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Research article

Intersection soft ideals and their quotients on KU-algebras

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Abstract: In this paper, we have discussed quotient structures of KU-algebras by using the concept of intersection soft ideals. In general, the soft sets are parameterized families of sets that are used to dealt with uncertainty. In particular, We have given the fundamental homomorphism theorem of quotient KU-algebras. A characterization of commutative quotient KU-algebras, implicative quotient KU-algebras and positive implicative quotient KU-algebras are also presented.

Keywords: KU-algebras; soft sets; intersection soft ideals; quotient KU-algebras
Mathematics Subject Classification: 06F35, 03G25

1. Introduction

It is well known that in classical algebraic structures or logical algebraic structures ideals play an important role in structural studies. Comparatively, the study of quotients on algebraic structures also has a significant role. The notion of the soft set was introduced by Molodtsov [13] in 1999. The concept of soft set is a generalization of fuzzy sets, that deals with uncertainty with the help of a parameterized family of sets. These sets are said to be soft sets because the boundary of the set depends on the parameters. The soft set is widely studied as a mathematical tool (like the rough sets) for dealing with uncertainties, which is an attribute of information. The algebraic structures of set theory that are related to uncertainties have been studied intensively by many authors. Feng [4] studied soft rough approximations in multicriteria groups decision-making problems. For other soft set related concepts and properties the reader can see [5, 6, 9]. More concepts based on intersection soft set theory and related terminologies in algebraic structures can be found in [10].

Prabpayak and Leerawat [15] introduced KU-algebras in 2009. The homomorphisms on KU-algebra and some of its related properties are given in [15, 16]. Later, several authors widely studied KU-algebras in different directions e.g. in fuzzy, neutrosophic and intuitionistic context, and also in a soft and rough sense, etc. Naveed et al. [14] introduced the concept of cubic KU-ideals of KU-algebras. Recently, Moin and Ali introduced roughness in KU-algebras [11]. Ali et al. [1] studied pseudo-metric on KU-algebras. Concepts based on n-ary block codes related to KU-algebras are discussed by Ali et al. [2]. KU-Algebras containing (α, β)-US soft sets were defined and studied by Moin et al. [12]. Further, Ali et al. defined and studied the extension of KU-algebras [3]. KU-algebra is the recent interest for many researchers working towards logical algebras.

Applications of soft sets can be seen in many directions e.g. decision-making problems introduced by Maji [19], to diagnose the prostate cancer risk by Yuksel [18] and in lattices by MA et al. [20]. Akram et al. [21] studied parameter reductions in N-soft sets and their applications in decision-making. Borzooei et al. [22] recently studied soft set theory in Hoops.

In this article, motivated by above works and work done in various directions using soft set ([7, 8]), we have considered KU-algebras and have introduced quotient KU-algebras by using soft sets that are induced by intersection soft ideals of a KU-algebra. We have studied the fundamental homomorphism theorem of quotient KU-algebras, and proved some characterizations related to quotient KU-algebras.

2. Preliminaries

In this section, we shall consider concepts based on KU-algebras, KU-subalgebras, KU-ideals, and soft sets with important terminologies examples and some related results.

Definition 2.1. *[15] By a KU-algebra we mean an algebra* $(K, \circ, 1)$ *of type* $(2, 0)$ *with a single binary operation* $\circ$ *that satisfies the following identities: for any* $x, y, z \in K$,
(ku1) $(x \circ y) \circ [(y \circ z) \circ (x \circ z)] = 1$,
(ku2) $x \circ 1 = 1$,
(ku3) $1 \circ x = x$,
(ku4) $x \circ y = y \circ x = 1$ *implies* $x = y$.

In what follows, let $(K, \circ, 1)$ denote a KU-algebra unless otherwise specified. For simplicity we will call K as a KU-algebra. The element 1 of K is called *constant* which is the fixed element of K. Partial order " $\leq$ " in K is denoted by the condition $x \leq y$ if and only if $y \circ x = 1$.

Lemma 2.2. *[15]* $(K, \circ, 1)$ *is a KU-algebra if and only if it satisfies:*
(ku5) $(y \circ z) \circ (x \circ z) \leq x \circ y$,
(ku6) $1 \leq x$,
(ku7) $x \leq y$, $y \leq x$ *implies* $x = y$,

Lemma 2.3. *In a KU-algebra, the following identities are true [17]:*
(1) $z \circ z = 1$,
(2) $z \circ (x \circ z) = 1$,
(3) $z \circ (y \circ x) = y \circ (z \circ x)$, *for all* $x, y, z \in X$,
(4) $y \circ [(y \circ x) \circ x] = 1$.

