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Research article

Global well-posedness to the Cauchy problem of 2D inhomogeneous incompressible magnetic Bénard equations with large initial data and vacuum

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Abstract: In this paper, we are concerned with the Cauchy problem of inhomogeneous incompressible magnetic Bénard equations with vacuum as far-field density in $\mathbb{R}^2$. We prove that if the initial density and magnetic field decay not too slowly at infinity, the system admits a unique global strong solution. Note that the initial data can be arbitrarily large and the initial density can contain vacuum states and even has compact support. Moreover, we extend the result of [16,17] to the global one.

Keywords: magnetic Bénard equations; global solution; large initial data; vacuum
Mathematics Subject Classification: 35Q35, 35B65, 76D03

1. Introduction

In this paper, we consider the Cauchy problem of the following 2D density-dependent magnetic Bénard equations

$$\begin{cases} \rho_t + u\cdot\nabla\rho = 0, \\ \rho u_t + \rho u\cdot\nabla u + \nabla p = \mu\Delta u + b\cdot\nabla b + \rho\theta e_2, \\ \rho\theta_t + \rho u\cdot\nabla\theta = \kappa\Delta\theta + \rho u\cdot e_2, \\ b_t + u\cdot\nabla b = \nu\Delta b + b\cdot\nabla u, \\ \mathrm{div} u = \mathrm{div} b = 0. \end{cases} \tag{1.1}$$

which is equipped the following initial conditions and far-field behavior:

$$\begin{cases} (\rho, \rho u, \rho\theta, b)(x, 0) = (\rho_0, \rho u_0, \rho\theta_0, b_0)(x) & \text{for } x \in \mathbb{R}^2, \\ (\rho, u, \theta, b)(x, \cdot) \to (0, 0, 0, 0), & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

where ρ, u, θ, b and p denote the density, velocity, temperature, magnetic field, and pressure of the fluid, respectively. $\mu > 0$ is the viscosity coefficient, $\kappa > 0$ is the heat conductivity coefficient, and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. $e_2 = (0,1)^T$, where T is the transpose.

The magnetic Bénard equations (1.1) illuminates the heat convection phenomenon under the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas (see [10, 11] for details). If we ignore the Rayleigh-Bénard convection term $u \cdot e_2$, system (1.1) recovers the inhomogeneous incompressible MHD equations (i.e., $\theta \equiv 0$). Let us review some previous works about the standard incompressible MHD equations. In the absence of vacuum, Abidi-Paicu [1] established the local and global (with small initial data) existence of strong solutions in the framework of Besov spaces. Chen et al. [2] proved a global solution for the global well-posedness to the 3D Cauchy problem for the bounded density. In the presence of vacuum, imposing the following compatibility condition,

$$-\mu\Delta u_0 + \nabla p_0 - b_0 \cdot \nabla b_0 = \sqrt{\rho_0} g \tag{1.3}$$

for some $(p_0, g) \in H^1 \times L^2$. Chen et al. [3] obtained the unique local strong solutions to the 3D Cauchy problem with general initial data. Song [13] studied the local well-posedness of strong solutions without additional compatibility condition (1.3), which extended the main result of [3]. Recently, Gao-Li [4] shown the global strong solutions with vacuum in bounded domain, provided that initial data is suitable small. Later on, Zhang-Yu [15] extended this result to the whole space. For the 2D case, Huang-Wang [5] investigated the global existence of strong solution with general large data in bounded domain provided that the compatibility condition (1.3) holds. Recently, Lv et al. [8] showed the global existence of strong solutions to the 2D Cauchy problem with the large data and vacuum.

Let us go back to the system (1.1). Very recently, by weighted energy method, Zhong [16] showed the local existence of strong solutions to the Cauchy problem of (1.1) in $\mathbb{R}^2$. However, the global existence of strong solution to the 2D Cauchy problem of (1.1) with vacuum and general initial data is not addressed. In fact, this is the main aim of this paper.

Before stating the main results, we first explain the notations and conventions used throughout this paper. For $R > 0$. Set

$$B_R := \{x \in \mathbb{R}^2 | |x| < R\}, \quad \int f dx = \int_{\mathbb{R}^2} f dx, \quad \mu = \kappa = \nu = 1.$$

Moreover, for $1 \le r \le \infty$ and $k \ge 1$, the standard Sobolev spaces are defined as follows:

$$L^r = L^r(\mathbb{R}^2), \quad W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}.$$

Without loss of generality, we assume that initial density ρ_0 satisfies

$$\int \rho_0 dx = 1, \tag{1.4}$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \ge \frac{1}{2} \int \rho_0 dx = \frac{1}{2}. \tag{1.5}$$

Throughout this paper, always denote

$$\bar{x} := (\mathrm{e} + |x|^2)^{1/2} \log^{1+\sigma_0}(\mathrm{e} + |x|^2), \tag{1.6}$$

with $\sigma_0 > 0$ fixed. The main result of this paper is stated as the following theorem:

Theorem 1.1. *In addition to* (1.4) *and* (1.5)*, assume that the initial data* $(\rho_0, u_0, \theta_0, b_0)$ *satisfies for any given numbers* $a > 1$ *and* $q > 2$*,*

$$\begin{cases} \rho_0 \geq 0,\ \bar{x}^a\rho_0 \in L^1 \cap H^1 \cap W^{1,q},\ \operatorname{div} u_0 = \operatorname{div} b_0 = 0, \\ \nabla u_0, \nabla\theta_0, \nabla b_0 \in L^2,\ \sqrt{\rho_0}u_0,\ \sqrt{\rho_0}\theta_0 \in L^2,\ \bar{x}^a b_0 \in L^2, \\ b_0 \in L^4. \end{cases} \tag{1.7}$$

Then the problems (1.1) *and* (1.2) *has a unique global strong solution* $(\rho \geq 0, u, \theta, b, p)$ *satisfying that for any* $0 < T < \infty$*,*

$$\begin{cases} 0 \leq \rho \in C([0,T]; L^1 \cap H^1 \cap W^{1,q}), \\ \bar{x}^a\rho \in L^\infty(0,T; L^1 \cap H^1 \cap W^{1,q}), \\ \sqrt{\rho}u, \nabla u, \bar{x}^{-1}u,\ \sqrt{t}\sqrt{\rho}u_t,\ \sqrt{t}\nabla p,\ \sqrt{t}\nabla^2 u \in L^\infty(0,T; L^2), \\ \sqrt{\rho}\theta, \nabla\theta, \bar{x}^{-1}\theta,\ \sqrt{t}\sqrt{\rho}\theta_t,\ \sqrt{t}\nabla^2\theta \in L^\infty(0,T; L^2), \\ b, \bar{x}^{\frac{a}{2}}b, \nabla b,\ \sqrt{t}b_t,\ \sqrt{t}\nabla^2 b \in L^\infty(0,T; L^2), \\ \nabla u, \nabla\theta \in L^2(0,T; H^1) \cap L^{\frac{q+1}{q}}(0,T; W^{1,q}), \\ \nabla p \in L^2(0,T; L^2) \cap L^{\frac{q+1}{q}}(0,T; L^q), \\ \sqrt{t}\nabla u,\ \sqrt{t}\nabla\theta \in L^2(0,T; W^{1,q}), \\ \sqrt{\rho}u_t,\ \sqrt{\rho}w_t \in L^2(\mathbb{R}^2 \times (0,T)), \\ \sqrt{t}\bar{x}^{\frac{a}{2}}\nabla b,\ \sqrt{t}\nabla b_t,\ \sqrt{t}\nabla u_t,\ \sqrt{t}\nabla\theta_t \in L^2(\mathbb{R}^2 \times (0,T)), \end{cases} \tag{1.8}$$

and

$$\inf_{0\leq t\leq T} \int_{B_{N_1}} \rho(x,t)dx \geq \frac{1}{4}, \tag{1.9}$$

for some positive constant N_1 *depending only on* $\sqrt{\rho_0}u_0$*,* N_0*, and* T*.*

