



Research article

Construction of some algebras of logics by using intuitionistic fuzzy filters on hoops

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Abstract: In this paper, we define the notions of intuitionistic fuzzy filters and intuitionistic fuzzy implicative (positive implicative, fantastic) filters on hoops. Then we show that all intuitionistic fuzzy filters make a bounded distributive lattice. Also, by using intuitionistic fuzzy filters we introduce a relation on hoops and show that it is a congruence relation, then we prove that the algebraic structure made by it is a hoop. Finally, we investigate the conditions that quotient structure will be different algebras of logics such as Brouwerian semilattice, Heyting algebra and Wajesberg hoop.

Keywords: Hoop; intuitionistic fuzzy (implicative, positive implicative, fantastic) filter; Brouwerian semilattice; Heyting algebra; Wajesberg hoop

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1. Introduction

Hoops are algebraic structure which are introduced by B. Bosbach in [5, 6] are naturally ordered commutative residuated integral monoids. In the last years, some mathematician studied hoop theory in different fields [1–3, 5–10, 17, 19]. Most of these results have a very deep relation with fuzzy logic. Particularly, by using of some theorems and notions of finite basic hoops, in [1] the authors could find a short proof for completeness theorem for propositional basic logic, which is introduced by Hájek in [11]. BL-algebras, the familiar cases of hoops, are the algebraic structures corresponding to basic

logic. In algebra of logics, there are some sub-algebras that are very important and have a fundamental role in these algebraic structures. They are very similar to normal subgroups in group theory and ideals in ring theory which we called them filters and by using these notion we can introduce a congruence relation on algebraic structures and study the quotient structure that is made by them. Kondo, in [14], introduced different kinds of filters such as implicative, positive implicative and fantastic filters of hoops and investigated some properties of them. Borzooei and Aaly Kologani in [7], investigated these filters deeply and they introduced some equivalent characterizations of these filters on hoops. Also, they studied the relation among these filters and they showed some equivalent characterizations of these filters.

Zadeh in [21] introduced the notion of fuzzy sets and different kinds of operations on them. After that mathematicians studied on them and applied it to diverse fields. Actually, fuzzy mathematics have reached by studying of fuzzy subsets and their application to mathematical contexts. Nowadays, fuzzy algebra is an important branch of mathematics and mathematicians studied it in different fields. For example, Rosenfeld in [18] studied fuzzy sub-groups in 1971. After using the concept of fuzzy sets to group theory and defined the notion of fuzzy subgroups, in [21] by Rosenfeld, the different fuzzy algebraic concepts has been growing very fast [12, 13, 16] and applied in other algebraic structures such as lattices, semigroups, rings, ideals, modules and vector spaces. Moreover, the concepts related to fuzzy sets have been used in various fields, including its use in fuzzy graphs and its application in decision theory. Borzooei and Aaly Kologani in [8], studied the notions of fuzzy filter of hoops and the relation among them and characterized some properties of them. Also, they defined a congruence relation on hoops by a fuzzy filter and proved that the quotient structure of this relation is a hoop.

Atanassov for the first time introduced the term, an intuitionistic fuzzy set [4] that is an extended form of a fuzzy set. These are the sets containing elements having degrees of membership and non-membership. Intuitionistic fuzzy sets are more adaptable and real intending to the uncertainty and vagueness than the conventional fuzzy sets. The foremost critical property of intuitionistic fuzzy sets not shared by the fuzzy sets is that modular operators can be characterized over intuitionistic fuzzy sets. The intuitionistic fuzzy sets have basically higher depicting conceivable outcomes than fuzzy sets. Also, there are a lot of applications of intuitionistic fuzzy sets in decision making, pattern recognition, medical diagnosis, neural models, image processing, market prediction, color region extraction, and others. In the last years, some mathematician studied intuitionistic fuzzy sets in different fields [15, 20]. In decision-making problems, the use of fuzzy approaches is ubiquitous. The purpose of these intuitionistic fuzzy sets is, to provide a new approach with useful mathematical tools to address the fundamental problem of decision-making. The generality of the fuzzy set is given special importance, illustrating how many interesting decision-making problems can be formulated as a problem of fuzzy sets. These applied contexts provide solid evidence of the wide applications of fuzzy sets approach to model and research decision-making problems.

Now, in the following we define the notions of intuitionistic fuzzy filters and intuitionistic fuzzy implicative (positive implicative, fantastic) filters on hoops. Then we show that all intuitionistic fuzzy filters make a bounded distributive lattice. Also, by using intuitionistic fuzzy filters we introduce a relation on hoops and show that it is a congruence relation, then we prove that the algebraic structure made by it is a hoop. Finally, we investigate the conditions that quotient structure will be different algebras of logics such as Brouwerian semilattice, Heyting algebra and Wajesberg hoop.

2. Preliminaries

In this section, we refer to the basic concepts and properties required in the field of hoop and fuzzy sets that we will use in the following sections.

A hoop is an algebraic structure $(H, *, \rightarrow\!, 1)$ such that, for any $d, s, q \in H$:

- (HP1) $(H, *, 1)$ is a commutative monoid,
- (HP2) $d \rightarrow\! d = 1$,
- (HP3) $(d * s) \rightarrow\! q = d \rightarrow\! (s \rightarrow\! q)$,
- (HP4) $d * (d \rightarrow\! s) = s * (s \rightarrow\! d)$.

On hoop H , we define the relation \leq by $d \leq s$ iff $d \rightarrow\! s = 1$. Obviously, (H, \leq) is a poset. A hoop H is said to be bounded if H has a least element such as 0, it means that for all $d \in H$, we have $0 \leq d$. Define $d^0 = 1$, $d^n = d^{n-1} * d$, for any $n \in \mathbb{N}$. If H is bounded, then we can introduce a unary operation “ $'$ ” on H such that $d' = d \rightarrow\! 0$, for any $d \in H$. The bounded hoop H is said to have the double negation property, or (DNP), for short if $(d')' = d$, for any $d \in H$ (see [1]).

The next proposition provides some properties of hoop.

Proposition 2.1. [5, 6] Suppose $(H, *, \rightarrow\!, 1)$ is a hoop and $d, s, q \in H$. Then:

- (i) (H, \leq) is a \wedge -semilattice with $d \wedge s = d * (d \rightarrow\! s)$.
- (ii) $d * s \leq q$ iff $d \leq s \rightarrow\! q$.
- (iii) $d * s \leq d, s$ and $d^n \leq d$, for any $n \in \mathbb{N}$.
- (iv) $d \leq s \rightarrow\! d$.
- (v) $1 \rightarrow\! d = d$.
- (vi) $d \rightarrow\! 1 = 1$.
- (vii) $d * (d \rightarrow\! s) \leq s$.
- (viii) $d \leq (d \rightarrow\! s) \rightarrow\! s$.
- (ix) $d \rightarrow\! s \leq (s \rightarrow\! q) \rightarrow\! (d \rightarrow\! q)$.
- (x) $(d \rightarrow\! s) * (s \rightarrow\! q) \leq d \rightarrow\! q$.
- (xi) $d \leq s$ implies $d * q \leq s * q$.
- (xii) $d \leq s$ implies $q \rightarrow\! d \leq q \rightarrow\! s$.
- (xiii) $d \leq s$ implies $s \rightarrow\! q \leq d \rightarrow\! q$.
- (xiv) $((d \rightarrow\! s) \rightarrow\! s) \rightarrow\! s = d \rightarrow\! s$.

Proposition 2.2. [5, 6] Suppose H is a bounded hoop and $d, s \in H$. Then:

- (i) $d \leq d''$.
- (ii) $d * d' = 0$.
- (iii) $d' \leq d \rightarrow\! s$.

Proposition 2.3. [10] Let H be a hoop and $d, s \in H$. Define a binary operation \sqcup on H as $d \sqcup s = ((d \rightarrow\! s) \rightarrow\! s) \wedge ((s \rightarrow\! d) \rightarrow\! d)$. Then for any $d, s, q \in H$, the next conditions are equivalent:

- (i) \sqcup is associative,
- (ii) $d \leq s$ implies $d \sqcup q \leq s \sqcup q$,
- (iii) $d \sqcup (s \wedge q) \leq (d \sqcup s) \wedge (d \sqcup q)$,
- (iv) \sqcup is the join operation on H .

Remark 2.4. A hoop H is said to be a \sqcup -hoop if \sqcup satisfies in one of the conditions of Proposition 2.3. Note that any \sqcup -hoop (H, \sqcup, \wedge) is a distributive lattice (see [10]).

A non-empty subset K of H is said to be a filter of H if (F1): $d, s \in K$ implies $d * s \in K$, (F2): $d \in K$ and $d \leq s$ imply $s \in K$, for any $d, s \in H$. This definition is equal with, if $1 \in K$ and if $d, d \rightarrow s \in K$, then $s \in K$, for any $d, s \in H$. We use $\mathcal{F}(H)$ to denote the set of all filters of H . Clearly, $1 \in K$, for all $K \in \mathcal{F}(H)$. A proper filter is a filter which is not equal to H . Obviously, a filter is proper iff $0 \notin H$ if H is a bounded hoop. Assume $\emptyset \neq K \subseteq H$ such that $1 \in K$. Then K is called an implicative filter of H if, $d \rightarrow ((s \rightarrow q) \rightarrow s) \in K$ and $d \in K$ imply $s \in K$, for any $d, s, q \in H$, is called a positive implicative filter of H if, $(d * s) \rightarrow q \in K$ and $d \rightarrow s \in K$ imply $d \rightarrow q \in K$, for any $d, s, q \in H$, is called a fantastic filter of H if $q \rightarrow (s \rightarrow d) \in K$ and $q \in K$ imply $((d \rightarrow s) \rightarrow s) \rightarrow d \in K$, for any $d, s, q \in H$ (see [10, 14]).

Let H be a set. A fuzzy set ς on H is a map $\varsigma : H \rightarrow [0, 1]$. Let $\varrho_B, \varsigma_B : H \rightarrow [0, 1]$ be two fuzzy sets on H and mapping $\mathcal{B} : H \rightarrow [0, 1] \times [0, 1]$ is defined by $\mathcal{B}(d) = (\varsigma_B(d), \varrho_B(d))$, for any $d \in H$. Then $\mathcal{B} = (\varsigma_B, \varrho_B)$ is called an intuitionistic fuzzy set or an IF-set of H if $\varsigma_B(x) + \varrho_B(d) \leq 1$, for all $d \in H$ or denoted by $\mathcal{B} = \{\langle d, \varsigma_B(d), \varrho_B(d) \rangle \mid d \in H\}$. The family of all intuitionistic fuzzy sets on H will be denoted by $\mathcal{IFS}(H)$. Let $\mathcal{B} = (\varsigma_B, \varrho_B), \mathcal{C} = (\varsigma_C, \varrho_C) \in \mathcal{IFS}(H)$. Then we define $\mathcal{B} \cap \mathcal{C} = (\varsigma_B \wedge \varsigma_C, \varrho_B \vee \varrho_C)$, $\mathcal{B} \cup \mathcal{C} = (\varsigma_B \vee \varsigma_C, \varrho_B \wedge \varrho_C)$ and $\mathcal{B} \subseteq \mathcal{C}$ iff $\varsigma_B \leq \varsigma_C, \varrho_B \geq \varrho_C$. Let $\mathcal{B} = (\varsigma_B, \varrho_B)$ and $\mathcal{C} = (\varsigma_C, \varrho_C)$ be two IF-sets on H . Then, for any $d \in H$, we define a relation between them as follows:

$$\mathcal{B} \leq \mathcal{C} \text{ iff } (\varsigma_B(d) < \varsigma_C(d)) \text{ or } (\varsigma_B(d) = \varsigma_C(d) \text{ and } \varrho_B(d) \leq \varrho_C(d))$$

A fuzzy set ς on hoop H is called a fuzzy filter of H if for all $d, s \in H$, $\varsigma(d) \leq \varsigma(1)$ and

$$\varsigma(d) \wedge \varsigma(d \rightarrow s) = \min\{\varsigma(d), \varsigma(d \rightarrow s)\} \leq \varsigma(s) \text{ (See [8])}$$

Proposition 2.5. [8] Let ς be a fuzzy filter on hoop H . Then, for any $d, s \in H$, $d \leq s$ implies $\varsigma(d) \leq \varsigma(s)$.

Theorem 2.6. [8] Let ς be a fuzzy set on hoop H and for $r \in [0, 1]$ $\varsigma_r = \{d \in H \mid \varsigma(d) \geq r\}$. Then ς is a fuzzy filter of H iff for any $r \in [0, 1]$, $\varsigma_r \neq \emptyset$ is a filter of H .

Theorem 2.7. [8] Let ς be a fuzzy filter on hoop H and fuzzy relation \approx_ς on hoop H is defined by

$$d \approx_\varsigma s \text{ iff } \varsigma(d \rightarrow s) \wedge \varsigma(s \rightarrow d) = \varsigma(1), \text{ for any } d, s \in H.$$

Then \approx_ς is a congruence relation on H .

