Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 12104–12113.
DOI:10.3934/math.2021702
Received: 31 March 2021
Accepted: 17 August 2021
Published: 20 August 2021

Research article

Some results involving multiple matrices

Sheng Dong[1], Qingwen Wang[2], Ming Zhao[3] and Zhiqiang Li[4,*]

[1] School of Mathematics, Zhengzhou University of Aeronautics, Zhengzhou 450046, China
[2] Department of Mathematics, Shanghai University, Shanghai 200444, China
[3] Department of public basic education, Henan Police College, Zhengzhou 450046, China
[4] School of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450046, China

* **Correspondence:** Email: lizhiqiang@amss.ac.cn.

Abstract: In this paper, we first give an alternative proof for a result of Mao in [Linear Algebra Appl., 589 (2020) 96–102], then we present some results involving multiple positive definite matrices and multiple sector matrices.

Keywords: multiple matrices; positive definite matrices; sector matrices
Mathematics Subject Classification: 15A45, 15A60

1. Introduction

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. We denote by $A^{(k)}, k = 1, \ldots, n-1$, the k-th leading principal submatrices of $A \in \mathbb{M}_n$.

Let $A, B \in \mathbb{M}_n$ be positive definite. It is well known that

$$\det(A + B) \geq \det A + \det B. \tag{1.1}$$

In [4], Haynsworth proved the following refinement of (1.1),

$$\det(A + B) \geq \Big(1 + \sum_{k=1}^{n-1} \frac{\det B^{(k)}}{\det A^{(k)}}\Big) \det A + \Big(1 + \sum_{k=1}^{n-1} \frac{\det A^{(k)}}{\det B^{(k)}}\Big) \det B. \tag{1.2}$$

Later, Hartfiel [5] obtained an improvement of (1.2) as follows,

$$\begin{aligned}\det(A + B) \geq &\Big(1 + \sum_{k=1}^{n-1} \frac{\det B^{(k)}}{\det A^{(k)}}\Big) \det A + \Big(1 + \sum_{k=1}^{n-1} \frac{\det A^{(k)}}{\det B^{(k)}}\Big) \det B \\ &+ (2^n - 2n)\sqrt{\det AB}.\end{aligned} \tag{1.3}$$

And the author also gave an interesting result,

$$\det(A+B) \geq \det A + \det B + (2^n - 2)\sqrt{\det AB}. \tag{1.4}$$

Extending results from two matrices case to multiple matrices case is a natural thought. Recently, by making use a result of Lin [11], Hou and Dong [7] gave the following extensions of Hartfiel's results (1.3) and (1.4) to the case of three matrices.

Theorem 1. *Let $A, B, C \in \mathbb{M}_n$ be positive definite matrices. Then*

$$\begin{aligned}\det(A+B+C) \geq & \Big(1 + \sum_{k=1}^{n-1} \frac{\det B^{(k)} + \det C^{(k)}}{\det A^{(k)}}\Big) \det A \\ & + \Big(1 + \sum_{k=1}^{n-1} \frac{\det A^{(k)} + \det C^{(k)}}{\det B^{(k)}}\Big) \det B \\ & + \Big(1 + \sum_{k=1}^{n-1} \frac{\det A^{(k)} + \det B^{(k)}}{\det C^{(k)}}\Big) \det C \\ & + (2^n - 2n)(\sqrt{\det AB} + \sqrt{\det AC} + \sqrt{\det BC}).\end{aligned} \tag{1.5}$$

Theorem 2. *Let $A, B, C \in \mathbb{M}_n$ be positive definite matrices. Then*

$$\begin{aligned}\det(A+B+C) \geq & \det A + \det B + \det C \\ & + (2^n - 2)(\sqrt{\det AB} + \sqrt{\det AC} + \sqrt{\det BC}).\end{aligned} \tag{1.6}$$

Very recently, Li et al. [9] extended Haynsworth's result (1.2) to multiple matrices by induction. At the same time, by making use of [1, Corollary 4.4], Mao [16] gave the following extension of Hartfiel's inequality (1.3) for multiple matrices.

