|. \tag{3.6}$$

In order to prove the uniform convergence of $u_{xx}(t,x)$, we need to show that $U_n(t)$ represent a continuous function. Due to Lemma 2.6, Lemma 2.7 and Eq (3.2), we obtain

$$t|U_n(t)| \leq \sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|}\Big(|{}_1\rho_n|t^{\omega_1(1-\nu_1)} + \sum_{j=2}^{n} m_j|{}_j\rho_n|t^{\omega_j(1-\nu_j)} + \|f\|_{C^1([0,T])}t\Big). \tag{3.7}$$

As $\lambda_n = n^2$, we can deduced that $t|U_n(t)|$ is convergent uniformly. Consequently, $t|U_n(t)|$ represents a continuous function due to WSMT. Based on (3.6) and (3.7), we conclude that $t|u_{xx}(t,x)|$, given by Eq (3.5), is convergent uniformly. Hence, WSMT ensures the continuity of $t|u_{xx}(t,x)|$.

Similarly, we will establish the continuity of $t^{\nu_1+1}\ {}^{H}D^{\nu_j,\omega_j}_{0_+,x}u(t,x), \quad j=1,2,...,n, \quad n\in\mathbb{N}$. For this, consider the following expression

$$\sum_{n=1}^{\infty}{}^{H}D^{\nu_j,\omega_j}_{0_+,x}U_n(t)X_n(x)=\sum_{n=1}^{\infty}X_n(x)\Big(\lambda_nU_n(t)+f_n(t)\Big)$$
$$=\sum_{n=1}^{\infty}\lambda_nU_nX_n(x)+f(t,x).$$

The uniform convergence of $\sum_{n=1}^{\infty}\lambda_nU_nX_n(x)$ can be proved by using Eq (3.2) and Lemma 2.6. Furthermore, $f(t,x)$ is given source term. Hence, by Lemma 2.3, we obtain

$${}^{H}D^{\nu_j,\omega_j}_{0_+,x}u(t,x)=\sum_{n=1}^{\infty}{}^{H}D^{\nu_j,\omega_j}_{0_+,x}U_n(t)X_n(x).$$

□

3.3. Uniqueness of DP's solution

In this subsection, we will discuss the uniqueness of the solution of the DP.

Theorem 3.2. *Consider two classic solution sets of the DP are $u(t,x)$ and $\tilde{u}(t,x)$. If $u(t,x_0)=\tilde{u}(t,x_0)$ for certain $x_0\in(0,\pi)$ then*

$$u(t,x)=\tilde{u}(t,x), \quad (t,x)\in\Omega.$$

Proof. Consider the following functions

$$U_n(t)=\int_0^{\pi}u(t,x)X_n(x)dx, \quad \text{and} \quad \tilde{U}_n(t)=\int_0^{\pi}\tilde{u}(t,x)X_n(x)dx. \tag{3.8}$$

In (4.10), we apply multi-term HFDs of the 2nd equation, we obtain

$${}^{H}D^{\nu_1,\omega_1}_{0_+,t}\tilde{U}_n(t)+\sum_{j=2}^{n}m_j{}^{H}D^{\nu_j,\omega_j}_{0_+,t}\tilde{U}_n(t)=\int_0^{\pi}\Big({}^{H}D^{\nu_1,\omega_1}_{0_+,t}\tilde{u}(t,x)+\sum_{j=2}^{n}m_j{}^{H}D^{\nu_j,\omega_j}_{0_+,t}\tilde{u}(t,x)\Big)X_n(x)dx.$$

As, $\tilde{u}(t,x)$ is a classic solution and the above relation is satisfying by interchanging of fractional derivatives and integral. From (1.1), we have the following fractional differential equation

$${}^{H}D^{\nu_1,\omega_1}_{0_+,t}\tilde{U}_n(t)+\sum_{j=2}^{n}m_j{}^{H}D^{\nu_j,\omega_j}_{0_+,t}\tilde{U}_n(t)=\lambda_n\tilde{U}_n(t)+f_n(t).$$