Theorem 2.8. [8] Suppose H is a hoop and $\frac{H}{\approx_\varsigma} = \{[e]_\varsigma \mid e \in H\}$. Define the operations \otimes and \hookrightarrow on $\frac{H}{\approx_\varsigma}$ as follows:

$$[e]_\varsigma \otimes [k]_\varsigma = [e * k]_\varsigma \text{ and } [e]_\varsigma \hookrightarrow [k]_\varsigma = [e \rightarrow k]_\varsigma$$

Then $(\frac{H}{\approx_\varsigma}, \otimes, \hookrightarrow, [1]_\varsigma)$ is a hoop.

Notation. In the following, we will consider in this article H as a hoop and ς and ϱ as fuzzy sets. Moreover, the set of all fuzzy filters of H and anti-fuzzy filter of H are denoted by $FF(H)$ and $AFF(H)$, respectively.

3. Construction of hoops by anti-fuzzy filters

In this section, the concept of anti-fuzzy filter on hoop H is defined and some related results are investigated.

A complement of ς is the fuzzy set ς^c which is defined by, $\varsigma^c(d) = 1 - \varsigma(d)$, for any $d \in H$.

Definition 3.1. ϱ is called an anti-fuzzy filter of H if for any $d, s \in H$:

(AFF1) $\varrho(d * s) \leq \max\{\varrho(d), \varrho(s)\} = \varrho(d) \vee \varrho(s)$,

(AFF2) if $d \leq s$, then $\varrho(s) \leq \varrho(d)$.

Example 3.2. Let $H = \{0, e, k, 1\}$ be a chain such that $0 \leq e \leq k \leq 1$. Define the operations $*$ and \Rightarrow on H as follows:

\Rightarrow	0	e	k	1	*	0	e	k	1
0	1	1	1	1	0	0	0	0	0
e	e	1	1	1	e	0	0	e	e
k	0	e	1	1	k	0	e	k	k
1	0	e	k	1	1	0	e	k	1

Then $(H, *, \Rightarrow, 0, 1)$ is a bounded hoop. Define $\varrho : H \rightarrow H$ such that $\varrho(1) = 0$, $\varrho(0) = \varrho(e) = 0.5$ and $\varrho(k) = 0.3$. Then ϱ is an anti-fuzzy filter of H .

Remark 3.3. (1) The following statements hold:

(i) $\varsigma \in FF(H)$ iff $\varsigma^c \in AFF(H)$.

(ii) $\varrho \in AFF(H)$ iff $\varrho^c \in FF(H)$.

(2) It is easy to see that $\varsigma(d) \leq \varsigma(1)(\varrho(1) \leq \varrho(d))$ and $\varsigma(0) \leq \varsigma(d)(\varrho(d) \leq \varrho(0))$, for any $d \in H$.

Theorem 3.4. Suppose for any $r \in [0, 1]$, we have $\varrho_r = \{d \in H \mid \varrho(d) \leq r\}$. Then $\varrho \in AFF(H)$ iff for any $r \in [0, 1]$, $\varrho_r \neq \emptyset$ is a filter of H .

Proof. The proof is clear. \square

In the next proposition we investigate some properties of anti-fuzzy filters of hoops.

Proposition 3.5. Assume $\varrho(1) \leq \varrho(d)$, for all $d \in H$. Then for all $d, s, q \in H$, the next statements are equivalent:

(i) $\varrho \in AFF(H)$,

(ii) If $d \leq s \Rightarrow q$, then $\varrho(q) \leq \varrho(d) \vee \varrho(s)$,

(iii) $\varrho(d \Rightarrow q) \leq \varrho(d \Rightarrow s) \vee \varrho(s \Rightarrow q)$,

(iv) $\varrho(s * q) \leq \varrho(d * q) \vee \varrho(d \Rightarrow s)$,

(v) $\varrho(d \Rightarrow q) \leq \varrho((d \Rightarrow s) \Rightarrow q) \vee \varrho(s)$.

Proof. (i) \Rightarrow (ii) Let $\varrho \in AFF(H)$ and $d, s, q \in H$ such that $d \leq s \Rightarrow q$. Then $\varrho(s \Rightarrow q) \leq \varrho(d)$. By Proposition 2.1(vii), $s * (s \Rightarrow q) \leq q$, since $\varrho \in AFF(H)$, we have $\varrho(q) \leq \varrho(s * (s \Rightarrow q)) \leq \varrho(s) \vee \varrho(s \Rightarrow q)$. Hence, $\varrho(q) \leq \varrho(s) \vee \varrho(d)$.

(ii) \Rightarrow (iii) By Proposition 2.1(ix), $d \Rightarrow s \leq (s \Rightarrow q) \Rightarrow (d \Rightarrow q)$. Then by (ii),

$$\varrho(d \Rightarrow q) \leq \varrho(s \Rightarrow q) \vee \varrho(d \Rightarrow s).$$

(iii) \Rightarrow (i) Let $d \leq s$, then $d \rightarrow s = 1$. Thus, by (iii) and assumption,

$$\varrho(s) = \varrho(1 \rightarrow s) \leq \varrho(1 \rightarrow d) \vee \varrho(d \rightarrow s) = \varrho(d) \vee \varrho(1) = \varrho(d).$$

Also, from $d * s \leq d * s$, by Proposition 2.1(ii), $d \leq s \rightarrow (d * s)$, then $\varrho(s \rightarrow (d * s)) \leq \varrho(d)$. Let $d = 1$ and $q = d * s$ in (iii). Then

$$\varrho(d * s) = \varrho(1 \rightarrow (d * s)) \leq \varrho(1 \rightarrow s) \vee \varrho(s \rightarrow (d * s)) \leq \varrho(s) \vee \varrho(d).$$

Hence, $\varrho \in \text{AFF}(H)$.

(ii) \Rightarrow (iv) By Proposition 2.1(vii) and (xi), $(q * d) * (d \rightarrow s) \leq q * s$. Then $q * d \leq (d \rightarrow s) \rightarrow (q * s)$. Thus, by (ii), $\varrho(q * s) \leq \varrho(q * d) \vee \varrho(d \rightarrow s)$.

(iv) \Rightarrow (v) Let $d, s, q \in H$. By Proposition 2.1(iii) and (iv), $d * s \leq s \leq d \rightarrow s$. Then by Proposition 2.1(xiii) and (HP3),

$$(d \rightarrow s) \rightarrow q \leq (d * s) \rightarrow q = s \rightarrow (d \rightarrow q).$$

Thus, by Proposition 2.1(vii), $s * ((d \rightarrow s) \rightarrow q) \leq d \rightarrow q$. Also, if $d \leq s$, let $q = 1$ in (iv), then by assumption $\varrho(s) = \varrho(1 * s) \leq \varrho(d * 1) \vee \varrho(d \rightarrow s) = \varrho(d)$, and so $\varrho(d \rightarrow q) \leq \varrho(s * ((d \rightarrow s) \rightarrow q))$. Moreover, if $q = d$ and $d = 1$ in (iv), then $\varrho(d * s) \leq \varrho(d) \vee \varrho(s)$. Hence,

$$\varrho(d \rightarrow q) \leq \varrho(s * ((d \rightarrow s) \rightarrow q)) \leq \varrho(s) \vee \varrho((d \rightarrow s) \rightarrow q)).$$

(v) \Rightarrow (ii) Let $d \leq s$. Then $d \rightarrow s = 1$. It is enough to choose $d = 1, q = s$ and $s = d$ in (v). Thus, by assumption,

$$\varrho(s) = \varrho(1 \rightarrow s) \leq \varrho((1 \rightarrow d) \rightarrow s) \vee \varrho(d) = \varrho(d \rightarrow s) \vee \varrho(d) = \varrho(1) \vee \varrho(d) = \varrho(d).$$

So, if $d \leq s \rightarrow q$, then $\varrho(s \rightarrow q) \leq \varrho(d)$. Now, let $d = 1$ in (v), then

$$\varrho(q) = \varrho(1 \rightarrow q) \leq \varrho((1 \rightarrow s) \rightarrow q) \vee \varrho(s) = \varrho(s \rightarrow q) \vee \varrho(s) \leq \varrho(d) \vee \varrho(s).$$

□

Theorem 3.6. Let $\varrho \in \text{AFF}(H)$ and fuzzy relation \approx_ϱ on H be defined by

$$d \approx_\varrho s \text{ iff } \varrho(d \rightarrow s) \vee \varrho(s \rightarrow d) = \varrho(1), \text{ for any } d, s \in H.$$

Then \approx_ϱ is a congruence relation on H .

Proof. Since $\varrho \in \text{AFF}(H)$, for all $d \in H$, we get

$$\varrho(d \rightarrow d) \vee \varrho(d \rightarrow d) = \varrho(1) \text{ iff } d \approx_\varrho d.$$

Hence \approx_ϱ is reflexive. Clearly, \approx_ϱ is symmetric. Now, suppose $d \approx_\varrho s$ and $s \approx_\varrho q$, for $d, s, q \in H$. Then $\varrho(d \rightarrow s) \vee \varrho(s \rightarrow d) = \varrho(1)$ and $\varrho(s \rightarrow q) \vee \varrho(q \rightarrow s) = \varrho(1)$. Thus,

$$\varrho(d \rightarrow s) \vee \varrho(s \rightarrow d) \vee \varrho(s \rightarrow q) \vee \varrho(q \rightarrow s) = \varrho(1).$$

Moreover, since ϱ is an anti-fuzzy filter, by Propositions 2.1(x), 3.5(iii) and Remark 3.3(2), we have

$$\begin{aligned}\varrho(d \rightarrow q) \vee \varrho(q \rightarrow d) &\leq \varrho((d \rightarrow s) * (s \rightarrow q)) \vee \varrho((q \rightarrow s) * (s \rightarrow d)) \\ &\leq [\varrho(d \rightarrow s) \vee \varrho(s \rightarrow q)] \vee [\varrho(q \rightarrow s) \vee \varrho(s \rightarrow d)] \\ &= \varrho(1).\end{aligned}$$

So, $d \approx_{\varrho} q$ and this means that \approx_{ϱ} is transitive. Hence, \approx_{ϱ} is an equivalence relation. Now, we prove that \approx_{ϱ} is a congruence relation. Let $d, s, q \in H$ such that $d \approx_{\varrho} s$. Then $\varrho(d \rightarrow s) \vee \varrho(s \rightarrow d) = \varrho(1)$. Since $s * q \leq s * q$, by Proposition 2.1(ii) and (xii), $d \rightarrow s \leq d \rightarrow (q \rightarrow (s * q))$. Thus by (HP3), $d \rightarrow s \leq (d * q) \rightarrow (s * q)$. Since ϱ is an anti-fuzzy filter, $\varrho(d \rightarrow s) \geq \varrho((d * q) \rightarrow (s * q))$. By the similar way, $\varrho(s \rightarrow d) \geq \varrho((s * q) \rightarrow (d * q))$. Thus,

$$\varrho(1) = \varrho(d \rightarrow s) \vee \varrho(s \rightarrow d) \geq \varrho((d * q) \rightarrow (s * q)) \vee \varrho((q * s) \rightarrow (q * d)).$$

Hence, $d * q \approx_{\varrho} s * q$. On the other hand, by Proposition 2.1(ix), $d \rightarrow s \leq (s \rightarrow q) \rightarrow (d \rightarrow q)$, for any $d, s, q \in H$. From ϱ is an anti-fuzzy filter, $\varrho(d \rightarrow s) \geq \varrho((s \rightarrow q) \rightarrow (d \rightarrow q))$. By the similar way, $\varrho(s \rightarrow d) \geq \varrho((d \rightarrow q) \rightarrow (s \rightarrow q))$. Then

$$\varrho(1) = \varrho(d \rightarrow s) \vee \varrho(s \rightarrow d) \geq \varrho((s \rightarrow q) \rightarrow (d \rightarrow q)) \vee \varrho((d \rightarrow q) \rightarrow (s \rightarrow q)),$$

and so $d \rightarrow q \approx_{\varrho} s \rightarrow q$. By the similar way, we can see $q \rightarrow d \approx_{\varrho} q \rightarrow s$. Therefore, \approx_{ϱ} is a congruence relation on H . \square

For any $e \in H$, $[e]_{\varrho}$ denotes the equivalence class of e with respect to \approx_{ϱ} . Clearly

$$[e]_{\varrho} = \{d \in H \mid e \approx_{\varrho} d\} = \{d \in H \mid \varrho(e \rightarrow d) \vee \varrho(d \rightarrow e) = \varrho(1)\}.$$

Theorem 3.7. Let $\frac{H}{\approx_{\varrho}} = \{[e]_{\varrho} \mid e \in H\}$ and operations \otimes and \hookrightarrow on $\frac{H}{\approx_{\varrho}}$ defined as follows:

$$[e]_{\varrho} \otimes [k]_{\varrho} = [e * k]_{\varrho} \text{ and } [e]_{\varrho} \hookrightarrow [k]_{\varrho} = [e \rightarrow k]_{\varrho}.$$

Then $(\frac{H}{\approx_{\varrho}}, \otimes, \hookrightarrow, [1]_{\varrho})$ is a hoop.