Theorem 3. *Suppose $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, are positive definite. Then*

$$\det\Big(\sum_{i=1}^{m} A_i\Big) \geq \sum_{i=1}^{m} \left(1 + \sum_{k=1}^{n-1} \frac{\sum_{j=1, j\neq i}^{m} \det A_j^{(k)}}{\det A_i^{(k)}}\right) \det A_i + (2^n - 2n) \sum_{1\leq i<j\leq m} \sqrt{\det A_i A_j}. \tag{1.7}$$

For any $A \in \mathbb{M}_n$, $W(A) = \{x^*Ax | x \in \mathbb{C}^n, x^*x = 1\}$ is defined as the numerical range of A. For $\theta \in [0, \frac{\pi}{2})$, the sector in the complex plane is given by $S_\theta = \{z \in \mathbb{C} | \Re z > 0, |\Im z| \leq (\Re z)\tan(\theta)\}$. A matrix A is called a sector matrix if $W(A) \subset S_\theta$. This class of matrices, as a natural generalization of positive definite matrix, has been extensively investigated; see [3, 12–14, 17] and reference therein.

In 2015, Lin [14] first extended (1.3) to sector matrix, his result was improved by Zheng et al. [17], Dong and Wang [2] at the same time. And their improvements are the special case of the following inequality [7], which is the generalization of Theorem 1 for sector matrices.

Theorem 4. *Let $A, B, C \in \mathbb{M}_n$, $W(A), W(B), W(C) \subset S_\theta$. Then*

$$\begin{aligned}\sec^n(\theta)|\det(A+B+C)| \geq& \Big(1+\sum_{k=1}^{n-1}\cos^k(\theta)\frac{|\det B^{(k)}|+|\det C^{(k)}|}{|\det A^{(k)}|}\Big)|\det A| \\ &+\Big(1+\sum_{k=1}^{n-1}\cos^k(\theta)\frac{|\det A^{(k)}|+|\det C^{(k)}|}{|\det B^{(k)}|}\Big)|\det B| \\ &+\Big(1+\sum_{k=1}^{n-1}\cos^k(\theta)\frac{|\det A^{(k)}|+|\det B^{(k)}|}{|\det C^{(k)}|}\Big)|\det C| \\ &+(2^n-2n)\Big(\sqrt{|\det AB|}+\sqrt{|\det AC|}+\sqrt{|\det BC|}\Big).\end{aligned}$$

In [17], Zheng et al. gave the following complement of Theorem 4.

Theorem 5. *Let $A, B, C \in \mathbb{M}_n$, $W(A), W(B), W(C) \subset S_\theta$. Then*

$$\begin{aligned}|\det(A+B+C)| \geq& \prod_{k=1}^{m}\Big(\frac{|\det A^{(k)}|}{|\det A^{(k-1)}|}+\frac{|\det B^{(k)}|}{|\det B^{(k-1)}|}\Big)\cos^k(\theta) \\ &+\prod_{k=1}^{m}\Big(\frac{|\det A^{(k)}|}{|\det A^{(k-1)}|}+\frac{|\det C^{(k)}|}{|\det C^{(k-1)}|}\Big)\cos^k(\theta) \\ &+\prod_{k=1}^{m}\Big(\frac{|\det B^{(k)}|}{|\det B^{(k-1)}|}+\frac{|\det C^{(k)}|}{|\det C^{(k-1)}|}\Big)\cos^k(\theta) \\ &-(|\det A|+|\det B|+|\det C|).\end{aligned}$$

The extension of Theorem 4 for multiple matrices is proved by Mao [16] as follows, which is the generation of Theorem 3 for sector matrices.

Theorem 6. *Suppose $A_i \in \mathbb{M}_n$, $W(A_i) \subset S_\theta, i = 1, \ldots, m$. Then*

$$\begin{aligned}\sec^n(\theta)|\det\Big(\sum_{i=1}^{m}A_i\Big)| \geq& \sum_{i=1}^{m}\Big(1+\sum_{k=1}^{n-1}\cos^k(\theta)\frac{\sum_{j=1,j\neq i}^{m}|\det A_j^{(k)}|}{|\det A_i^{(k)}|}\Big)|\det A_i| \\ &+(2^n-2n)\sum_{1\leq i<j\leq m}\sqrt{|\det A_iA_j|}.\end{aligned} \tag{1.8}$$

In this paper, we first give an alternative proof of Mao' result (1.7), then we present some interesting results for multiple positive definite matrices, this is done in section 2. In section 3, we show some generations of the results previously proved for sector matrices, and give an complement of Theorem 6 by extending Theorem 5 to multiple sector matrices.

