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Research article

Ideal theory on EQ-algebras

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Abstract: In this article, we introduce ideals and other special ideals on EQ-algebras, such as implicative ideals, primary ideals, prime ideals and maximal ideals. At first, we give the notion of ideal and its related properties on EQ-algebras, and give its equivalent characterizations. We discuss the relations between ideals and filters, and study the generating formula of ideals on EQ-algebras. Moreover, we study the properties of implicative ideals, primary ideals, prime ideals and maximal ideals and their relations. For example, we prove that every maximal ideal is prime and if prime ideals are implicative, then they are maximal in the EQ-algebra with the condition (DNP). Finally, we introduce the topological properties of prime ideals. We get that the set of all prime ideals is a compact T_0 topological space. Also, we transferred the spectrum of EQ-algebras to bounded distributive lattices and given the ideal reticulation of EQ-algebras.

Keywords: EQ-algebra; ideal; homomorphism; topological property
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1. Introduction

As we all know, logic is not only an important tool in mathematics and information science, but also a basic technology. Non-classical logic consists of fuzzy logic and multi-valued logic, they deal with uncertain information such as fuzziness and randomness. Therefore, all kinds of fuzzy logic algebras are widely introduced and studied, such as residuated lattices, BL-algebras, MV-algebras. The truth values in FTT were assumed to form either an IMTL-algebra, BL-algebra, or MV-algebra, all of them being special kinds of residuated lattices in which the basic operations are the monoidal operation (multiplication) and its residuum. The latter is a natural interpretation of implication in fuzzy logic; the equivalence is then interpreted by the biresiduum, a derived operation. The basic connective in FTT, however, is a fuzzy equality and, therefore, it is not natural to interpret it by a derived operation. In order to remove the defect, V. Novák [11] proposed EQ-algebra in 2006. EQ-algebras not only provide a wider structure of truth value algebras for fuzzy type theory, but also generalize residuated lattices. Higher order fuzzy logic [9, 10] is relative to classical higher order logic [1]. In 2009, V. Nov*á*k [12] gave the new concept of EQ-algebras, studied their properties in detail and gave their different subclasses, such as separated EQ-algebras, good EQ-algebras, involutive EQ-algebras, residuated EQ-algebras and so on. Since the universality and characteristics of EQ-algebras as the algebraic structure of truth value of high-order fuzzy logic, EQ-algebras have absorbed many scholars. It is not only a meaningful research object, but also a very popular research object.

Ideal theory plays a very important role in logic algebras. The concept of ideal has been proposed in a lot of algebraic structures. The concept of ideal on residuated lattices was put forward by Yi Liu and Ya Qin [7], their properties and equivalent characterizations were obtained. And, Dana Piciu introduced the notion of minimal prime ideal in residuated lattices and related properties were investigated in [15]. Lele and Nganou[5] proposed the notion of ideal on BL-algebras, which is a natural generalization of that of ideal in MV-algebras. Then, Akbar Paad [13] proposed the concept of integral ideals and maximal ideals in BL-algebras. Furthermore, Wenjuan Chen [2] mainly investigated ideals and congruences in quasi-pseudo-MV algebras. Kologani and Borzoei [4] discussed the relationships among associative ideals, maximal ideals and prime ideals on hoops. Akbar Paada introduced ideals on bounded equality algebras in [14]. Also, the notions of prime and Boolean ideals in equality algebras were introduced. Hailing Niu and Xiaolong Xin[8] studied ideals on semihoop algebras. However, the notion of ideal on EQ-algebras is missing, which may make it difficult for us to study the algebraic structure of logical systems. Futhermore, in MV-algebras and Boolean algebras, filters and ideals are dual, but ideals and filters are not dual in semihoops. So we want to investigate whether they have a dual relation. For these reasons, we will introduce the ideals on EQ-algebras in this paper.

The papers is organized as follows: In section 2, we give some basic results on EQ-algebras, which will be used in the other sections. In section 3, we introduce the definition of ideal, its generating formula and its equivalent characterizations, and we discuss the relationships between ideals and filters. Moreover, we derive congruence relations from ideals. In section 4, we introduce some special ideals, such as implicative ideals, primary ideals, prime ideals and maximal ideals. We discuss some related properties and their relations. In section 5, we introduce the topological properties of prime ideals, we obtain that the set of all prime ideals $I_P(\varepsilon)$ is a compact T_0 topological space, transfer the spectrum of EQ-algebras to bounded distributive lattices and give the ideal reticulation of EQ-algebras.