Taking Laplace transform in above equation and using Eq (3.8) and initial conditions (1.3), we have

$$\tilde{U}_n(t)=\left(\int_0^{\pi}{}_1\rho(x)X_n(x)dx\right)\mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(t;m_n,...,m_3,m_2,\lambda_n)$$

$$+\sum_{j=2}^{n}\left(\int_0^{\pi} {}_j\rho(x)X_n(x)dx\right)m_j\mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(t;m_n,...,m_3,m_2,\lambda_n)$$
$$+ f_n(t)*\mathcal{E}_{\nu,\nu_1}(t;m_n,...,m_3,m_2,\lambda_n).$$

In the similar lines, the expression $U_n(t)$ is obtained as

$$U_n(t)=\left(\int_0^{\pi} {}_1\rho(x)X_n(x)dx\right)\mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(t;m_n,...,m_3,m_2,\lambda_n)$$
$$+\sum_{j=2}^{n}\left(\int_0^{\pi} {}_j\rho(x)X_n(x)dx\right)m_j\mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(t;m_n,...,m_3,m_2,\lambda_n)$$
$$+ f_n(t)\mathcal{E}_{\nu,\nu_1}(t;m_n,...,m_3,m_2,\lambda_n).$$

Since, $u(t,x)=\tilde{u}(t,x)$. Hence, we have $U_n(t)=\tilde{U}_n(t)$. □

3.4. Stability of the DP's solution

The stability result for the DP solution will be presented in this subsection.

Theorem 3.3. *Under the assumptions of Theorem 3.1, the solution of the DP is continually based on the given current data, i.e., ${}_1\rho(x)$, ${}_j\rho(x)$.*

Proof. From (3.3), we have

$$|u(t,x)-\tilde{u}(t,x)|\le\sum_{n=1}^{\infty}\Bigg\{\Bigg(({}_1\rho_n-{}_1\tilde{\rho}_n)\mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(t;m_n,...,m_3,m_2,\lambda_n)$$
$$+\sum_{j=2}^{n}m_j({}_j\rho_n-{}_j\tilde{\rho}_n)\mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(t;m_n,...,m_3,m_2,\lambda_n)$$
$$+(f_n(t)-\tilde{f}_n(t))*\mathcal{E}_{\nu,\nu_1}(t;m_n,...,m_3,m_2,\lambda_n)\Bigg)\sin(nx)\Bigg\}.$$

Due to Cauchy Schwarz inequality together with Lemmas 2.6 and 2.7, we obtain

$$t|u(t,x)-\tilde{u}(t,x)|\le\sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|}\Bigg(\|{}_1\rho(x)-{}_1\tilde{\rho}(x)\|_{C^2([0,\pi])}t^{\omega_1(1-\nu_1)}+\sum_{j=2}^{n}m_j\|{}_j\rho(x)-{}_j\tilde{\rho}(x)\|_{C^2([0,\pi])}t^{\omega_2(1-\nu_2)}$$
$$+t\|f-\tilde{f}\|_{C^2([0,T])}\Bigg).$$

□

4. Inverse source problem (ISP)

We will look at the ISP in this section, which is the determination of a pair of functions $u(t,x)$, $f(x)$ for the particular system (1.1)-(1.3). For the complete recovery of the classic solution, we need some additional data usually known as over-specified condition and is given by

$$u(T,x)=\Psi(x),\qquad x\in[0,\pi]. \tag{4.1}$$

Let us define the classical solution of the ISP $\{u(t,x), f(x)\}$ such that $Tf(x) \in C([0,\pi])$, $tu(t,x) \in C(\bar{\Omega})$, $tu_{xx}(t,x) \in C(\bar{\Omega})$ and $t^{\nu_1+1}\ {}^{H}D_{0+,t}^{\nu_j,\omega_j}u(t,x) \in C(\bar{\Omega})$, $j = 1,2,...,n$, $n \in \mathbb{N}$ is called a classical solution. We investigated existence, uniqueness and stability results for the ISP's solution under given current data.