Remark 1.1. *We remark that Theorem 1.1 is proved without any smallness on the initial data. Moreover, the initial density can contain vacuum states and even has compact support. We also point out that Theorem 1.1 extends the result of Zhong [16] to the global one. In particular, when* $b = 0$*, the incompressible magnetic Bénard equations* (1.1) *reduces to the incompressible Bénard equations, Theorem 1.1 also extends Zhong [17] to the global one.*

We now make some comments on the key ingredients of the analysis in this paper. For the initial data satisfying (1.7), Zhong [16] recently established the local existence and uniqueness of strong solutions to the Cauchy problems (1.1) and (1.2) (see Lemma 2.1). Thus, to extend the local strong solution to be a global one, we need to obtain global a *priori* estimates on strong solutions to (1.1) and (1.2) in suitable higher norms. However, due to critically of Sobolev's inequality in $\mathbb{R}^2$, it seems difficult to bound $\|u\|_{L^p}$ just in term of $\|\sqrt{\rho}u\|_{L^2}$ and $\|\nabla u\|_{L^2}$ for any $p \geq 2$. Moreover, compared with [9], for the systems (1.1) and (1.2) here, the strong coupling terms and Rayleigh-Bénard convection terms, such as $u \cdot \nabla b$, $\rho u \cdot e_2$, and $\rho\theta e_2$, will bring out some new difficulties.

To overcome these difficulties mentioned above, some new ideas are needed. First, using the structure of the 2D magnetic equations, we multiply $(1.1)_4$ by $4|b|^2 b$ and thus obtain the useful a *priori* estimate on $L^2(\mathbb{R}^2\times(0,T))$-norm of $|b||\nabla b|$ (see (3.5)), which is crucial in deriving the $L^\infty(0,T;L^2(\mathbb{R}^2))$-norm of ∇u, $\nabla\theta$ and ∇b. Next, in order to derive the estimates on $L^\infty(0,T;L^2(\mathbb{R}^2))$-norm of ∇u, $\nabla\theta$, motivated by [9], multiplying $(1.1)_2$ and $(1.1)_3$ by $\dot{u} := u_t + u\cdot\nabla u$ and $\dot{\theta} := \theta_t + u\cdot\nabla\theta$ instead of usual u_t and θ_t respectively, we deduce that the key point to obtain the estimate on the $L^\infty(0,T;L^2(\mathbb{R}^2))$-norm of the gradient of the velocity u and temperature θ is to bound the terms

$$I_2 := \int p\partial_j u^i\partial_i u^j dx.$$

We find I_2 in fact can be bounded by $\|\nabla p\|_{L^2}\|\nabla u\|_{L^2}^2$ (see (3.8)), since $\partial_j u^i\partial_i u^j \in \mathcal{H}^1$ due to the fact that $\mathrm{div}u = 0$ and $\nabla^\perp\cdot\nabla u = 0$ (see Lemma 2.4). Moreover, the usual $L^2(\mathbb{R}^2\times(0,T))$-norm of b_t cannot be directly estimated due to the strong coupled term $u\cdot\nabla b$. Thus, we multiplying $(1.1)_4$ by Δb instead of usual b_t, the coupled term can be controlled after integration by parts. Thirdly, to tackle the difficulty caused by the lack of the Sobolev inequality, motivated by [8, 16, 17], by introducing a weighted function to the density, as well as a Hardy-type inequality in [7] by Lions, the $\|\rho^\eta v\|_\sigma$ ($\eta > 0$, $\sigma > \max\{2, \frac{2}{\eta}\}$) is controlled in term of $\|\sqrt{\rho}v\|_{L^2}$ and $\|\nabla v\|_{L^2}$ (see (3.18)), which plays an important role in bounding the Rayleigh-Bénard convection terms $\rho u\cdot e_2$ and $\rho\theta e_2$, and deriving the estimates on the $L^\infty(0,T;L^2(\mathbb{R}^2))$ of $\sqrt{t}\sqrt{\rho}u_t$ and $\sqrt{t}\sqrt{\rho}\theta_t$. Finally, with these a *priori* estimates on the velocity, temperature and magnetic field at hand, some useful spatial weighted estimates on both b, ∇u and $\nabla\theta$ are derived, which yields the bounded of $L^\infty(0,T;L^2(\mathbb{R}^2))$-norm of $\sqrt{t}\nabla^2 b$ (see Lemma 3.7).

The rest of the paper is organized as follows. In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to the a *priori* estimates. Finally, we give the proof of Theorem 1.1 in Section 4.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later.

We start with the local existence of strong solutions whose proof can be found in [16].

Lemma 2.1. *Assume that* $(\rho_0, u_0, \theta_0, b_0)$ *satisfies* (1.7). *Then there exists a small time* $T > 0$ *and a unique strong solution* (ρ, u, θ, b, p) *to the problems* (1.1) *and* (1.2) *in* $\mathbb{R}^2\times T$ *satisfying* (1.8) *and* (1.9).

Next, the following Gagliardo-Nirenberg inequalities will be stated, which see [12] for the detailed proof.

Lemma 2.2. *For all* $v \in C_0^\infty(\mathbb{R}^n)$, *integer* j, $0 \le j < m$, $1 \le r, q \le \infty$, *and* $\frac{j}{m} \le \vartheta \le 1$, *there exists a positive constant C depending only on j, m, n, p′, q, and r such that*

$$\|\nabla^j v\|_{L^{p'}} \le C\|\nabla^m v\|_{L^r}^{\vartheta}\|v\|_{L^q}^{1-\vartheta} \tag{2.1}$$

where

$$\frac{1}{p'} = \frac{j}{n} + \vartheta\Big(\frac{1}{r} - \frac{m}{n}\Big) + (1-\vartheta)\frac{1}{q},$$

and $m - j - \frac{n}{r}$ *is not a nonnegative integer. If* $1 < r < \infty$ *and* $m - j - \frac{n}{r}$ *is a nonnegative integer,* (2.1) *holds with* $\vartheta \in [\frac{j}{m}, 1)$.

As a key technical ingredient for our approach, we need the following weighted bounds for functions in the space $\tilde{D}^{1,2}(\mathbb{R}^2) \triangleq \{v \in H^1_{\mathrm{loc}}(\mathbb{R}^2) : \nabla v \in L^2(\mathbb{R}^2)\}$, whose proof can be found in [6, Lemma 2.4].

Lemma 2.3. *Let $\bar{x}$ be as in* (1.6). *Assume that $\rho \in L^1 \cap L^\infty$ be a non-negative function satisfying*

$$\int_{B_{N_1}} \rho dx \geq M_1, \quad \|\rho\|_{L^1\cap L^\infty} \leq M_2,$$

with $M_1, M_2 > 0$ and $B_{N_1} \subset \mathbb{R}^2$ ($N_1 \geq 1$). Then there exists $C = C(M_1, M_2, N_1) > 0$ such that

$$\|v\bar{x}^{-1}\|_{L^2} \leq C(\|\sqrt{\rho}v\|_{L^2} + \|\nabla v\|_{L^2}), \quad \forall\, v \in \tilde{D}^{1,2}(\mathbb{R}^2). \tag{2.2}$$

Moreover, for any $\eta > 0$ and $\sigma > \max\{2, \frac{2}{\eta}\}$, there exits $C = C(\sigma, \eta, M_1, M_2, N_1) > 0$ such that

$$\|v\bar{x}^{-\eta}\|_{L^\sigma} \leq C(\|\sqrt{\rho}v\|_{L^2} + \|\nabla v\|_{L^2}), \quad \forall\, v \in \tilde{D}^{1,2}(\mathbb{R}^2). \tag{2.3}$$

Finally, let $\mathcal{H}^1$ and BMO stand for the usual Hardy and BMO spaces (see [14, Section 4]). Then the following well-known facts play a key role in the proof of Lemma 3.2, whose proof can be found in [9].

