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Research article

Theoretical and numerical stability results for a viscoelastic swelling porous-elastic system with past history

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Abstract: The purpose of this paper is to establish a general stability result for a one-dimensional linear swelling porous-elastic system with past history, irrespective of the wave speeds of the system. First, we establish an explicit and general decay result under a wider class of the relaxation (kernel) functions. The kernel in our memory term is more general and of a broader class. Further, we get a better decay rate without imposing some assumptions on the boundedness of the history data considered in many earlier results in the literature. We also perform several numerical tests to illustrate our theoretical results. Our output extends and improves some of the available results on swelling porous media in the literature.

Keywords: swelling porous problem; viscoelastic; general decay; convex functions; finite element and Crank-Nicolson methods
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1. Introduction

Over the few last decades, a great number of scientists and mathematicians have got interested in the theory of elastic materials with voids (porous materials), see Figure 1. This theory is considered to be a simple extension of the classical theory of elasticity. See in this regard the concept of granular materials with voids introduced by Goodman and Cowin [1] and the idea of Nuziato and Cowin [2] related to the nonlinear theory of elastic materials with voids. The importance of materials with microstructure has been demonstrated by the very large number of published work and the various applications in the fields of petroleum industry, material science, biomedical, etc. Scientists investigated the coupling of macroscopic and microscopic structures in the porous material and its strength. Moreover, the stability has been studied by adding some dissipation mechanisms at the microscopic and/or macroscopic level. We start with the pioneer contribution of Quintanilla [3] in 2003, where he considered the following problem

$$\begin{cases} \rho_0 u_{tt} = \mu u_{xx} + \beta\phi_x, & x \in (0,\pi),\ t > 0, \\ \rho_0\kappa\phi_{tt} = \alpha\phi_{xx} - \beta u_x - \tau\phi_t - \xi\phi, & x \in (0,\pi),\ t > 0, \\ u(x,0) = u_0(x),\ \ \phi(x,0) = \phi_0(x), & x \in (0,\pi), \\ u_t(x,0) = u_1(x),\ \ \phi_t(x,0) = \phi_1(x), & x \in (0,\pi), \\ u(0,t) = u(\pi,t) = \phi(0,t) = \phi(\pi,t) = 0, & t \geq 0, \end{cases} \tag{1.1}$$

where u, ϕ, and $\rho_0, \kappa > 0$ are the longitudinal displacement, the volume fraction, the mass density, and the equilibrated inertia respectively; and $\mu, \alpha, \beta, \tau, \xi$ are positive constitutive constants which satisfy $\mu\xi > \beta^2$. He showed that the damping in the porous equation $(-\tau\phi_t)$ is not strong enough to establish an exponential decay. However, a slow decay was obtained. Casas and Quintanilla [4] looked into a thermo-porous system of the form

$$\begin{cases} \rho_0 u_{tt} = \mu u_{xx} + b\phi_x - \beta\theta_x, & x \in (0,\pi),\ t > 0, \\ \rho_0\kappa\phi_{tt} = \alpha\phi_{xx} - bu_x - \tau\phi_t - \xi\phi + m\theta, & x \in (0,\pi),\ t > 0, \\ c\theta_t = \kappa\theta_{xx} - \beta u_{xt} - m\phi_t, & x \in (0,\pi),\ t > 0, \\ u(x,0) = u_0(x),\ \ \phi(x,0) = \phi_0(x),\ \ \theta(x,0) = \theta_0(x) & x \in (0,\pi), \\ u_t(x,0) = u_1(x),\ \ \phi_t(x,0) = \phi_1(x), & x \in (0,\pi), \\ u(0,t) = u(\pi,t) = \phi(0,t) = \phi(\pi,t) = 0, & t \geq 0, \end{cases} \tag{1.2}$$

where θ is the temperature difference and the other parameters are as defined above. Under the same conditions, the authors showed that the presence of the macro-temperature and the porous dissipations acting simultaneously are able to stabilize the system exponentially. In addition, the same authors, in [5], considered (1.2), with $\tau = 0$, and proved that heat effect alone is not strong enough to obtain an exponential decay. In this case, a slow decay was established. But, adding a micro-temperature to the system (1.1) with $\tau = 0$ leads to an exponential decay. Managa and Quintanilla [6] studied diverse one-dimensional porous systems and established many slow and exponential decay results. The summary of their result is that the exponential decay can only be obtained if two dissipations from the macroscopic and microscopic equations are combined. Otherwise, only the slow decay can be obtained. In [7], Pamplona et al. studied the problem

$$\begin{cases} \rho_0 u_{tt} = \mu u_{xx} + b\phi_x - \beta\theta_x + \gamma u_{xxt}, & x \in (0,\pi),\ t > 0, \\ \rho_0\kappa\phi_{tt} = \alpha\phi_{xx} - bu_x - \tau\phi_t - \xi\phi + m\theta, & x \in (0,\pi),\ t > 0, \\ c\theta_t = \kappa\theta_{xx} - \beta u_{xt} - m\phi_t, & x \in (0,\pi),\ t > 0, \\ u(x,0) = u_0(x),\ \ \phi(x,0) = \phi_0(x),\ \ \theta(x,0) = \theta_0(x) & x \in (0,\pi), \\ u_t(x,0) = u_1(x),\ \ \phi_t(x,0) = \phi_1(x), & x \in (0,\pi), \\ u(0,t) = u(\pi,t) = \phi(0,t) = \phi(\pi,t) = 0, & t \geq 0. \end{cases} \tag{1.3}$$

and showed that combining the strong damping in the elastic equation with the macro-temperature effect is not enough to obtain an exponential decay. However, for regular enough solutions, the decay is polynomial. Managa and Quintanilla [8] discussed several systems with quasi-static micro-voids and proved many slow and exponential decay theorems. Rivera and Quintanilla [9] improved the slow-decay results obtained for some systems in [8] and proved several polynomial results. They also showed that the rate of the polynomial decay can be improved for more regular solutions. Soufyane [10] used a viscoelastic damping in the porous equation together with a macro-temperature effect and established an exponential (respect. polynomial) decay for relaxation functions of exponential (respect. polynomial) decay. A similar result was also obtained by Soufyane et al. [11, 12], for system (1.2) supplemented with the following boundary conditions:

$$
\begin{aligned}
&u(0,t) = \phi(0,t) = \theta(0,t) = \theta(L,t) = 0, && t \geq 0,\\
&u(L,t) = -\int_0^t g_1(t-s)\left[\mu u_x(L,s) + b\phi(L,s)\right] ds, && t \geq 0,\\
&\phi(L,t) = -\delta \int_0^t g_2(t-s)\phi_x(L,s)ds, && t \geq 0,
\end{aligned}
$$

where g_1 and g_2 are relaxation functions of general type. They obtained a general decay result, from which the usual exponential and polynomial decay rates are particular cases. Pamplona et al. [13] treated a one-dimensional porous elastic problem with history and proved, in the absence of the porous or the elastic dissipation, the lack of exponential decay. Messaoudi and Fareh [14, 15] considered the following viscoelastic porous system

$$
\begin{cases}
\rho_1 u_{tt} = k(\phi_x + \psi)_x + \theta_x, & \text{in } (0,1)\times \mathbb{R}_+\\
\rho_2 \psi_{tt} - \alpha\psi_{xx} + k(\phi_x + \psi) - \theta + \int_0^t g(t-s)\psi_{xx}(x,s)ds = 0, & \text{in } (0,1)\times \mathbb{R}_+\\
\rho_3 \theta_t - \kappa\theta_{xx} + \phi_{xt} + \psi_t = 0, & \text{in } (0,1)\times \mathbb{R}_+,
\end{cases}
\tag{1.4}
$$

together with initial and boundary conditions. Here, $\rho_1, \rho_2, \rho_3, k, \kappa, \alpha$ are positive constants and $g : [0, +\infty) \to (0, +\infty)$ is a non-increasing differentiable function satisfying

$$
g(0) > 0, \qquad \int_0^t g(s)ds = l > 0, \quad g'(t) \leq -\gamma(t)g(t), \quad \forall t \geq 0,
$$

where $\gamma(t)$ is a positive nonincreasing differentiable function. They established general decay results for the case of equal and non-equal speed of wave propagations. Also, Messaoudi and Fareh [16] considered the following

$$
\begin{cases}
\rho u_{tt} = \mu u_{xx} + b\phi_x - \beta\theta_x, & \text{in } (0,1)\times \mathbb{R}_+\\
J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi + m\theta - \tau\phi_t, & \text{in } (0,1)\times \mathbb{R}_+\\
c\theta_t = -q_x - \beta u_{tx} - m\phi_t, & \text{in } (0,1)\times \mathbb{R}_+\\
\tau_0 q_t + q + \kappa\theta_x = 0, & \text{in } (0,1)\times \mathbb{R}_+,
\end{cases}
\tag{1.5}
$$

together with initial and boundary conditions. They used the multiplier method and established an expoential decay result. See also a similar result by Han and Xu [17], where they established the well posedness and proved an exponential decay result using a very lengthy and detailed spectral theory approach. For the Cauchy problems, Said-Houari and Messaoudi [18] looked into the following

$$\begin{cases}\rho u_{tt} = \mu u_{xx} + b\phi_x - \beta\theta_x + \gamma u_{xxt},\\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - a\phi + m\theta,\\ c\theta_t = -q_x - \beta u_{tx} - m\phi_t,\\ \tau_0 q_t + q + \kappa\theta_x = 0,\end{cases} \qquad x \in \mathbb{R},\ t > 0 \tag{1.6}$$

together with initial data. Here ρ, μ, α, J, a are strictly positive constants such that $\mu a > b^2$ and b, β and m are different from zero. They showed that the decay rate of the system is very slow and of regularity-loss type. Apalara [19] considered

$$\begin{cases}\rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,1)\times\mathbb{R}_+\\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)ds = 0, & \text{in } (0,1)\times\mathbb{R}_+,\end{cases} \tag{1.7}$$

together with initial and boundary conditions where the relaxation function satisfied $g'(t) \le -\eta(t)g(t)$ and established a general decay result under the assumption of equal-speed wave propagations. After that, Feng and Yin [20] extended the result of [19] to the case of non-equal wave speeds. Magana and Quintanilla [6] considered

$$\begin{cases}\rho u_{tt} - \mu u_{xx} - \gamma u_{txx} = 0, & \text{in } (0,1)\times\mathbb{R}_+\\ J\phi_{tt} - \delta\delta\phi_{xx} + bu_x + \xi\phi = 0, & \text{in } (0,1)\times\mathbb{R}_+,\end{cases} \tag{1.8}$$

together with initial and boundary conditions and proved that the viscoelasticity damping is not strong enough to bring about exponential stability. However, they showed that the presence of both porous-viscosity and viscoelasticity stabilizes the system exponentially. Some other forms of damping were also considered in the literature. See [21] for thermal damping, [5] for thermal damping with microtemperatures, [4] for porous-elasticity with thermal damping, [22] for porous dissipation with microtemperatures. For more recent stability results related to porous problems, we refer the reader to [23, 24] and the references therein.