4.1. Series form solution

The Fourier process can be used to write the solution to the ISP (1.1)-(1.3) & (4.1)

$$u(t,x) = \sum_{n=1}^{\infty} X_n(x)U_n(t), \tag{4.2}$$

$$f(x) = \sum_{n=1}^{\infty} X_n(x)f_n, \tag{4.3}$$

where $U_n(t)$ and f_n are unknowns that satisfy the fractional linear equation

$${}^{H}D_{0+,t}^{\nu_1,\omega_1}U_n(t) + \sum_{j=2}^{n} m_j {}^{H}D_{0+,t}^{\nu_2,\omega_2}U_n(t) = \lambda_n U_n(t) + f_n.$$

Employing Laplace transform and the initial conditions (1.3), we obtain

$$\mathcal{L}\{U_n(t);s\} = \frac{s^{\omega_1(\nu_1-1)}{}_1\rho_n}{s^{\nu_1} + \sum\limits_{j=2}^{n} m_j s^{\nu_j} - \lambda_n} + \frac{\sum\limits_{j=2}^{n} m_j s^{\omega_j(\nu_j-1)}{}_j\rho_n}{s^{\nu_1} + \sum\limits_{j=2}^{n} m_j s^{\nu_j} - \lambda_n} + \frac{s^{-1}f_n}{s^{\nu_1} + \sum\limits_{j=2}^{n} m_j s^{\nu_j} - \lambda_n}$$

where ${}_j\rho_n = \langle {}_j\rho(x), X_n(x)\rangle$, $j = 1,2,...,n$, $n \in \mathbb{N}$.
Due to Lemma 2.5, we obtain

$$\begin{aligned} U_n(t) =& {}_1\rho_n \mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(t; m_n,...,m_3,m_2,\lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(t; m_n,...,m_3,m_2,\lambda_n) \\ &+ f_n \mathcal{E}_{\nu,\nu_1+1}(t; m_n,...,m_3,m_2,\lambda_n), \end{aligned} \tag{4.4}$$

Now, we will use the over-specified condition (4.1) to evaluate the space dependent source term, we have

$$\begin{aligned} f_n =& \frac{1}{\mathcal{E}_{\nu,\nu_1+1}(T; m_n,...,m_3,m_2,\lambda_n)}\Big\{\Psi_n - \Big({}_1\rho_n \mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(T; m_n,...,m_3,m_2,\lambda_n) \\ &+ \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(T; m_n,...,m_3,m_2,\lambda_n)\Big)\Big\}, \end{aligned} \tag{4.5}$$

where $\Psi_n = \langle \Psi(x), X_n(x)\rangle$.
As a consequence, the ISP solution, i.e., $\{u(t,x), f(x)\}$ is expressed by

$$u(t,x) = \sum_{n=1}^{\infty} \Big({}_1\rho_n \mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(t; m_n,...,m_3,m_2,\lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(t; m_n,...,m_3,m_2,\lambda_n)$$

$$+ f_n \mathcal{E}_{\nu,\nu_1+1}(t; m_n, ..., m_3, m_2, \lambda_n)\Big) \sin(nx), \tag{4.6}$$

where f_n is given by (4.5), and we have

$$f(x) = \sum_{n=1}^{\infty} \Big[\frac{1}{\mathcal{E}_{\nu,\nu_1+1}(T; m_n, ..., m_3, m_2, \lambda_n)} \Big\{ \Psi_n - \Big({}_1\rho_n \mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(T; m_n, ..., m_3, m_2, \lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(T; m_n, ..., m_3, m_2, \lambda_n) \Big) \Big\} \Big] \sin(nx). \tag{4.7}$$

4.2. Existence of ISP's solution

The existence of the set solution of the ISP given by (4.6)-(4.7) will be investigated in this subsection.

Theorem 4.1. *Let ${}_1\rho(x)$, ${}_j\rho(x)$ & $\Psi(x)$ satisfy conditions below*

1. ${}_j\rho \in C^2([0,\pi])$, *such that* ${}_j\rho(0) = 0 = {}_j\rho(\pi)$, $j = 1, 2, ..., n$, $n \in \mathbb{N}$
2. $\Psi \in C^2([0,\pi])$, *such that* $\Psi(0) = 0 = \Psi(\pi)$.

Then, there exists a classic solution of the ISP.