Lemma 2.4. *(i) There is a positive constant C such that*

$$\|E \cdot B\|_{\mathcal{H}^1} \leq C\|E\|_{L^2}\|B\|_{L^2}, \tag{2.4}$$

for all $E \in L^2$ and $B \in L^2$ with

$$\mathrm{div}E = 0, \quad \nabla^\perp B = 0 \quad \textit{in } \mathcal{D}'.$$

(ii) There is a positive constant C such that for all $v \in \widetilde{D}^{1,2}(\mathbb{R}^2)$, it holds

$$\|v\|_{\mathrm{BMO}} \leq C\|\nabla v\|_{L^2}.$$

3. A priori estimates

In this section, we will establish some necessary a *priori* bounds for strong solutions (ρ, u, θ, b, p) to the Cauchy problems (1.1) and (1.2) to extend the local strong solution. Thus, let $T > 0$ be a fixed time and (ρ, u, θ, b, p) be the strong solution to (1.1) and (1.2) on $\mathbb{R}^2 \times (0, T]$ with initial data $(\rho_0, u_0, \theta_0, b_0)$ satisfying (1.4)–(1.6). In what follows, we will use the convention that C denotes a generic positive constant depending on initial data and T.

We begin with the following standard energy estimate and the estimate on the $L^\infty(0, T; L^1 \cap L^\infty)$-norm of the density.

Lemma 3.1. *Under the assumption of Theorem 1.1, it holds that*

$$\begin{aligned}\sup_{0\leq t\leq T} &(\|\rho\|_{L^1\cap L^\infty} + \|\sqrt{\rho}u\|^2_{L^2} + \|\sqrt{\rho}\theta\|^2_{L^2} + \|b\|^2_{L^2}) \\ &+ \int_0^T (\|\nabla u\|^2_{L^2} + \|\nabla\theta\|^2_{L^2} + \|\nabla b\|^2_{L^2})dt \leq C.\end{aligned} \tag{3.1}$$

Proof. First, it follows from the transport equation $(1.1)_1$ and making use of $(1.1)_4$ (see Lions [7, Theorem 2.1]) that

$$\sup_{0\le t\le T}\|\rho\|_{L^1\cap L^\infty}\le C. \tag{3.2}$$

Next, multiplying the Eqs $(1.1)_{2,3,4}$ by (u,θ,b) and integrating by parts over $\mathbb{R}^2$, one obtains by using $\mathrm{div}u=\mathrm{div}b=0$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2+\|b\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2\\&\le C\int\rho|u||\theta|dx\le C\|\sqrt{\rho}u\|_{L^2}^2+C\|\sqrt{\rho}\theta\|_{L^2}^2,\end{aligned} \tag{3.3}$$

which together with Gronwall's inequality yields (3.1) and completes the proof of lemma. □

Lemma 3.2. *Under the assumption of Theorem 1.1, it holds that*

$$\begin{aligned}&\sup_{0\le t\le T}(\|b\|_{L^4}^4+\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\\&\quad+\int_0^T(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2)dt\\&\quad+\int_0^T(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2)dt\le C.\end{aligned} \tag{3.4}$$

Proof. (1). Multiplying $(1.1)_4$ by $4|b|^2b$ and integrating the resulting equation over $\mathbb{R}^2$, one has

$$\begin{aligned}&\frac{d}{dt}\int|b|^4dx+12\int|b|^2|\nabla b|^2dx\\&\le C\|\nabla u\|_{L^2}\||b|^2\|_{L^4}^2\\&\le C\|\nabla u\|_{L^2}\||b|^2\|_{L^2}\|\nabla|b|^2\|_{L^2}\\&\le\||b||\nabla b|\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|b\|_{L^4}^4,\end{aligned}$$

which together with Gronwall's inequality and (3.1) yields that

$$\sup_{0\le t\le T}\|b\|_{L^4}^4+\int_0^T\||b||\nabla b|\|_{L^2}^2dt\le C. \tag{3.5}$$

(2). Multiplying $(1.1)_2$ by $\dot{u}:=u_t+u\cdot\nabla u$ and integrating by parts over $\mathbb{R}^2$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\nabla u|^2dx+\int\rho|\dot{u}|^2dx\\&=\int\Delta u\cdot(u\cdot\nabla u)dx-\int\nabla p\cdot\dot{u}dx+\int b\cdot\nabla b\cdot\dot{u}dx\\&\quad+\int\rho\theta e_2\cdot\dot{u}dx=\sum_{i=1}^4 I_i.\end{aligned} \tag{3.6}$$

It follows from integration by parts and Hölder's inequality that

$$I_1 = -\int \partial_i u^j \partial_i u^k \partial_k u^j dx \le C\|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}, \tag{3.7}$$

Notice that $\mathrm{div}(\partial_j u) = \partial_j \mathrm{div} u = 0$ and $\nabla^\perp \cdot (\nabla u^j) = 0$, we infer from Lemma 2.4 that

$$|I_2| \le \left| \int p \partial_j u^i \partial_i u^j dx \right| \le C\|p\|_{\mathrm{BMO}} \|\partial_j u^i \partial_i u^j\|_{\mathcal{H}^1} \le C\|\nabla p\|_{L^2} \|\nabla u\|_{L^2}^2. \tag{3.8}$$

In view of $(1.1)_4$, Hölder's and Gagliardo-Nirenberg inequalities, we deduce after integrating by parts that

$$\begin{aligned}
I_3 &= -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \int b_t\cdot\nabla u\cdot b dx + \int b\cdot\nabla u\cdot b_t dx \\
&\quad + \int b\cdot\nabla b\cdot(u\cdot\nabla u) dx \\
&= -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \int(\Delta b - u\cdot\nabla b + b\cdot\nabla u)\cdot\nabla u\cdot b dx \\
&\quad + \int b\cdot\nabla u\cdot(\Delta b - u\cdot\nabla b + b\cdot\nabla u) dx - \int b^i \partial_i u^j \partial_j u^k b^k dx \\
&\quad - \int b^i u^j \partial_i \partial_j u^k b^k dx \\
&= -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \int(\Delta b + b\cdot\nabla u)\cdot\nabla u\cdot b dx \\
&\quad + \int b\cdot\nabla u\cdot(\Delta b + b\cdot\nabla u) dx - \int b^i \partial_i u^j \partial_j u^k b^k dx \\
&\le -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \delta\|\nabla^2 b\|_{L^2}^2 + C\||b||\nabla u|\|_{L^2}^2 \\
&\le -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \delta\|\nabla^2 b\|_{L^2}^2 + C\|b\|_{L^4}^2 \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \\
&\le -\frac{d}{dt}\int b\cdot\nabla u\cdot b dx + \delta\|\nabla^2 b\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^2}.
\end{aligned} \tag{3.9}$$

For the estimates of I_4, we derive

$$I_4 \le C\|\sqrt{\rho}\dot{u}\|_{L^2} \|\sqrt{\rho}\theta\|_{L^2} \le \varepsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C.$$

Combining the above estimates yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2} + N(t)) + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \\
&\le \delta\|\nabla^2 b\|_{L^2}^2 + C(\|\nabla^2 u\|_{L^2} + \|\nabla p\|_{L^2})(\|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2}^2),
\end{aligned} \tag{3.10}$$

where $N(t) := 2\int b\cdot\nabla u\cdot b dx$. We get by direct computations

$$|N(t)| \le C\|\nabla u\|_{L^2} \|b\|_{L^4}^2 \le C\|\nabla u\|_{L^2} \|b\|_{L^2} \|\nabla b\|_{L^2} \le \frac{1}{2}\|\nabla u\|_{L^2}^2 + c_1\|\nabla b\|_{L^2}^2.$$