Swelling (also called expansive) soils have been characterized under porous media theory. They contain clay minerals that attract and absorb water, which may lead to increase pressure. In architectural and civil engineering, swelling soils are considered to be sources of problems and harms. If the pressure of the soil is higher compared to the main structure, it could result in heaving. The more the initial dry density of a soil, the more its potential to swell due to capillary action that accompanies absorption of underground water or shrinkage due to dryness as a result of changes in weather condition. Swelling soils causes serious engineering problems. Estimates indicate that about 20–25% of land area in the United State is covered with such problematic soils with the accompanied economic loss of 5.5 to 7 billions USD in 2003 [25]. Hence, it is crucial to study the ways to annihilate or at least minimize such damages. Reader is referred to [26–32] for other details concerning swelling soil. As established by Ieşan [33] and simplified by Quintanilla [34], the basic field equations for the linear theory of swelling porous elastic soils are mathematically given by

$$\begin{cases}\rho_z z_{tt} = P_{1x} - G_1 + F_1\\ \rho_u u_{tt} = P_{2x} + G_2 + F_1,\end{cases} \tag{1.9}$$

where the constituents z and u represent the displacement of the fluid and the elastic solid material, respectively. The positive constant coefficients ρ_z and ρ_u are the densities of each constituent. The functions (P_1, G_1, F_1) represent the partial tension, internal body forces, and eternal forces acting on the displacement, respectively. Similar definition holds for (P_2, G_2, F_2) but acting on the elastic solid. In addition, the constitutive equations of partial tensions are given by

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}}_{\text{A}} \begin{bmatrix} z_x \\ u_x \end{bmatrix}, \tag{1.10}$$

where a_1, a_3 are positive constants and $a_2 \neq 0$ is a real number. The matrix A is positive definite in the sense that $a_1a_3 \geq a_2^2$.

Quintanilla [34] investigated (1.9) by taking

$$G_1 = G_2 = \xi(z_t - u_t), \quad F_1 = a_3 z_{xxt}, \quad F_2 = 0,$$

where ξ is a positive coefficient, with initial and homogeneous Dirichlet boundary conditions and obtained an exponential stability result. Similarly, Wang and Guo [35] considered (1.9) with initial and some mixed boundary conditions, taking

$$G_1 = G_2 = 0, \quad F_1 = -\rho_z \gamma(x) z_t, \quad F_2 = 0,$$

where $\gamma(x)$ is an internal viscous damping function with a positive mean. They used the spectral method to establish an exponential stability result. Ramos et al. [36] looked into the following swelling porous elastic soils

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, & \text{in } (0, L) \times \mathbb{R}_+ \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \gamma(t) g(u_t) = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \tag{1.11}$$

and established an exponential decay rate provided that the wave speeds of the system are equal. Recently, Apalara [37] considered the following

$$\begin{aligned} &\rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \qquad \text{in } (0,1) \times (0, \infty) \\ &\rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \int_0^t g(t-s) u_{xx}(x, s) ds = 0, \text{ in } (0,1) \times (0, \infty) \\ &z(x, 0) = z_0(x),\ z_t(x, 0) = z_1(x),\ u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ x \in [0, 1] \\ &u(0, t) = u(1, t) = z(0, t) = z(1, t) = 0 \qquad t \geq 0, \end{aligned} \tag{1.12}$$

where the relaxation function satisfies the condition

$$g'(t) \leq -\xi(t) g(t), \quad t \geq 0, \tag{1.13}$$

and established a general decay result. For more results in porous elasticity system, porous-thermo-elasticity systems, Timoshenko system and other systems, we refer the reader to see [38–42].

Figure 1. Cracking and uneven foundation.

Motivated by all the above works and the other works of some viscoelastic problems with infinite memory [43–52], we consider the following problem

$$\begin{aligned}
&\rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \qquad \text{in } (0,1)\times(0,\infty)\\
&\rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \int_0^\infty g(s) u_{xx}(x,t-s)ds = 0, \ \text{in } (0,1)\times(0,\infty)\\
&z(x,0) = z_0(x),\ z_t(x,0) = z_1(x),\ u(x,-t) = u_0(x,t),\ u_t(x,0) = u_1(x),\ x\in[0,1]\\
&u(0,t) = u(1,t) = z(0,t) = z(1,t) = 0 \qquad t \geq 0.
\end{aligned} \tag{1.14}$$

where the solution is (z,u) such that u and u represent the displacement of the fluid and the elastic solid material. The positive constant coefficients ρ_u and ρ_z are the densities of each constituent. The coefficients a_1, a_2 and a_3 are positive constants satisfying specific conditions.

Our main objectives

We consider Problem (1.14) and intend to establish a four fold objective:

- extend many earlier works such as the ones in [37] from finite memory to infinite memory.
- prove a general decay estimate for the solution of Problem (1.14) with a wider class of relaxation functions; that is
$$g'(t) \leq -\xi(t)\Psi(g(t)), \quad t \geq 0, \tag{1.15}$$
where ξ and Ψ are two functions that satisfy some conditions to be specified later.
- get a better decay rate without imposing some assumptions on the boundedness of initial data considered in many papers in the literature such as the ones assumed in [48, 53, 54].
- perform some numerical tests to illustrate our theoretical results.

We establish our result by using the multiplier method and some convexity properties. In fact, the result in the present paper is essential to engineers and architects in order to plan and to ensure safe construction.

The rest of this paper is organized as follows. In section 2, we present some assumptions and material needed for our work. Some technical lemmas are presented and proved in section 3. We state and prove our main decay result and provide some conclusions in section 4. The numerical illustrations are presented in section 5.

2. Assumptions

In this section, we state some assumptions needed in the proof of our results. Throughout this paper, c is used to denote a generic positive constant. For the relaxation function g, we assume the following:

(A) $g : [0, +\infty) \to (0, +\infty)$ is a C^1 nonincreasing function satisfying

$$g(0) > 0 \qquad\qquad and \qquad\qquad 0 < \ell := \int_0^{+\infty} g(s)ds < a_3 - \frac{a_2^2}{a_1}, \tag{2.1}$$

In addition, there exists a C^1 function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0, r]$ for some $r > 0$ with $\Psi(0) = \Psi'(0) = 0$, $\lim_{s\to+\infty} \Psi'(s) = +\infty$, $s \mapsto s\Psi'(s)$ and $s \mapsto s\,(\Psi')^{-1}(s)$ are convex on $(0, r]$. Moreover, there exists a positive nonincreasing differentiable function ξ such that

$$g'(t) \le -\xi(t)\Psi(g(t)), \quad \forall t \ge 0, \tag{2.2}$$

Remark 2.1. *[55] If Ψ is a strictly increasing, strictly convex C^2 function over $(0, r]$ and satisfying $\Psi(0) = \Psi'(0) = 0$, then it has an extension $\overline{\Psi}$, that is also strictly increasing and strictly convex C^2 over $(0, \infty)$. For example, if $\Psi(r) = a, \Psi'(r) = b, \Psi''(r) = c$, and for $t > r$, $\overline{\Psi}$ can be defined by*

$$\overline{\Psi}(t) = \frac{c}{2}t^2 + (b - cr)t + \left(a + \frac{c}{2}r^2 - br\right). \tag{2.3}$$

For simplicity, in the rest of this paper, we use Ψ instead of $\overline{\Psi}$

Remark 2.2. *Since Ψ is strictly convex on $(0, r]$ and $\Psi(0) = 0$, then*

$$\Psi(\theta z) \le \theta\Psi(z),\ 0 \le \theta \le 1 \text{ and } z \in (0, r]. \tag{2.4}$$

3. Technical lemmas

In this section, we establish some essential lemmas needed for the proof of our stability result.

Lemma 3.1. *The energy functional E, defined by*

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_z z_t^2 + a_1 z_x^2 + \rho_u u_t^2 + \left(a_3 - \int_0^\infty g(s)ds\right)u_x^2 + 2a_2 z_x u_x\right]dx + \frac{1}{2}(g \circ u_x)(t), \tag{3.1}$$

where

$$(g \circ v)(t) = \int_0^1 \int_0^\infty g(s)|v(t) - v(t-s)|^2\, dsdx, \tag{3.2}$$

and satisfies

$$E'(t) = \frac{1}{2}(g' \circ u_x)(t) \le 0. \tag{3.3}$$

Proof. By multiplying the first two equations of (1.14) by z_t and u_t, respectively, and then, integrating by parts over $(0, 1)$, we end up with (3.3). □

Lemma 3.2. *([55]) For all $u \in H_0^1([0, 1])$,*

$$\int_0^1 \left(\int_0^\infty g(s)|u_x(t) - u_x(t-s)|ds\right)^2 dx \le C_\alpha (h \circ u_x)(t), \tag{3.4}$$

for any $0 < \alpha < 1$,

$$C_\alpha = \left(\int_0^t \frac{g^2(s)}{\alpha g(s) - g'(s)}ds\right) \qquad and \qquad h(t) = \alpha g(t) - g'(t). \tag{3.5}$$

Lemma 3.3. *There exists a positive constant M_1 such that*

$$\int_0^1 \int_t^\infty g(s)\left(u_x(t) - u_x(t-s)\right)^2 dsdx \le M_1 h_0(t), \tag{3.6}$$

where $h_0(t) = \int_0^\infty g(t+s)\left(1 + \|u_{0x}(s)\|^2\right) ds.$

Proof.