Definition 2.4. *A mapping* $f : K \to K'$ *of a KU-algebra is called a KU-homomorphism if* $f(x \circ y) = f(x) \circ f(y)$ *for all* $x, y \in K$.

Definition 2.5. *[15] A non-empty subset I of a KU-algebra K is called a KU-ideal of K if it satisfies the following conditions:*
(1) $1 \in I$,
(2) $(\forall x \in K)(\forall y \in I)$ $(y \circ x \in I \Rightarrow x \in I)$.

A KU-algebra K is said to be *commutative* if $x \vee y = y \vee x$ for all $x, y \in K$ where $y \vee x = (x \circ y) \circ y$. A KU-algebra K is said to be *positive implicative* if $(z \circ y) \circ (z \circ x) = z \circ (y \circ x)$ for all $x, y \in K$. A KU-algebra K is positive implicative if and only if the following equality holds:

$$y \circ x = y \circ (y \circ x) \quad \forall\, x, y \in K. \tag{2.1}$$

A KU-algebra K is said to be *implicative* if $x = (x \circ y) \circ x \quad \forall x, y \in K$.

Definition 2.6. *[13] A pair $(\mathcal{K}, E)$ is called a soft set over a universal set X, where $\mathcal{K}$ is a function given by: $\mathcal{K} : E \to \mathcal{P}(X)$ and E is the set of parameters. For $\varepsilon \in E$, $\mathcal{K}(\varepsilon)$ is the set of approximate ε-elements of the soft set $(\mathcal{K}, E)$.*

Here is an example for soft set in a topological spaces.

Example 2.7. *Let (K, τ) be a topological space, i.e., τ is a family of subsets of the set K called the open sets of K. Then, the family of open neighborhoods $N(x)$ of point x, where $N(x) = \{V \in \tau | x \in V\}$, may be considered as the soft set $(N(x), \tau)$.*

Definition 2.8. *[13] Let A be a non-empty subset of E. Then a soft set $(\mathcal{K},\ E)$ over X satisfying the condition: $\mathcal{K}(x) = \emptyset$ for all $x \notin A$ is called the A-soft set over X and is denoted by $\mathcal{K}_A$, so an A-soft set $\mathcal{K}_A$ over X is a function $\mathcal{K}_A : E \to \mathcal{P}(X)$ such that $\mathcal{K}_A(x) = \emptyset$ for all $x \notin A$. A soft set over X can be followed by the set of ordered pairs: $\mathcal{K}_A = \{(x, \mathcal{K}_A(x)) : x \in E, \mathcal{K}_A(x) \in \mathcal{P}(X)\}$. Clearly, a soft set is not a set.*

3. Quotient KU-algebras

Through out the text by K we mean a KU-algebra unless otherwise specified.

Definition 3.1. *A soft set $(\mathcal{K}, E)$ over K is called an intersection soft ideal of K if it satisfies:*

$$\mathcal{K}(x) \subseteq \mathcal{K}(1) \quad \forall x \in K, \tag{3.1}$$

$$\mathcal{K}(y \circ x) \cap \mathcal{K}(y) \subseteq \mathcal{K}(x) \quad \forall x, y \in K. \tag{3.2}$$

Lemma 3.2. *Every intersection soft ideal $(\mathcal{K}, E)$ of K satisfies:*

$$x \leq y \Rightarrow \mathcal{K}(y) \subseteq \mathcal{K}(x) \;\; \forall\, x, y \in K, \tag{3.3}$$

$$y \circ x \leq z \Rightarrow \mathcal{K}(y) \cap \mathcal{K}(z) \subseteq \mathcal{K}(x) \;\; \forall\, x, y, z \in K. \tag{3.4}$$

Proof. For any $x, y, z \in K$ if $x \leq y$, then $y \circ x = 1$. By Definition 3.1,

$$\mathcal{K}(y \circ x) \cap \mathcal{K}(y) \subseteq \mathcal{K}(x) \Rightarrow \mathcal{K}(1) \cap \mathcal{K}(y) \subseteq \mathcal{K}(x) \Rightarrow \mathcal{K}(y) \subseteq \mathcal{K}(x).$$