(3). Multiplying $(1.1)_3$ by $\dot\theta := \theta_t + u\cdot\nabla\theta$ and integrating by parts over $\mathbb{R}^2$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla\theta|^2dx+\int\rho|\dot\theta|^2dx\\
&=-\int\partial_i\theta\partial_iu^j\partial_j\theta dx+\int\rho u\cdot e_2\cdot\dot\theta dx\\
&\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^4}^2+C\|\sqrt{\rho}\dot\theta\|_{L^2}\|\sqrt{\rho}u\|_{L^2}\\
&\le\kappa\|\sqrt{\rho}\dot\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}+C.
\end{aligned}\tag{3.11}$$

(4). Multiplying $(1.1)_4$ by Δb and integrating by parts over $\mathbb{R}^2$, we infer from Hölder's inequality, Gagliardo-Nirenberg inequality, and (3.5) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla b|^2dx+\int|\nabla^2b|^2dx\\
&\le C\int|\nabla u||\nabla b|^2dx+\int|b||\nabla u||\nabla^2b|dx\\
&\le C\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}^2+C\|b\|_{L^4}\|\nabla u\|_{L^4}\|\nabla^2b\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}\|\nabla b\|_{L^2}\|\nabla^2b\|_{L^2}+C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2u\|_{L^2}^{\frac{1}{2}}\|\nabla^2b\|_{L^2}\\
&\le\delta\|\nabla^2b\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\nabla b\|_{L^2}^2+C\|\nabla^2u\|_{L^2}\|\nabla u\|_{L^2}.
\end{aligned}\tag{3.12}$$

(5). It follows from the standard $L^{p'}$-estimates of Stokes equations that for any $p'\in[2,\infty)$,

$$\|\nabla^2u\|_{L^{p'}}+\|\nabla p\|_{L^{p'}}\le C\|\rho\dot u\|_{L^{p'}}+C\|\rho\theta e_2\|_{L^{p'}}+\||b||\nabla b|\|_{L^{p'}}.\tag{3.13}$$

which combined with (3.1) gives

$$\begin{aligned}
\|\nabla^2u\|_{L^2}+\|\nabla p\|_{L^2}&\le C\|\sqrt{\rho}\dot u\|_{L^2}+C\||b||\nabla b|\|_{L^2}+C\|\sqrt{\rho}\theta\|_{L^2}\\
&\le C\|\sqrt{\rho}\dot u\|_{L^2}+C\||b||\nabla b|\|_{L^2}+C.
\end{aligned}\tag{3.14}$$

On the other hand, in view of the standard estimate of elliptic system, one obtains

$$\|\nabla^2\theta\|_{L^2}\le C\|\rho\dot\theta\|_{L^2}+C\|\rho u\cdot e_2\|_{L^2}.\tag{3.15}$$

Adding (3.10) + $(c_1+\frac{1}{2})\times$ (3.12) + (3.11) altogether for enough large constant $c_1>0$, it follows from (3.14) and (3.15) that

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+4\|\sqrt{\rho}\dot u\|_{L^2}^2+4\|\sqrt{\rho}\dot\theta\|_{L^2}+4\Big(c_1+\frac{1}{2}\Big)\|\nabla^2b\|_{L^2}^2\\
&\le\varepsilon\|\sqrt{\rho}\dot u\|_{L^2}^2+\eta\|\sqrt{\rho}\dot\theta\|_{L^2}^2+\delta\|\nabla^2b\|_{L^2}^2+C\|\nabla u\|_{L^2}^2\|\nabla b\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}\\
&\quad+C(\|\nabla^2u\|_{L^2}+\|\nabla p\|_{L^2})(\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}^2)+C\\
&\le\varepsilon\|\sqrt{\rho}\dot u\|_{L^2}^2+\kappa\|\sqrt{\rho}\dot\theta\|_{L^2}^2+\delta\|\nabla^2b\|_{L^2}^2+\varepsilon\|\nabla^2u\|_{L^2}^2+\kappa\|\nabla^2\theta\|_{L^2}^2\\
&\quad+C\|\nabla u\|_{L^2}^2(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+C\\
&\le C\|\nabla u\|_{L^2}^2(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+C\||b||\nabla b|\|_{L^2}^2\\
&\quad+\varepsilon\|\sqrt{\rho}\dot u\|_{L^2}^2+\kappa\|\sqrt{\rho}\dot\theta\|_{L^2}^2+\delta\|\nabla^2b\|_{L^2}^2+C,
\end{aligned}$$

which together with (3.1), (3.5), Gronwall's inequality, and choosing ε, κ, δ small enough, one obtains

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\int_0^T(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2)dt\le C. \tag{3.16}$$

(6). It follows from [9, Lemma 3.4] and (1.5) that

$$\sup_{0\le t\le T}\|\bar{x}^a\rho\|_{L^1}\le C,\quad \inf_{0\le t\le T}\int_{B_{N_0}}\rho dx\ge\frac{1}{4}, \tag{3.17}$$

which along with (3.1) and (2.3) entails that for any $\eta>0$ and $\sigma>\max\{2,\frac{2}{\eta}\}$, there is a constant $\bar{C}(\sigma,\eta)>0$ such that

$$\begin{aligned}\|\rho^\eta v\|_{L^\sigma}&\le\|\rho^\eta\bar{x}^{\frac{3a}{4\sigma}}\|_{L^{\frac{4\sigma}{3}}}\|v\bar{x}^{-\frac{3a}{4\sigma}}\|_{L^{4\sigma}}\\&\le\|\rho\|_{L^\infty}^{\eta-\frac{3}{4\sigma}}\|\rho\bar{x}^a\|_{L^1}^{\frac{3}{4\sigma}}\|v\bar{x}^{-\frac{3a}{4\sigma}}\|_{L^{4\sigma}}\\&\le\bar{C}(\eta,\sigma)(\|\sqrt{\rho}v\|_{L^2}+\|\nabla v\|_{L^2})\quad\text{for all } v\in\tilde{D}^{1,2}.\end{aligned} \tag{3.18}$$

In particular, this together with (2.3) and (3.1) yields

$$\|\rho^\eta u\|_{L^\sigma}+\|\rho^\eta\theta\|_{L^\sigma}+\|u\bar{x}^{-\eta}\|_{L^\sigma}+\|\theta\bar{x}^{-\eta}\|_{L^\sigma}\le C(1+\|\nabla u\|_{L^2}+\|\nabla\theta\|_{L^2}). \tag{3.19}$$

Thus, we infer from (3.14)–(3.16), (3.5) and (3.1), Hölder's and and Garliardo-Nirenberg inequalities that

$$\begin{aligned}&\int_0^T(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2)dt\\&\le C\int_0^T(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\sqrt{\rho}|u||\nabla u|\|_{L^2}^2+\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\|\sqrt{\rho}|u||\nabla\theta|\|_{L^2}^2)dt\\&\le C\int_0^T(\|\sqrt{\rho}u\|_{L^4}\|\nabla u\|_{L^4}+\|\sqrt{\rho}u\|_{L^4}\|\nabla\theta\|_{L^4})dt+C\\&\le C\int_0^T(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2)dt+C\\&\le C\int_0^T(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\||b||\nabla b|\|_{L^2}^2)dt+C\\&\le C,\end{aligned} \tag{3.20}$$

(7). We infer from (3.14), (3.15), (3.19), (3.1), (3.4), and Gagliardo-Nirenberg inequality that

$$\begin{aligned}&\|\nabla^2 u\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2\\&\le C\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+C\||b||\nabla b|\|_{L^2}^2+C\|\sqrt{\rho}\theta\|_{L^2}^2+C\|\sqrt{\rho}u\|_{L^2}^2\\&\le C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\sqrt{\rho}|u||\nabla u|\|_{L^2}^2+C\|\sqrt{\rho}|u||\nabla\theta|\|_{L^2}^2\\&\quad+C\|b\|_{L^4}^2\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}+C\\&\le C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C\|\sqrt{\rho}u\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u\|_{L^6}^{\frac{3}{2}}(\|\nabla u\|_{L^4}^2+\|\nabla\theta\|_{L^4}^2)\end{aligned}$$

$$
\begin{aligned}
&\quad + C\|\nabla^2 b\|_{L^2}^2 + C \\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} + C\|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2} \\
&\quad + C\|\nabla^2 b\|_{L^2}^2 + C \\
&\le \frac{1}{2}(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2) + C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla^2 b\|_{L^2}^2 + C,
\end{aligned}
$$

which yields to

$$
\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 \le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla^2 b\|_{L^2}^2 + C. \tag{3.21}
$$