2. Results involving multiple positive definite matrices

In [16], the way that Mao proved Theorem 3 is original. Considering that alternative proof may provide new perspectives to the elegant result, we take the chance to do it here.

The following lemma is useful for our new proof.

Lemma 1. *[9, Lemma 6] Suppose $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, are positive definite. Let $A_i^{(k)}, k = 1, \ldots, n-1$, denote the k-th leading principal submatrices of A_i. Then*

$$\det\Big(\big(\sum_{i=1}^{m} A_i\big)/\big(\sum_{i=1}^{m} A_i^{(k)}\big)\Big) \geq \sum_{i=1}^{m} \frac{\det A_i}{\det A_i^{(k)}}.$$

Proof of Theorem 3. We prove the theorem by induction on n.

For $n = 2$, we get

$$\begin{aligned}\det\big(\sum_{i=1}^{m} A_i\big) &= \det\big(\sum_{i=1}^{m} A_i^{(1)}\big) \cdot \det\Big(\big(\sum_{i=1}^{m} A_i\big)/\big(\sum_{i=1}^{m} A_i^{(1)}\big)\Big)\\ &\geq \det\big(\sum_{i=1}^{m} A_i^{(1)}\big) \cdot \sum_{i=1}^{m} \frac{\det A_i}{\det A_i^{(1)}}\\ &\geq \big(\sum_{i=1}^{m} \det A_i^{(1)}\big) \cdot \sum_{i=1}^{m} \frac{\det A_i}{\det A_i^{(1)}}\\ &= \sum_{i=1}^{m}\left(1 + \frac{\sum_{j=1, j\neq i}^{m} \det A_j^{(1)}}{\det A_i^{(1)}}\right)\det A_i,\end{aligned}$$

where the first inequality is by Lemma 1. This proves (1.7) for $n = 2$.

Suppose the inequality (1.7) holds for all A_i of order less than or equal to $n-1$. If $A_i, i = 1, \ldots, m$, are of order n, by Lemma 1, we have

$$\begin{aligned}\det\big(\sum_{i=1}^{m} A_i\big) &= \det\big(\sum_{i=1}^{m} A_i^{(n-1)}\big) \cdot \det\Big(\big(\sum_{i=1}^{m} A_i\big)/\big(\sum_{i=1}^{m} A_i^{(n-1)}\big)\Big)\\ &\geq \det\big(\sum_{i=1}^{m} A_i^{(n-1)}\big) \cdot \sum_{i=1}^{m} \frac{\det A_i}{\det A_i^{(n-1)}}.\end{aligned}$$

By the induction hypothesis

$$\begin{aligned}\det\big(\sum_{i=1}^{m} A_i^{(n-1)}\big) \geq& \sum_{i=1}^{m}\left(1 + \sum_{k=1}^{n-2} \frac{\sum_{j=1, j\neq i}^{m} \det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)}\\ &+ (2^{n-1} - 2(n-1)) \sum_{1\leq i<j\leq m} \sqrt{\det A_i^{(n-1)} \det A_j^{(n-1)}},\end{aligned}$$