Proof. We have $[e]_{\varrho} = [k]_{\varrho}$ and $[m]_{\varrho} = [z]_{\varrho}$ iff $e \approx_{\varrho} k$ and $m \approx_{\varrho} z$. Since \approx_{ϱ} is the congruence relation on H , we get that all above operations are well-defined. Thus, by routine calculation, we can see that $\frac{H}{\approx_{\varrho}}$ is a hoop. Now, we define a binary relation on $\frac{H}{\approx_{\varrho}}$ by

$$[e]_{\varrho} \leq [k]_{\varrho} \text{ iff } \varrho(e \rightarrow k) = \varrho(1), \text{ for any } e, k \in H.$$

Easily we can see $(\frac{H}{\approx_{\varrho}}, \leq)$ is a partial order monoid. \square

Example 3.8. Let H be the hoop as in Example 3.2. Define a map $\varrho : H \rightarrow H$ by $\varrho(1) = \varrho(k) = 0.3$ and $\varrho(0) = \varrho(e) = 0.5$. Then, we have $[0]_{\varrho} = \{0\}$, $[e]_{\varrho} = \{e\}$ and $[k]_{\varrho} = [1]_{\varrho} = \{k, 1\}$. Hence, $\frac{H}{\approx_{\varrho}} = \{[0]_{\varrho}, [e]_{\varrho}, [1]_{\varrho}\}$, which by the operations defining in Theorem 3.7 is a hoop.

4. Construction of hoops and distributive lattices by intuitionistic fuzzy filters

In the following, we define the concept of intuitionistic fuzzy filter on hoop H and investigate some related results.

Definition 4.1. An intuitionistic fuzzy set or IF-set $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ on H is called an intuitionistic fuzzy filter or an IF-filter of H if, for any $d, s \in H$, it satisfies the next conditions:

(IFF1) if $d \leq s$, then $\varsigma_{\mathcal{B}}(d) \leq \varsigma_{\mathcal{B}}(s)$ and $\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(d)$,

(IFF2) $\varsigma_{\mathcal{B}}(d * s) \geq \varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(s)$,

(IFF3) $\varrho_{\mathcal{B}}(d * s) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(s)$.

The family of all intuitionistic fuzzy filters of H will be denoted by $\mathcal{IFF}(H)$.

Example 4.2. Let H be the hoop as in Example 3.2. Define $\varsigma_{\mathcal{B}}(1) = 0.9$, $\varsigma_{\mathcal{B}}(0) = \varsigma_{\mathcal{B}}(e) = 0.5$, $\varsigma_{\mathcal{B}}(k) = 0.7$ and $\varrho_{\mathcal{B}}(1) = 0.1$, $\varrho_{\mathcal{B}}(0) = \varrho_{\mathcal{B}}(e) = 0.5$ and $\varrho_{\mathcal{B}}(k) = 0.3$. Then $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an intuitionistic fuzzy filter on H .

Proposition 4.3. An IF-set $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ on A is an IF-filter iff $\varsigma_{\mathcal{B}} \in FF(H)$ and $\varrho_{\mathcal{B}} \in AFF(H)$ such that $\varsigma_{\mathcal{B}}(d) + \varrho_{\mathcal{B}}(d) \leq 1$.

Proof. By Definitions 3.1, 4.1 and Proposition 2.5, the proof is clear. \square

In the next example, we show that the condition $\varsigma_{\mathcal{B}}(d) + \varrho_{\mathcal{B}}(d) \leq 1$, for any $d \in H$ is necessary.

Example 4.4. Let H be the hoop as in Example 3.2. Define

$$\varsigma_{\mathcal{B}}(1) = 1, \varsigma_{\mathcal{B}}(e) = \varsigma_{\mathcal{B}}(0) = 0.5, \varsigma_{\mathcal{B}}(k) = 0.7, \varrho_{\mathcal{B}}(1) = 0.7, \varrho_{\mathcal{B}}(e) = \varrho_{\mathcal{B}}(0) = 0.9, \varrho_{\mathcal{B}}(k) = 0.8.$$

Clearly, $\varsigma_{\mathcal{B}} \in FF(H)$ and $\varrho_{\mathcal{B}} \in AFF(H)$ such that $\varsigma_{\mathcal{B}}(1) + \varrho_{\mathcal{B}}(1) \geq 1$. Hence, $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is not an IF-set of H .

In the next proposition we prove that by a fuzzy filter (anti-fuzzy filter) on A we can make an IF-filter.

Proposition 4.5. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-set on H . Then \mathcal{B} is an IF-filter iff $\mathcal{B}_C = (\varsigma_{\mathcal{B}}, \varsigma_{\mathcal{B}}^c)$ and ${}_C\mathcal{B} = (\varrho_{\mathcal{B}}^c, \varrho_{\mathcal{B}})$ are IF-filters of H .

Proof. (\Rightarrow) Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-filter of H . By Definition 4.1, $\varsigma_{\mathcal{B}} \in FF(H)$ and $\varrho_{\mathcal{B}} \in AFF(H)$. Thus, by Remark 3.3, $\varsigma_{\mathcal{B}}^c \in AFF(H)$ and $\varrho_{\mathcal{B}}^c \in FF(H)$. Also, from $\varsigma_{\mathcal{B}} + \varsigma_{\mathcal{B}}^c \leq 1$ and $\varrho_{\mathcal{B}} + \varrho_{\mathcal{B}}^c \leq 1$, obviously, $\mathcal{B}_C = (\varsigma_{\mathcal{B}}, \varsigma_{\mathcal{B}}^c)$ and ${}_C\mathcal{B} = (\varrho_{\mathcal{B}}^c, \varrho_{\mathcal{B}})$ are IF-filters of H .

(\Leftarrow) Let $\mathcal{B}_C = (\varsigma_{\mathcal{B}}, \varsigma_{\mathcal{B}}^c)$ and ${}_C\mathcal{B} = (\varrho_{\mathcal{B}}^c, \varrho_{\mathcal{B}})$ be IF-filters of H . Then $\varsigma_{\mathcal{B}} \in FF(H)$ and $\varrho_{\mathcal{B}} \in AFF(H)$. From, $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-set on H , then $\varsigma_{\mathcal{B}} + \varrho_{\mathcal{B}} \leq 1$. Thus, by Proposition 4.3, \mathcal{B} is an IF-filter of H . \square

Theorem 4.6. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-set on H and for any $r \in [0, 1]$,

$$\mathcal{B}_r = \{d \in H \mid \varsigma_{\mathcal{B}}(d) \geq r \text{ and } \varrho_{\mathcal{B}}(d) \leq r\}.$$

Then \mathcal{B} is an IF-filter of H iff for any $r \in [0, 1]$, $\mathcal{B}_r \neq \emptyset$ is a filter of H .

Proof. By Theorems 2.6, 3.4 and Proposition 4.3, the proof is clear. \square

Now, we mainly investigate the lattice of all IF-filters by introducing the notion of tip-extended pair of IF-sets.

Proposition 4.7. *Let $\{\mathcal{B}_i = (\varsigma_{\mathcal{B}_i}, \varrho_{\mathcal{B}_i})\}_{i \in I}$ be a set of IF-filters of H and fuzzy sets $\bigwedge_{i \in I} \varsigma_{\mathcal{B}_i}$ and $\bigvee_{i \in I} \varrho_{\mathcal{B}_i}$ on H are defined as follows:*

$$(\bigwedge_{i \in I} \varsigma_{\mathcal{B}_i})(d) = \bigwedge_{i \in I} \{\varsigma_{\mathcal{B}_i}(d) \mid i \in I\}, \quad (\bigvee_{i \in I} \varrho_{\mathcal{B}_i})(d) = \bigvee_{i \in I} \{\varrho_{\mathcal{B}_i}(d) \mid i \in I\},$$

and IF-set $\bigcap_{i \in I} \mathcal{B}_i$ on H , are defined by $\bigcap_{i \in I} \mathcal{B}_i = (\bigwedge_{i \in I} \varsigma_{\mathcal{B}_i}, \bigvee_{i \in I} \varrho_{\mathcal{B}_i})$. Then $\mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-filter of H , too.

Proof. Let $\{\mathcal{B}_i = (\varsigma_{\mathcal{B}_i}, \varrho_{\mathcal{B}_i})\}_{i \in I}$ be a set of IF-filters of H and $\mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ such that $\varsigma_{\mathcal{B}} = \bigwedge_{i \in I} \varsigma_{\mathcal{B}_i}$ and $\varrho_{\mathcal{B}} = \bigvee_{i \in I} \varrho_{\mathcal{B}_i}$. Let $d, s, q \in H$ such that $d \leq s \Rightarrow q$. Then

$$\varsigma_{\mathcal{B}}(q) = (\bigwedge_{i \in I} \varsigma_{\mathcal{B}_i})(q) = \bigwedge_{i \in I} (\varsigma_{\mathcal{B}_i}(q)) \geq \bigwedge_{i \in I} (\varsigma_{\mathcal{B}_i}(d) \wedge \varsigma_{\mathcal{B}_i}(s)) = \bigwedge_{i \in I} \varsigma_{\mathcal{B}_i}(d) \wedge \bigwedge_{i \in I} \varsigma_{\mathcal{B}_i}(s) = \varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(s).$$

$$\varrho_{\mathcal{B}}(q) = (\bigvee_{i \in I} \varrho_{\mathcal{B}_i})(q) = \bigvee_{i \in I} (\varrho_{\mathcal{B}_i}(q)) \leq \bigvee_{i \in I} (\varrho_{\mathcal{B}_i}(d) \vee \varrho_{\mathcal{B}_i}(s)) = \bigvee_{i \in I} \varrho_{\mathcal{B}_i}(d) \vee \bigvee_{i \in I} \varrho_{\mathcal{B}_i}(s) = \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(s).$$

□

In the next example we show that $(\bigvee_{i \in I} \varsigma_{\mathcal{B}_i}, \bigvee_{i \in I} \varrho_{\mathcal{B}_i})$ is not an IF-filter, in general.

Example 4.8. *Let H be as Example 4.4 and;*

$\varsigma_{\mathcal{B}_1}(1) = 0.4$, $\varsigma_{\mathcal{B}_1}(e) = \varsigma_{\mathcal{B}_1}(0) = 0.2$, $\varsigma_{\mathcal{B}_1}(k) = 0.3$, $\varrho_{\mathcal{B}_1}(1) = 0.6$, $\varrho_{\mathcal{B}_1}(e) = \varrho_{\mathcal{B}_1}(0) = 0.8$ and $\varrho_{\mathcal{B}_1}(k) = 0.7$

$\varsigma_{\mathcal{B}_2}(1) = 0.5$, $\varsigma_{\mathcal{B}_2}(e) = \varsigma_{\mathcal{B}_2}(0) = 0.3$, $\varsigma_{\mathcal{B}_2}(k) = 0.4$, $\varrho_{\mathcal{B}_2}(1) = 0.5$, $\varrho_{\mathcal{B}_2}(e) = \varrho_{\mathcal{B}_2}(0) = 0.7$ and $\varrho_{\mathcal{B}_2}(k) = 0.6$

$\varsigma_{\mathcal{B}_3}(1) = 1$, $\varsigma_{\mathcal{B}_3}(e) = \varsigma_{\mathcal{B}_3}(0) = 0.5$, $\varsigma_{\mathcal{B}_3}(k) = 0.7$, $\varrho_{\mathcal{B}_3}(1) = 0$, $\varrho_{\mathcal{B}_3}(e) = \varrho_{\mathcal{B}_3}(0) = 0.5$ and $\varrho_{\mathcal{B}_3}(k) = 0.3$

$\varsigma_{\mathcal{B}_4}(1) = 0.3$, $\varsigma_{\mathcal{B}_4}(e) = \varsigma_{\mathcal{B}_4}(0) = 0.1$, $\varsigma_{\mathcal{B}_4}(k) = 0.2$, $\varrho_{\mathcal{B}_4}(1) = 0.7$, $\varrho_{\mathcal{B}_4}(e) = \varrho_{\mathcal{B}_4}(0) = 0.9$ and $\varrho_{\mathcal{B}_4}(k) = 0.8$

Then $(\bigvee \varsigma_{\mathcal{B}}(1), \bigvee \varrho_{\mathcal{B}}(1)) = (1, 0.7)$ such that $\varsigma_{\mathcal{B}} + \varrho_{\mathcal{B}} \geq 1$, which is a contradiction with definition of IF-set.