This together with (3.16) and (3.20) leads to

$$
\int_0^T (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2)dt \le C. \tag{3.22}
$$

Thus, it follows from (3.16), (3.20), and (3.22) that (3.4) holds. The proof of Lemma 3.2 is completed. □

Lemma 3.3. *Under the assumption of Theorem 1.1, it holds that*

$$
\sup_{0\le t\le T} \|\bar{x}^a b\|_{L^2}^2 + \int_0^T \|\bar{x}^a \nabla b\|_{L^2}^2 dt \le C. \tag{3.23}
$$

Proof. Multiplying $(1.1)_4$ by $\bar{x}^a b$ and integrating by parts over $\mathbb{R}^2$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \bar{x}^a|b|^2dx + \int \bar{x}^a|\nabla b|^2dx \\
&= \frac{1}{2}\int |b|^2\Delta\bar{x}^a dx + \int b\cdot\nabla u\cdot b\bar{x}^a dx + \frac{1}{2}\int |b|^2 u\cdot\nabla\bar{x}^a dx \\
&= N_1 + N_2 + N_3.
\end{aligned} \tag{3.24}
$$

It follows from (3.19), (3.4), Hölder's and Gagliardo-Nirenberg inequalities that

$$
\begin{aligned}
N_1 &\le C\int |b|^2\bar{x}^a(|\bar{x}^{-1}\nabla x|^2 + |\bar{x}^{-1}\nabla^2\bar{x}|)dx \le C\|\bar{x}^{\frac{a}{2}}b\|_{L^2}^2, \\
N_2 &\le C\|\bar{x}^{\frac{a}{2}}b\|_{L^4}^2\|\nabla u\|_{L^2} \\
&\le C\|\bar{x}^{\frac{a}{2}}b\|_{L^2}(\|\bar{x}^{\frac{a}{2}}b\|_{L^2} + \|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^\infty}) \\
&\le \frac{1}{4}\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 + C\|\bar{x}^{\frac{a}{2}}b\|_{L^2}^2, \\
N_3 &\le C\int |b|^2\bar{x}^a\bar{x}^{-\frac{3}{4}}u\bar{x}^{-\frac{1}{4}}\log^{1+\sigma_0}(e+|x|^2)dx \\
&\le C\|\bar{x}^{\frac{a}{2}}b\|_{L^4}\|\bar{x}^{\frac{a}{2}}b\|_{L^2}\|u\bar{x}^{-\frac{3}{4}}\|_{L^4} \\
&\le C\|\bar{x}^{\frac{a}{2}}b\|_{L^2}(\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2} + \|\bar{x}^{\frac{a}{2}}b\|_{L^2}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^\infty}) \\
&\le \frac{1}{4}\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 + C\|\bar{x}^{\frac{a}{2}}b\|_{L^2}^2,
\end{aligned}
$$

where we use the fact that $\bar{x}^{-1}\nabla\bar{x}$ and $\bar{x}^{-1}\nabla^2\bar{x}$ are uniformly bounded on $\mathbb{R}^2$ since $|\bar{x}^{-1}\nabla\bar{x}| \le C/(1+|x|)$ and $|\bar{x}^{-1}\nabla^2\bar{x}| \le C/(1+|x|^2)$, and $(e+y)^{-\alpha}\log(e+y) \le \alpha^{-1}$ for $\alpha > 0$ and $y \ge 0$.

Substituting $N_1 - N_3$ into (3.24), we obtain that

$$\frac{d}{dt}\|\bar{x}^{\frac{a}{2}}b\|_{L^2}^2 + \|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 \le C\|\bar{x}^{\frac{a}{2}}b\|_{L^2}^2, \tag{3.25}$$

which together with Gronwall's inequality yields (3.23). The proof of Lemma 3.3 is completed. □

Lemma 3.4. *Under the assumption of Theorem 1.1, it holds that*

$$\sup_{0\le t\le T} t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2 + \|b_t\|_{L^2}^2) + \int_0^T t(\|\nabla u_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2)dt \le C. \tag{3.26}$$

Proof. Firstly, differentiating $(1.1)_2$, $(1.1)_3$ with respect to t respectively, we have

$$\rho u_{tt} + \rho u\cdot\nabla u_t - \Delta u_t + \nabla p_t = -\rho_t(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u + (\rho\theta e_2)_t + (b\cdot\nabla b)_t, \tag{3.27}$$

$$\rho\theta_{tt} + \rho u\cdot\nabla\theta_t - \Delta\theta_t = -\rho_t(\theta_t + u\cdot\nabla\theta) - \rho u_t\cdot\nabla\theta + (\rho u\cdot e_2)_t. \tag{3.28}$$

Multiplying (3.27), (3.28) by u_t, θ_t respectively, and integrating it by parts over $\mathbb{R}^2$, it implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2) + \|\nabla u_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^2\\
&= \int \rho u\cdot\nabla u_t\cdot u_t dx - \int \rho u_t\cdot\nabla u\cdot u_t dx - \int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)dx\\
&+ \int (b\cdot\nabla b)_t\cdot u_t dx + \int(\rho\theta e_2)_t\cdot u_t dx + \int \rho u\cdot\nabla\theta_t\theta_t dx - \int\rho u_t\cdot\nabla\theta\theta_t dx\\
&- \int\rho u\cdot\nabla(u\cdot\nabla\theta\theta_t)dx + \int(\rho u\cdot e_2)_t\theta_t dx\\
&\le C\int\rho|u||u_t||\nabla u_t|dx + C\int\rho|u_t|^2|\nabla u|dx + C\int|b_t||\nabla u_t||b|dx\\
&+ C\int\rho|u|(|u_t||\nabla u|^2 + |u||\nabla^2 u| + |u||\nabla u||\nabla u_t|)dx + C\int\rho|u_t||\nabla\theta||\theta_t|dx\\
&+ C\int\rho(|\theta_t||u_t| + |u||\nabla\theta||u_t| + |u||\theta||\nabla u_t|)dx + C\int\rho|u||\nabla\theta_t||\theta_t|dx\\
&+ C\int\rho|u|(|\nabla u||\nabla\theta||\theta_t| + |u||\nabla^2\theta||\theta_t| + |u||\nabla\theta||\nabla\theta_t|)dx\\
&+ C\int\rho(|u_t||\theta_t| + |u||\nabla u||\theta_t| + |u|^2|\nabla\theta_t|)dx =: \sum_{i=1}^9 Z_i.
\end{aligned} \tag{3.29}$$