we get

$$\det\Big(\sum_{i=1}^{m}A_i\Big)\geq\sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)}\cdot\sum_{h=1}^{m}\frac{\det A_h}{\det A_h^{(n-1)}}$$
$$+(2^{n-1}-2(n-1))\sum_{1\leq i<j\leq m}\sqrt{\det A_i^{(n-1)}\det A_j^{(n-1)}}\cdot\sum_{h=1}^{m}\frac{\det A_h}{\det A_h^{(n-1)}}$$
$$=\sum_{i=1,h=1,i=h}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)}\cdot\frac{\det A_h}{\det A_h^{(n-1)}}$$
$$+\sum_{i=1,h=1,i\neq h}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i^{(n-1)}\cdot\frac{\det A_h}{\det A_h^{(n-1)}}$$
$$+(2^{n-1}-2n+2)\sum_{1\leq i<j\leq m}\sqrt{\det A_i^{(n-1)}\det A_j^{(n-1)}}\cdot\sum_{h=1}^{m}\frac{\det A_h}{\det A_h^{(n-1)}}$$
$$\geq\sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-2}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\cdot\det A_i$$
$$+\sum_{i=1,h=1,i\neq h}^{m}\det A_i^{(n-1)}\cdot\frac{\det A_h}{\det A_h^{(n-1)}}$$
$$+\sum_{i=1,h=1,i\neq h}^{m}\sum_{k=1}^{n-2}\frac{\det A_h^{(k)}}{\det A_i^{(k)}}\det A_i^{(n-1)}\cdot\frac{\det A_h}{\det A_h^{(n-1)}}$$
$$+(2^{n-1}-2n+2)\sum_{1\leq i<j\leq m}\sqrt{\det A_i^{(n-1)}\det A_j^{(n-1)}}\cdot\left(\frac{\det A_i}{\det A_i^{(n-1)}}+\frac{\det A_j}{\det A_j^{(n-1)}}\right)$$
$$\geq\sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\cdot\det A_i+2(n-2)\sum_{1\leq i<h\leq m}\sqrt{\det A_i\det A_h}$$
$$+(2^{n}-4n+4)\sum_{1\leq i<j\leq m}\sqrt{\det A_i\det A_j}$$
$$=\sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i+(2^{n}-2n)\sum_{1\leq i<j\leq m}\sqrt{\det A_i\det A_j}.$$

This completes the proof. □

The next theorem is an interesting result for multiple positive definite matrices. Clearly, when m=2, (2.1) becomes (1.4); when m=3, (2.1) reduces to (1.6).

Theorem 7. *Let $A_i\in\mathbb{M}_n, i=1,\ldots,m$, be positive definite. Then*

$$\det\Big(\sum_{i=1}^{m}A_i\Big)\geq\sum_{i=1}^{m}\det A_i+(2^{n}-2)\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j}. \tag{2.1}$$

Proof. By (1.7), we get

$$
\begin{aligned}
&\det\Big(\sum_{i=1}^{m} A_i\Big)\\
&\geq \sum_{i=1}^{m}\left(1+\sum_{k=1}^{n-1}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\right)\det A_i + (2^n-2n)\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j}\\
&= \sum_{i=1}^{m}\det A_i + \sum_{i=1}^{m}\left(\sum_{k=1}^{n-1}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\cdot\det A_i\right) + (2^n-2n)\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j}\\
&= \sum_{i=1}^{m}\det A_i + \sum_{k=1}^{n-1}\left(\sum_{i=1}^{m}\frac{\sum_{j=1,j\neq i}^{m}\det A_j^{(k)}}{\det A_i^{(k)}}\cdot\det A_i\right) + (2^n-2n)\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j}\\
&= \sum_{i=1}^{m}\det A_i + \sum_{k=1}^{n-1}\left(\sum_{i,j=1,i\neq j}^{m}\Big(\frac{\det A_j^{(k)}}{\det A_i^{(k)}}\cdot\det A_i + \frac{\det A_i^{(k)}}{\det A_j^{(k)}}\cdot\det A_j\Big)\right)\\
&\quad + (2^n-2n)\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j}\\
&\geq \sum_{i=1}^{m}\det A_i + \sum_{k=1}^{n-1}\sum_{i,j=1,i\neq j}^{m} 2\sqrt{\det A_iA_j} + (2^n-2n)\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j}\\
&= \sum_{i=1}^{m}\det A_i + (2^n-2)\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j},
\end{aligned}
$$

the proof is completed. □

Remark 1. We mention that Theorem 7 can be obtained by inequality (1.4) and the following inequality, which is a special case of [1, Corollary 4.4].

$$\det\Big(\sum_{i=1}^{m} A_i\Big) + (m-2)\sum_{i=1}^{m}\det A_i \geq \sum_{1\leq i<j\leq m}\det(A_i+A_j). \tag{2.2}$$

The following is equivalent to Theorem 7.