$$\begin{aligned}
&\int_0^1 \int_t^\infty g(s)\|u_x(t) - u_x(t-s)\|^2 ds \le 2\|u_x(t)\|^2 \int_t^\infty g(s)ds + 2\int_t^\infty g(s)\|u_x(t-s)\|^2 ds \\
&\le 2\sup_{s\ge 0}\|u_x(s)\|^2 \int_0^\infty g(t+s)ds + 2\int_0^\infty g(t+s)\|u_x(-s)\|^2 ds \\
&\le \left(\frac{8a_2^2}{a_1}\sup_{s\ge 0} E(s)\right)\int_0^\infty g(t+s)ds + 2\int_0^\infty g(t+s)\|u_{0x}(s)\|^2 ds \\
&\le \left(\frac{8a_2^2}{a_1}E(0)\right)\int_0^\infty g(t+s)ds + 2\int_0^\infty g(t+s)\|u_{0x}(s)\|^2 ds \\
&\le M_1 \int_0^\infty g(t+s)\left(1 + \|u_{0x}(s)\|^2\right) ds,
\end{aligned} \tag{3.7}$$

where $M_1 = \max\left\{2, \left(\frac{8a_2^2}{a_1}E(0)\right)\right\}$. □

Lemma 3.4. *The functional*

$$F_1(t) := \rho_u \int_0^1 u_t u dx - \frac{a_2}{a_1}\rho_z \int_0^1 z_t u dx$$

satisfies, for any $\varepsilon_1 > 0$ and some constant a_0,

$$F_1'(t) \le -\frac{a_0}{2}\int_0^1 u_x^2 dx + \varepsilon_1 \int_0^1 z_t^2 dx + \left(\rho_u + \frac{a_2^2\rho_z^2}{4\varepsilon_1 a_1^2}\right)\int_0^1 u_t^2 dx + \frac{C_\alpha}{2a_0}(h \circ u_x)(t), \tag{3.8}$$

where $a_0 = a_3 - \frac{a_2^2}{a_1} - \int_0^\infty g(s)ds.$

Proof. Direct computations using integration by parts gives

$$F_1'(t) = -\left(a_3 - \frac{a_2^2}{a_1}\right)\int_0^1 u_x^2 dx + \rho_u \int_0^1 u_t^2 dx - \frac{a_2\rho_z}{a_1}\int_0^1 z_t u_t dx + \int_0^1 u_x \int_0^\infty g(s)u_x(t-s)dsdx. \tag{3.9}$$

Applying Young's inequality, we get for $\varepsilon_1 > 0$,

$$-\frac{a_2\rho_z}{a_1}\int_0^1 z_t u_t dx \le \varepsilon_1 \int_0^1 z_t^2 dx + \frac{a_2^2\rho_z^2}{4\varepsilon_1 a_1^2}\int_0^1 u_t^2 dx, \tag{3.10}$$

and for $\varepsilon_2 > 0$,

$$\begin{aligned}
\int_0^1 u_x \int_0^\infty g(s)u_x(t-s)dsdx &= \int_0^\infty g(s)ds \int_0^1 u_x^2 dx - \int_0^1 u_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx \\
&\le \left(\int_0^\infty g(s)ds + \frac{\varepsilon_2}{2}\right)\int_0^1 u_x^2 dx + \frac{C_\alpha}{2\varepsilon_2}(h \circ u_x)(t).
\end{aligned} \tag{3.11}$$

Combining (3.9)–(3.11), we end up with

$$F_1'(t) \le -\left(a_3 - \frac{a_2^2}{a_1} - \int_0^\infty g(s)ds - \frac{\varepsilon_2}{2}\right)\int_0^1 u_x^2 dx + \varepsilon_1 \int_0^1 z_t^2 dx + \left(\rho_u + \frac{a_2^2\rho_z^2}{\varepsilon_1 a_1^2}\right)\int_0^1 u_t^2 dx + \frac{C_\alpha}{2\varepsilon_2}(h \circ u_x)(t). \tag{3.12}$$

Using assumption (A), taking $\varepsilon_2 = a_0$, where $a_0 = a_3 - \frac{a_2^2}{a_1} - \int_0^\infty g(s)ds$, we obtain (3.8). □

Lemma 3.5. *Assume that* (A) *holds. Then, the functional*

$$F_2(t) := -\rho_u \int_0^1 u_t \int_0^\infty g(s)(u(t) - u(t-s))dsdx$$

satisfies, for any $\delta_2, \delta_3 > 0$,

$$F_2'(t) \le -\frac{\rho_u c_0}{2}\int_0^1 u_t^2 dx + \delta_1 \int_0^1 u_x^2 dx + \delta_3 a_2^2 \int_0^1 z_x^2 dx + \left[\frac{cC_\alpha}{\delta_1} + \frac{c}{\delta_2}(1 + C_\alpha) + \frac{C_\alpha}{\delta_3} + C_\alpha\right](h \circ u_x)(t). \tag{3.13}$$

Proof. Differentiating F_2, taking into account (1.14), and using integrating by parts, we obtain

$$\begin{aligned}
F_2'(t) &= -\rho_u \int_0^\infty g(s)ds \int_0^1 u_t^2 dx + a_3 \int_0^1 u_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx \\
&- \int_0^1 \int_0^\infty g(s)u_x(s)ds \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx \\
&- \rho_u \int_0^1 u_t \int_0^\infty g'(s)(u(t) - u(t-s))dsdx + a_2 \int_0^1 z_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx \\
&= -\rho_u \int_0^\infty g(s)ds \int_0^1 u_t^2 dx + \int_0^1 \left(\int_0^\infty g(s)(u_x(t) - u_x(t-s))\right)^2 dx \\
&+ \left(a_3 - \int_0^\infty g(s)ds\right)\int_0^1 u_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx \\
&+ a_2 \int_0^1 z_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx - \rho_u \int_0^1 u_t \int_0^\infty g'(s)(u(t) - u(t-s))dsdx.
\end{aligned} \tag{3.14}$$

Using Young's inequality and Lemma 3.2, for any $\delta_1 > 0$, we obtain

$$\left(a_3 - \int_0^\infty g(s)ds\right)\int_0^1 u_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))dsdx \le \delta_1 \int_0^1 u_x^2(t)dx + \frac{cC_\alpha}{\delta_1}(h \circ u_x)(t). \tag{3.15}$$

Similarly, we can get for any $\delta_2 > 0$

$$-\quad \rho_u \int_0^1 u_t(t) \int_0^\infty g'(s)(z(t) - z(t-s))dsdx$$

$$
\begin{aligned}
&= \rho_u \int_0^1 u_t(t) \int_0^\infty h(s)(u(t)-u(t-s))dsdx - \rho_u \int_0^1 u_t(t) \int_0^\infty \alpha g(s)(u(t)-u(t-s))dsdx \\
&\le \delta_2 \int_0^1 u_t^2(t)dx + \rho_u^2 \frac{\left(\int_0^\infty h(s)\,ds\right)}{2\delta_2}(h \circ u)(t) + \rho_u^2 \frac{\alpha^2}{2\delta_2} \int_0^1 \left(\int_0^\infty g(s)(u(t)-u(t-s)ds\right)^2 dx \\
&\le \delta_2 \int_0^1 u_t^2(t)dx + \frac{c}{2\delta_2}(h \circ u_x)(t) + \frac{\alpha^2 c C_\alpha}{2\delta_2}(h \circ u_x)(t) \\
&\le \delta_2 \int_0^1 u_t^2(t)dx + \frac{c}{\delta_2}(1 + C_\alpha)(h \circ u_x)(t).
\end{aligned}
$$

Using Young's inequality and performing similar calculations as in (3.15), we obtain for any $\delta_3 > 0$

$$
a_2 \int_0^1 u_x \int_0^t g(t-s)(u_x(t) - u_x(s))dsdx \le \delta_3 a_2^2 \int_0^1 u_x^2\, dx + \frac{C_\alpha}{\delta_3}(h \circ u_x)(t). \tag{3.16}
$$

Combining all the above estimates, we get

$$
\begin{aligned}
F_2'(t) \le &-\rho_u \left(\int_0^\infty g(s)ds - \delta_2\right) \int_0^1 u_t^2 dx + \delta_2 \int_0^1 u_x^2 dx + \delta_3 a_2^2 \int_0^1 z_x^2 dx \\
&+ \left[\frac{cC_\alpha}{\delta_1} + \frac{c}{\delta_2}(1 + C_\alpha) + \frac{C_\alpha}{\delta_3} + C_\alpha\right](h \circ u_x)(t).
\end{aligned} \tag{3.17}
$$

By taking $\delta_2 = \frac{\rho_u c_0}{2}$, we end up with the desired inequality (3.13). □

Lemma 3.6. *Assume that* (A) *holds. Then the functional*

$$
F_3(t) := a_2 \int_0^1 (\rho_z u z_t - \rho_u z u_t)\, dx
$$

satisfies

$$
\begin{aligned}
F_3'(t) \le &-\frac{a_2^2}{2} \int_0^1 z_x^2 dx + \left(a_2^2 + \frac{3}{2}(a_1^2 + a_3^2) + \frac{3}{2}\ell\right) \int_0^1 u_x^2 dx + \frac{3}{2(1-\ell)}(h \circ u_x)(t) \\
&+ 2\eta_1 \int_0^1 u_t^2 dx + \frac{a_2^2}{\eta_1}(\rho_u^2 + \rho_z^2) \int_0^1 z_t^2 dx.
\end{aligned} \tag{3.18}
$$

Proof. Direct computations give

$$
\begin{aligned}
F_3'(t) = &\, a_2^2 \int_0^1 z_x^2 dx - a_2^2 \int_0^1 u_x^2 dx + a_2 (a_3 - a_1) \int_0^1 u_x z_x dx - a_2 \int_0^1 z_x \int_0^\infty g(s) u_x(t-s) dsdx \\
&+ a_2(\rho_z - \rho_u) \int_0^1 u_t z_t dx.
\end{aligned} \tag{3.19}
$$

Using Young's inequality,

$$
a_2 (a_1 - a_3) \int_0^1 u_x z_x dx \le \frac{a_2^2}{3} \int_0^1 z_x^2 dx + \frac{3}{2}\left(a_1^2 + a_3^2\right)^2 \int_0^1 u_x^2 dx.
$$

Similarly, we can get for any $\eta_1 > 0$,

$$a_2(\rho_z - \rho_u)\int_0^1 u_t z_t dx \leq \frac{\eta_1}{2}\int_0^1 u_t^2 dx + \left(\frac{a_2^2\rho_z^2}{\eta_1} + \frac{a_2^2\rho_u^2}{\eta_1}\right)\int_0^1 z_t^2 dx. \tag{3.20}$$