By using Hölder's, Gagliardo-Nirenberg inequalities, (3.1), (3.4), (3.18), and (3.19), one gets

$$\begin{aligned}
Z_1 &\le C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\\
&\le C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}+\|\nabla u_t\|_{L^2}^{\frac{1}{2}})\|\nabla u_t\|_{L^2}\\
&\leq \frac{1}{12}\|\nabla u_t\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2,\\
Z_2 &\leq C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^4}^2\\
&\leq C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{3}{2}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\\
&\leq C(\|\nabla u_t\|_{L^2}^{\frac{3}{2}}+\|\sqrt{\rho}u_t\|_{L^2}^{\frac{3}{2}})\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\\
&\leq \frac{1}{12}\|\nabla u_t\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2,\\
Z_3 &\leq C\|\nabla u_t\|_{L^2}\|b_t\|_{L^4}\|b\|_{L^4}\\
&\leq C\|\nabla u_t\|_{L^2}\|b_t\|_{L^2}^{\frac{1}{2}}\|\nabla b_t\|_{L^2}^{\frac{1}{2}}\\
&\leq \frac{1}{12}\|\nabla u_t\|_{L^2}^2+\delta\|\nabla b_t\|_{L^2}^2+C\|b_t\|_{L^2}^2,\\
Z_4 &\leq C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u\|_{L^4}^2+C\|\rho^{\frac{1}{4}}u\|_{L^{12}}^2\|\sqrt{\rho}u_t\|_{L^3}\|\nabla^2 u\|_{L^2}\\
&\quad +C\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^8}^2\|\nabla u\|_{L^4}\\
&\leq C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}+C\|\nabla u_t\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\\
&\leq C(\|\sqrt{\rho}u_t\|_{L^2}+\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{1}{2}})\|\nabla^2 u\|_{L^2}+C\|\nabla u_t\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\\
&\leq \frac{1}{12}\|\nabla u_t\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\nabla^2 u\|_{L^2}^2+C,\\
Z_5 &\leq C\|\sqrt{\rho}u_t\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^3}\|\nabla\theta\|_{L^2}\\
&\leq C(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u_t\|_{L^2})(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}})\\
&\leq \frac{1}{12}\|\nabla u_t\|_{L^2}^2+\frac{1}{8}\|\nabla\theta_t\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2,\\
Z_6 &\leq C\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2}+C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla\theta\|_{L^2}\\
&\quad +C\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}\theta\|_{L^3}\\
&\leq C\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2}+C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^6}^{\frac{1}{2}}\\
&\quad +C\|\nabla u_t\|_{L^2}(\|\sqrt{\rho}u\|_{L^2}+\|\nabla u\|_{L^2})\|\sqrt{\rho}\theta\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_{L^6}^{\frac{1}{2}}\\
&\leq \frac{1}{12}\|\nabla u_t\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2+C,\\
Z_7 &\leq C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^3}\|\nabla\theta_t\|_{L^2}\\
&\leq C(\|\sqrt{\rho}u\|_{L^2}+\|\nabla u\|_{L^2})\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}\\
&\leq C(\|\sqrt{\rho}\theta_t\|_{L^2}+\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}})\|\nabla\theta_t\|_{L^2}\\
&\leq \frac{1}{8}\|\nabla\theta_t\|_{L^2}^2+C\|\sqrt{\rho}\theta_t\|_{L^2}^2,\\
Z_8 &\leq C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^3}\|\nabla u\|_{L^4}\|\nabla\theta\|_{L^4}\\
&\leq C\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C(\|\sqrt{\rho}\theta_t\|_{L^2} + \|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\nabla\theta_t\|_{L^2}^{\frac{1}{2}})\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\\
&\le \frac{1}{8}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|\nabla^2\theta\|_{L^2}^2,\\
Z_9 &\le C\|\sqrt{\rho}\theta_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} + C\|\sqrt{\rho}u\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^3}\|\nabla u\|_{L^2}\\
&\quad + C\|\sqrt{\rho}u\|_{L^3}\|\sqrt{\rho}u\|_{L^6}\|\nabla\theta_t\|_{L^2}\\
&\le C\|\sqrt{\rho}\theta_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2} + C(\|\sqrt{\rho}u\|_{L^2} + \|\nabla u\|_{L^2})\|\sqrt{\rho}\theta_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\theta_t\|_{L^6}^{\frac{1}{2}}\\
&\quad + C\|\sqrt{\rho}u\|_{L^2}^{\frac{1}{2}}(\|\sqrt{\rho}u\|_{L^2} + \|\nabla u\|_{L^2})^{\frac{3}{2}}\|\nabla\theta_t\|_{L^2}\\
&\le \frac{1}{8}\|\nabla\theta_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C.
\end{aligned}
$$

Putting all above estimates into (3.29), we thus obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2) + \|\nabla u_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^2\\
&\le 2\delta\|\nabla b_t\|_{L^2}^2 + C(\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) + C.
\end{aligned}
\tag{3.30}
$$

Next, differentiating $(1.1)_4$ with respect to t gives

$$
b_{tt} - b_t\cdot\nabla u - b\cdot\nabla u_t + u_t\cdot\nabla b + u\cdot\nabla b_t = \Delta b_t. \tag{3.31}
$$

Multiplying (3.31) by b_t, and integrating it by parts over $\mathbb{R}^2$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|b_t\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2\\
&= \int b\cdot\nabla u_t\cdot b_t dx + \int b_t\cdot\nabla u\cdot b_t dx + \int u_t\cdot\nabla b_t\cdot b dx\\
&\le C\|\nabla u_t\|_{L^2}\|b_t\|_{L^4}\|b\|_{L^4} + C\|\nabla u\|_{L^2}\|b_t\|_{L^4}^2 + C\|\nabla b_t\|_{L^2}\||u_t||b|\|_{L^2}\\
&\le C\|\nabla u_t\|_{L^2}\|b_t\|_{L^2}^{\frac{1}{2}}\|\nabla b_t\|_{L^2}^{\frac{1}{2}} + C\|\nabla b_t\|_{L^2}\|u_t\bar{x}^{-\frac{a}{4}}\|_{L^8}^2\|\bar{x}^{\frac{a}{2}}b\|_{L^2}\|b\|_{L^4}\\
&\quad + C\|b_t\|_{L^2}\|\nabla b_t\|_{L^2}\\
&\le \frac{1}{2}\|\nabla b_t\|_{L^2}^2 + c_2\|\nabla u_t\|_{L^2}^2 + c_2\|\sqrt{\rho}u_t\|_{L^2}^2,
\end{aligned}
$$

which leads to

$$
\frac{d}{dt}\|b_t\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2 \le c_2\|\nabla u_t\|_{L^2}^2 + c_2\|\sqrt{\rho}u_t\|_{L^2}^2, \tag{3.32}
$$

for enough large constant $c_2 > 0$.

Moreover, multiplying (3.30) by $c_2 + 1$ and adding the resulting inequality with (3.32), and choosing δ suitable small, one obtains

$$
\begin{aligned}
&\frac{d}{dt}((c_2+1)\|\sqrt{\rho}u_t\|_{L^2}^2+(c_2+1)\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|b_t\|_{L^2}^2)\\
&\quad+\|\nabla u_t\|_{L^2}^2+(c_2+1)\|\nabla\theta_t\|_{L^2}^2+\|\nabla b_t\|_{L^2}^2\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|b_t\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2)+C\\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|b_t\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2)+C. \qquad (3.33)
\end{aligned}
$$

Multiplying (3.33) by t, we obtain (3.26) after using Gronwall's inequality, (3.5), (3.4) and (3.16). The proof of Lemma 3.4 is completed. □

Lemma 3.5. *Under the assumption of Theorem 1.1, it holds that*

$$
\begin{aligned}
\sup_{0\le t\le T}\|\rho\|_{H^1\cap W^{1,q}}&+\int_0^T\Big(\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}}+\|\nabla^2\theta\|_{L^q}^{\frac{q+1}{q}}+\|\nabla p\|_{L^q}^{\frac{q+1}{q}}\Big)dt\\
&+\int_0^T t(\|\nabla^2 u\|_{L^q}^2+\|\nabla p\|_{L^q}^2+\|\nabla^2\theta\|_{L^q}^2)dt\le C. \qquad (3.34)
\end{aligned}
$$

Proof. First, it follows from the mass equation $(1.1)_1$ that $|\nabla\rho|^r$ satisfies for any $r\ge 2$,

$$
(|\nabla\rho|^r)_t+\operatorname{div}(|\nabla\rho|^r u)+r|\nabla\rho|^r(\nabla\rho)^{tr}\nabla u(\nabla\rho)=0, \qquad (3.35)
$$

which together with integrating by parts over $\mathbb{R}^2$ implies

$$
\frac{d}{dt}\|\nabla\rho\|_{L^r}\le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^r}. \qquad (3.36)
$$