Corollary 1. *Let* $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, *be positive definite. Then*

$$\det\Big(\sum_{i=1}^{m} A_i\Big) \geq \Big(\sum_{i=1}^{m}\sqrt{\det A_i}\Big)^2 + (2^n-4)\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j}.$$

Proof. Note that

$$\Big(\sum_{i=1}^{m}\sqrt{\det A_i}\Big)^2 = \sum_{i=1}^{m}\det A_i + 2\sum_{1\leq i<j\leq m}\sqrt{\det A_iA_j}.$$

By Theorem 7, the desired result follows. □

The next result can be derived from Theorem 7.

Corollary 2. *Let $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, be positive definite. Then*

$$\det\Big(\sum_{i=1}^{m} A_i\Big) \geq \sum_{i=1}^{m} \det A_i + m(m-1)(2^{n-1}-1)(\prod_{i=1}^{m} \det A_i)^{\frac{1}{m}}.$$

Proof. By the arithmetic-geometric inequality, we get

$$\begin{aligned}\sum_{1\leq i<j\leq m} \sqrt{\det A_i A_j} \geq& \frac{m(m-1)}{2}\Big((\prod_{i=1}^{m} \sqrt{\det A_i})^{m-1}\Big)^{\frac{1}{m(m-1)/2}}\\ =& \frac{m(m-1)}{2}(\prod_{i=1}^{m} \det A_i)^{\frac{1}{m}}.\end{aligned}$$

The desired result follows by Theorem 7 and the inequality above. □

3. Results involving multiple sector matrices

We present two lemmas which are going to be needed in our proofs. The first lemma is known as the Ostrowski-Taussky inequality.

Lemma 2. *[6, Theorem 7.8.19] Let $A \in \mathbb{M}_n$. If $\Re(A)$ is positive definite, then*

$$\det(\Re A) \leq |\det A|.$$

The next lemma is a reverse of the Ostrowski-Taussky inequality proved by Lin [14].

Lemma 3. *[14, Lemma 2.6] Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\theta$. Then*

$$\sec^n(\theta)\det(\Re A) \geq |\det A|.$$

Now, we give the following generation of Theorem 7 for sector matrices.

Theorem 8. *Let $A_i \in \mathbb{M}_n, W(A_i) \subset S_\theta, i = 1, \ldots, m$. Then*

$$|\det\Big(\sum_{i=1}^{m} A_i\Big)| \geq \cos^n(\theta)\sum_{i=1}^{m} |\det A_i| + (2^n-2)\cos^n(\theta)\sum_{1\leq i<j\leq m} \sqrt{|\det A_i A_j|}.$$

Proof. Compute

$$\begin{aligned}|\det\Big(\sum_{i=1}^{m} A_i\Big)| \geq& \det\Big(\Re(\sum_{i=1}^{m} A_i)\Big)\\ =& \det\Big(\sum_{i=1}^{m} \Re A_i\Big)\\ \geq& \sum_{i=1}^{m} \det \Re A_i + (2^n-2)\sum_{1\leq i<j\leq m} \sqrt{\det \Re A_i \det \Re A_j}\\ \geq& \cos^n(\theta)\sum_{i=1}^{m} |\det A_i| + (2^n-2)\cos^n(\theta)\sum_{1\leq i<j\leq m} \sqrt{|\det A_i A_j|},\end{aligned}$$

where the first inequality above is by Lemma 2; the second is due to Theorem 7 and the last inequality holds by Lemma 3. □

Similarly to Theorem 8, the generations of Corollary 1 and Corollary 2 for sector matrices are as follows.

Corollary 3. *Let $A_i \in \mathbb{M}_n, W(A_i) \subset S_\theta, i = 1, \ldots, m$. Then*

$$|\det\Big(\sum_{i=1}^{m} A_i\Big)| \geq \cos^n(\theta)\Big(\sum_{i=1}^{m} \sqrt{|\det A_i|}\Big)^2 + (2^n - 4)\cos^n(\theta) \sum_{1\leq i<j\leq m} \sqrt{|\det A_i A_j|}.$$