Hence if $y \circ x \leq z \Rightarrow \mathcal{K}(z) \subseteq \mathcal{K}(y \circ x) \Rightarrow \mathcal{K}(z) \cap \mathcal{K}(y) \subseteq \mathcal{K}(y \circ x) \cap \mathcal{K}(y) \subseteq \mathcal{K}(x)$. □

Let $(\mathcal{K}, E)$ be an intersection soft ideal of K. For any $x, y \in K$, we define a binary operation "∇" on K as follows:

$$x \nabla y \Leftrightarrow \mathcal{K}(y \circ x) = \mathcal{K}(x \circ y) = \mathcal{K}(1). \tag{3.5}$$

Lemma 3.3. *The relation "∇" is an equivalence relation on K.*

Proof. By the definition of the relation, "∇" is reflexive and symmetric. Further $x, y, z \in K$ is such that $x\nabla y$ and $y\nabla z$. Then $\mathcal{K}(y \circ x) = \mathcal{K}(x \circ y) = \mathcal{K}(1)$ and $\mathcal{K}(z \circ y) = \mathcal{K}(y \circ z) = \mathcal{K}(1)$. Since $(y \circ x) \circ (z \circ x) \leq z \circ y$ *and* $(y \circ z) \circ (x \circ z) \leq x \circ y$, it follows from (3.4) that

$$K(1) = \mathcal{K}(y \circ x) \cap \mathcal{K}(z \circ y) \subseteq \mathcal{K}(z \circ x) \subseteq \mathcal{K}(1)$$

and

$$K(1) = \mathcal{K}(y \circ z) \cap \mathcal{K}(x \circ y) \subseteq \mathcal{K}(x \circ z) \subseteq \mathcal{K}(1).$$

Hence $\mathcal{K}(z \circ x) = \mathcal{K}(1) = \mathcal{K}(x \circ z)$, and so $x\nabla z$. Therefore "∇" is an equivalence relation on K. □

Lemma 3.4. *For any $x, y \in K$, if $x\nabla y$ then $z \circ x\nabla z \circ y$ and $x \circ z\nabla y \circ z$ for all $z \in K$.*

Proof. Let $x, y, z \in K$ be such that $x\nabla y$. Then $\mathcal{K}(y \circ x) = \mathcal{K}(x \circ y) = \mathcal{K}(1)$. Since $(z \circ y) \circ (z \circ x) \leq y \circ x$ *and* $(z \circ x) \circ (z \circ y) \leq x \circ y$, it follows from (3.3) that

$$\mathcal{K}(y \circ x) = \mathcal{K}(1) \subseteq \mathcal{K}((z \circ y) \circ (z \circ x)) \subseteq K(1)$$

and

$$K(1) = \mathcal{K}(x \circ y) \subseteq \mathcal{K}((z \circ x) \circ (z \circ y)) \subseteq \mathcal{K}(1).$$

In this way we get $\mathcal{K}((z \circ y) \circ (z \circ x)) = \mathcal{K}(1) = \mathcal{K}((z \circ x) \circ (z \circ y))$, and so $z \circ x\nabla z \circ y$. Similarly, we have $x \circ z\nabla y \circ z$. □

The following result follows from Lemma 3.4 and the transitivity of ∇.

Lemma 3.5. *For any $x, y, u, v \in K$, if $x\nabla y$ and $u\nabla v$ then $u \circ x\nabla v \circ y$.*

Summarizing the lemmas above, we know that the operation "∇" is a congruence relation on K. Denote by $\mathcal{K}_x$ and $K/\mathcal{K}$ the set of all equivalence classes containing x and the set of all equivalence classes of K, respectively, that is,

$$\mathcal{K}_x := \{y \in K | y\nabla x\} \text{ and } K/\mathcal{K} := \{\mathcal{K}_x | x \in X\}.$$

Define a binary operation $\diamond$ on $X/\mathcal{K}$ as follows:

$$\mathcal{K}_x \diamond \mathcal{K}_y = \mathcal{K}_{x \circ y}$$

for all $\mathcal{K}_x, \mathcal{K}_y \in K/\mathcal{K}$. Then this operation is well-defined by Lemma 3.5.