Proof. We can demonstrate the uniform convergence of the series representations of $f(x)$, $u(t,x)$, $u_{xx}(t,x)$,
${}^{H}D_{0_+,t}^{\nu_j,\omega_j} u(t,x)$ $j = 1, 2, ..., n$, $n \in \mathbb{N}$ to investigate that the ISP solution is classic.
Using Lemma 2.6 and Eq (4.7), we get

$$|f(x)| \leq \sum_{n=1}^{\infty} \Big(\frac{|\lambda_n|}{C_0} |\Psi_n| - |{}_1\rho_n| T^{\omega_1(1-\nu_1)-1} + \sum_{j=2}^{n} m_j |{}_j\rho_n| T^{\omega_j(1-\nu_j)} \Big).$$

Due to Lemma (2.8), we obtain

$$T|f(x)| \leq \sum_{n=1}^{\infty} \frac{D_0}{|\lambda_n|} \Big(\frac{|\lambda_n| T}{C_0} \|\Psi''(x)\|_{C^2([0,\pi])} - \|{}_1\rho''(x)\|_{C^2([0,\pi])} T^{\omega_1(1-\nu_1)} + \sum_{j=2}^{n} m_j \|{}_j\rho''(x)\|_{C^2([0,\pi])} T^{\omega_j(1-\nu_j)} \Big). \tag{4.8}$$

Since, $\lambda_n = n^2$. By Eq (4.8), we can conclude that $T|f(x)|$ convergent uniformly in Ω. Hence, by virtue of WSMT, $T|f(x)|$ shows a continuous function.
Further, we will derive that $u(t,x)$ given by (4.6) shows a continuous function. By using Lemma 2.6 and Eq (4.4), we have

$$t|u(t,x)| \leq \sum_{n=1}^{\infty} \frac{C_0}{|\lambda_n|} \Big(|{}_1\rho_n| t^{\omega_1(1-\nu_1)} + \sum_{j=2}^{n} m_j |{}_j\rho_n| t^{\omega_j(1-\nu_j)} + |f_n| \Big). \tag{4.9}$$

By $\lambda_n = n^2$, the series involved in (4.9) is convergent uniformly. Consequently, by WSMT, $t|u(t,x)|$ shows a continuous function.

Next, we are going to discuss that $u_{xx}(t,x)$ represent a continuous function. Due to (3.6) and (4.4), we deduce that $t|u_{xx}(t,x)|$, given by Eq (3.5), is convergent uniformly. Hence, by virtue of WSMT shows the continuity of $t|u_{xx}(t,x)|$.

Similarly, we can establish the continuity of ${}^{H}D_{0_+,x}^{\nu_j,\omega_j}u(x,t), \quad j=1,2,...,n.$ □

4.3. Uniqueness of ISP's solution

We will report about the solution's uniqueness in this subsection.

Theorem 4.2. *Consider* $\{u(t,x), f(x)\}$ *and* $\{\tilde{u}(t,x), \tilde{f}(x)\}$ *are two classic ISP's solution sets. If* $u(t,x_0)=\tilde{u}(t,x_0)$ *for some* $x_0 \in (0,\pi)$*, then we have*

$$u(t,x)=\tilde{u}(t,x) \qquad t\in[0,T] \quad \Rightarrow \quad f(x)=\tilde{f}(x) \qquad \forall \quad x\in(0,\pi).$$

Proof. Let us define the following functions

$$U_n(t)=\int_0^\pi u(t,x)X_n(x)dx, \quad \text{and} \quad \tilde{U}_n(t)=\int_0^\pi \tilde{u}(t,x)X_n(x)dx. \tag{4.10}$$

In (4.10), we take multi-term HFDs of the second equation, we have

$${}^{H}D_{0_+,t}^{\nu_1,\omega_1}\tilde{U}_n(t)+\sum_{j=2}^{n}m_j\,{}^{H}D_{0_+,t}^{\nu_j,\omega_j}\tilde{U}_n(t)=\int_0^\pi\Big({}^{H}D_{0_+,t}^{\nu_1,\omega_1}\tilde{u}(t,x)+\sum_{j=2}^{n}m_j\,{}^{H}D_{0_+,t}^{\nu_j,\omega_j}\tilde{u}(t,x)\Big)X_n(x)dx.$$