Let L be an IF-set on H . The intersection of all IF-filters containing L is called the generated IF-filter by L , denoted as $\langle L \rangle$.

Theorem 4.9. *Let $L = (\varsigma_L, \varrho_L)$ be an IF-set on H and $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is defined on H by;*

$$\varsigma_{\mathcal{B}}(d) = \bigvee_{e_1 * e_2 * \dots * e_n \leq d} \{\varsigma_L(e_1) \wedge \varsigma_L(e_2) \wedge \dots \wedge \varsigma_L(e_n)\},$$

and

$$\varrho_{\mathcal{B}}(d) = \bigwedge_{e_1 * e_2 * \dots * e_n \leq d} \{\varrho_L(e_1) \vee \varrho_L(e_2) \vee \dots \vee \varrho_L(e_n)\},$$

for all $d \in H$, $e_i \in H$, $1 \leq i \leq n$ and $n \in \mathbb{N}$. Then $\mathcal{B} = \langle L \rangle$.

Proof. First, we verify that \mathcal{B} is an IF-filter. For all $d, s \in H$, such that $d \leq s$, the definition of \mathcal{B} yields that $\varsigma_{\mathcal{B}}(d) \leq \varsigma_{\mathcal{B}}(s)$ and $\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(d)$. For all $d, s, e_i, k_j, m_l \in H$, $1 \leq i \leq n$, $1 \leq j \leq m$, $1 \leq l \leq k$ and $n, m, k \in \mathbb{N}$, we have

$$\begin{aligned}
\varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(s) &= \bigvee_{e_1 * e_2 * \dots * e_n \leq d} \{\varsigma_L(e_1) \wedge \varsigma_L(e_2) \wedge \dots \wedge \varsigma_L(e_n)\} \\
&\wedge \bigvee_{k_1 * k_2 * \dots * k_m \leq s} \{\varsigma_L(k_1) \wedge \varsigma_L(k_2) \wedge \dots \wedge \varsigma_L(k_m)\} \\
&= \bigvee_{e_1 * e_2 * \dots * e_n \leq d, k_1 * k_2 * \dots * k_m \leq s} \bigvee_{\varsigma_L(e_1) \wedge \varsigma_L(e_2) \wedge \dots \wedge \varsigma_L(e_n)} \\
&\quad \wedge \varsigma_L(k_1) \wedge \varsigma_L(k_2) \wedge \dots \wedge \varsigma_L(k_m)\} \\
&\leq \bigvee_{m_1 * m_2 * \dots * m_k \leq d * s} \{\varsigma_L(m_1) \wedge \varsigma_L(m_2) \wedge \dots \wedge \varsigma_L(m_k)\} \\
&= \varsigma_{\mathcal{B}}(d * s)
\end{aligned}$$

and

$$\begin{aligned}
\varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(s) &= \bigwedge_{e_1 * e_2 * \dots * e_n \leq d} \{\varrho_L(e_1) \vee \varrho_L(e_2) \vee \dots \vee \varrho_L(e_n)\} \\
&\vee \bigwedge_{k_1 * k_2 * \dots * k_m \leq s} \{\varrho_L(k_1) \vee \varrho_L(k_2) \vee \dots \vee \varrho_L(k_m)\} \\
&= \bigwedge_{e_1 * e_2 * \dots * e_n \leq d, k_1 * k_2 * \dots * k_m \leq s} \bigwedge_{\varrho_L(e_1) \vee \varrho_L(e_2) \vee \dots \vee \varrho_L(e_n)} \\
&\quad \vee \varrho_L(k_1) \vee \varrho_L(k_2) \vee \dots \vee \varrho_L(k_m)\} \\
&\geq \bigwedge_{m_1 * m_2 * \dots * m_k \leq d * s} \{\varrho_L(m_1) \vee \varrho_L(m_2) \vee \dots \vee \varrho_L(m_k)\} \\
&= \varrho_{\mathcal{B}}(d * s).
\end{aligned}$$

Thus \mathcal{B} is an IF-filter. Secondly, let C be an IF-filter such that $L \subseteq C$. By definition of IF-filter, for all $d, s, e_i \in H$, $1 \leq i \leq n$ and $n \in \mathbb{N}$, it holds that

$$\begin{aligned}
\varsigma_{\mathcal{B}}(d) &= \bigvee_{e_1 * e_2 * \dots * e_n \leq d} \{\varsigma_L(e_1) \wedge \varsigma_L(e_2) \wedge \dots \wedge \varsigma_L(e_n)\} \\
&\leq \bigvee_{e_1 * e_2 * \dots * e_n \leq d} \{\varsigma_C(e_1) \wedge \varsigma_C(e_2) \wedge \dots \wedge \varsigma_C(e_n)\} \\
&\leq \bigvee_{e_1 * e_2 * \dots * e_n \leq d} \{\varsigma_C(e_1 * e_2 * \dots * e_n)\} \\
&\leq \varsigma_C(d)
\end{aligned}$$

and

$$\begin{aligned}
 \varrho_{\mathcal{B}}(d) &= \bigwedge_{e_1 * e_2 * \dots * e_n \leq d} \{\varrho_L(e_1) \vee \varrho_L(e_2) \vee \dots \vee \varrho_L(e_n)\} \\
 &\geq \bigwedge_{e_1 * e_2 * \dots * e_n \leq d} \{\varrho_C(e_1) \vee \varrho_C(e_2) \vee \dots \vee \varrho_C(e_n)\} \\
 &\geq \bigwedge_{e_1 * e_2 * \dots * e_n \leq d} \{\varrho_C(e_1 * e_2 * \dots * e_n)\} \\
 &\geq \varrho_C(d)
 \end{aligned}$$

and hence, $\mathcal{B} \subseteq C$. Thus, $\mathcal{B} = \langle L \rangle$. \square

Associating with the above results, similar to the proof of ([15, Theorems 10–12]), we define the operations \sqcap and \sqcup on IF-filters of H in this way, $\mathcal{B} \sqcap C = \mathcal{B} \cap C$ and $\mathcal{B} \sqcup C = \langle \mathcal{B} \cup C \rangle$, for any $\mathcal{B}, C \in IFF(H)$.

Theorem 4.10. *$(IFF(H), \sqcap, \sqcup, \emptyset, H)$ is a bounded distributive lattice.*

Proof. The proof is similar to the proof of [15, Theorems 10–12]. \square

Theorem 4.11. *Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-filter of H and fuzzy relation $\approx_{\mathcal{B}}$ on H , for any $d, s \in H$, is defined by*

$$d \approx_{\mathcal{B}} s \text{ iff } \varsigma_{\mathcal{B}}(d \rightarrow s) \wedge \varsigma_{\mathcal{B}}(s \rightarrow d) = \varsigma_{\mathcal{B}}(1), \varrho_{\mathcal{B}}(d \rightarrow s) \vee \varrho_{\mathcal{B}}(s \rightarrow d) = \varrho_{\mathcal{B}}(1).$$

Then $\approx_{\mathcal{B}}$ is a congruence relation on H .

Proof. The proof is similar to the proof of Theorems 2.7 and 3.6. \square

Theorem 4.12. *Let $\frac{H}{\approx_{\mathcal{B}}} = \{[e]_{\mathcal{B}} \mid e \in H\}$ and operations \odot and \hookrightarrow on $\frac{H}{\approx_{\mathcal{B}}}$ are defined as follows:*

$$[e]_{\mathcal{B}} \odot [k]_{\mathcal{B}} = [e * k]_{\mathcal{B}} \text{ and } [e]_{\mathcal{B}} \hookrightarrow [k]_{\mathcal{B}} = [e \rightarrow k]_{\mathcal{B}}.$$

Also, we define a binary relation on $\frac{H}{\approx_{\mathcal{B}}}$ by

$$[e]_{\mathcal{B}} \leq [k]_{\mathcal{B}} \text{ iff } \varsigma_{\mathcal{B}}(e \rightarrow k) = \varsigma_{\mathcal{B}}(1) \text{ and } \varrho_{\mathcal{B}}(e \rightarrow k) = \varrho_{\mathcal{B}}(1), \text{ for any } e, k \in H.$$

Clearly, $(\frac{H}{\approx_{\mathcal{B}}}, \leq)$ is a poset. Then $(\frac{H}{\approx_{\mathcal{B}}}, \odot, \hookrightarrow, [1]_{\mathcal{B}})$ is a hoop.

Proof. The proof is similar to the proof of Theorems 2.8 and 3.7. \square

Example 4.13. *Let H be the hoop as in Example 3.2. Define $\varsigma_{\mathcal{B}}(0) = \varsigma_{\mathcal{B}}(e) = 0.5$, $\varsigma_{\mathcal{B}}(1) = \varsigma_{\mathcal{B}}(k) = 0.7$ and $\varrho_{\mathcal{B}}(0) = \varrho_{\mathcal{B}}(e) = 0.5$ and $\varrho_{\mathcal{B}}(1) = \varrho_{\mathcal{B}}(k) = 0.3$. Then $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an intuitionistic fuzzy filter on H and $\frac{H}{\approx_{\mathcal{B}}} = \{[0]_{\mathcal{B}}, [e]_{\mathcal{B}}, [1]_{\mathcal{B}}\}$ with the operations defining in Theorem 4.12 is a hoop.*

5. Construction of Brouwerian semilattice by intuitionistic fuzzy (positive) implicative filters

Here, we define the notions of intuitionistic fuzzy (positive) implicative filters on hoops and some related results are investigated. We find some equivalence characterizations of them.

Definition 5.1. An IF-set $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ on H is called an intuitionistic fuzzy implicative filter or an IF-implicative filter of H if for all $d, s, q \in H$, it satisfies the next conditions:

(IFIF₁) If $d \leq s$, then $\varsigma_{\mathcal{B}}(d) \leq \varsigma_{\mathcal{B}}(s)$ and $\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(d)$,

(IFIF₂) $\varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(d \twoheadrightarrow ((s \twoheadrightarrow q) \twoheadrightarrow s)) \leq \varsigma_{\mathcal{B}}(s)$ and $\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \twoheadrightarrow ((s \twoheadrightarrow q) \twoheadrightarrow s))$.

Example 5.2. Assume $H = \{0, e, k, 1\}$ is a chain such that $0 \leq e \leq k \leq 1$. Define two binary operations $*$ and \twoheadrightarrow on H as follows:

\twoheadrightarrow	0	e	k	1	*	0	e	k	1
0	1	1	1	1	0	0	0	0	0
e	0	1	1	1	e	0	e	e	e
k	0	k	1	1	k	0	e	e	k
1	0	e	k	1	1	0	e	k	1

Clearly, $(H, *, \twoheadrightarrow, 0, 1)$ is a hoop. Define $\varsigma_{\mathcal{B}}$ on H by $\varsigma_{\mathcal{B}}(1) = \varsigma_{\mathcal{B}}(k) = \varsigma_{\mathcal{B}}(e) = r_2$ and $\varsigma_{\mathcal{B}}(0) = r_1$ such that $0 \leq r_1 < r_2 \leq 1$ and $\varrho_{\mathcal{B}} = 1 - \varsigma_{\mathcal{B}}$. One easily verify that $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-implicative filter.

Theorem 5.3. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-set on H . Then \mathcal{B} is an IF-implicative filters of H iff for any $r \in [0, 1]$, $\mathcal{B}_r \neq \emptyset$ is an implicative filter of H .

Proof. The proof is similar to the proof of Theorem 4.6. \square

Theorem 5.4. Every IF-implicative filter of H is an IF-filter.

Proof. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-implicative filter. By (IFIF₁), obviously (IFF1) holds. Since $d * s \leq d * s$, by Proposition 2.1(ii), $s \leq d \twoheadrightarrow (d * s)$. Then by (IFIF₁), $\varrho_{\mathcal{B}}(d \twoheadrightarrow (d * s)) \leq \varrho_{\mathcal{B}}(s)$. Also, since $\varrho_{\mathcal{B}}$ is an IF-implicative filter, it is enough to choose $q = 1$, then

$$\varrho_{\mathcal{B}}(d * s) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \twoheadrightarrow (((d * s) \twoheadrightarrow 1) \twoheadrightarrow (d * s))) = \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \twoheadrightarrow (d * s)) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(s).$$

Similarly, we can see $\varsigma_{\mathcal{B}}(d * s) \geq \varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(s)$. Hence, $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-filter. \square

In the next example we can see that the converse of the previous theorem does not hold.