Exploiting Young's inequality again, we obtain

$$\begin{aligned} a_2\int_0^1 z_x \int_0^\infty g(s)u_x(t-s)dsdx \leq & \frac{a_2^2}{6}\int_0^1 z_x^2 dx \\ & + \frac{3}{2}\int_0^1 \left(\int_0^\infty g(s)\left(|u_x(t-s)| - |u_x(t) + |u_x(t)|\right)ds\right)^2 dx. \end{aligned}$$

Recalling that $\int_0^\infty g(s)ds = \ell$ and exploiting Young's and Cauchy-Schwarz inequalities and using $(a+b)^2 \leq (1+\eta)a^2 + (1+\frac{1}{\eta})b^2,\ \forall \eta > 0$, we obtain

$$\begin{aligned} & \tfrac{3}{2} \int_0^1 \left(\int_0^\infty g(s)\left(|u_x(t-s) - |u_x(t)| + |u_x(t)|\right)ds\right)^2 dx \\ \leq\ & \tfrac{3}{2}\ (1+\eta_2)\ell^2\int_0^1 u_x^2 dx + \frac{3}{2}(1+\frac{1}{\eta_2})\int_0^1\left(\int_0^\infty g(s)|u_x(t) - u_x(t-s)|ds\right)^2 dx. \end{aligned} \tag{3.21}$$

By taking $\eta_2 = \frac{1-\ell}{\ell}$, and aiming to lemma 3.2, we obtain

$$a_2\int_0^1 z_x(t)\int_0^t g(t-s)u_x(s)dsdx \ \leq\ \frac{a_2^2}{6}\int_0^1 u_x^2\, dx + \frac{3}{2\ell}\int_0^1 z_x^2 dx + \frac{3}{2(1-\ell)}C_\alpha(h\circ u_x)(t).$$

Estimate (3.18) follows by combining all the above estimates. □

Lemma 3.7. *The functional*

$$F_4(t) := -\rho_z\int_0^1 z_t z dx$$

satisfies, for any $\varepsilon_3 > 0$,

$$F_4'(t) \leq -\rho_z\int_0^1 z_t^2 dx + \left(a_1 + \frac{a_2^2}{\varepsilon_3}\right)\int_0^1 z_x^2 dx + \varepsilon_3\int_0^1 u_x^2 dx. \tag{3.22}$$

Proof. It is straightforward to see that

$$F_4'(t) = -\rho_z\int_0^1 z_t^2 dx + a_1\int_0^1 z_x^2 dx + a_2\int_0^1 z_x u_x dx.$$

Using Young's inequality and the fact that $a_1a_3 > a_2^2$, we end up with (3.22). □

Lemma 3.8. *([55]) Under the assumption* (*A*)*, the functional*

$$F_5(t) := \int_\Omega\int_0^t r(t-s)|u_x(s)|^2 dsdx, \tag{3.23}$$

satisfies the estimate

$$F_5'(t) \leq -\frac{1}{2}(g \circ u_x)(t) + 3\ell \int_0^1 u_x^2\, dx + \frac{1}{2}\int_0^1 \int_t^{+\infty} g(s)\,(u_x(t) - u_x(t-s)ds)^2\, dsdx, \tag{3.24}$$

where $r(t) = \int_t^{+\infty} g(s)ds$.

Proof. The proof can be achieved by following the same calculations in [40]. □

Lemma 3.9. *The functional* $\mathcal{L}$ *defined by*

$$\mathcal{L}(t) = \mu E(t) + \sum_{i=1}^{4} \mu_i F_i(t), \tag{3.25}$$

satisfies, for suitable choice of $\mu, \mu_1, \mu_2, \mu_3, \mu_4$ *and for all* $t \geq 0$,

$$\mathcal{L}'(t) \leq -4\ell \int_0^1 u_x^2\, dx - \int_0^1 z_x^2\, dx - \int_0^1 u_t^2\, dx - \int_0^1 z_t^2\, dx + \frac{1}{4}(g \circ u_x)(t), \tag{3.26}$$

and

$$\mathcal{L}(t) \sim E(t). \tag{3.27}$$

Proof. By taking the derivative of the functional $\mathcal{L}$ and using the above estimates, we get

$$\begin{aligned}
\mathcal{L}'(t) \ \leq\ & \frac{\mu\alpha}{2}(g \circ u_x)(t) - \left(\mu_3 \frac{a_2^2}{2} - \mu_2 \delta_3 a_2^2 - \mu_4\left[a_1 + \frac{a_2^2}{\epsilon_3}\right]\right) \int_0^1 z_x^2 dx \\
- & \left(\mu_1 \frac{a_0}{2} - \mu_2 \delta_1 - \mu_3\left[\frac{3}{2}(a_1^2 + a_3^2 + \ell) + a_2^2\right] - \mu_4 \epsilon_3\right) \int_0^1 u_x^2\, dx \\
- & \left(\frac{\mu_2 \rho_z c_0}{2} - \mu_1\left[\rho_u + \frac{a_2^2 \rho_z^2}{4\epsilon_1 a_1^2}\right] - 2\eta_1 \mu_3\right) \int_0^1 u_t^2 dx \\
- & \left(\mu_4 \rho_z - \mu_1 \epsilon_1 - \mu_3\left[\frac{a_2^2 \rho_z^2}{\eta_1} + \frac{a_2^2 \rho_u^2}{\eta_1}\right]\right) \int_0^1 z_t^2 dx \\
- & \left(\frac{\mu}{2} - \mu_1 \frac{C_\alpha}{2a_0} - \mu_2\left[\frac{cC_\alpha}{\delta_1} + \frac{c}{\delta_2}(1 + C_\alpha) + \frac{C_\alpha}{\delta_3} + C_\alpha\right] - \frac{3\mu_3}{2(1-\ell)} C_\alpha\right)(h \circ u_x)(t).
\end{aligned}$$

Firstly, we select μ_4 large enough such that

$$\beta_1 := \mu_4 \rho_z - \mu_3\left[\frac{a_2^2 \rho_z^2}{\eta_1} + \frac{a_2^2 \rho_u^2}{\eta_1}\right] - 1 > 0. \tag{3.28}$$

Next, we take μ_3 large enough such that

$$\beta_2 := \mu_3 \frac{a_2^2}{2} - \mu_4\left[a_3 + \frac{a_2^2}{\epsilon_3}\right] - 1 > 0, \tag{3.29}$$

and then, we choose μ_1 so large that

$$\beta_3 := \mu_1 \frac{a_0}{2} - \mu_3 \left[\frac{3}{2}(a_1^2 + a_3^2 + \ell) + a_2^2 \right] - 4\ell > 0. \tag{3.30}$$

After fixing μ_1, we select μ_2 large enough such that

$$\beta_4 := \frac{\mu_2 \rho_z c_0}{2} - \mu_1 \left[\rho_u + \frac{a_2^2}{4\epsilon_1 a_1^2} \rho_z^2 \right] - 1 > 0. \tag{3.31}$$

Now, for any fixed $\mu_1, \mu_2, \mu_3, \mu_4 > 0$, we pick $\delta_3 < \frac{\beta_2}{2\mu_2 a_2^2}$, $\delta_1 < \frac{\beta_3}{4\mu_2}$, $\epsilon_3 < \frac{\beta_3}{4\mu_4}$, $\epsilon_1 < \frac{\beta_1}{2\mu_1}$ and $\eta_1 < \frac{\beta_4}{4\mu_3}$, so that the following estimates are satisfied

$$\mu_3 \frac{a_2^2}{2} - \mu_2 \delta_3 a_2^2 - \mu_4 \left[a_3 + \frac{a_2^2}{\epsilon_3} \right] > 1, \tag{3.32}$$

$$\mu_1 \frac{a_0}{2} - \mu_2 \delta_1 - \mu_3 \left[\frac{3}{2}(a_1^2 + a_3^2 + \ell) + a_2^2 \right] - \mu_4 \epsilon_3 > 4\ell, \tag{3.33}$$

$$\mu_4 \rho_z - \mu_1 \epsilon_1 - \mu_3 \left[\frac{a_2^2 \rho_z^2}{\eta_1} + \frac{a_2^2 \rho_u^2}{\eta_1} \right] > 1. \tag{3.34}$$

Since $\frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$, using the Lebesgue dominated convergence theorem, we can get

$$\alpha C_\alpha = \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)}\, ds \longrightarrow 0, \qquad \text{as} \qquad \alpha \to 0. \tag{3.35}$$

Hence, there exists some $0 < \alpha^* < 1$, such that if $\alpha < \alpha^*$, then

$$\alpha C_\alpha < \frac{1}{8\left(\frac{\mu_1}{2a_0} + \mu_2 \left[\frac{c}{\delta_1} + \frac{c}{\delta_2} + \frac{1}{\delta_3} + 1 \right] + \mu_3 \frac{3}{2(1-\ell)} \right)}. \tag{3.36}$$

By putting $\alpha = \frac{1}{2\mu}$ and choosing μ sufficiently large such that

$$\frac{\mu}{4} - \frac{c\mu_2}{\delta_2} > 0, \tag{3.37}$$

which gives

$$\frac{\mu}{2} - \frac{c\mu_2}{\delta_2} - C_\alpha \left(\frac{\mu_1}{2a_0} + \mu_2 \left[\frac{c}{\delta_1} + \frac{c}{\delta_2} + \frac{1}{\delta_3} + 1 \right] + \frac{3\mu_3}{2(1-\ell)} \right) > 0. \tag{3.38}$$

Hence, we conclude that (3.26) holds. Moreover, we can choose μ even larger (if needed) so that (3.27) is satisfied, which means that, for some constants $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t).$$

□

Lemma 3.10. *Assume that* (*A*) *holds. Then, the energy functional satisfies, for all* $t \in \mathbb{R}^+$, *the following estimate*

$$\int_0^t E(s)ds < \tilde{m}h_1(t), \tag{3.39}$$

where $h_1(t) = \left(1 + \int_0^t h_0(s)ds\right)$ *and* h_0 *is defined in Lemma* (3.3).