Corollary 4. *Let $A_i \in \mathbb{M}_n, W(A_i) \subset S_\theta, i = 1, \ldots, m$. Then*

$$|\det\Big(\sum_{i=1}^{m} A_i\Big)| \geq \cos^n(\theta)\sum_{i=1}^{m} |\det A_i| + m(m-1)(2^{n-1} - 1)\cos^n(\theta)|\prod_{i=1}^{m} \det A_i|^{\frac{1}{m}}.$$

For $A \in \mathbb{M}_n$, recall the Cartesian decomposition (see, e.g., [6, p.7]) $A = \Re A + i\Im A$, where $\Re A = \frac{1}{2}(A + A^*)$, $\Im A = \frac{1}{2i}(A - A^*)$. If $\Re A$ and $\Im A$ are positive definie, We say A is accretive-dissipative. For more details about this class of matrices, we refer to [8, 10, 15]. Note that if A is accretive-dissipative, then $W(e^{-i\pi/4}A) \subset S_{\pi/4}$. Thus, we have the following corollaries.

Corollary 5. *Let $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, be accretive-dissipative. Then*

$$|\det\Big(\sum_{i=1}^{m} A_i\Big)| \geq 2^{-\frac{n}{2}}\sum_{i=1}^{m} |\det A_i| + (2^{\frac{n}{2}} - 2^{1-\frac{n}{2}}) \sum_{1\leq i<j\leq m} \sqrt{|\det A_i A_j|}.$$

Corollary 6. *Let $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, be accretive-dissipative. Then*

$$|\det\Big(\sum_{i=1}^{m} A_i\Big)| \geq 2^{-\frac{n}{2}}\Big(\sum_{i=1}^{m} \sqrt{|\det A_i|}\Big)^2 + (2^{\frac{n}{2}} - 2^{2-\frac{n}{2}}) \sum_{1\leq i<j\leq m} \sqrt{|\det A_i A_j|}.$$

Corollary 7. *Let $A_i \in \mathbb{M}_n, i = 1, \ldots, m$, be accretive-dissipative. Then*

$$|\det\Big(\sum_{i=1}^{m} A_i\Big)| \geq 2^{-\frac{n}{2}}\sum_{i=1}^{m} |\det A_i| + m(m-1)(2^{\frac{n}{2}-1} - 2^{-\frac{n}{2}})|\prod_{i=1}^{m} \det A_i|^{\frac{1}{m}}.$$

Next, We give the following complement of Theorem 6. Clearly, when m=3, Theorem 9 becomes Theorem 5.

Theorem 9. *Let $A_i \in \mathbb{M}_n, W(A_i) \subset S_\theta, i = 1, \ldots, m$. Then*

$$|\det\Big(\sum_{i=1}^{m} A_i\Big)| \geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{m} \Big(\frac{\left|\det A_i^{(k)}\right|}{\left|\det A_i^{(k-1)}\right|} + \frac{\left|\det A_j^{(k)}\right|}{\left|\det A_j^{(k-1)}\right|}\Big)\cos^k(\theta) - (m-2)\sum_{i=1}^{m} |\det A_i|.$$

Proof. Let $B, C \in \mathbb{M}_n$ be positive definite. Haynsworth [4] proved

$$\frac{\det(B^{(k)} + C^{(k)})}{\det(B^{(k-1)} + C^{(k-1)})} \geq \frac{\det B^{(k)}}{\det B^{(k-1)}} + \frac{\det C^{(k)}}{\det C^{(k-1)}}.$$

Taking products for k from 1 to n yields

$$\det(B + C) \geq \prod_{k=1}^{m} \Big(\frac{\det B^{(k)}}{\det B^{(k-1)}} + \frac{\det C^{(k)}}{\det C^{(k-1)}} \Big). \tag{3.1}$$