From (1.1), we have the following fractional differential equation

$${}^{H}D_{0_+,t}^{\nu_1,\omega_1}\tilde{U}_n(t)+\sum_{j=2}^{n}m_j\,{}^{H}D_{0_+,t}^{\nu_j,\omega_j}\tilde{U}_n(t)=\lambda_n\tilde{U}_n(t)+\tilde{f}_n. \tag{4.11}$$

Using Laplace transform, Eq (4.10) and initial conditions (1.3), we obtain

$$\begin{aligned}\tilde{U}_n(t)=&\left(\int_0^\pi {}_1\rho(x)X_n(x)dx\right)\mathcal{E}_{\boldsymbol{\nu},\nu_1+\omega_1(1-\nu_1)}(t;m_n,...,m_3,m_2,\lambda_n)\\&+\sum_{j=2}^{n}\left(\int_0^\pi {}_j\rho(x)X_n(x)dx\right)m_j\mathcal{E}_{\boldsymbol{\nu},\nu_1+\omega_j(1-\nu_j)}(t;m_n,...,m_3,m_2,\lambda_n)\\&+\tilde{f}_n\mathcal{E}_{\boldsymbol{\nu},\nu_1+1}(t;m_n,...,m_3,m_2,\lambda_n).\end{aligned}$$

On the similar lines, we can be obtained $U_n(t)$

$$\begin{aligned}U_n(t)=&\left(\int_0^\pi {}_1\rho(x)X_n(x)dx\right)\mathcal{E}_{\boldsymbol{\nu},\nu_1+\omega_1(1-\nu_1)}(t;m_n,...,m_3,m_2,\lambda_n)\\&+\sum_{j=2}^{n}\left(\int_0^\pi {}_j\rho(x)X_n(x)dx\right)m_j\mathcal{E}_{\boldsymbol{\nu},\nu_1+\omega_j(1-\nu_j)}(t;m_n,...,m_3,m_2,\lambda_n)\\&+f_n\mathcal{E}_{\boldsymbol{\nu},\nu_1+1}(t;m_n,...,m_3,m_2,\lambda_n).\end{aligned}$$

Since, $u(t,x) = \tilde{u}(t,x)$, we have

$$U_n(t) = \tilde{U}_n(t),$$

and hence

$$f_n \mathcal{E}_{\nu,\nu_1+1}(t; m_n, ..., m_3, m_2, \lambda_n) = \tilde{f}_n \mathcal{E}_{\nu,\nu_1+1}(t; m_n, ..., m_3, m_2, \lambda_n)$$

which implies

$$\left(f_n - \tilde{f}_n\right) \mathcal{E}_{\nu,\nu_1+1}(t; m_n, ..., m_3, m_2, \lambda_n) = 0.$$

Taking Laplace transform technique, we obtain

$$\frac{\left(f_n - \tilde{f}_n\right)}{s\left(s^{\nu_1} + \sum_{j=2}^{n} m_j s^{\nu_j} - \lambda_n\right)} = 0, \quad Re\ s > 0, \quad \Rightarrow \quad \frac{f_n - \tilde{f}_n}{\Phi - \lambda_n} = 0, \tag{4.12}$$

where $s^{\nu_1} + \sum_{j=2}^{n} m_j s^{\nu_j} = \Phi$. Just one appropriate disc should be taken. D_1, which only involves λ_1, and then integrating (4.12) around the disc using the Cauchy integral formula, we get the following only one unique source term

$$f_1 = \tilde{f}_1.$$

To obtain the n^{th} terms, we take different disks, we have

$$f_n = \tilde{f}_n, \forall \quad n \in \mathbb{N}, \qquad \Rightarrow \qquad f(x) = \tilde{f}(x),$$

respectively. □

4.4. *Stability of ISP's solution*

The stability result for the ISP solution will be addressed in this subsection.

Theorem 4.3. *Under the supposition of Theorem 4.1, the ISP's solution is constantly relies on the given original and final data, i.e.,* ${}_1\rho(x)$, ${}_j\rho(x)$ *and* $\Psi(x)$.