Example 5.5. Assume $H = \{0, e, k, 1\}$ is a chain, where $0 \leq e \leq k \leq 1$. Define two binary operations $*$ and \twoheadrightarrow on H as follows:

\twoheadrightarrow	0	e	k	1	*	0	e	k	1
0	1	1	1	1	0	0	0	0	0
e	e	1	1	1	e	0	0	e	e
k	0	e	1	1	k	0	e	e	k
1	0	e	k	1	1	0	e	k	1

Hence $(H, *, \rightarrow\!, 0, 1)$ is a hoop. Define $\varsigma_{\mathcal{B}} \in FF(H)$ by $\varsigma_{\mathcal{B}}(1) = r_1$, $\varsigma_{\mathcal{B}}(k) = r_2$ and $\varsigma_{\mathcal{B}}(e) = \varsigma_{\mathcal{B}}(0) = r_3$ such that $0 \leq r_3 < r_2 < r_1 \leq 1$ and $\varrho_{\mathcal{B}} = 1 - \varsigma_{\mathcal{B}}$. One easily verify that $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-filter of H but \mathcal{B} is not an IF-implicative filter. Because $r_2 = \varsigma_{\mathcal{B}}(k) = \varsigma_{\mathcal{B}}(k) \wedge \varsigma_{\mathcal{B}}(k \rightarrow ((e \rightarrow 0) \rightarrow e)) \not\leq \varsigma_{\mathcal{B}}(e) = r_3$.

Theorem 5.6. Suppose H is bounded and $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-filter of H . Then the next statements are equivalent, for all $d, s \in H$,

- (i) \mathcal{B} is an IF-implicative filter of H ,
- (ii) $\varsigma_{\mathcal{B}}((d \rightarrow s) \rightarrow d) \leq \varsigma_{\mathcal{B}}(d)$ and $\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d) \geq \varrho_{\mathcal{B}}(d)$,
- (iii) $\varsigma_{\mathcal{B}}(d \rightarrow q) \geq \varsigma_{\mathcal{B}}(d \rightarrow (q' \rightarrow s)) \wedge \varsigma_{\mathcal{B}}(s \rightarrow q)$ and $\varrho_{\mathcal{B}}(d \rightarrow q) \leq \varrho_{\mathcal{B}}(d \rightarrow (q' \rightarrow s)) \vee \varrho_{\mathcal{B}}(s \rightarrow q)$,
- (iv) $\varsigma_{\mathcal{B}}((d * s') \rightarrow s) \leq \varsigma_{\mathcal{B}}(d \rightarrow s)$ and $\varrho_{\mathcal{B}}((d * s') \rightarrow s) \geq \varrho_{\mathcal{B}}(d \rightarrow s)$.

Proof. In this proof, we just prove the items of $\varrho_{\mathcal{B}}$. Also, since \mathcal{B} is an IF-filter of H , by Definition 4.1, $\varsigma_{\mathcal{B}} \in FF(H)$ and $\varrho_{\mathcal{B}} \in AFF(H)$. Then as we notice, $\varrho_{\mathcal{B}}(1) \leq \varrho_{\mathcal{B}}(d)$ and $\varsigma_{\mathcal{B}}(1) \geq \varsigma_{\mathcal{B}}(d)$, for any $d \in H$.

(i) \Rightarrow (ii) It is enough to let $d = 1$ in $(IFIF_2)$. Then

$$\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(1) \vee \varrho_{\mathcal{B}}(1 \rightarrow ((s \rightarrow q) \rightarrow s)) = \varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s).$$

(ii) \Rightarrow (i) Let $d, s, q \in H$. Then by Proposition 2.1(vii),

$$d * (d \rightarrow ((s \rightarrow q) \rightarrow s)) \leq (s \rightarrow q) \rightarrow s.$$

Since $\varrho_{\mathcal{B}} \in AFF(H)$, we get

$$\varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \rightarrow ((s \rightarrow q) \rightarrow s)).$$

By (ii),

$$\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \rightarrow ((s \rightarrow q) \rightarrow s)).$$

Hence, \mathcal{B} is an IF-implicative filter of H .

(i) \Rightarrow (iv) Let $d, s \in H$. Then by Proposition 2.1(iv) and (xii), $s \leq d \rightarrow s$, and so $(d \rightarrow s)' \leq s'$. Again, by Proposition 2.1(xiii), $s' \rightarrow (d \rightarrow s) \leq (d \rightarrow s)' \rightarrow (d \rightarrow s)$. Since $\varrho_{\mathcal{B}} \in AFF(H)$, $\varrho_{\mathcal{B}}((d \rightarrow s)' \rightarrow (d \rightarrow s)) \leq \varrho_{\mathcal{B}}(s' \rightarrow (d \rightarrow s))$. Moreover, since \mathcal{B} is an IF-implicative filter of H , we have

$$\varrho_{\mathcal{B}}(d \rightarrow s) \leq \varrho_{\mathcal{B}}(1) \vee \varrho_{\mathcal{B}}(1 \rightarrow (((d \rightarrow s) \rightarrow 0) \rightarrow (d \rightarrow s))) \leq \varrho_{\mathcal{B}}(1) \vee \varrho_{\mathcal{B}}(s' \rightarrow (d \rightarrow s)) = \varrho_{\mathcal{B}}((s' * d) \rightarrow s).$$

(iv) \Rightarrow (iii) Let $d, s, q \in H$. Then by Proposition 2.1(x),

$$(d \rightarrow (q' \rightarrow s)) * (s \rightarrow q) = ((d * q') \rightarrow s) * (s \rightarrow q) \leq (d * q') \rightarrow q.$$

Since $\varrho_{\mathcal{B}} \in AFF(H)$,

$$\varrho_{\mathcal{B}}((d * q') \rightarrow q) \leq \varrho_{\mathcal{B}}((d * q') \rightarrow s) \vee \varrho_{\mathcal{B}}(s \rightarrow q),$$

by (iv), $\varrho_{\mathcal{B}}(d \rightarrow q) \leq \varrho_{\mathcal{B}}((d * q') \rightarrow q)$. Hence, $\varrho_{\mathcal{B}}(d \rightarrow q) \leq \varrho_{\mathcal{B}}(d \rightarrow (q' \rightarrow s)) \vee \varrho_{\mathcal{B}}(s \rightarrow q)$.

(iii) \Rightarrow (i) Let $d = 1$ and $q = s$ in (iii). Then $\varrho_{\mathcal{B}}(1 \rightarrow s) \leq \varrho_{\mathcal{B}}(1 \rightarrow (s' \rightarrow s)) \vee \varrho_{\mathcal{B}}(s \rightarrow s)$, and so $\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(s' \rightarrow s) \vee \varrho_{\mathcal{B}}(s \rightarrow s) = \varrho_{\mathcal{B}}(s' \rightarrow s)$. Also, by Proposition 2.1(vii),

$$d * (d \rightarrow ((s \rightarrow q) \rightarrow s)) \leq (s \rightarrow q) \rightarrow s.$$

Since \mathcal{B} is an IF-filter of H , by Definition 4.1, $\varrho_{\mathcal{B}} \in AFF(H)$, then $\varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \rightarrow ((s \rightarrow q) \rightarrow s))$. Moreover, by Proposition 2.1(xiii) and Proposition 2.2(iii), $s' \leq s \rightarrow q$, and so $(s \rightarrow q) \rightarrow s \leq s' \rightarrow s$, thus, $\varrho_{\mathcal{B}}(s' \rightarrow s) \leq \varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s)$. Hence, $\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \rightarrow ((s \rightarrow q) \rightarrow s))$. \square

Definition 5.7. Assume $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-set of H . Then \mathcal{B} is said to be an intuitionistic fuzzy positive implicative filter or an IF-positive implicative filter of H if, for all $d, s, q \in H$,

(IFPIF₁) if $d \leq s$, then $\varsigma_{\mathcal{B}}(d) \leq \varsigma_{\mathcal{B}}(s)$ and $\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(d)$,

(IFPIF₂) $\varsigma_{\mathcal{B}}(d \rightarrow s) \wedge \varsigma_{\mathcal{B}}(d \rightarrow (s \rightarrow q)) \leq \varsigma_{\mathcal{B}}(d \rightarrow q)$ and $\varrho_{\mathcal{B}}(d \rightarrow s) \leq \varrho_{\mathcal{B}}(d \rightarrow s) \vee \varrho_{\mathcal{B}}(d \rightarrow (s \rightarrow q))$.

Example 5.8. Assume $H = \{0, e, k, 1\}$ is a chain where $0 \leq e \leq k \leq 1$. Define two binary operations $*$ and \rightarrow on H as follows:

\rightarrow	0	e	k	1	*	0	e	k	1
0	1	1	1	1	0	0	0	0	0
e	0	1	1	1	e	0	e	e	e
k	0	e	1	1	k	0	e	k	k
1	0	e	k	1	1	0	e	k	1

Hence $(H, *, \rightarrow, 0, 1)$ is a hoop. Define $\varsigma_{\mathcal{B}} \in FF(H)$ by $\varsigma_{\mathcal{B}}(1) = r_2$ and $\varsigma_{\mathcal{B}}(k) = \varsigma_{\mathcal{B}}(e) = \varsigma_{\mathcal{B}}(0) = r_1$ such that $0 \leq r_1 < r_2 \leq 1$ and let $\varrho_{\mathcal{B}} = 1 - \varsigma_{\mathcal{B}}$. Clearly, \mathcal{B} is an IF-positive implicative filter.

Theorem 5.9. Let \mathcal{B} be an IF-set of H . Then \mathcal{B} is an IF-positive implicative filter of H iff for any $r \in [0, 1]$, $\mathcal{B}_r \neq \emptyset$ is a positive implicative filter of H .

Proof. The proof is similar to the proof of Theorem 4.6. \square

Theorem 5.10. Each IF-positive implicative filter of H is an IF-filter.

Proof. Assume $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-positive implicative filter of H . Then by (IFPIF₁), obviously, (IFF1) holds. Moreover, by Proposition 2.1(v) and (IFPIF₂), we have

$$\varrho_{\mathcal{B}}(d * s) \leq \varrho_{\mathcal{B}}(1 \rightarrow s) \vee \varrho_{\mathcal{B}}(1 \rightarrow (s \rightarrow (d * s))) = \varrho_{\mathcal{B}}(s) \vee \varrho_{\mathcal{B}}(s \rightarrow (d * s)).$$

Also, by Proposition 2.1(ii), $d \leq s \rightarrow (d * s)$. Then by (IFPIF₁), $\varrho_{\mathcal{B}}(s \rightarrow (d * s)) \leq \varrho_{\mathcal{B}}(d)$. Hence, $\varrho_{\mathcal{B}}(d * s) \leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(s)$. Similarly, $\varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(s) \leq \varsigma_{\mathcal{B}}(d * s)$. Therefore, \mathcal{B} is an IF-filter of H . \square

Next example shows that the converse of the previous theorem does not hold.

Example 5.11. According to Example 5.2, introduce $\varsigma_{\mathcal{B}}$ by $\varsigma_{\mathcal{B}}(1) = r_3$, $\varsigma_{\mathcal{B}}(k) = \varsigma_{\mathcal{B}}(e) = r_2$ and $\varsigma_{\mathcal{B}}(0) = r_1$ such that $0 \leq r_1 < r_2 < r_3 \leq 1$. Routine calculation shows $\varsigma_{\mathcal{B}} \in FF(H)$ and $\varrho_{\mathcal{B}} = 1 - \varsigma_{\mathcal{B}} \in AFF(H)$. So, $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-filter of H but $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is not an IF-positive implicative filter. Because

$$r_2 = \varsigma_{\mathcal{B}}(k) = \varsigma_{\mathcal{B}}(k \rightarrow e) \not\leq \varsigma_{\mathcal{B}}(k \rightarrow (k \rightarrow e)) \wedge \varsigma_{\mathcal{B}}(k \rightarrow k) = \varsigma_{\mathcal{B}}(1) = r_3.$$

Theorem 5.12. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-filter of H . Then for any $d, s, q \in H$ the next conditions are equivalent:

- (i) $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-positive implicative filter of H ,
- (ii) $\varsigma_{\mathcal{B}}(d \rightarrow (d \rightarrow s)) \leq \varsigma_{\mathcal{B}}(d \rightarrow s)$, $\varrho_{\mathcal{B}}(d \rightarrow s) \leq \varrho_{\mathcal{B}}(d \rightarrow (d \rightarrow s))$,
- (iii) $\varsigma_{\mathcal{B}}(q \rightarrow (s \rightarrow d)) \leq \varsigma_{\mathcal{B}}((q \rightarrow s) \rightarrow (q \rightarrow d))$, $\varrho_{\mathcal{B}}((q \rightarrow s) \rightarrow (q \rightarrow d)) \leq \varrho_{\mathcal{B}}(q \rightarrow (s \rightarrow d))$,
- (iv) $\varsigma_{\mathcal{B}}(q) \wedge \varsigma_{\mathcal{B}}(q \rightarrow (s \rightarrow (s \rightarrow d))) \leq \varsigma_{\mathcal{B}}(s \rightarrow d)$, $\varrho_{\mathcal{B}}(s \rightarrow d) \leq \varrho_{\mathcal{B}}(q) \vee \varrho_{\mathcal{B}}(q \rightarrow (s \rightarrow (s \rightarrow d)))$,
- (v) $\varsigma_{\mathcal{B}}(d \rightarrow d^2) = \varsigma_{\mathcal{B}}(1)$ and $\varrho_{\mathcal{B}}(d \rightarrow d^2) = \varrho_{\mathcal{B}}(1)$.