Theorem 3.6. *If $(\mathcal{K}, E)$ is an intersection soft ideal of K, then the quotient algebra $Q/\mathcal{K} := (K/\mathcal{K}, \diamond, \mathcal{K}_1)$ is a KU-algebra.*

Proof. Let $\mathcal{K}_x, \mathcal{K}_y, \mathcal{K}_z \in K/\mathcal{K}$. Then

$$\begin{aligned}
&(\mathcal{K}_y \diamond \mathcal{K}_z) \diamond ((\mathcal{K}_z \diamond \mathcal{K}_x) \diamond (\mathcal{K}_y \diamond \mathcal{K}_x)) = \mathcal{K}_{(y \circ z) \circ ((z \circ x) \circ (y \circ x))} = \mathcal{K}_1,\\
&\mathcal{K}_x \diamond \mathcal{K}_1 = \mathcal{K}_{x \circ 1} = \mathcal{K}_1,\\
&\mathcal{K}_1 \diamond \mathcal{K}_x = \mathcal{K}_{1 \circ x} = \mathcal{K}_x \quad \forall x \in K.
\end{aligned}$$

If $\mathcal{K}_x \diamond \mathcal{K}_y = \mathcal{K}_1$ and $\mathcal{K}_y \diamond \mathcal{K}_x = \mathcal{K}_1$, then $\mathcal{K}_{x \circ y} = \mathcal{K}_1 = \mathcal{K}_{y \circ x}$. Hence $\mathcal{K}(y \circ x) = \mathcal{K}(1) = \mathcal{K}(x \circ y)$, and so $x\nabla y$. Thus $\mathcal{K}_x = \mathcal{K}_y$. Therefore $Q/\mathcal{K} := (K/\mathcal{K}, \diamond, \mathcal{K}_1)$ is a KU-algebra. □

Proposition 3.7. *Let $Q := (K, \circ_K, 1_K)$ and $\mathcal{R} := (L, \circ_L, 1_L)$ be KU-algebras and let $\nu : K \to L$ be an epimorphism. If $(\bar{k}, L)$ is an intersection soft ideal of $\mathcal{R}$, then $(\bar{k} \circ \nu, K)$ is an intersection soft ideal of Q.*

Proof. For any $x \in K$, we have

$$(\bar{k} \circ \nu)(x) = \bar{k}(\nu(x)) \subseteq \bar{k}(1_L) = \bar{k}(\nu(1_K)) = (\bar{k} \circ \nu)(1_K)$$

and $(\bar{k} \circ \nu)(x) = \bar{k}(\nu(x)) \supseteq \bar{k}(\nu(x) \circ_L a) \cap \bar{k}(a)$ for any $a \in L$. Let y be any preimage of a under ν. Then

$$\begin{aligned}(\bar{k} \circ \nu)(x) &\supseteq \bar{k}(\nu(x) \circ_L a) \cap \bar{k}(a) = \bar{k}(\nu(x) \circ_L \nu(y)) \cap \bar{k}(\nu(y)) \\ &= \bar{k}(\nu(x \circ_K y)) \cap \bar{k}(\nu(y)) = (\bar{k} \circ \nu)(x \circ_K y) \cap (\bar{k} \circ \nu)(y).\end{aligned}$$

Therefore $(\bar{k} \circ \nu, K)$ is an intersection soft ideal of K. □

The fundamental homomorphism theorem for quotient KU-algebras is as follows:

Proposition 3.8. *Let $Q := (K, \circ_K, 1_K)$ and $\mathcal{R} := (L, \circ_L, 1_L)$ be KU-algebras and let $\nu : K \to L$ be an epimorphism. If $(\bar{k}, L)$ is an intersection soft ideal of L, then the quotient algebra $Q/(\bar{k} \circ \nu) := (K/(\bar{k} \circ \nu), \diamond_K, (\bar{k} \circ \nu)_{1_K})$ is isomorphic to the quotient algebra $\mathcal{R}/\bar{k} := (L/\bar{k}, \diamond_L, \bar{k}_{1_L})$.*