2. Preliminaries

In the following, we give some basic results about EQ-algebras.

Definition 2.1. *[12] An algebra* $\varepsilon = (E, \wedge, \odot, \sim, 1)$ *of type* $(2, 2, 2, 0)$ *is said to be an EQ-algebra, if for each* $p, q, w, u \in E$*, it satisfies:*
$(E1)$ $(E, \wedge, 1)$ *is a commutative idempotent monoid;*
$(E2)$ $(E, \odot, 1)$ *is a commutative monoid and* $\odot$ *is isotone w.r.t. "* $\leq$ *", where* $p \leq q$ *is defined by* $p \wedge q = p$*;*
$(E3)$ $((p \wedge q) \sim w) \odot (u \sim p) \leq w \sim (u \wedge q)$*;*
$(E4)$ $(p \sim q) \odot (w \sim u) \leq (p \sim w) \sim (q \sim u)$*;*
$(E5)$ $(p \wedge q \wedge w) \sim p \leq (p \wedge q) \sim p$*;*

$(E6)$ $(p \wedge q) \sim p \leq (p \wedge q \wedge w) \sim (p \wedge w)$;
$(E7)$ $p \odot q \leq p \sim q$.

Definition 2.2. *[12] Let* ε *be an EQ-algebra. We call it is*
(1) *good if* $p \sim 1 = p$ *for each* $p \in E$;
(2) *separated if* $p \sim q = 1$ *implies* $p = q$ *for each* $p, q \in E$;
(3) *residuated if for each* $p, q, w \in E$, $(p \odot q) \wedge w = (p \odot q)$ *if and only if* $p \wedge ((q \wedge w) \sim q) = p$.

Proposition 2.3. *[3, 12] Let* ε *be an EQ-algebra, define* $p \to q := (p \wedge q) \sim p$. *Then for each* $p, q, w \in E$, *we have:*
(1) $w \odot (p \wedge q) \leq (w \odot p) \wedge (w \odot q)$;
(2) $p \sim q \leq p \to q$, $p \sim q = q \sim p$;
(3) *if* $p \leq q$, *then* $p \to q = 1$;
(4) $(p \sim q) \odot (q \sim w) \leq p \sim w$, $p \sim q \leq (p \wedge w) \sim (q \wedge w)$;
(5) $(p \to q) \odot (q \to w) \leq p \sim w \leq p \to w$;
(6) $p \to q \leq (w \to p) \to (w \to q)$, $p \to q \leq (q \to w) \to (p \to w)$;
(7) *if* $p \leq q$, *then* $p \sim q = q \to p$, $w \to p \leq w \to q$ *and* $q \to w \leq p \to w$;
(8) $w \to (p \wedge q) \leq (w \to p) \wedge (w \to q)$.
If ε *is good, then*
(9) $p \leq (p \sim q) \sim q$ *and* $p \leq (p \to q) \to q$;
(10) $p \to (q \to w) = q \to (p \to w)$.

Proposition 2.4. *[3] Assume that* ε *is an EQ-algebra. Then* ε *is residuated if and only if* ε *is separated and* $(p \odot q) \to w = p \to (q \to w)$ *for every* $p, q, w \in E$.

Definition 2.5. *[12] Assume that* G *is a subset of an EQ-algebra. Then* G *is said to be a filter if for each* $p, q, u \in E$ *it satisfies:*
$(F1)$ $1 \in G$;
$(F2)$ $p, q \in G$ *imply* $p \odot q \in G$;
$(F3)$ $p, p \to q \in G$ *imply* $q \in G$;
$(F4)$ $p \to q \in G$ *imply* $(p \odot u) \to (q \odot u) \in G$.

Definition 2.6. *[6] Let* G *be a filter of an EQ-algebra. Then* G *is called a positive implicative filter if* $(p \odot q) \to u \in G$ *and* $p \to q \in G$ *implies* $p \to u \in G$ *for each* $p, q, u \in E$.