Proof. In this proof, we just prove the items of $\varrho_{\mathcal{B}}$. Also, since \mathcal{B} is an IF-filter of H , by Definition 4.1, $\varsigma_{\mathcal{B}} \in FF(H)$ and $\varrho_{\mathcal{B}} \in AFF(H)$. Then as we notice, $\varrho_{\mathcal{B}}(1) \leq \varrho_{\mathcal{B}}(d)$ and $\varsigma_{\mathcal{B}}(1) \geq \varsigma_{\mathcal{B}}(d)$, for any $d \in H$. (i) \Rightarrow (ii) Let $q = s$ and $s = d$ in (IFPIF₂). Then

$$\varrho_{\mathcal{B}}(d \rightarrow s) \leq \varrho_{\mathcal{B}}(d \rightarrow d) \vee \varrho_{\mathcal{B}}(d \rightarrow (d \rightarrow s)) = \varrho_{\mathcal{B}}(1) \vee \varrho_{\mathcal{B}}(d \rightarrow (d \rightarrow s)) = \varrho_{\mathcal{B}}(d \rightarrow (d \rightarrow s)).$$

(ii) \Rightarrow (iii) Since $q \rightarrow s \leq q \rightarrow s$, by Proposition 2.1(vii), $q * (q \rightarrow s) \leq s$. Also, by (HP3),

$$q \rightarrow (q \rightarrow ((q \rightarrow s) \rightarrow d)) = q \rightarrow [(q * (q \rightarrow s)) \rightarrow d] = [q * (q \rightarrow s)] \rightarrow [q \rightarrow d] \geq s \rightarrow (q \rightarrow d).$$

Since $\varrho_{\mathcal{B}} \in AFF(H)$,

$$\varrho_{\mathcal{B}}(s \rightarrow (q \rightarrow d)) \geq \varrho_{\mathcal{B}}(q \rightarrow (q \rightarrow ((q \rightarrow s) \rightarrow d))).$$

Also, by (ii),

$$\varrho_{\mathcal{B}}(q \rightarrow (q \rightarrow ((q \rightarrow s) \rightarrow d))) \geq \varrho_{\mathcal{B}}(q \rightarrow ((q \rightarrow s) \rightarrow d)).$$

Hence, by (HP3),

$$\varrho_{\mathcal{B}}(q \rightarrow (s \rightarrow d)) \geq \varrho_{\mathcal{B}}(q \rightarrow (q \rightarrow ((q \rightarrow s) \rightarrow d))) \geq \varrho_{\mathcal{B}}(q \rightarrow ((q \rightarrow s) \rightarrow d)) = \varrho_{\mathcal{B}}((q \rightarrow s) \rightarrow (q \rightarrow d)).$$

(iii) \Rightarrow (iv) By Proposition 2.1(vii), $q * (q \rightarrow (s \rightarrow (s \rightarrow d))) \leq s \rightarrow (s \rightarrow d)$. Since $\varrho_{\mathcal{B}} \in AFF(H)$,

$$\varrho_{\mathcal{B}}(s \rightarrow (s \rightarrow d)) \leq \varrho_{\mathcal{B}}(q) \vee \varrho_{\mathcal{B}}(q \rightarrow (s \rightarrow (s \rightarrow d))).$$

So it is enough to choose $q = s$ in (iii). Then

$$\varrho_{\mathcal{B}}(1 \rightarrow (s \rightarrow d)) \leq \varrho_{\mathcal{B}}(s \rightarrow (s \rightarrow d)) \leq \varrho_{\mathcal{B}}(q) \vee \varrho_{\mathcal{B}}(q \rightarrow (s \rightarrow (s \rightarrow d))).$$

(iv) \Rightarrow (v) Let $q = 1$, $s = d$ and $d = d^2$ in (iv). Then

$$\varrho_{\mathcal{B}}(d \rightarrow d^2) \leq \varrho_{\mathcal{B}}(1) \vee \varrho_{\mathcal{B}}(1 \rightarrow (d \rightarrow (d \rightarrow d^2))) = \varrho_{\mathcal{B}}(1),$$

and so $\varrho_{\mathcal{B}}(d \rightarrow d^2) = \varrho_{\mathcal{B}}(1)$.

(v) \Rightarrow (i) By Proposition 2.1(vii),

$$(d \rightarrow s) * (s \rightarrow (d \rightarrow q)) \leq d \rightarrow (d \rightarrow q) = d^2 \rightarrow q.$$

Since $\varrho_{\mathcal{B}} \in AFF(H)$, we have

$$\varrho_{\mathcal{B}}(d^2 \rightarrow q) \leq \varrho_{\mathcal{B}}(d \rightarrow s) \vee \varrho_{\mathcal{B}}(s \rightarrow (d \rightarrow q)).$$

By Proposition 2.1(vii), $(d \rightarrow d^2) * (d^2 \rightarrow q) \leq d \rightarrow q$. Then

$$\varrho_{\mathcal{B}}(d \rightarrow q) \leq \varrho_{\mathcal{B}}(d \rightarrow d^2) \vee \varrho_{\mathcal{B}}(d^2 \rightarrow q).$$

By (v), $\varrho_{\mathcal{B}}(d \rightarrow d^2) = \varrho_{\mathcal{B}}(1)$, hence, $\varrho_{\mathcal{B}}(d \rightarrow q) \leq \varrho_{\mathcal{B}}(d^2 \rightarrow q)$. As we prove,

$$\varrho_{\mathcal{B}}(d \rightarrow q) \leq \varrho_{\mathcal{B}}(d^2 \rightarrow q) \leq \varrho_{\mathcal{B}}(d \rightarrow s) \vee \varrho_{\mathcal{B}}(s \rightarrow (d \rightarrow q)).$$

□

Corollary 5.13. Let $d^2 = d$, for all $d \in H$. Then any IF-filter of H is an IF-positive implicative filter.

Proof. Let $d, s, q \in H$ and $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-filter of H . If, for all $d \in H$, $d^2 = d$, then $\varrho_{\mathcal{B}}(d \rightarrow d^2) = \varrho_{\mathcal{B}}(d \rightarrow d) = \varrho_{\mathcal{B}}(1)$ and also, $\varsigma_{\mathcal{B}}(d \rightarrow d^2) = \varsigma_{\mathcal{B}}(1)$. Thus, by Theorem 5.12(v), \mathcal{B} is an IF-positive implicative filter. \square

Theorem 5.14. Every IF-implicative filter of H is an IF-positive implicative filter.

Proof. In this proof, we just prove the sentence of $\varrho_{\mathcal{B}}$. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-implicative filter of H . Then by (IFIF₁), clearly (IFPIF₁) holds. Now, let $d \in H$. Then by Theorem 5.4, \mathcal{B} is an IF-filter. Thus, by Theorem 5.6(ii), Proposition 2.1(xiv), (HP2) and (HP3),

$$\begin{aligned}\varrho_{\mathcal{B}}(d \rightarrow d^2) &\leq \varrho_{\mathcal{B}}(((d \rightarrow d^2) \rightarrow d^2) \rightarrow (d \rightarrow d^2)) \\ &\leq \varrho_{\mathcal{B}}(1) \vee \varrho_{\mathcal{B}}(1 \rightarrow (((d \rightarrow d^2) \rightarrow d^2) \rightarrow (d \rightarrow d^2))) \\ &= \varrho_{\mathcal{B}}(d \rightarrow (((d \rightarrow d^2) \rightarrow d^2) \rightarrow d^2)) \\ &= \varrho_{\mathcal{B}}(d \rightarrow (d \rightarrow d^2)) \\ &= \varrho_{\mathcal{B}}(d^2 \rightarrow d^2) \\ &= \varrho_{\mathcal{B}}(1)\end{aligned}$$

By the similar way, $\varsigma_{\mathcal{B}}(d \rightarrow d^2) = \varsigma_{\mathcal{B}}(1)$. Hence, by Theorem 5.12(v), \mathcal{B} is an IF-positive implicative filter. \square

The next example shows that the converse of the previous theorem does not hold, in general.

Example 5.15. According to Example 5.8, we see that \mathcal{B} is an IF-positive implicative filter, but it is not an IF-implicative filter. Because by Theorem 5.6(iv), $r_2 = \varsigma_{\mathcal{B}}(1) = \varsigma_{\mathcal{B}}(0 \rightarrow e) = \varsigma_{\mathcal{B}}((k * e') \rightarrow e) \neq \varsigma_{\mathcal{B}}(k \rightarrow e) = \varsigma_{\mathcal{B}}(e) = r_1$.

Theorem 5.16. Let H be bounded with (DNP). Then every IF-positive implicative filter of H is an IF-implicative filter.

Proof. Let H be bounded with (DNP) and $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-positive implicative filter of H . Then by Theorem 5.10, \mathcal{B} is an IF-filter. Thus, by Proposition 2.1(vii),

$$d * (d \rightarrow ((s \rightarrow q) \rightarrow s)) \leq (s \rightarrow q) \rightarrow s,$$

and so

$$\varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(d \rightarrow ((s \rightarrow q) \rightarrow s)) \leq \varsigma_{\mathcal{B}}((s \rightarrow q) \rightarrow s),$$

and

$$\varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \rightarrow ((s \rightarrow q) \rightarrow s)) \geq \varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s).$$

Since H has (DNP), by (HP3), we have,

$$\varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s) = \varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s'') = \varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow (s' \rightarrow 0)) = \varrho_{\mathcal{B}}(s' \rightarrow ((s \rightarrow q) \rightarrow 0)).$$

By the similar way,

$$\varsigma_{\mathcal{B}}((s \rightarrow q) \rightarrow s) = \varsigma_{\mathcal{B}}(s' \rightarrow ((s \rightarrow q) \rightarrow 0)).$$

Moreover, since \mathcal{B} is an IF-positive implicative filter of H , by Theorem 5.12(iii), Proposition 2.2(ii) and (DNP), we get

$$\begin{aligned}
 \varrho_{\mathcal{B}}(s) &= \varrho_{\mathcal{B}}(1 \rightarrow s) \\
 &= \varrho_{\mathcal{B}}((0 \rightarrow q) \rightarrow s'') \\
 &= \varrho_{\mathcal{B}}(((s * s') \rightarrow q) \rightarrow s'') \\
 &= \varrho_{\mathcal{B}}((s' \rightarrow (s \rightarrow q)) \rightarrow (s' \rightarrow 0)) \\
 &\leq \varrho_{\mathcal{B}}(s' \rightarrow ((s \rightarrow q) \rightarrow 0)) \\
 &= \varrho_{\mathcal{B}}((s \rightarrow q) \rightarrow s) \\
 &\leq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \rightarrow ((s \rightarrow q) \rightarrow s))
 \end{aligned}$$

By the similar way, $\varsigma_{\mathcal{B}}(s) \geq \varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(d \rightarrow ((s \rightarrow q) \rightarrow s))$. Therefore, by Theorem 5.6(ii), \mathcal{B} is an IF-implicative filter. \square

Theorem 5.17. *Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-set of H . Then \mathcal{B} is an IF-implicative filter of H iff \mathcal{B} is an IF-positive implicative filter of H and $\varsigma_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \leq \varsigma_{\mathcal{B}}((s \rightarrow d) \rightarrow d)$ and $\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \geq \varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow d)$, for all $d, s \in H$.*

Proof. (\Rightarrow) Let \mathcal{B} be an IF-implicative filter. Then by Theorem 5.14, \mathcal{B} is an IF-positive implicative filter. Now, suppose $d, s \in H$. Then by Proposition 2.1(iv) and (viii), $d, s \leq (s \rightarrow d) \rightarrow d$. Since $s \leq (s \rightarrow d) \rightarrow d$, by Proposition 2.1(xii), we have $(d \rightarrow s) \rightarrow s \leq (d \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d)$. Moreover, since $d \leq (s \rightarrow d) \rightarrow d$, by Proposition 2.1(xiii), $d \rightarrow s \geq ((s \rightarrow d) \rightarrow d) \rightarrow s$, and so

$$(d \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d) \leq (((s \rightarrow d) \rightarrow d) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d).$$

Thus, we have,

$$(d \rightarrow s) \rightarrow s \leq (d \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d) \leq (((s \rightarrow d) \rightarrow d) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d).$$

Since \mathcal{B} is an IF-implicative filter, by Theorem 5.4, \mathcal{B} is an IF-filter. Thus, by Theorem 5.6(ii),

$$\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \geq \varrho_{\mathcal{B}}(((s \rightarrow d) \rightarrow d) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d) \geq \varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow d).$$

Hence, $\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \geq \varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow d)$.