Next, one gets from Gagliardo-Nirenberg inequality that

$$
\|\nabla u\|_{L^\infty}\le C\|\nabla u\|_{L^2}^{\frac{q-2}{2(q-1)}}\|\nabla^2 u\|_{L^q}^{\frac{q}{2(q-1)}}. \qquad (3.37)
$$

On the one hand, it is easy to check that

$$
\begin{aligned}
\|\nabla^2 u\|_{L^q}+\|\nabla p\|_{L^q}&\le C(\|\rho u_t\|_{L^q}+\|\rho u\cdot\nabla u\|_{L^q}+\|b\cdot\nabla b\|_{L^q}+\|\rho\theta e_2\|_{L^q})\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\rho u_t\|_{L^{q^2}}^{\frac{q^2-2q}{q^2-2}}+C\|\rho u\|_{L^{2q}}\|\nabla u\|_{L^{2q}}\\
&\quad+C\|b\|_{L^{2q}}\|\nabla b\|_{L^{2q}}+C\|\rho\theta\|_{L^q}\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\rho u_t\|_{L^{q^2}}^{\frac{q^2-2q}{q^2-2}}+C\|\nabla u\|_{L^2}^{\frac{1}{q}}\|\nabla^2 u\|_{L^2}^{\frac{q-1}{q}}\\
&\quad+C\|b\|_{L^2}^{\frac{1}{q}}\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}^{\frac{q-1}{q}}+C\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\nabla u_t\|_{L^2}^{\frac{q^2-2q}{q^2-2}}+C\|\sqrt{\rho}u_t\|_{L^2}\\
&\quad+C\|\nabla^2 u\|_{L^2}^{\frac{q-1}{q}}+C\|\nabla^2 b\|_{L^2}^{\frac{q-1}{q}}+C, \qquad (3.38)
\end{aligned}
$$

which together with (3.4) and (3.26) implies that

$$\begin{aligned}
&\int_0^T \Big(\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + \|\nabla p\|_{L^q}^{\frac{q+1}{q}}\Big)dt \\
&\le C \sup_{0\le t\le T} (t\|\sqrt{\rho}u_t\|_{L^2})^{\frac{q^2-1}{q(q^2-2)}} \int_0^T t^{-\frac{q+1}{2q}} (t\|\nabla u_t\|_{L^2})^{\frac{(q-2)(q+1)}{2(q^2-2)}} dt \\
&\quad + C\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^{\frac{q+1}{q}} dt + C\int_0^T (\|\nabla^2 u\|_{L^2}^{\frac{q^2-1}{q^2}} + \|\nabla^2 b\|_{L^2}^{\frac{q^2-1}{q^2}})dt + C \\
&\le C\int_0^T t^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}} dt + C\int_0^T (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2)dt + C \\
&\le C.
\end{aligned} \tag{3.39}$$

On the other hand, it follows from (3.4) and (3.26) that

$$\begin{aligned}
&\int_0^T t\Big(\|\nabla^2 u\|_{L^q}^2 + \|\nabla p\|_{L^q}^2\Big)dt \\
&\le C\int_0^T t(\|\rho u_t\|_{L^q}^2 + C\|\rho u\cdot\nabla u\|_{L^q}^2 + \|b\cdot\nabla b\|_{L^2}^2 + \|\rho\theta\|_{L^q}^2)dt \\
&\le C\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2 dt + C\int_0^T t\|\nabla u_t\|_{L^2}^2 dt \\
&\quad + C\int_0^T (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2)dt + C \\
&\le C.
\end{aligned} \tag{3.40}$$

Thanks to (3.37), (3.39) and (3.40), we immediately obtain

$$\int_0^T \|\nabla u\|_{L^\infty} dt \le C. \tag{3.41}$$

Thus, applying Gronwall's inequality to (3.36) gives

$$\sup_{0\le t\le T} \|\nabla\rho\|_{L^2\cap L^q} \le C. \tag{3.42}$$

Finally, similar to (3.39) and (3.40), we obtain from $(1.1)_3$ by L^q-estimates to elliptic equations that

$$\int_0^T \Big(\|\nabla^2\theta\|_{L^q}^{\frac{q+1}{q}} + t\|\nabla^2\theta\|_{L^q}^2\Big)dt \le C, \tag{3.43}$$

which together with (3.39), (3.40), and (3.42) yields (3.34) and completes the proof of lemma. □

Next, the following high order weighted estimates on the density has been proven in [9, Lemma 3.6]. We omit the detailed proof here for simplicity.

Lemma 3.6. *Under the assumption of Theorem 1.1, it holds that*

$$\sup_{0\le t\le T} \|\bar{x}^a\rho\|_{L^1\cap H^1\cap W^{1,q}} \le C. \tag{3.44}$$

Lemma 3.7. *Under the assumption of Theorem 1.1, it holds that*

$$\sup_{0\le t\le T} t(\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2) + \int_0^T t\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}^2 dt \le C. \tag{3.45}$$

Proof. First, multiplying $(1.1)_4$ by $\bar{x}^a\Delta b$ and integrating by parts over $\mathbb{R}^2$ lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 + \|\bar{x}^{\frac{a}{2}}\Delta b\|_{L^2}^2\\
&\le C\int |\nabla b||b||\nabla u||\nabla \bar{x}^a|dx + C\int |\nabla b|^2|u||\nabla\bar{x}^a|dx\\
&\quad + C\int |\nabla b||\Delta b||\nabla\bar{x}^a|dx + C\int |b||\nabla u||\Delta b|\bar{x}^a dx\\
&\quad + C\int |\nabla u||\nabla b|^2\bar{x}^a dx =: \sum_{i=1}^5 Q_i.
\end{aligned} \tag{3.46}$$

By using Hölder's inequality, Gagliardo-Nirenberg inequality, (3.23), (3.4) and (3.19), one obtains

$$\begin{aligned}
Q_1 &\le C\|\bar{x}^{\frac{a}{2}}b\|_{L^4}\|\nabla u\|_{L^4}\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^\infty}\\
&\le C\|\bar{x}^{\frac{a}{2}}b\|_{L^2}^{\frac{1}{2}}(\|\bar{x}^{\frac{a}{2}}b\|_{L^2} + \|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^\infty})^{\frac{1}{2}}\\
&\quad \times \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}\\
&\le C\|\nabla^2 u\|_{L^2}^2 + C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 + C,\\
Q_2 &\le C\||\nabla b|^{2-\frac{2}{3a}}\bar{x}^{a-\frac{1}{3}}\|_{L^{\frac{6a}{6a-2}}}\|u\bar{x}^{-\frac{1}{3}}\|_{L^{6a}}\||\nabla b|^{\frac{2}{3a}}\|_{L^{6a}}\\
&\le C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^{\frac{6a-2}{3a}}\|\nabla b\|_{L^4}^{\frac{2}{3a}}\\
&\le C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 + C\|\nabla b\|_{L^4}^2\\
&\le C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 + C\|\nabla^2 b\|_{L^2}^2 + C,\\
Q_3 &\le C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^\infty}\\
&\le \frac{1}{4}\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}^2 + C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2,\\
Q_4 &\le C\|\bar{x}^{\frac{a}{2}}b\|_{L^4}\|\nabla u\|_{L^4}\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}\\
&\le C\|\bar{x}^{\frac{a}{2}}b\|_{L^2}^{\frac{1}{2}}(\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^{\frac{1}{2}} + \|\bar{x}^{\frac{a}{2}}b\|_{L^2}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^\infty})\\
&\quad \times \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}\\
&\le \frac{1}{4}\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2 + C,\\
Q_5 &\le C\|\nabla u\|_{L^\infty}\|\bar{x}^a\nabla b\|_{L^2}^2\\
&\le C\Big(1 + \|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}}\Big)\|\bar{x}^a\nabla b\|_{L^2}^2,
\end{aligned}$$