Proof. Due to (4.6), we have the following expression

$$\begin{aligned}
|u(t,x) - \tilde{u}(t,x)| \le \sum_{n=1}^{\infty} \Bigg\{ \Bigg(& ({}_1\rho_n - {}_1\tilde{\rho}_n) \mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(t; m_n, ..., m_3, m_2, \lambda_n) \\
& + \sum_{j-2}^{n} m_j ({}_j\rho_n - {}_j\tilde{\rho}_n) \mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(t; m_n, ..., m_3, m_2, \lambda_n) \\
& + (f_n - \tilde{f}_n) \mathcal{E}_{\nu,\nu_1+1)}(t; m_n, ..., m_3, m_2, \lambda_n) \Bigg) \sin(nx) \Bigg\},
\end{aligned}$$

where f_n is given by Eq (4.5) and $\tilde{f}_n$ is

$$\tilde{f}_n = \frac{1}{\mathcal{E}_{\nu,\nu_1+1}(T; m_n, ..., m_3, m_2, \lambda_n)} \Big\{ \tilde{\Psi}_n - \Big({}_1\tilde{\rho}_n \mathcal{E}_{\nu,\nu_1+\omega_1(1-\nu_1)}(T; m_n, ..., m_3, m_2, \lambda_n)$$

$$+\sum_{j=2}^{n} {}_j\tilde{\rho}_n m_j \mathcal{E}_{\nu,\nu_1+\omega_j(1-\nu_j)}(T; m_n, ..., m_3, m_2, \lambda_n)\Big)\Big\}.$$

Thanks to Cauchy Schwarz inequality together with Lemmas 2.6, we have

$$t|u(t,x)-\tilde{u}(t,x)| \leq \sum_{n=1}^{\infty} \frac{C_0}{|\lambda_n|}\Big(\|{}_1\rho - {}_1\tilde{\rho}\|_{C^2([0,\pi])} t^{\omega_1(1-\nu_1)} + \sum_{j=2}^{m} m_j \|{}_j\rho - {}_j\tilde{\rho}\|_{C^2([0,\pi])} t^{\omega_j(1-\nu_j)}$$
$$+ \|f-\tilde{f}\|_{C^2([0,\pi])}\Big).$$

Similarly, we can be proved the stability of $f(x)$. □

5. Particular cases

In this section, we will look at some specific cases for time FDE, as defined by (1.1).

Case-I: For $\omega_j = 1, j = 1, 2, ..., n,\quad n \in \mathbb{N}$ (1.1) reduces to the following system

$$ {}^C D^{\nu_1}_{0_+,t} u(t,x) + \sum_{j=2}^{n} m_j {}^C D^{\nu_j}_{0_+,t} u(t,x) = u_{xx}(t,x) + F(t,x), \quad (t,x) \in \Omega, \tag{5.1}$$

In Caputo sense we need only one initial condition

$$u(0,x) = {}_1\rho(x)$$

and boundary condition (1.2). The solution of the DP in this case with given source term $F(t,x)$ has the following form

$$u(t,x) = \sum_{n=1}^{\infty}\Big({}_1\rho_n \mathcal{E}_{\nu,1}(t; m_n, ..., m_3, m_2, \lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,1}(t; m_n, ..., m_3, m_2, \lambda_n)$$
$$+ f_n(t) * \mathcal{E}_{\nu,\nu_1}(t; m_n, ..., m_3, m_2, \lambda_n)\Big) \sin(nx).$$

The ISP's solution, i.e., $\{u(t,x), f(x)\}$ for Eq (5.1) and over-specified (4.1) can be determined by substituting $\omega_j = 1,\ j = 1, 2, ..., n$ in (4.6) and (4.7), we have

$$u(t,x) = \sum_{n=1}^{\infty}\Big({}_1\rho_n \mathcal{E}_{\nu,1}(t; m_n, ..., m_3, m_2, \lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,1}(t; m_n, ..., m_3, m_2, \lambda_n)$$
$$+ f_n \mathcal{E}_{\nu,\nu_1+1}(t; m_n, ..., m_3, m_2, \lambda_n)\Big) \sin(nx),$$

$$f(x) = \sum_{n=1}^{\infty}\Big[\frac{1}{\mathcal{E}_{\nu,\nu_1+1}(T; m_n, ..., m_3, m_2, \lambda_n)}\Big\{\Psi_n - \Big({}_1\rho_n \mathcal{E}_{\nu,1}(T; m_n, ..., m_3, m_2, \lambda_n)$$
$$+ \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,1}(T; m_n, ..., m_3, m_2, \lambda_n)\Big)\Big\}\Big] \sin(nx).$$