(\Leftarrow) Let $d, s \in H$. Then by Proposition 2.1(viii), $d \leq (d \rightarrow s) \rightarrow s$, and by Proposition 2.1(xii), $(d \rightarrow s) \rightarrow d \leq (d \rightarrow s) \rightarrow ((d \rightarrow s) \rightarrow s)$. Since \mathcal{B} is an IF-positive implicative filter, by Theorem 5.12(ii), we have

$$\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow ((d \rightarrow s) \rightarrow s)) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d).$$

Also, by assumption,

$$\varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow d) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d).$$

On the other hand, by Proposition 2.1(iv), $s \leq d \rightarrow s$ and by Proposition 2.1(xiii), $(d \rightarrow s) \rightarrow d \leq s \rightarrow d$. Since \mathcal{B} is an IF-positive implicative filter, by Theorem 5.10, \mathcal{B} is an IF-filter, thus, $\varrho_{\mathcal{B}}(s \rightarrow d) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d)$. Hence,

$$\begin{aligned}
\varrho_{\mathcal{B}}(d) &\leq \varrho_{\mathcal{B}}(s \rightarrow d) \vee \varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow d) \\
&\leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d) \vee \varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow d) \\
&= \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d)
\end{aligned}$$

Therefore, \mathcal{B} is an IF-implicative filter. \square

Theorem 5.18. *Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-filter of H . Then \mathcal{B} is an IF-positive implicative filter iff $\frac{H}{\approx_{\mathcal{B}}}$ is a Brouwerian semilattice.*

Proof. (\Rightarrow) Let \mathcal{B} be an IF-filter. Then by Theorem 4.12, $\frac{H}{\approx_{\mathcal{B}}}$ is well-defined. Since $\frac{H}{\approx_{\mathcal{B}}}$ is a hoop, then by Proposition 2.1(i), $\frac{H}{\approx_{\mathcal{B}}}$ is a \wedge -semilattice. Now, it is enough to prove that

$$[d]_{\mathcal{B}} \wedge [s]_{\mathcal{B}} \leq [q]_{\mathcal{B}} \text{ iff } [d]_{\mathcal{B}} \leq [s]_{\mathcal{B}} \hookrightarrow [q]_{\mathcal{B}}, \text{ for all } d, s, q \in H.$$

Let $[d]_{\mathcal{B}} \wedge [s]_{\mathcal{B}} \leq [q]_{\mathcal{B}}$. Then by Proposition 2.1(iii), $[d]_{\mathcal{B}} \otimes [s]_{\mathcal{B}} \leq [d]_{\mathcal{B}} \wedge [s]_{\mathcal{B}} \leq [q]_{\mathcal{B}}$. Thus, $[d]_{\mathcal{B}} \otimes [s]_{\mathcal{B}} \leq [q]_{\mathcal{B}}$. Since $\frac{H}{\approx_{\mathcal{B}}}$ is a hoop, by Proposition 2.1(ii), we get $[d]_{\mathcal{B}} \leq [s]_{\mathcal{B}} \hookrightarrow [q]_{\mathcal{B}}$.

Conversely, suppose $[d]_{\mathcal{B}} \leq [s]_{\mathcal{B}} \hookrightarrow [q]_{\mathcal{B}}$. Then $[d]_{\mathcal{B}} \hookrightarrow ([s]_{\mathcal{B}} \hookrightarrow [q]_{\mathcal{B}}) = [1]_{\mathcal{B}}$, so $\varsigma_{\mathcal{B}}(d \rightarrow (s \rightarrow q)) = \varsigma_{\mathcal{B}}(1)$ and $\varrho_{\mathcal{B}}(d \rightarrow (s \rightarrow q)) = \varrho_{\mathcal{B}}(1)$. Since \mathcal{B} is an IF-positive implicative filter, by Theorem 5.6(iii), we have

$$\varsigma_{\mathcal{B}}((d \rightarrow s) \rightarrow (d \rightarrow q)) \geq \varsigma_{\mathcal{B}}(d \rightarrow (s \rightarrow q)) = \varsigma_{\mathcal{B}}(1),$$

and

$$\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow (d \rightarrow q)) \leq \varrho_{\mathcal{B}}(d \rightarrow (s \rightarrow q)) = \varrho_{\mathcal{B}}(1).$$

Thus, $\varsigma_{\mathcal{B}}((d \rightarrow s) \rightarrow (d \rightarrow q)) = \varsigma_{\mathcal{B}}(1)$ and $\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow (d \rightarrow q)) = \varrho_{\mathcal{B}}(1)$, and so $[d \rightarrow s]_{\mathcal{B}} \leq [d \rightarrow q]_{\mathcal{B}}$. Hence, $[d]_{\mathcal{B}} \hookrightarrow [s]_{\mathcal{B}} \leq [d]_{\mathcal{B}} \hookrightarrow [q]_{\mathcal{B}}$. Since $\frac{H}{\approx_{\mathcal{B}}}$ is a hoop, by Proposition 2.1(ii) and (i), we have

$$[d]_{\mathcal{B}} \wedge [s]_{\mathcal{B}} = [d]_{\mathcal{B}} \otimes ([d]_{\mathcal{B}} \hookrightarrow [s]_{\mathcal{B}}) \leq [q]_{\mathcal{B}}.$$

Therefore, $\frac{H}{\approx_{\mathcal{B}}}$ is a Brouwerian semilattice.

(\Leftarrow) Since \mathcal{B} is an IF-filter, by (IFF1), $\varsigma_{\mathcal{B}}(d) \leq \varsigma_{\mathcal{B}}(1)$ and $\varrho_{\mathcal{B}}(1) \leq \varrho_{\mathcal{B}}(d)$, for all $d \in H$. By assumption, $\frac{H}{\approx_{\mathcal{B}}}$ is a Brouwerian semilattice, define $[d]_{\mathcal{B}} \otimes [s]_{\mathcal{B}} = [d]_{\mathcal{B}} \wedge [s]_{\mathcal{B}}$, for all $d, s \in H$. Since $[d]_{\mathcal{B}} \leq [d]_{\mathcal{B}}$, we have

$$[d]_{\mathcal{B}} \leq [d]_{\mathcal{B}} \wedge [d]_{\mathcal{B}} = [d]_{\mathcal{B}} \otimes [d]_{\mathcal{B}} = [d * d]_{\mathcal{B}}.$$

So, $[d]_{\mathcal{B}} \leq [d^2]_{\mathcal{B}}$. Then $[d]_{\mathcal{B}} \hookrightarrow [d^2]_{\mathcal{B}} = [1]_{\mathcal{B}}$, and so, $\varsigma_{\mathcal{B}}(d \rightarrow d^2) = \varsigma_{\mathcal{B}}(1)$ and $\varrho_{\mathcal{B}}(d \rightarrow d^2) = \varrho_{\mathcal{B}}(1)$. Hence, by Theorem 5.12(v), \mathcal{B} is an IF-positive implicative filter. \square

Example 5.19. *Let H be the hoop and $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-positive implicative filter of H as in Example 5.8. Then $\frac{H}{\approx_{\mathcal{B}}} = \{[0]_{\mathcal{B}}, [e]_{\mathcal{B}}, [k]_{\mathcal{B}}, [1]_{\mathcal{B}}\}$ is a Brouwerian semilattice.*

6. Construction of Heyting semilattice and Wajesberg hoops by intuitionistic fuzzy fantastic filters

In the following, the concept of intuitionistic fuzzy fantastic filter on hoops is defined and some related results are investigated.

Definition 6.1. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-set of H . Then \mathcal{B} is called an intuitionistic fuzzy fantastic filter or an IF-fantastic filter of H if, for all $d, s, q \in H$,

(IFFF₁) if $d \leq s$, then $\varsigma_{\mathcal{B}}(d) \leq \varsigma_{\mathcal{B}}(s)$ and $\varrho_{\mathcal{B}}(s) \leq \varrho_{\mathcal{B}}(d)$,

(IFFF₂) $\varsigma_{\mathcal{B}}(q) \wedge \varsigma_{\mathcal{B}}(q \rightarrow (s \rightarrow d)) \leq \varsigma_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow d)$ and $\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow d) \leq \varrho_{\mathcal{B}}(q) \vee \varrho_{\mathcal{B}}(q \rightarrow (s \rightarrow d))$.

Example 6.2. According to Example 5.5, routine calculation shows that \mathcal{B} is an IF-fantastic filter.

Theorem 6.3. Let \mathcal{B} be an IF-set of H . Then \mathcal{B} is an IF-fantastic filter of H iff for any $r \in [0, 1]$, $\mathcal{B}_r \neq \emptyset$ is a fantastic filter.

Proof. The proof is similar to the proof of Theorem 4.6. \square

Theorem 6.4. Every IF-fantastic filter of H is an IF-filter.

Proof. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-fantastic filter and $d, s \in H$. Then (IFF1) holds. Since \mathcal{B} is an IF-fantastic filter, then by Proposition 2.1(v), we have

$$\varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \rightarrow s) \geq \varrho_{\mathcal{B}}(d) \vee \varrho_{\mathcal{B}}(d \rightarrow (1 \rightarrow s)) \geq \varrho_{\mathcal{B}}((s \rightarrow 1) \rightarrow 1) \rightarrow s).$$

By the similar way, we can see that $\varsigma_{\mathcal{B}}(d) \wedge \varsigma_{\mathcal{B}}(d \rightarrow s) \leq \varsigma_{\mathcal{B}}(s)$. Therefore, \mathcal{B} is an IF-filter of H . \square

In the next example we show that the converse of the previous theorem does not hold, in general.

Example 6.5. According to Example 5.5, \mathcal{B} is an IF-filter but it is not an IF-fantastic filter. Because

$$r_1 = \varsigma_{\mathcal{B}}(1) = \varsigma_{\mathcal{B}}(1) \wedge \varsigma_{\mathcal{B}}(1 \rightarrow (e \rightarrow k)) \not\leq \varsigma_{\mathcal{B}}((k \rightarrow e) \rightarrow e) \rightarrow k) = \varsigma_{\mathcal{B}}(k) = r_2.$$

Theorem 6.6. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-filter of H . Then the next statements are equivalent, for all $d, s \in H$,

- (i) \mathcal{B} is an IF-fantastic filter of H ,
- (ii) $\varsigma_{\mathcal{B}}(s \rightarrow d) \leq \varsigma_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow d)$ and $\varrho_{\mathcal{B}}(s \rightarrow d) \geq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow d)$,
- (iii) $\varsigma_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d)) = \varsigma_{\mathcal{B}}(1)$ and $\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d)) = \varrho_{\mathcal{B}}(1)$.

Proof. In this proof, we just prove the items of $\varrho_{\mathcal{B}}$. Also, since \mathcal{B} is an IF-filter of H , by Definition 4.1, $\varsigma_{\mathcal{B}} \in FF(H)$ and $\varrho_{\mathcal{B}} \in AFF(H)$. Then as we notice, $\varrho_{\mathcal{B}}(1) \leq \varrho_{\mathcal{B}}(d)$ and $\varsigma_{\mathcal{B}}(1) \geq \varsigma_{\mathcal{B}}(d)$, for any $d \in H$.

(i) \Rightarrow (ii) Suppose \mathcal{B} is an IF-fantastic filter and $d, s \in H$. Let $q = 1$. Then

$$\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow d) \leq \varrho_{\mathcal{B}}(1) \vee \varrho_{\mathcal{B}}(1 \rightarrow (s \rightarrow d)) = \varrho_{\mathcal{B}}(s \rightarrow d).$$

(ii) \Rightarrow (i) Let $d, s, q \in H$. Since \mathcal{B} is an IF-filter, we get $\varrho_{\mathcal{B}}(s \rightarrow d) \leq \varrho_{\mathcal{B}}(q) \vee \varrho_{\mathcal{B}}(q \rightarrow (s \rightarrow d))$. Thus, by (ii),

$$\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow d) \leq \varrho_{\mathcal{B}}(s \rightarrow d) \leq \varrho_{\mathcal{B}}(q) \vee \varrho_{\mathcal{B}}(q \rightarrow (s \rightarrow d)).$$

(ii) \Rightarrow (iii) Let $d, s \in H$. Then by Proposition 2.1(viii), $d \leq (s \rightarrow d) \rightarrow d$, thus, by Proposition 2.1(xiii), $((s \rightarrow d) \rightarrow d) \rightarrow s \leq (((s \rightarrow d) \rightarrow d) \rightarrow s) \rightarrow s$, and so

$$(((s \rightarrow d) \rightarrow d) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d) \leq ((d \rightarrow s) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d).$$

Since \mathcal{B} is an IF-filter,

$$\varrho_{\mathcal{B}}(((s \rightarrow d) \rightarrow d) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d) \geq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d).$$

Now, since \mathcal{B} is an IF-fantastic filter, we have

$$\begin{aligned} \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d)) &\leq \varrho_{\mathcal{B}}(((s \rightarrow d) \rightarrow d) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d) \\ &\leq \varrho_{\mathcal{B}}(s \rightarrow ((s \rightarrow d) \rightarrow d)) \\ &= \varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow (s \rightarrow d)) \\ &= \varrho_{\mathcal{B}}(1) \end{aligned}$$

Hence, (iii) holds.