Proof. Let $\mathcal{F}(t) = \mathcal{L}(t) + F_5(t)$, then using (3.26) and (3.23), we obtain for all $t \in \mathbb{R}^+$,

$$\begin{aligned} \mathcal{F}'(t) &\leq -\ell \int_0^1 u_x^2\, dx - \int_0^1 z_x^2\, dx - \int_0^1 u_t^2\, dx - \int_0^1 z_t^2\, dx - \frac{1}{4}(g \circ u_x)(t) \\ &\leq -\lambda E(t) + \frac{1}{2}\int_0^1 \int_t^{+\infty} g(s)\,(u_x(t) - u_x(t-s))^2\, dsdx, \end{aligned} \tag{3.40}$$

where λ is some positive constant. Therefore,

$$\begin{aligned} \lambda \int_0^t E(s)ds &\leq \mathcal{F}(0) - \mathcal{F}(t) + \frac{M_1}{2}\int_0^t \int_0^{+\infty} g(\tau + s)\,(1 + |u_{0x}(s)|ds)^2\, d\tau ds \\ &\leq F(0) + \frac{M_1}{2}\int_0^t h_0(s)ds. \end{aligned} \tag{3.41}$$

Therefore, (3.39) is established with $\tilde{m} = \max\left\{\frac{F(0)}{\lambda}, \frac{M_1}{2\lambda}\right\}$. □

Corollary 3.11. *There exists* $0 < q_0 < 1$ *such that, for all* $t \geq 0$, *we have the following estimate:*

$$\int_0^t g(s) \int_0^1 |u_x(t) - u_x(t-s)|^2 dxds \leq \frac{1}{q(t)} \Psi^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right), \tag{3.42}$$

where Ψ *is defined in Remark* (2.1),

$$\mu(t) := -\int_0^t g'(s) \int_0^1 |u_x(t) - u_x(t-s)|^2 dxds \leq -cE'(t), \tag{3.43}$$

and

$$q(t) := \frac{q_0}{h_1(t)} < 1. \tag{3.44}$$

Proof. Using (3.1) and (3.39), we have

$$\begin{aligned} q(t)\int_0^1 \int_0^t (u_x(t) - u_x(t-s))^2\, dsdx &\leq 2q(t)\int_0^1 \int_0^t \left(|u_x(t)|^2 + |u_x(t-s)|^2\right) dsdx \\ &\leq \left(\frac{4a_2^2 q(t)}{a_1}\right)\int_0^t (E(t) + E(t-s))\, dsdx \\ &\leq \left(\frac{8a_2^2 q(t)}{a_1}\right)\int_0^t E(s)dsdx \leq \left(\frac{8a_2^2 q(t)}{a_1}\right)\tilde{m}h_1(t), \forall t \in \mathbb{R}^+. \end{aligned} \tag{3.45}$$

Thanks to (3.39), then for all $t \geq 0$ and for $0 < q_0 < \min\left\{1, (\frac{a_1}{8a_2^2\tilde{m}})\right\}$, we have

$$q(t)\int_0^t \int_0^1 |u_x(t) - u_x(t-s)|^2 dxds < 1.$$

Then, the rest of the proof of (3.42) is straightforward as the one in [56]. □

4. A decay result

In this section, we state and prove a new general decay result for our problem (1.14). We introduce the following functions:

$$\Psi_1(t) := \int_t^1 \frac{1}{s\Psi'(s)}ds, \tag{4.1}$$

$$\Psi_2(t) = t\Psi'(t), \quad \Psi_3(t) = t(\Psi')^{-1}(t), \quad \Psi_4(t) = \Psi_3^*(t). \tag{4.2}$$

Further, we introduce the class S of functions $\chi : \mathbb{R}_+ \to \mathbb{R}_+^*$ satisfying for fixed $c_1, c_2 > 0$ (should be selected carefully in (4.18)):

$$\chi \in C^1(\mathbb{R}_+), \quad \chi \leq 1, \quad \chi' \leq 0, \tag{4.3}$$

and

$$c_2\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] \leq c_1\left(\Psi_2\Big(\frac{\Psi_5(t)}{\chi(t)}\Big) - \frac{\Psi_2\left(\Psi_5(t)\right)}{\chi(t)}\right), \tag{4.4}$$

where $d > 0$, c is a generic positive constant which may change from line to line, h_0 and q are defined in (3.6) and (3.44) and

$$\Psi_5(t) = \Psi_1^{-1}\Big(c_1\int_0^t \xi(s)ds\Big). \tag{4.5}$$

Remark 4.1. *According to the properties of* Ψ *introduced in* (*A*)*,* Ψ_2 *is convex increasing and defines a bijection from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$*,* Ψ_1 *is decreasing defines a bijection from* $(0,1]$ *to* $\mathbb{R}_+$*, and* Ψ_3 *and* Ψ_4 *are convex and increasing functions on* $(0,r]$*. Then the set S is not empty because it contains* $\chi(s) = \varepsilon\Psi_5(s)$ *for any* $0 < \varepsilon \leq 1$ *small enough. Indeed,* (4.3) *is satisfied (since* (4.1) *and* (4.5)*). On the other hand, we have* $q(t)h_0(t)$ *is nonincreasing,* $0 < \Psi_5 \leq 1$*, and* Ψ' *and* Ψ_4 *are increasing, then* (4.4) *is satisfied if*

$$c_2\Psi_4\Big[\frac{c}{d}q_0h_0(0)\Big] \leq \frac{c_1}{\varepsilon}\Big(\Psi'\Big(\frac{1}{\varepsilon}\Big) - \Psi'(1)\Big)$$

which holds, for $0 < \varepsilon \leq 1$ *small enough, since* $\lim_{t\to+\infty}\Psi'(t) = +\infty$*. But with the choice* $\chi = \varepsilon\Psi_5$*,* (4.6) *(below) does not lead to any stability estimate. The idea is to choose* χ *satisfy* (4.3) *and* (4.4) *such that* (4.6) *gives the best possible decay rate for E.*

Theorem 4.2. *Assume that* (*A*) *holds, then there exists a strictly positive constant C such that, for any* χ *satisfying* (4.3) *and* (4.4)*, the solution of* (1.14) *satisfies, for all* $t \geq 0$*,*

$$E(t) \leq \frac{C\Psi_5(t)}{\chi(t)q(t)}. \tag{4.6}$$

Proof. We start by combining (3.3), (3.6), (3.26), and (3.42), then, for some $m > 0$ and for any $t \geq 0$, we have

$$L'(t) \leq -mE(t) + \frac{c}{q(t)}\Psi^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right) + ch_0(t). \tag{4.7}$$

Without loss of generality, one can assume that $E(0) > 0$. For $\varepsilon_0 < r$, let the functional $\mathcal{F}$ defined by

$$\mathcal{F}(t) := \Psi'\left(\varepsilon_0\frac{E(t)q(t)}{E(0)}\right)L(t),$$

which satisfies $\mathcal{F} \sim E$. By noting that $\Psi'' \geq 0$, $q' \leq 0$, $E' \leq 0$ and using (4.7), we get

$$\begin{aligned}\mathcal{F}'(t) &= \varepsilon_0 \frac{(qE)'(t)}{E(0)} \Psi'' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) L(t) + \Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) L'(t) \\ &\leq -mE(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{c}{q(t)} \Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \Psi^{-1} \left(\frac{q(t)\mu(t)}{\xi(t)}\right) \\ &+ ch_0(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right).\end{aligned} \tag{4.8}$$

Let Ψ^* be the convex conjugate of Ψ in the sense of Young (see [57]), then

$$\Psi^*(s) = s(\Psi')^{-1}(s) - \Psi\left[(\Psi')^{-1}(s)\right], \quad \text{if } s \in (0, \Psi'(r)] \tag{4.9}$$

and Ψ^* satisfies the following generalized Young inequality

$$AB \leq \Psi^*(A) + \Psi(B), \quad \text{if } A \in (0, \Psi'(r)], \ B \in (0, r]. \tag{4.10}$$

So, with $A = \Psi'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)$ and $B = \Psi^{-1}\left(\frac{q(t)\mu(t)}{\xi(t)}\right)$ and using (3.3) and (4.8)–(4.10), we arrive at

$$\begin{aligned}\mathcal{F}'(t) &\leq -mE(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{c}{q(t)} \Psi^* \left(\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\right) + c\left(\frac{\mu(t)}{\xi(t)}\right) \\ &+ ch_0(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \\ &\leq -mE(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)} \Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + c\left(\frac{\mu(t)}{\xi(t)}\right) \\ &+ ch_0(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right).\end{aligned} \tag{4.11}$$

So, multiplying (4.11) by $\xi(t)$ and using (3.43) and the fact that $\varepsilon_0 \frac{E(t)q(t)}{E(0)} < r$ gives

$$\begin{aligned}\xi(t)\mathcal{F}'(t) &\leq -m\xi(t)E(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + c\xi(t)\varepsilon_0 \frac{E(t)}{E(0)} \Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \\ &+ c\mu(t) + c\xi(t)h_0(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \\ &\leq -\varepsilon_0 \left(\frac{mE(0)}{\varepsilon_0} - c\right) \xi(t) \frac{E(t)}{E(0)} \Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) - cE'(t) + c\xi(t)h_0(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right).\end{aligned} \tag{4.12}$$

Consequently, recalling the definition of Ψ_2 and choosing ε_0 so that $k = (\frac{mE(0)}{\varepsilon_0} - c) > 0$, we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}\mathcal{F}_1'(t) &\leq -k\varepsilon_0\xi(t) \left(\frac{E(t)}{E(0)}\right) \Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + c\xi(t)h_0(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \\ &= -k\frac{\xi(t)}{q(t)} \Psi_2 \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + c\xi(t)h_0(t)\Psi' \left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right),\end{aligned} \tag{4.13}$$

where $\mathcal{F}_1 = \xi\mathcal{F} + cE \sim E$ and satisfies for some $\alpha_1, \alpha_2 > 0$.

$$\alpha_1 \mathcal{F}_1(t) \leq E(t) \leq \alpha_2 \mathcal{F}_1(t). \tag{4.14}$$

Since $\Psi_2'(t) = \Psi'(t) + t\Psi''(t)$, then, using the strict convexity of Ψ on $(0, r]$, we find that $\Psi_2'(t), \Psi_2(t) > 0$ on $(0, r]$.