Proof. Using Theorem 3.6 and Proposition 3.7, $Q/(\bar{k} \circ \nu) := (K/(\bar{k} \circ \nu), \diamond_K, (\bar{k} \circ \nu)_{1_K})$ and $\mathcal{R}/\bar{k} := (L/\bar{k}, \diamond_Y, \bar{k}_{1_L})$ are KU-algebras. Define a function

$$\zeta : K/(\bar{k} \circ \nu) \to L/\bar{k}, (\bar{k} \circ \nu)_x \to \bar{k}_{\nu(x)}$$

for all $x \in K$, the function ζ is well-defined. Also for $(\bar{k} \circ \nu)_x = (\bar{k} \circ \nu)_y$ for all $x, y \in K$ we have

$$\begin{aligned}(\bar{k}(\nu(y) \circ_L \nu(x))) &= \bar{k}(\nu(y \circ_K x)) = (\bar{k} \circ \nu)(y \circ_K x) \\ &= (\bar{k} \circ \nu)(1_K) = \bar{k}(\nu(1_K)) = \bar{k}(1_L)\end{aligned}$$

and

$$\begin{aligned}\bar{k}(\nu(y) \circ_L \nu(x)) &= \bar{k}(\nu(y \circ_K x)) = (\bar{k} \circ \nu)(y \circ_K x) \\ &= (\bar{k} \circ \nu)(1_K) = \bar{k}(\nu(1_K)) = \bar{k}(1_L),\end{aligned}$$

and so $\bar{k}_{\nu(x)} = \bar{k}_{\nu(y)}$. For any $(\bar{k} \circ \nu)_x, (\bar{k} \circ \nu)_y \in X/(\bar{k} \circ \nu)$, we have

$$\begin{aligned}\zeta(\bar{k} \circ \nu)_y \diamond_K (\bar{k} \circ \nu)_x &= \zeta(\bar{k} \circ \nu)_{y \circ_K x} = \bar{k}\nu(y \circ_K x) \\ &= \bar{k}_{\nu(y)} \circ_L \nu(x) = \bar{k}_{\nu(y)} \diamond_L \bar{k}_{\nu(x)} = \zeta((\bar{k} \circ \nu)_y) \diamond_L \zeta(\bar{k} \circ \nu)_x.\end{aligned}$$

Hence ζ is a homomorphism. Let $\bar{k}_a \in L/\bar{k}$. Then there exists $x \in K$ such that $\nu(x) = a$ since ν is surjective. Hence $\zeta((\bar{k} \circ \nu)_x) = \bar{k}_{\nu(x)} = \bar{k}_a$, and so ζ is surjective. Let $x, y \in K$ be such that $\bar{k}_{\nu(x)} = \bar{k}_{\nu(y)}$. Then

$$\begin{aligned}(\bar{k} \circ \nu)(y \circ_K x) &= \bar{k}(\nu(y \circ_K x)) = \bar{k}(\nu(y) \circ_L \nu(x)) \\ &= \bar{k}(1_L) = \bar{k}(\nu(1_K)) = (\bar{k} \circ \nu)(1_K)\end{aligned}$$

and

$$\begin{aligned}(\bar{k} \circ \nu)(y \circ_K x) &= \bar{k}(\nu(y \circ_K x)) = \bar{k}(\nu(y) \circ_L \nu(x)) \\ &= \bar{k}(1_L) = \bar{k}(\nu(1_K)) = (\bar{k} \circ \nu)(1_K).\end{aligned}$$

It follows that $(\bar{k} \circ \nu)_x = (\bar{k} \circ \nu)_y$ and that ζ is injective. This completes the proof. □

The homomorphism $\sigma : K \to K/\mathcal{K},\ x \to \mathcal{K}_x$, is called the natural homomorphism of K onto $K/\mathcal{K}$. In Proposition 3.8, if we define natural homomorphisms $\sigma_K : K \to K/\bar{k} \circ \nu$ and $\sigma_L : L \to L/\bar{k}$ then it is easy to show that $\zeta \circ \sigma_K = \sigma_L \circ \nu$, i.e., the following diagram commutes.

$$\begin{tikzcd} K \arrow[r, "\nu"] \arrow[d, "\sigma_K"'] & L \arrow[d, "\sigma_L"] \\ K/(\bar{k} \circ \nu) \arrow[r, "\zeta"'] & L/\bar{k} \end{tikzcd}$$

Here is a characterization given for a commutative quotient KU-algebra.