(iii) \Rightarrow (ii) Let $d, s \in H$ such that

$$\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d)) = \varrho_{\mathcal{B}}(1).$$

Then by (HP3),

$$\varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow (((d \rightarrow s) \rightarrow s) \rightarrow d)) = \varrho_{\mathcal{B}}(1).$$

Now, since \mathcal{B} is an IF-filter, we get

$$\varrho_{\mathcal{B}}(((d \rightarrow s) \rightarrow s) \rightarrow d) \leq \varrho_{\mathcal{B}}(s \rightarrow d) \vee \varrho_{\mathcal{B}}((s \rightarrow d) \rightarrow (((d \rightarrow s) \rightarrow s) \rightarrow d)) = \varrho_{\mathcal{B}}(s \rightarrow d).$$

□

Theorem 6.7. *Every IF-implicative filter of H is an IF-fantastic filter.*

Proof. Let $d, s \in H$ and $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-implicative filter of H . Then by Proposition 2.1(iv), $d \leq ((d \rightarrow s) \rightarrow s) \rightarrow d$, thus by Proposition 2.1(xiii), $((d \rightarrow s) \rightarrow s) \rightarrow d \rightarrow s \leq d \rightarrow s$, and so by Proposition 2.1(xiii) and (vii),

$$\begin{aligned} s \rightarrow d &\leq ((d \rightarrow s) * ((d \rightarrow s) \rightarrow s)) \rightarrow d \\ &= (d \rightarrow s) \rightarrow ((d \rightarrow s) \rightarrow s) \rightarrow d \\ &\leq (((d \rightarrow s) \rightarrow s) \rightarrow d) \rightarrow s \rightarrow ((d \rightarrow s) \rightarrow s) \rightarrow d \end{aligned}$$

Moreover, since \mathcal{B} is an IF-implicative filter,

$$\begin{aligned} \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d) &\leq \varrho_{\mathcal{B}}(((d \rightarrow s) \rightarrow s) \rightarrow d) \rightarrow (((d \rightarrow s) \rightarrow s) \rightarrow d) \\ &\leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow ((d \rightarrow s) \rightarrow s) \rightarrow d) \\ &= \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow ((d \rightarrow s) \rightarrow d)) \\ &\leq \varrho_{\mathcal{B}}(s \rightarrow d) \end{aligned}$$

Hence, by Theorem 6.6(ii), \mathcal{B} is an IF-fantastic filter. □

The next example shows that the converse of the previous theorem may not be true, in general.

Example 6.8. According to Example 4.4, define an IF-set \mathcal{B} by $\varsigma_{\mathcal{B}}(1) = \varsigma_{\mathcal{B}}(k) = r_1$ and $\varsigma_{\mathcal{B}}(e) = \varsigma_{\mathcal{B}}(0) = r_2$ such that $0 \leq r_2 < r_1 \leq 1$ and $\varrho_{\mathcal{B}} = 1 - \varsigma_{\mathcal{B}}$. By routine calculation, \mathcal{B} is an IF-fantastic filter but it is not an IF-implicative filter. Because by Theorem 5.6(ii),

$$\varsigma_{\mathcal{B}}((e' \rightarrow e) \rightarrow e) = \varsigma_{\mathcal{B}}((e \rightarrow e) \rightarrow e) = \varsigma_{\mathcal{B}}(e) = r_2 \neq r_1 = \varsigma_{\mathcal{B}}(1).$$

Theorem 6.9. $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-implicative filter of H iff $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ is an IF-positive implicative filter and IF-fantastic filter of H .

Proof. (\Rightarrow) By Theorems 6.7 and 5.14, the proof is clear.

(\Leftarrow) Let $d, s \in H$. Then by Proposition 2.1(viii) and (xii),

$$(d \rightarrow s) \rightarrow d \leq (d \rightarrow s) \rightarrow ((d \rightarrow s) \rightarrow s).$$

Since \mathcal{B} is an IF-positive implicative filter, by Theorem 5.10, \mathcal{B} is an IF-filter of H , we have

$$\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d) \geq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow ((d \rightarrow s) \rightarrow s)).$$

Also, since \mathcal{B} is an IF-positive implicative filter, by Theorem 5.12(ii), we get

$$\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow ((d \rightarrow s) \rightarrow s)) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d).$$

On the other side, by Proposition 2.1(iv) and (xiii), $(d \rightarrow s) \rightarrow d \leq s \rightarrow d$. Since \mathcal{B} is an IF-filter of H , $\varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d) \geq \varrho_{\mathcal{B}}(s \rightarrow d)$. Moreover, since \mathcal{B} is an IF-fantastic filter, by Theorem 6.6(ii),

$$\varrho_{\mathcal{B}}(((d \rightarrow s) \rightarrow s) \rightarrow d) \leq \varrho_{\mathcal{B}}(s \rightarrow d) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d).$$

From \mathcal{B} is an IF-filter, then

$$\varrho_{\mathcal{B}}(d) \leq \varrho_{\mathcal{B}}(((d \rightarrow s) \rightarrow s) \rightarrow d) \vee \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow s) \leq \varrho_{\mathcal{B}}((d \rightarrow s) \rightarrow d).$$

Thus, by Theorem 5.6(ii), \mathcal{B} is an IF-implicative filter. \square

Theorem 6.10. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-filter of H . Then \mathcal{B} is an IF-fantastic filter of H iff $\frac{H}{\approx_{\mathcal{B}}}$ is a Wajesberg hoop.

Proof. (\Rightarrow) Since \mathcal{B} is an IF-filter, by Theorem 4.12, $\frac{H}{\approx_{\mathcal{B}}}$ is well-defined and is a hoop. Since \mathcal{B} is an IF-fantastic filter of H , by Theorem 6.6(iii), we get

$$\varsigma_{\mathcal{B}}(((d \rightarrow s) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d)) = \varsigma_{\mathcal{B}}(1),$$

and

$$\varrho_{\mathcal{B}}(((d \rightarrow s) \rightarrow s) \rightarrow ((s \rightarrow d) \rightarrow d)) = \varrho_{\mathcal{B}}(1).$$

Then by Theorem 4.12, $[(d \rightarrow s) \rightarrow s]_{\mathcal{B}} \leq [(s \rightarrow d) \rightarrow d]_{\mathcal{B}}$. By the similar way, $[(s \rightarrow d) \rightarrow d]_{\mathcal{B}} \leq [(d \rightarrow s) \rightarrow s]_{\mathcal{B}}$. Thus

$$([d]_{\mathcal{B}} \hookrightarrow [s]_{\mathcal{B}}) \hookrightarrow [s]_{\mathcal{B}} = ([s]_{\mathcal{B}} \hookrightarrow [d]_{\mathcal{B}}) \hookrightarrow [d]_{\mathcal{B}}.$$

Therefore, $\frac{H}{\approx_{\mathcal{B}}}$ is a Wajesberg hoop.

(\Leftarrow) Let \mathcal{B} be an IF-filter of H . Then by (IFF1), $\varsigma_{\mathcal{B}}(d) \leq \varsigma_{\mathcal{B}}(1)$ and $\varrho_{\mathcal{B}}(d) \geq \varrho_{\mathcal{B}}(1)$, for all $d \in H$. Moreover, since $\frac{H}{\approx_{\mathcal{B}}}$ is a Wajesberg hoop, we have $([d]_{\mathcal{B}} \hookrightarrow [s]_{\mathcal{B}}) \hookrightarrow [s]_{\mathcal{B}} = ([s]_{\mathcal{B}} \hookrightarrow [d]_{\mathcal{B}}) \hookrightarrow [d]_{\mathcal{B}}$, for any $[d]_{\mathcal{B}}, [s]_{\mathcal{B}} \in \frac{H}{\approx_{\mathcal{B}}}$. Thus, $\varsigma_{\mathcal{B}}(((d \Rightarrow s) \Rightarrow s) \Rightarrow ((s \Rightarrow d) \Rightarrow d)) = \varsigma_{\mathcal{B}}(1)$ and $\varrho_{\mathcal{B}}(((d \Rightarrow s) \Rightarrow s) \Rightarrow ((s \Rightarrow d) \Rightarrow d)) = \varrho_{\mathcal{B}}(1)$, thus, by Theorem 6.6(iii), \mathcal{B} is an IF-fantastic filter of H . \square

Example 6.11. Let H be the hoop as in Example 5.5. Define $\varsigma_{\mathcal{B}}(1) = \varsigma_{\mathcal{B}}(k) = r_1$, $\varsigma_{\mathcal{B}}(0) = \varsigma_{\mathcal{B}}(e) = r_2$, $\varrho_{\mathcal{B}}(1) = \varrho_{\mathcal{B}}(k) = 1 - r_1$ and $\varrho_{\mathcal{B}}(0) = \varrho_{\mathcal{B}}(e) = 1 - r_2$, where $0 \leq r_2 \leq r_1 \leq 1$. Then $\frac{H}{\approx_{\mathcal{B}}} = \{[0]_{\mathcal{B}}, [e]_{\mathcal{B}}, [1]_{\mathcal{B}}\}$ is a Wajesberg hoop.

Theorem 6.12. Let $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-filter of H . Then \mathcal{B} is an IF-implicative filter of H iff $\frac{H}{\approx_{\mathcal{B}}}$ is a Heyting semilattice that has Wajesberg property.

Proof. (\Rightarrow) Let \mathcal{B} be an IF-filter. Then by Theorem 4.12, $\frac{H}{\approx_{\mathcal{B}}}$ is well-defined and is a hoop. Since \mathcal{B} is an IF-implicative filter, by Theorem 5.14, we have \mathcal{B} is an IF-positive implicative filter. Hence by Theorem 5.18, $\frac{H}{\approx_{\mathcal{B}}}$ is a Brouwerian semilattice. On the other side, by Theorem 6.7, \mathcal{B} is an IF-fantastic filter, and by Theorem 6.10, $\frac{H}{\approx_{\mathcal{B}}}$ is a Wajesberg hoop. Thus, by Proposition 2.3, we define

$$[d]_{\mathcal{B}} \sqcup [s]_{\mathcal{B}} = ([d]_{\varsigma} \hookrightarrow [s]_{\mathcal{B}}) \hookrightarrow [s]_{\mathcal{B}}.$$

Then it is easy to see that \sqcup is a join operation, and so by Remark 2.4, $H/\approx_{\mathcal{B}}$ is a distributive lattice. Therefore, $\frac{H}{\approx_{\mathcal{B}}}$ is a Heyting semilattice.

(\Leftarrow) Since $\frac{H}{\approx_{\mathcal{B}}}$ is a Heyting semilattice, then $\frac{H}{\approx_{\mathcal{B}}}$ is a Brouwerian semilattice. Thus, by Theorem 5.18, \mathcal{B} is an IF-positive implicative filter. Moreover, by assumption, $\frac{H}{\approx_{\mathcal{B}}}$ is a Wajesberg hoop, by Theorem 6.10, \mathcal{B} is an IF-fantastic filter. Thus, by Theorem 6.9, \mathcal{B} is an IF-implicative filter of H . \square

Example 6.13. Let H be the hoop and $\mathcal{B} = (\varsigma_{\mathcal{B}}, \varrho_{\mathcal{B}})$ be an IF-implicative filter of H as in Example 5.2. Then $\frac{H}{\approx_{\mathcal{B}}} = \{[0]_{\mathcal{B}}, [1]_{\mathcal{B}}\}$ is a Heyting semilattice that has Wajesberg property.

7. Conclusions

In decision problems, the use of fuzzy approaches is ubiquitous. Given the importance of fuzzy concepts in solving decision problems, we decided to use these concepts, intuitionistic fuzzy sets, in a specific logical algebra to provide a new approach with useful mathematical tools to address the fundamental decision problem. In this paper, the concept of anti-fuzzy filter of hoops is defined and the concepts of intuitionistic fuzzy filters, intuitionistic fuzzy (positive) implicative and intuitionistic fuzzy fantastic filters of hoops are introduced and the properties and equivalent characterizations of them are discussed. Moreover, it was proved that all intuitionistic fuzzy filters make a bounded distributive

lattice. Also, the relations between different kinds of intuitionistic fuzzy filters are investigated and studied that under which conditions they are equivalent. Also, a congruence relation on hoops is defined by an intuitionistic fuzzy filter and proved the new structure is a hoop. Finally, the conditions that quotient structure will be Brouwerian semilattice, Heyting algebra and Wajesberg hoop are investigated.

Conflict of interest

The authors declare that there is no conflict of interest.

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