Let $d > 0$. Using the general Young's inequality (4.10) on the last term in (4.13) with $A = \Psi'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)$ and $B = [\frac{c}{d} h_0(t)]$, we have

$$\begin{aligned} ch_0(t)\Psi'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) &= \frac{d}{q(t)}\left[\frac{c}{d}q(t)h_0(t)\right]\left(\Psi'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\right) \\ &\le \frac{d}{q(t)}\Psi_3\left(\Psi'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\right) + \frac{d}{q(t)}\Psi_3^*\left[\frac{c}{d}q(t)h_0(t)\right] \\ &\le \frac{d}{q(t)}\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\left(\Psi'\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right)\right) + \frac{d}{q(t)}\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] \\ &\le \frac{d}{q(t)}\Psi_2\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{d}{q(t)}\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right]. \end{aligned} \tag{4.15}$$

Now, combining (4.13) and (4.15) and choosing d small enough so that $k_1 = (k - d) > 0$, we arrive at

$$\begin{aligned} \mathcal{F}_1'(t) &\le -k\frac{\xi(t)}{q(t)}\Psi_2\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}\Psi_2\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] \\ &\le -k_1\frac{\xi(t)}{q(t)}\Psi_2\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right]. \end{aligned} \tag{4.16}$$

Using the equivalent property in (4.14) and the nonincreasing of Ψ_2, we have, for some $d_0 = \frac{\alpha_1}{E(0)} > 0$,

$$\Psi_2\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) \ge \Psi_2\left(d_0 \mathcal{F}_1(t)q(t)\right).$$

Letting $\mathcal{F}_2(t) := d_0\mathcal{F}_1(t)q(t)$ and recalling $q' \le 0$, then we arrive at,

$$\mathcal{F}_2'(t) \le d_0 q(t)\left(-k_1\frac{\xi(t)}{q(t)}\Psi_2\left(\varepsilon_0 \frac{E(t)q(t)}{E(0)}\right) + \frac{d\xi(t)}{q(t)}\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right]\right). \tag{4.17}$$

Then, (4.17) becomes for some constant $c_1 = d_0k_1 > 0$ and $c_2 = d_0 d > 0$,

$$\mathcal{F}_2'(t) \le -c_1\xi(t)\Psi_2(\mathcal{F}_2(t)) + c_2\xi(t)\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right]. \tag{4.18}$$

Since $d_0q(t)$ is is nonincreasing. Using the equivalent property $\mathcal{F}_1 \sim E$ implies that there exists $b_0 > 0$ such that $\mathcal{F}_2(t) \ge b_0E(t)q(t)$. Let $t \in \mathbb{R}_+$ and $\chi(t)$ satisfying (4.3) and (4.4).
If

$$b_0 q(t)E(t) \le 2\frac{\Psi_5(t)}{\chi(t)}, \tag{4.19}$$

then, we have

$$E(t) \le \frac{2}{b_0}\frac{\Psi_5(t)}{\chi(t)q(t)}. \tag{4.20}$$

If

$$b_0 q(t)E(t) > 2\frac{\Psi_5(t)}{\chi(t)}, \tag{4.21}$$

then, for any $0 \le s \le t$, we get

$$b_0 q(s)E(s) > 2\frac{\Psi_5(t)}{\chi(t)}, \tag{4.22}$$

since, $q(t)E(t)$ is nonincreasing function. Therefore, for any $0 \le s \le t$, we have

$$\mathcal{F}_2(s) > 2\frac{\Psi_5(t)}{\chi(t)}. \tag{4.23}$$

Using (2.4), $0 < \chi \le 1$ and the fact that Ψ_2 is convex, we get, for any $0 < \epsilon_1 \le 1$,

$$\begin{aligned}\Psi_2\Big(\epsilon_1\chi(s)\mathcal{F}_2(s) - \epsilon_1\Psi_5(s)\Big) &= \Psi_2\Big(\epsilon_1\chi(s)\mathcal{F}_2(s) - \frac{\epsilon_1\chi(s)\Psi_5(s)}{\chi(s)}\Big)\\ &\le \epsilon_1\chi(s)\Psi_2\Big(\mathcal{F}_2(s) - \frac{\Psi_5(s)}{\chi(s)}\Big).\end{aligned} \tag{4.24}$$

Recalling the definition of Ψ_2, that is $\Psi_2(t) = t\Psi'(t)$, we obtain

$$\begin{aligned}&\Psi_2\Big(\epsilon_1\chi(s)\mathcal{F}_2(s) - \epsilon_1\Psi_5(s)\Big) \le \epsilon_1\chi(s)\Big(\mathcal{F}_2(s) - \frac{\Psi_5(s)}{\chi(s)}\Big)\Psi'\Big(\mathcal{F}_2(s) - \frac{\Psi_5(s)}{\chi(s)}\Big)\\ &\le \epsilon_1\chi(s)\mathcal{F}_2(s)\Psi'\Big(\mathcal{F}_2(s) - \frac{\Psi_5(s)}{\chi(s)}\Big) - \epsilon_1\chi(s)\frac{\Psi_5(s)}{\chi(s)}\Psi'\Big(\mathcal{F}_2(s) - \frac{\Psi_5(s)}{\chi(s)}\Big).\end{aligned} \tag{4.25}$$

Now, using (4.23) and the fact that Ψ' is increasing, for any $0 \le s \le t$, we arrive at

$$\Psi'\Big(\mathcal{F}_2(s) - \frac{\Psi_5(s)}{\chi(s)}\Big) < \Psi'\Big(\mathcal{F}_2(s)\Big), \quad \Psi'\Big(\mathcal{F}_2(s) - \frac{\Psi_5(s)}{\chi(s)}\Big) > \Psi'\Big(\frac{\Psi_5(s)}{\chi(s)}\Big). \tag{4.26}$$

Therefore, we have

$$\Psi_2\Big(\epsilon_1\chi(s)\mathcal{F}_2(s) - \epsilon_1\Psi_5(s)\Big) \le \epsilon_1\chi(s)\mathcal{F}_2(s)\Psi'\Big(\mathcal{F}_2(s)\Big) - \epsilon_1\chi(s)\frac{\Psi_5(s)}{\chi(s)}\Psi'\Big(\frac{\Psi_5(s)}{\chi(s)}\Big). \tag{4.27}$$

Now, we let

$$\mathcal{F}_3(s) = \epsilon_1\chi(s)\mathcal{F}_2(s) - \epsilon_1\Psi_5(s), \tag{4.28}$$

where ϵ_1 small enough so that $\mathcal{F}_3(0) \le 1$. Then, (4.27) becomes, for any $0 \le s \le t$,

$$\Psi_2\Big(\mathcal{F}_3(s)\Big) \le \epsilon_1\chi(t)\Psi_2\Big(\mathcal{F}_2(s)\Big) - \epsilon_1\chi(t)\Psi_2\Big(\frac{\Psi_5(s)}{\chi(s)}\Big). \tag{4.29}$$

Further, we get

$$\mathcal{F}_3'(t) = \epsilon_1\chi'(t)\mathcal{F}_2(t) + \epsilon_1\chi(s)\mathcal{F}_2'(t) - \epsilon_1\Psi_5'(t). \tag{4.30}$$

Since $\chi' \le 0$ and using (4.18), then for any $0 \le s \le t$, $0 < \epsilon_1 \le 1$, we obtain

$$\begin{aligned}&\mathcal{F}_3'(t) \le \epsilon_1\chi(s)\mathcal{F}_2'(t) - \epsilon_1\Psi_5'(t)\\ &\le -c_1\epsilon_1\xi(t)\chi(t)\Psi_2(\mathcal{F}_2(t)) + c_2\epsilon_1\xi(t)\chi(s)\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] - \epsilon_1\Psi_5'(t).\end{aligned} \tag{4.31}$$

Then, using (4.29), we get

$$\begin{aligned}\mathcal{F}_3'(t) &\leq -c_1\xi(t)\Psi_2(\mathcal{F}_3(t)) - c_1\epsilon_1\xi(t)\chi(t)\Psi_2\Big(\frac{\Psi_5(s)}{\chi(s)}\Big)\\&+ c_2\epsilon_1\xi(t)\chi(t)\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] - \epsilon_1\Psi_5'(t).\end{aligned}\tag{4.32}$$

From the definition of Ψ_1 and Ψ_5, we have

$$\Psi_1\left(\Psi_5(s)\right) = c_1\int_0^s \xi(\tau)d\tau,$$

hence,

$$\Psi_5'(s) = -c_1\xi(s)\Psi_2\left(\Psi_5(s)\right).\tag{4.33}$$

Now, we have

$$\begin{aligned}&c_2\epsilon_1\xi(t)\chi(t)\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] - c_1\epsilon_1\xi(t)\chi(t)\Psi_2\Big(\frac{\Psi_5(s)}{\chi(s)}\Big) - \epsilon_1\Psi_5'(t)\\&= c_2\epsilon_1\xi(t)\chi(t)\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] - c_1\epsilon_1\xi(t)\chi(t)\Psi_2\Big(\frac{\Psi_5(s)}{\chi(s)}\Big) + c_1\epsilon_1\xi(t)\Psi_2\left(\Psi_5(t)\right)\\&= \epsilon_1\xi(t)\chi(t)\Big(c_2\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] - c_1\Psi_2\Big(\frac{\Psi_5(s)}{\chi(s)}\Big) + c_1\frac{\Psi_2\left(\Psi_5(t)\right)}{\chi(t)}\Big).\end{aligned}\tag{4.34}$$

Then, according to (4.4), we get

$$\epsilon_1\xi(t)\chi(t)\Big(c_2\Psi_4\left[\frac{c}{d}q(t)h_0(t)\right] - c_1\Psi_2\Big(\frac{\Psi_5(s)}{\chi(s)}\Big) - c_1\frac{\Psi_2\left(\Psi_5(t)\right)}{\chi(t)}\Big) \leq 0.$$

Then (4.32) gives

$$\mathcal{F}_3'(t) \leq -c_1\xi(t)\Psi_2(\mathcal{F}_3(t)).\tag{4.35}$$

Thus from (4.35) and the definition of Ψ_1 and Ψ_2 in (4.1) and (4.2), we obtain

$$\Big(\Psi_1\left(\mathcal{F}_3(t)\right)\Big)' \geq c_1\xi(t).\tag{4.36}$$

Integrating (4.36) over $[0, t]$, we get

$$\Psi_1\left(\mathcal{F}_3(t)\right) \geq c_1\int_0^t \xi(s)ds + \Psi_1\left(\mathcal{F}_3(0)\right).\tag{4.37}$$

Since Ψ_1 is decreasing, $\mathcal{F}_3(0) \leq 1$ and $\Psi_1(1) = 0$, then

$$\mathcal{F}_3(t) \leq \Psi_1^{-1}\Big(c_1\int_0^t \xi(s)ds\Big) = \Psi_5(t).\tag{4.38}$$