Then, we have

$$\begin{aligned}
|\det\Big(\sum_{i=1}^{m} A_i\Big)| &\geq \det\Big(\Re(\sum_{i=1}^{m} A_i)\Big) \\
&= \det\Big(\sum_{i=1}^{m} \Re A_i\Big) \\
&\geq \sum_{1\leq i<j\leq m} \det(\Re A_i + \Re A_j) - (m-2)\sum_{i=1}^{m} \det \Re A_i \\
&\geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{m} \Big(\frac{\det \Re A_i^{(k)}}{\det \Re A_i^{(k-1)}} + \frac{\det \Re A_j^{(k)}}{\det \Re A_j^{(k-1)}} \Big) - (m-2)\sum_{i=1}^{m} \det \Re A_i \\
&\geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{m} \Big(\frac{\det \Re A_i^{(k)}}{\left|\det A_i^{(k-1)}\right|} + \frac{\det \Re A_j^{(k)}}{\left|\det A_j^{(k-1)}\right|} \Big) - (m-2)\sum_{i=1}^{m} |\det A_i| \\
&\geq \sum_{1\leq i<j\leq m} \prod_{k=1}^{m} \Big(\frac{\left|\det A_i^{(k)}\right|}{\left|\det A_i^{(k-1)}\right|} + \frac{\left|\det A_j^{(k)}\right|}{\left|\det A_j^{(k-1)}\right|} \Big) \cos^k(\theta) \\
&\quad - (m-2)\sum_{i=1}^{m} |\det A_i|,
\end{aligned}$$

where the first inequality is by Lemma 2; the second is by (2.2); the third is by (3.1); the fourth is by Lemma 2 and the last inequality is due to Lemma 3. □

Acknowledgments

The work was supported by National Natural Science Foundation of China (NNSFC) [grant number 11971294], the Key Research Project of Henan Higher Education Institutions under Grant 20A120003 and the Young Talents Fund of HUEL under Grant hncjzfdxqnbjrc201607.

Conflict of interest

We declare no conflict of interest.

References

1. W. Berndt, S. Sra, Hlawka-Popoviciu inequalities on positive definite tensors, *Linear Algebra Appl.*, **486** (2015), 317–327.
2. S. Dong, Q. Wang, More generalizations of Hartfiel's inequality and the Brunn-Minkowski inequality, *B. Iran. Math. Soc.*, **47** (2021), 21–29.
3. X. Fu, Y. Liu, Rotfel'd inequality for partitioned matrices with numerical ranges in a sector, *Linear Multilinear Algebra*, **64** (2016), 105–109.
4. E. V. Haynsworth, Applications of an inequality for the Schur complement, *P. Am. Math. Soc.*, **24** (1970), 512–516.
5. D. J. Hartfiel, An extension of Haynsworth's determinant inequality, *P. Am. Math. Soc.*, **41** (1973), 463–465.
6. R. A. Horn, C. R. Johnson, *Matrix analysis*, 2 Eds., Cambridge: Cambridge University Press, 2013.
7. L. Hou, S. Dong, An Extension of Hartfiel's determinant inequality, *Math. Inequal. Appl.*, **21** (2018), 1105–1110.
8. A. George, Kh. D. Ikramov, On the properties of Accretive-Dissipative Matrices, *Math. Notes*, **77** (2005), 767–776.
9. Q. Li, Q. Wang, S. Dong, Extending a result of Haynsworth, *J. Math. Inequal.*, **14** (2020), 845–852.
10. M. Lin, Fischer type determinantal inequalities for accretive-dissipative matrices, *Linear Algebra Appl.*, **438** (2013), 2808–2812.
11. M. Lin, A determinantal inequality for positive definite matrices, *Electron J. Linear Algebra*, **27** (2014), 821–826.
12. M. Lin, Some inequalities for sector matrices, *Oper. Matrices*, **10** (2016), 915–921.
13. S. Drury, M. Lin, Singular value inequalities for matrices with numerical ranges in a sector, *Oper. Matrices*, **8** (2014), 1143–1148.
14. M. Lin, Extension of a result of Hanynsworth and Hartfiel, *Arch. Math.*, **1** (2015), 93–100.
15. M. Lin, D. Zhou, Norm inequalities for accretive-dissipative operator matrices, *J. Math. Anal. Appl.*, **407** (2013), 436–442.
16. Y. Mao, Extensions of Hartfiel's inequality to multiple matrices, *Linear Algebra Appl.*, **589** (2020), 96–102.
17. Y. Zheng, X. Jiang, X. Chen, F. Alsaadi, More extensions of a determinant inequality of Hartfiel, *Appl. Math. Comput.*, **369** (2020), 124827.

AIMS Press