3. Ideals on EQ-algebras

In this part, we introduce the definition of ideals, their generating formulas and their equivalent characterizations, and we discuss the relationships between them and filters. Moreover, we derive congruence relations from ideals.

Suppose that ε is an EQ-algebra. Define two binary operations on ε in the following:
(1) $p \oplus q = p^- \to q$, where $p^- = p \to 0$.
(2) $p \vee_1 q = ((p \to q) \to q) \wedge ((q \to p) \to p)$.
we can easily check that $p \vee_1 q$ is an upper bound of $\{p, q\}$. If $p^{--} = p$ for every $p \in E$, we call that ε satisfies the double negation property (DNP, for short).

Proposition 3.1. *Let ε be a residuated EQ-algebra with the operation $\vee_1$, then for each $p, q, u \in E$, we have*
(1) $(\vee_{1i\in I} p_i) \odot q = \vee_{1i\in I}(p_i \odot q)$;
(2) $(\vee_{1i\in I} p_i) \to q = \wedge_{i\in I}(p_i \to q)$, $q \to (\wedge_{i\in I} p_i) = \wedge_{i\in I}(q \to p_i)$;
(3) *For any* $n \in N$, $(p \vee_1 q)^n \to u := ((p \vee_1 q) \odot (p \vee_1 q) \odot \cdots \odot (p \vee_1 q)) \to u = \bigwedge\{(a_1 \odot (a_2 \odot \cdots a_n) \to u \mid a_i \in \{p, q\}\}$.

Proof. (1) It is obvious that $\vee_{1i\in I}(p_i \odot q) \leq (\vee_{1i\in I} p_i) \odot q$. Conversely, since $(p_i \odot q) \leq \vee_{1i\in I}(p_i \odot q)$, then $p_i \leq q \to \vee_{1i\in I}(p_i \odot q)$ for each $i \in I$. Thus $\vee_{1i\in I} p_i \leq q \to \vee_{1i\in I}(p_i \odot q)$, and so $(\vee_{1i\in I} p_i) \odot q \leq \vee_{1i\in I}(p_i \odot q)$. Hence $(\vee_{1i\in I} p_i) \odot q = \vee_{1i\in I}(p_i \odot q)$.
(2) $u \leq (\vee_{1i\in I} p_i) \to q$ iff $(\vee_{1i\in I} p_i) \odot u \leq q$ iff $\vee_{1i\in I}(p_i \odot u) \leq q$ iff $p_i \odot u \leq q$, for any $i \in I$ iff $u \leq p_i \to q$, for any $i \in I$ iff $u \leq \wedge_{i\in I}(p_i \to q)$, hence $(\vee_{1i\in I} p_i) \odot q = \wedge_{i\in I}(p_i \odot q)$. Analogously, we can prove the second equation.
(3) If $n = 1$, then $(p \vee_1 q) \to u = (p \to u) \wedge (q \to u)$ by (2). Assume that the equality holds for n. Thus
$(p \vee_1 q)^{n+1} \to u$
$=(p \vee_1 q) \odot (p \vee_1 q)^n \to u$
$=(p \vee_1 q) \to ((p \vee_1 q)^n \to u)$ (by proposition 2.4)
$=(p \vee_1 q) \to \bigwedge\{(a_1 \odot (a_2 \odot \cdots a_n) \to u \mid a_i \in \{p, q\}\}$ (by assumption)
$=(p \to \bigwedge\{(a_1 \odot (a_2 \odot \cdots a_n) \to u \mid a_i \in \{p, q\}\}) \wedge (q \to \bigwedge\{(a_1 \odot (a_2 \odot \cdots a_n) \to u \mid a_i \in \{p, q\}\})$ (by (2))
$=\bigwedge\{p \to (a_1 \odot (a_2 \odot \cdots a_n) \to u) \mid a_i \in \{p, q\}\} \wedge \bigwedge\{q \to (a_1 \odot (a_2 \odot \cdots a_n) \to u) \mid a_i \in \{p, q\}\}$ (by (2))
$=\bigwedge\{(p \odot a_1 \odot a_2 \odot \cdots a_n) \to u \mid a_i \in \{p, q\}\} \wedge \bigwedge\{(q \odot a_1 \odot a_2 \odot \cdots a_n) \to u \mid a_i \in \{p, q\}\}$ (by proposition 2.4)
$=\bigwedge\{(a_1 \odot (a_2 \odot \cdots a_n \odot a_{n+1}) \to u \mid a_i \in \{p, q\}\}$.
Therefore, the conclusion holds. □