Recalling that $\mathcal{F}_3(t) = \epsilon_1\chi(t)\mathcal{F}_2(t) - \epsilon_1\Psi_5(t)$, we have

$$\mathcal{F}_2(t) \leq \frac{(1+\epsilon_1)}{\epsilon_1}\frac{\Psi_5(t)}{\chi(t)}.\tag{4.39}$$

Similarly, recall that $\mathcal{F}_2(t) := d_0\mathcal{F}_1(t)q(t)$, then

$$\mathcal{F}_1(t) \leq \frac{(1+\epsilon_1)}{d_0\epsilon_1}\frac{\Psi_5(t)}{\chi(t)q(t)}. \tag{4.40}$$

Since $\mathcal{F}_1 \sim E$, then for some $b > 0$, we have $E(t) \leq b\mathcal{F}_1$; which gives

$$E(t) \leq \frac{b(1+\epsilon_1)}{d_0\epsilon_1}\frac{\Psi_5(t)}{\chi(t)q(t)}. \tag{4.41}$$

From (4.20) and (4.41), we obtain the following estimate

$$E(t) \leq c_3\Big(\frac{\Psi_5(t)}{\chi(t)q(t)}\Big), \tag{4.42}$$

where $c_3 = \max\{\frac{2}{b_0}, \frac{b(1+\epsilon_1)}{d_0\epsilon_1}\}$. □

Example 1: Let $g(t) = \frac{a}{(1+t)^\nu}$, where $\nu > 1$ and $0 < a < \nu - 1$ so that (A) is satisfied. In this case $\xi(t) = \nu a^{\frac{-1}{\nu}}$ and $\Psi(t) = t^{\frac{\nu+1}{\nu}}$. Then, there exist positive constants $a_i (i = 0, ..., 3)$ depending only on a, ν such that

$$\Psi_4(t) = a_0 t^{\frac{\nu+1}{\nu}}, \quad \Psi_2(t) = a_1 t^{\frac{\nu+1}{\nu}}, \quad \Psi_1(t) = a_2(t^{\frac{-1}{\nu}} - 1), \;\; \Psi_5(t) = (a_3 t + 1)^{-\nu}. \tag{4.43}$$

We will discuss two cases:
Case 1: if

$$m_0(1+t)^r \leq 1 + \|u_{0x}\|^2 \leq m_1(1+t)^r, \tag{4.44}$$

where $0 < r < \nu - 1$ and $m_0, m_1 > 0$, then we have, for some positive constants $a_i (i = 4, ..., 7)$ depending only on a, ν, m_0, m_1, r, the following:

$$a_4(1+t)^{-\nu+1+r} \leq h_0(t) \leq a_5(1+t)^{-\nu+1+r}, \tag{4.45}$$

$$\frac{q_0}{q(t)} \geq a_6 \begin{cases} 1 + \ln(1+t), & \nu - r = 2; \\ 2, & \nu - r > 2; \\ (1+t)^{-\nu+r+2}, & 1 < \nu - r < 2\,. \end{cases} \tag{4.46}$$

$$\frac{q_0}{q(t)} \leq a_7 \begin{cases} 1 + \ln(1+t), & \nu - r = 2; \\ 2, & \nu - r > 2; \\ (1+t)^{-\nu+r+2}, & 1 < \nu - r < 2\,. \end{cases} \tag{4.47}$$

We notice that condition (4.4) is satisfied if

$$(t+1)^\nu q(t)h_0(t)\chi(t) \leq a_8\Big(1 - (\chi)^{\frac{1}{\nu}}\Big)^{\frac{\nu}{\nu+1}}, \tag{4.48}$$

where $a_8 > 0$ depending on a, ν, c_1 and c_2. Choosing $\chi(t)$ as the following

$$\chi(t) = \lambda \begin{cases} (1+t)^{-p}, & p = r + 1 \quad \nu - r \geq 2; \\ (1+t)^{-p}, & p = \nu - 1, \quad 1 < \nu - r < 2\,. \end{cases} \tag{4.49}$$

with $0 < \lambda \leq 1$, so that (4.3) is valid. Moreover, using (4.45) and (4.46), we see that (4.48) is satisfied if $0 < \lambda \leq 1$ is small enough, and then, (4.4) is satisfied. Hence (4.6) and (4.47) imply that, for any $t \in \mathbb{R}_+$

$$E(t) \leq a_9 \begin{cases} \Big(1+\ln(1+t)\Big)(1+t)^{-(\nu-r-1)}, & \nu - r = 2; \\ (1+t)^{-(\nu-r-1)}, & \nu - r > 2; \\ (1+t)^{-(\nu-r-1)}, & 1 < \nu - r < 2\,. \end{cases} \tag{4.50}$$

Thus, the estimate (4.50) gives $\lim_{t\to+\infty} E(t) = 0$.

Case 2: if $m_0 \leq 1 + \|u_{0x}\|^2 \leq m_1$. That is $r = 0$ in (4.44) (as it was assumed in [48, 53, 54], then (4.50) holds with $r = 0$.

5. Numerical results

In this section, some numerical experiments are performed to illustrate the energy decay results in Theorem 4.2. For this purpose, we use a finite element scheme in space, where for the time discretization, we use the Crank-Nicolson method, which is a second-order method in time that has the property to be unconditionally stable. The spatial interval $(0, L) = (0, 1)$ is subdivided into 100 subintervals, where the temporal interval $(0, T)$ is subdivided into N with a time step $\Delta t = T/N$. The solution at the time step $n + 1$ is given by

$$\begin{cases} (\mathbb{M} + \gamma_1\mathbb{R})\, Z^{n+1} = (2\mathbb{M} - \gamma_1\mathbb{R})\, Z^n - \mathbb{M}Z^{n-1} - \beta_1\mathbb{R}U^n \\ (\mathbb{M} + \gamma_2\mathbb{R})\, U^{n+1} = (2\mathbb{M} - \gamma_2\mathbb{R})\, U^n - \mathbb{M}U^{n-1} - \beta_2\mathbb{R}Z^{n+1} + \alpha_2\mathbb{R}\int_0^T g(s)U(t^{n+1} - s)ds \end{cases}, \tag{5.1}$$

where

$$\gamma_1 = \frac{a_1\Delta t^2}{2\rho_z}, \quad \beta_1 = \frac{a_2\Delta t^2}{\rho_z},$$
$$\gamma_2 = \frac{a_3\Delta t^2}{2\rho_u}, \quad \beta_2 = \frac{a_2\Delta t^2}{\rho_u}, \quad \alpha_2 = \frac{\Delta t^2}{\rho_u},$$

and the matrices $\mathbb{M}$ and $\mathbb{R}$ are the mass matrix and the stiffness matrix respectively.

We run our code for N time steps ($N = T/\Delta t$) using the following initial conditions:

$$z_0(x) = \sin(\pi x), \quad u_0(x) = 2\sin(\pi x),$$
$$z_1(x) = u_1(x) = \sin(\frac{\pi}{2}x).$$

For the numerical experiments, we choose the functional g as in the Example 1 in the previous section

$$g(t) = \frac{a}{(1+t)^\nu},$$

where $\nu > 1$ and $0 < a < \nu - 1$.

Based on the Example 1, we perform three tests in order to the energy satisfies

$$E(t) \leq E_h(t) \quad \text{for any } t \in \mathbb{R}_+,$$

where E_h given by (4.50) and the tests done as follows:

- **Test 1:** For the first numerical test, we choose the following entries:

$$\rho_z = \rho_u = a_1 = a_2 = 1, \quad a_3 = 2,$$

and

$$a = 1, \quad \nu = 4, \quad r = 2,$$

where

$$E_h(t) = a_9 \left(\frac{1 + ln(1+t)}{(1+t)^{\nu - r - 1}} \right).$$

- **Test 2:** For the second test, we choose the following entries:

$$\rho_z = \frac{1}{2}, \quad \rho_u = \frac{1}{4}, \quad a_1 = a_2 = 1, \quad a_3 = 2,$$

and

$$a = 1, \quad \nu = 6, \quad r = 3,$$

where

$$E_h(t) = a_9 \left(\frac{1}{(1+t)^{\nu - r - 1}} \right).$$

- **Test 3:** Then, we consider the last case of (4.50) by choosing the following quantities:

$$\rho_z = \rho_u = a_1 = a_2 = 1, \quad a_3 = 2,$$

and

$$a = 1, \quad \nu = 3, \quad r = 3/2,$$

where

$$E_h(t) = a_9 \left(\frac{1}{(1+t)^{\nu - r - 1}} \right).$$

- **Test 4:** Finally, we consider the case whene $r = 0$ and keeping the same quantities of Test 3:

$$\rho_z = \rho_u = a_1 = a_2 = 1, \quad a_3 = 2,$$

and

$$a = 1, \quad \nu = 3, \quad r = 0,$$

where

$$E_h(t) = a_9 \left(\frac{1}{(1+t)^{\nu - r - 1}} \right).$$

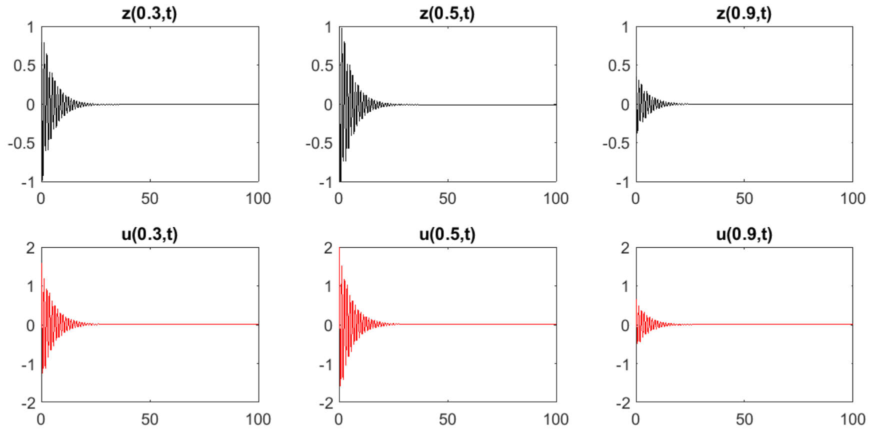

Figure 2. Test 1: the cross section of the approximate solution (z, u).