On similar way, we can show the regularity of the solution DP and ISP.
Case-II: For $\omega_j = 0,\ \ j = 1, 2, ..., n,\ \ \ n \in \mathbb{N}$, the Eq (1.1) becomes

$$ {}^{RL}D^{\nu_1}_{0_+,t}u(t,x) + \sum_{j=2}^{n} m_j {}^{RL}D^{\nu_j}_{0_+,t}u(t,x) = u_{xx}(t,x) + F(t,x), \quad (t,x) \in \Omega, \tag{5.2} $$

In this case we need only one initial condition

$$ \lim_{t\to 0} J^{(1-\nu_j)}_{0_+,t}u(t,x) = {}_j\rho(x), \qquad j = 1, 2, ..., n \quad n \in \mathbb{N}, $$

and boundary condition (1.2). The solution of the DP in this case is given by

$$ \begin{aligned} u(t,x) = \sum_{n=1}^{\infty} \Big(& {}_1\rho_n \mathcal{E}_{\nu,\nu_1}(t; m_n, ..., m_3, m_2, \lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,\nu_1}(t; m_n, ..., m_3, m_2, \lambda_n) \\ & + f_n(t) * \mathcal{E}_{\nu,\nu_1}(t; m_n, ..., m_3, m_2, \lambda_n)\Big) \sin(nx). \end{aligned} $$

The ISP's solution, i.e., $\{u(t,x), f(x)\}$ for Eq (5.2) and over-specified (4.1) can be determined by substituting $\omega_j = 0,\ \ j = 1, 2, ..., n$ in (4.6) and (4.7), we have

$$ \begin{aligned} u(t,x) = \sum_{n=1}^{\infty} \Big(& {}_1\rho_n \mathcal{E}_{\nu,\nu_1}(t; m_n, ..., m_3, m_2, \lambda_n) + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,\nu_1}(t; m_n, ..., m_3, m_2, \lambda_n) \\ & + f_n \mathcal{E}_{\nu,\nu_1+1}(t; m_n, ..., m_3, m_2, \lambda_n)\Big) \sin(nx), \end{aligned} $$

$$ \begin{aligned} f(x) = \sum_{n=1}^{\infty} \Big[& \frac{1}{\mathcal{E}_{\nu,\nu_1+1}(T; m_n, ..., m_3, m_2, \lambda_n)} \Big\{ \Psi_n - \Big({}_1\rho_n \mathcal{E}_{\nu,\nu_1}(T; m_n, ..., m_3, m_2, \lambda_n) \\ & + \sum_{j=2}^{n} {}_j\rho_n m_j \mathcal{E}_{\nu,\nu_1}(T; m_n, ..., m_3, m_2, \lambda_n)\Big)\Big\}\Big] \sin(nx). \end{aligned} $$

On the similar lines, we can present regularity solution for the DP and ISP.

6. Conclusions

Time-fractional diffusion equation involves multi-term time fractional derivative so-called HFD of different orders ν_j, $0 < \nu_n < ... < \nu_2 < \nu_1 < 1$ and type $0 \le \omega_n \le ... \le \omega_2 \le \omega_1 \le 1$ is considered. Under certain conditions on the given data, the formal solution of the DP obtained by the Fourier method is used to prove the classic solution of the DP (see Theorem 3.1). The DP solution has been shown to be unique and stable. The second ISP is the Identification of a space-dependent source-term from an over-specified condition at some T. The formal solution of the ISP is constructed with the help of the eigenfunction expansion method. Under some assumptions about the data, the ISP's series solution is shown to be a classic solution (see Theorem 4.1). It has also been shown to be unique and stable.

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Conflict of interest

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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