Theorem 3.9. *Let $(\mathcal{K}, K)$ be an intersection soft ideal of a KU-algebra Q. Then the quotient KU-algebra $Q/\mathcal{K} := (K/\mathcal{K}, \diamond, \mathcal{K}_1)$ is commutative if and only if the following assertion is valid:*

$$\mathcal{K}(y \circ x) = \mathcal{K}(((x \circ y) \circ y) \circ x) \ \ \forall x, y \in K. \tag{3.6}$$

Proof. Let $Q/\mathcal{K} := (K/\mathcal{K}, \diamond, \mathcal{K}_1)$ is a commutative KU-algebra and let $x, y \in K$.
Then $\mathcal{K}_x \vee \mathcal{K}_y = \mathcal{K}_y \vee \mathcal{K}_x$, and so

$$\mathcal{K}_{(x\circ y)\circ x} = \mathcal{K}_x \diamond (\mathcal{K}_x \diamond \mathcal{K}_y) = \mathcal{K}_y \diamond (\mathcal{K}_y \diamond \mathcal{K}_x) = \mathcal{K}_{(x\circ y)\circ y}$$

It follows from Lemma 2.3(3) and (3.1) that

$$\mathcal{K}(y \circ x) \supseteq \mathcal{K}(1) = \mathcal{K}(((x \circ y) \circ y) \circ ((y \circ x) \circ x)) = \mathcal{K}((y \circ x) \circ (((x \circ y) \circ y \circ x)))$$

and from (3.2) that

$$\mathcal{K}(y \circ x) = \mathcal{K}((y \circ x) \circ (((x \circ y) \circ y) \circ x)) \cap \mathcal{K}(y \circ x) \supseteq \mathcal{K}(((x \circ y) \circ y) \circ x).$$

Since $y \circ x \leq ((x \circ y) \circ y) \circ x$, we have $\mathcal{K}(y \circ x) \supseteq \mathcal{K}(((x \circ y) \circ y) \circ x)$ by using (3.3). Therefore $\mathcal{K}(y \circ x) = \mathcal{K}(((x \circ y) \circ y) \circ x)$ for all $x, y \in K$.

Conversely, let $(\mathcal{K}, K)$ be an intersection soft ideal of a KU-algebra Q which satisfies (3.6). For any $x, y \in K$, let $t := y \circ x$. Then $(y \circ (t \circ x)) = 1$, and so $\mathcal{K}(y \circ (t \circ x)) = \mathcal{K}(1)$. If we put $a := t \circ x$, then

$$\begin{aligned} &\mathcal{K}(((((y \circ x) \circ x) \circ y) \circ y) \circ ((y \circ x) \circ x)) = \mathcal{K}((((t \circ x) \circ y) \circ y) \circ (t \circ t)) \\ &= \mathcal{K}(((a \circ y) \circ y) \circ a) = \mathcal{K}(y \circ a) = \mathcal{K}(y \circ (t \circ x)) = \mathcal{K}(1) \end{aligned}$$

by using (3.6), Lemma 2.3(3), Lemma 2.3(1) and Definition 2.1(1) Note that,

$$\begin{aligned} &(((((y \circ x) \circ x) \circ y) \circ y) \circ ((y \circ x) \circ x)) \circ (((x \circ y) \circ y) \circ ((y \circ x) \circ x)) \\ &\leq ((x \circ y) \circ y) \circ ((((y \circ x) \circ x) \circ y) \circ y) \leq (((y \circ x) \circ x) \circ y) \circ (x \circ y) \leq x \circ ((y \circ x) \circ x) = 1. \end{aligned}$$

It follows from (3.4) that,

$$\mathcal{K}(1) = \mathcal{K}(((((y \circ x) \circ x) \circ y) \circ y) \circ ((y \circ x) \circ x)) \cap \mathcal{K}(1) \supseteq \mathcal{K}(((x \circ y) \circ y) \circ ((y \circ x) \circ x)).$$

Thus $\mathcal{K}(((x \circ y) \circ y) \circ ((y \circ x) \circ x)) = \mathcal{K}(1)$. Similarly, we have,

$$\mathcal{K}(((y \circ x) \circ x) \circ ((x \circ y) \circ y)) = \mathcal{K}(1).$$

Hence $\mathcal{K}_y \vee \mathcal{K}_x = \mathcal{K}_x \diamond (\mathcal{K}_x \diamond \mathcal{K}_y) = \mathcal{K}_{(x\circ_y)\circ x} = \mathcal{K}_{(x\circ y)\circ y} = \mathcal{K}_y \diamond (\mathcal{K}_{y\diamond\mathcal{K}_x}) = \mathcal{K}_{x\vee\mathcal{K}_y}$, and therefore $Q/\mathcal{K} := (X/\mathcal{K}, \diamond, \mathcal{K}_1)$ is a commutative KU-algebra. $\square$

Here is a characterization given for positive implicative quotient KU-algebra.