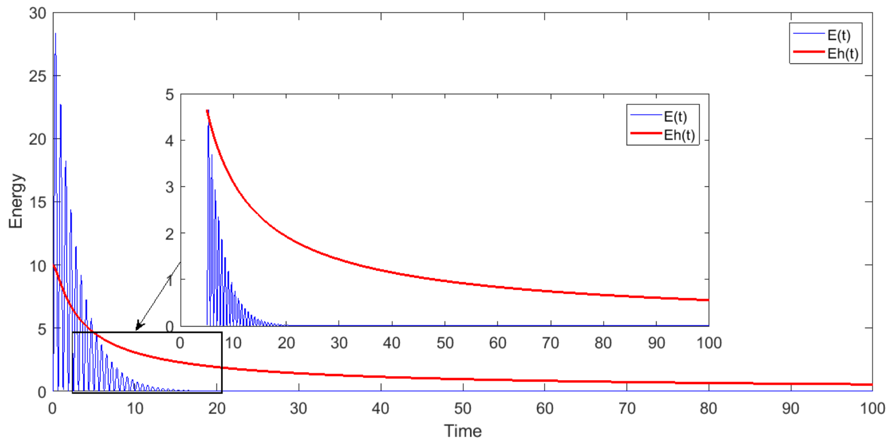

Figure 3. Test 1: energy decay.

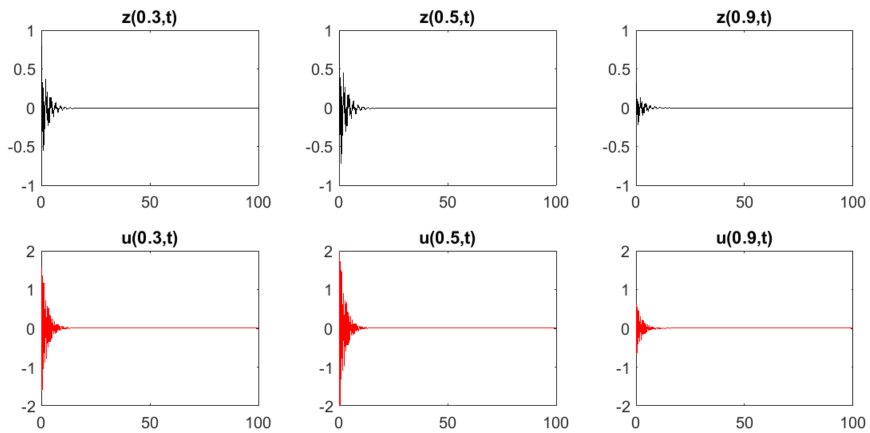

Figure 4. Test 2: the cross section of the approximate solution (z, u).

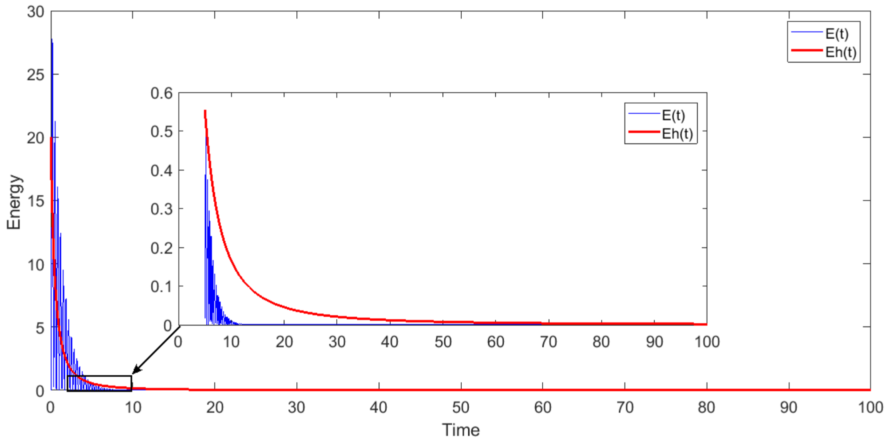

Figure 5. Test 2: energy decay.

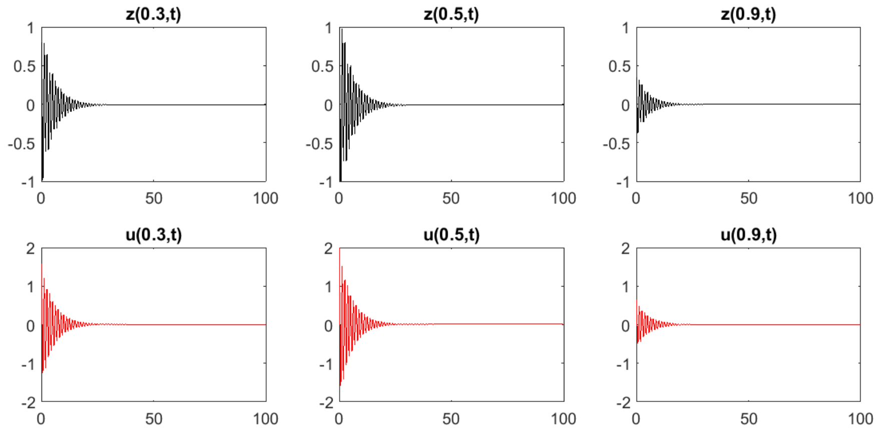

Figure 6. Test 3: the cross section of the approximate solution (z, u).

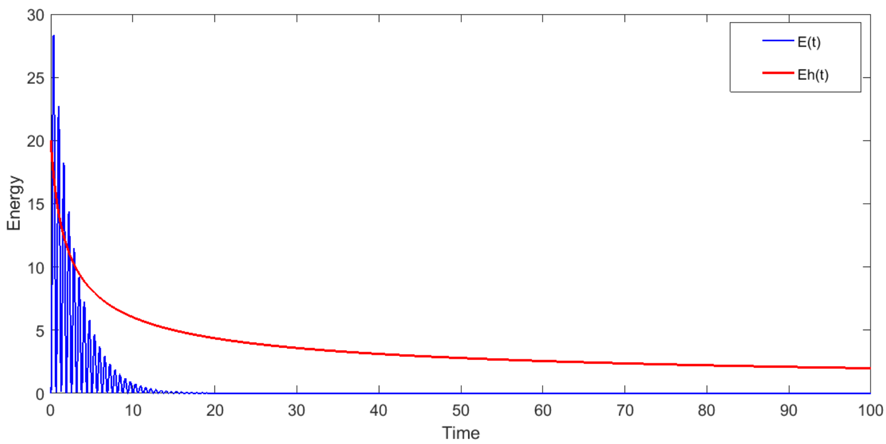

Figure 7. Test 3: energy decay.

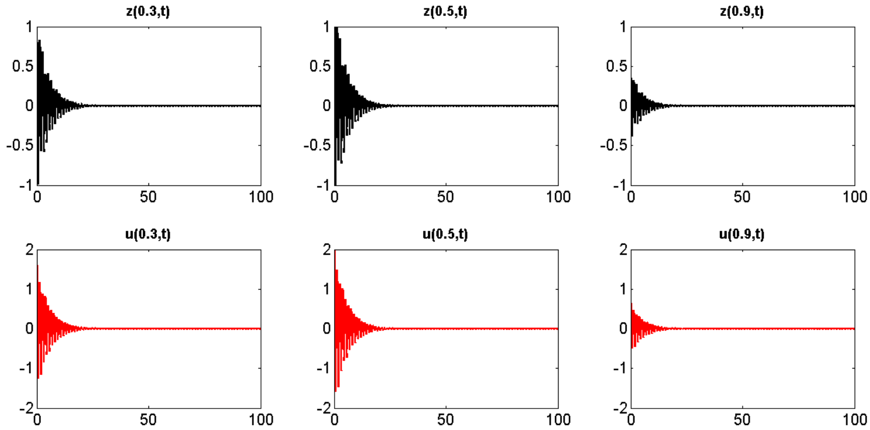

Figure 8. Test 4: the cross section of the approximate solution (z, u).

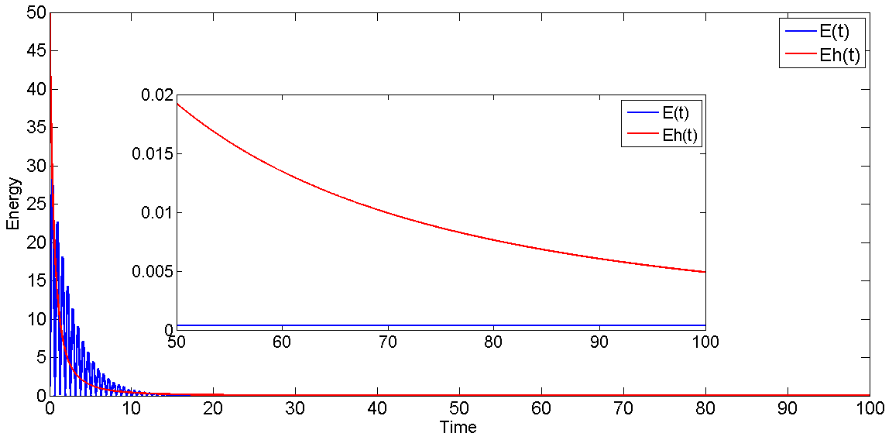

Figure 9. Test 4: energy decay.

In Figure 2, we show the cross section cuts of the numerical solution (z, u) at $x = 0.3$, $x = 0.5$ and at $x = 0.9$ for Test 1. Figure 3 presents the evolution in time of the system energy $E(t)$ compared with $E_h(t)$ for Test 1, we made a zoom on the first part to show the difference between the curves. Next, by following the same process, we present the rest of numerical results in Figures 4 and 5 for Test 2, in Figures 6 and 7 for Test 3 and in Figures 8 and 9 for Test 4.

The computational simulations show the decay behaviour for the solutions and the energy of the system (1.14), we examined four tests based on Example 1 (in both cases $r \neq 0$ and $r = 0$). As a conclusion, we observed that the energy decay uniformly for all tests and satisfies (4.50) in Example 1, and moreover independently of the waves speeds of the system.

6. Conclusions

In this work, we considered a viscoelastic swelling porous elastic system with an infinite memory term. We proved a general decay result with a large class of the relaxation functions associated with the memory term. The proof is based on the multiplier method and some properties of convex functions and without imposing some assumptions on the boundedness of the history data considered in many earlier results in the literature. We tested our decay results numerically through several numerical tests and we found the energy decay uniformly for all tests and satisfies our decay theory. Our result generalizes some earlier works on swelling porous media in the literature and it is significant to engineers and architects as it might help to attenuate the harmful effects of swelling soils swiftly.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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