Substituting the above estimates into (3.46), we have

$$\frac{1}{2}\frac{d}{dt}\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2+\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}^2$$
$$\le C(1+\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}})\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2+C\|\nabla^2 u\|_{L^2}^2+C\|\nabla^2 b\|_{L^2}^2+C$$
$$\le C(1+\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}})\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\nabla^2 b\|_{L^2}^2+C, \tag{3.47}$$

due to the following fact that

$$\begin{aligned}\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}^2 &= \int \bar{x}^a|\Delta b|^2dx-\int \bar{x}^a\partial_i\partial_j b\cdot\partial_j b\bar{x}^{-1}\log^{1+\sigma_0}(e+|x|^2)dx\\ &\quad+\int \bar{x}^a\partial_i\partial_i b\cdot\partial_j b\bar{x}^{-1}\log^{1+\sigma_0}(e+|x|^2)dx\\ &\le C\|\bar{x}^{\frac{a}{2}}\Delta b\|_{L^2}^2+\frac{1}{2}\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}^2+C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2.\end{aligned} \tag{3.48}$$

Thus, multiplying (3.47) by t, we deduce from Gronwall's inequality, (3.4) and (3.34) that

$$\sup_{0\le t\le T} t\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2+C\int_0^T t\|\bar{x}^{\frac{a}{2}}\nabla^2 b\|_{L^2}^2dt\le C. \tag{3.49}$$

Next, it follows from $(1.1)_4$, (3.19), (3.4), and Gagliardo-Nirenberg inequality that

$$\begin{aligned}\|\nabla^2 b\|_{L^2}^2 &\le C\|b_t\|_{L^2}^2+C\||u||\nabla b|\|_{L^2}^2+C\||b||\nabla u|\|_{L^2}^2\\ &\le C\|b_t\|_{L^2}^2+C\|u\bar{x}^{-\frac{a}{2}}\|_{L^8}^2\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}\|\nabla b\|_{L^4}+C\|b\|_{L^4}^2\|\nabla u\|_{L^4}^2\\ &\le C\|b_t\|_{L^2}^2+C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2+C\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}+C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\\ &\le \frac{1}{2}\|\nabla^2 b\|_{L^2}^2+C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2+C\|b_t\|_{L^2}^2+C\|\nabla^2 u\|_{L^2}^2+C\\ &\le \frac{1}{2}\|\nabla^2 b\|_{L^2}^2+C\|\bar{x}^{\frac{a}{2}}\nabla b\|_{L^2}^2+C\|b_t\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2+C,\end{aligned} \tag{3.50}$$

which together with (3.26) and (3.49) yields that (3.45) and completes the proof of lemma. □

4. Proof of Theorem 1.1

With a *priori* estimates in Section 3 at hand, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.1, there exists a $T_*>0$ such that the problems 1.1 and 1.2 has a unique strong solution (ρ,u,θ,b,p) on $\mathbb{R}^2\times(0,T_*]$. Now, we will extend the local solution to all time.

Set

$$T^*=\sup\{T\mid(\rho,u,\theta,b,p)\text{ is a strong solution on }\mathbb{R}^2\times(0,\mathrm{T}]\}. \tag{4.1}$$

First, for any $0<\tau<T_*<T\le T^*$ with T finite, one deduces from (3.1), (3.4), (3.26), (3.34) and (3.45) that for any $q>2$,

$$\nabla u,\nabla\theta,\nabla b,b\in C([\tau,T];L^2\cap L^q), \tag{4.2}$$

where one has used the standard embedding

$$L^\infty(\tau,T;H^1)\cap H^1(\tau,T;H^{-1})\hookrightarrow C(\tau,T;L^q) \text{ for any } q\in(2,\infty).$$

Moreover, it follows from (3.34) and (3.44) and [7, Lemma 2.3] that

$$\rho\in C([0,T];L^1\cap H^1\cap W^{1,q}). \tag{4.3}$$

Finally, if $T^*<\infty$, it follows from (4.2), (4.3), (3.1), (3.4), (3.34) and (3.45) that

$$(\rho,u,\theta,b)(x,T^*)=\lim_{t\to T^*}(\rho,u,\theta,b)(x,t)$$

satisfies the initial condition (1.7) at $t=T^*$. Thus, taking $(\rho,u,\theta,b)(x,T^*)$ as the initial data, Lemma 2.1 implies that one can extend the strong solutions beyond T^*. This contradicts the assumption of T^* in (4.1). The proof of Theorem 1.1 is completed. □

5. Conclusions

In this paper, we are concerned with the Cauchy problem of inhomogeneous incompressible magnetic Bénard equations with vacuum as far-field density in $\mathbb{R}^2$. Using the weighted function to the density, as well as the Hardy-type inequality, we have successfully established the time-uniform a priori estimates of solutions. Thus, we can extend the local strong solutions to the global one.

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Conflict of interest

The author declares that there are no conflicts of interest in this paper.

References

1. H. Abidi, M. Paicu, Global existence for the magnetohydrodynamic system in critical spaces, *P. Roy. Soc. Edinb. A*, **138** (2008), 447–476.
2. F. Chen, B. Guo, X. Zhai, Global solution to the 3-D inhomogeneous incompressible MHD system with discontinuous density, *Kinet. Relat. Mod.*, **12** (2019), 37–58.
3. Q. Chen, Z. Tan, Y. Wang, Strong solutions to the incompressible magnetohydrodynamic equations, *Math. Methods Appl. Sci.*, **34** (2011), 94–107.

4. H. Gong, J. Li, Global existence of strong solutions to incompressible MHD, *Commun. Pure Appl. Anal.,* **13** (2014), 1553–1561.

5. X. Huang, Y. Wang, Global strong solution to the 2D nonhomogeneous incompressible MHD system, *J. Differ. Equations*, **254** (2013), 511–527.

6. J. Li, Z. Xin, Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier-Stokes equations with vacuum, *Ann. PDE*, **5** (2019), 1–37.

7. P. Lions, *Mathematical topics in fluid mechanics, Vol. I: Incompressible models*, Oxford: Oxford University Press, 1996.

8. B. Lv, Z. Xu, X. Zhong, Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent magnetohydrodynamic equations with vacuum, *J. Math. Pures Appl.,* **108** (2017), 41–62.

9. B. Lv, X. Shi, X. Zhong, Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent Navier-Stokes equations with vacuum, *Nonlinearity*, **31** (2018), 2617–2632.

10. G. Mulone, S. Rionero, Necessary and sufficient conditions for nonlinear stability in the magnetic Bénard problem, *Arch. Ration. Mech. Anal.*, **166** (2003), 197–218.

11. M. Nakamura, On the magnetic Bénard problem, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **38** (1991), 359–393.

12. L. Nirenberg, On elliptic partial differential equations, In: S. Faedo, *Il principio di minimo e sue applicazioni alle equazioni funzionali*, Berlin: Springer, 2011.

13. S. Song, On local strong solutions to the three-dimensional nonhomogeneous incompressible magnetohydrodynamic equations with density-dependent viscosity and vacuum, *Z. Angew. Math. Phys.*, **69** (2018), 1–27.

14. E. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton: Princeton University Press, 1993.

15. P. Zhang, H. Yu, Global regularity to the 3D incompressible MHD equations, *J. Math. Anal. Appl.*, **432** (2015), 613–631.

16. X. Zhong, The local well-posedness to the density-dependent magnetic Bénard system with nonnegative density, *Commun. Math. Sci.*, **18** (2020), 725–750.

17. X. Zhong, Local strong solutions to the nonhomogeneous Bénard system with nonnegative density, *Rocky Mt. J. Math.,* **50** (2020), 1497–1516.

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