Theorem 3.10. *Let $(\mathcal{K}, K)$ be an intersection soft ideal of a KU-algebra Q. Then the quotient KU-algebra $Q/\mathcal{K} := (K/\mathcal{K}, \diamond, \mathcal{K}_1)$ is positive implicative if and only if the following assertion is valid:*

$$\mathcal{K}(y \circ x) = \mathcal{K}(y \circ (y \circ x)) \;\; \forall\, x, y \in K. \tag{3.7}$$

Proof. Assume that $Q/\mathcal{K} := (X/\mathcal{K}, \diamond, \mathcal{K}_1)$ is a positive implicative KU-algebra and let $x, y \in K$. Then $\mathcal{K}_{y \circ x} = \mathcal{K}_x \diamond \mathcal{K}_y = (\mathcal{K}_x \diamond \mathcal{K}_y) \diamond \mathcal{K}_y = \mathcal{K}_{(x \circ_y) \circ_y}$, and so $\mathcal{K}((y \circ (y \circ x)) \circ (y \circ x)) = \mathcal{K}(1)$. Since $(\mathcal{K}, K)$ is an intersection soft ideal of Q, it follows from (3.2) and (3.1) that

$$\begin{aligned}\mathcal{K}(y \circ x) &\supseteq \mathcal{K}((y \circ (y \circ x) \circ (y \circ x))) \cap \mathcal{K}(y \circ (y \circ x)) \\ &= \mathcal{K}(1) \cap \mathcal{K}(y \circ (y \circ x)) = \mathcal{K}(y \circ (y \circ x).\end{aligned}$$

Since $(y \circ (y \circ x)) \leq y \circ x$, we have $\mathcal{K}(y \circ x) \subseteq \mathcal{K}(y \circ (y \circ x))$ by (3.3). Therefore the equality (3.7) holds. Conversely, assume that $(\mathcal{K}, K)$ is an intersection soft ideal of a KU-algebra Q which satisfies (3.7). Let $x, y \in X$. Letting $b := y \circ (y \circ x)$ implies that $y \circ (y \circ (b \circ x)) = 1$, and hence $\mathcal{K}(1) = \mathcal{K}(y \circ (y \circ (b \circ x))) = \mathcal{K}(y \circ (b \circ x))$ by using (3.7), that is, $\mathcal{K}((y \circ (y \circ x)) \circ (y \circ x)) = \mathcal{K}(1)$. Since $(y \circ x) \circ (y \circ (y \circ x)) = 1$, we have $\mathcal{K}(y \circ x) \circ (y \circ (y \circ x)) = \mathcal{K}(1)$. Thus $\mathcal{K}_x \diamond \mathcal{K}_y = \mathcal{K}_{x \circ_L} = \mathcal{K}_{(x \circ_y) \circ_y} = (\mathcal{K}_x \diamond \mathcal{K}_y) \diamond \mathcal{K}_y$.
Therefore $Q/\mathcal{K} := (K/\mathcal{K}, \diamond, \mathcal{K}_1)$ is a positive implicative KU-algebra. □

A KU-algebra is implicative if and only if it is both positive implicative and commutative. In support of this statement we have the following corollary.

Corollary 3.11. *Let $(\mathcal{K}, K)$ be an intersection soft ideal of a KU-algebra Q. Then the quotient KU-algebra $Q/\mathcal{K} := (K/\mathcal{K}, \diamond, \mathcal{K}_1)$ is implicative if and only if it satisfies two conditions* (3.6) *and* (3.7).

4. Conclusions

In this paper, we have studied and discussed quotient structures of KU-algebras by applying intersection soft ideals. Results based on Quotient KU-algebras and characterization of commutative quotient KU-algebras, implicative quotient KU-algebras and positive implicative quotient KU-algebras are given. The fundamental homomorphism theorem of quotient KU-algebras in these cases has been discussed and examples are given in support. Intersection soft ideals and their quotients on other logical algebras can be study in the future based on the same line of motivation.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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