Proposition 3.2. *Assume that ε is an EQ-algebra. Then for every $p, q, u \in E$:*
(1) *if* $p \leq q$, *then* $p \oplus u \leq q \oplus u$;
(2) $p, q \leq p \oplus q$;
(3) $p \oplus p^- = 1$, $0 \oplus p = p$ *and* $p \oplus 0 = p^{--}$;
(4) $p \leq p^{--}$, $p^{---} = p^-$;
(5) $(p \oplus q) \oplus u \leq p \oplus (q \oplus u)$;
If ε is residuated with (DNP), then
(6) $p \odot p^- = 0$, $p \oplus q = 1$ *if and only if* $p^- \leq q$;
(7) $p^- \odot q^- = (p \oplus q)^-$;
(8) $p^- \oplus q^- = (p \odot q)^-$;
(9) $\oplus$ *is commutative and associative.*

Proof. The proofs of (1)–(4) and (6) are easy and we omit them.
(5) $(p \oplus q) \oplus u \leq p \oplus (q \oplus u)$ iff $(p^- \to q)^- \to u \leq p^- \to (q^- \to u)$ iff $p^- \odot ((p^- \to q)^- \to u) \leq q^- \to u$ iff $p^- \leq ((p^- \to q)^- \to u) \to (q^- \to u)$ iff $p^- \to (((p^- \to q)^- \to u) \to (q^- \to u)) = 1$.

Now $p^- \to (((p^- \to q)^- \to u) \to (q^- \to u)) \geq p^- \to (q^- \to (p^- \to q)^-) \geq p^- \to ((p^- \to q) \to q) = 1$. Therefore, we have $(p \oplus q) \oplus u \leq p \oplus (q \oplus u)$.
(7) Since $(p^- \odot q^-) \to (p \oplus q)^- = p^- \to (q^- \to (p \oplus q)^-) \geq p^- \to ((p \oplus q) \to q) = p^- \to ((p^- \to q) \to q) \geq p^- \to p^- = 1$, then $(p^- \odot q^-) \to (p \oplus q)^- = 1$. Thus $(p^- \odot q^-) \leq (p \oplus q)^-$. Conversely, $(p \oplus q)^- \leq (p^- \odot q^-)$ iff $(p^- \odot q^-)^- \leq (p \oplus q)^{--}$ iff $p^- \to q^{--} \leq p^- \to q$ iff $p^- \to q \leq p^- \to q$. Hence, $p^- \odot q^- = (p \oplus q)^-$.

(8) Since $(p^- \oplus q^-) \to (p \odot q)^- = (p^{--} \to q^-) \to (p \to q^-) \geq p \to p^{--} = 1$, then $(p^- \oplus q^-) \to (p \odot q)^- = 1$, and so $p^- \oplus q^- \leq (p \odot q)^-$. Conversely, $(p \odot q)^- \leq p^- \oplus q^-$ iff $p \to q^- \leq p^{--} \to q^-$ iff $p \to q^- \leq p \to q^-$. It follows that $p^- \oplus q^- = (p \odot q)^-$.
(9) Since $p \oplus q = p^- \to q = p^- \to q^{--} = p^- \to (q^- \to 0) = q^- \to (p^- \to 0) = q^- \to p^{--} = q^- \to p = q \oplus p$ for each $p, q \in E$, then $\oplus$ is commutative. Now, we prove the associativity, $p \oplus (q \oplus u) = p \oplus (q^- \to u) = p^- \to (q^- \to u) = (p^- \odot q^-) \to u = (p^- \odot q^-)^{--} \to u = (p^- \to q^{--})^- \to u = (p^- \to q)^- \to u = (p \oplus q) \oplus u$. □

Proposition 3.3. *Suppose that ε is residuated, then for each a, $p_i \in E$, $i = 1, 2, \cdots n$, $n \in N$, $a \wedge (p_1 \oplus p_2 \oplus \cdots \oplus p_n) \leq (a \wedge p_1) \oplus (a \wedge p_2) \oplus \cdots \oplus (a \wedge p_n)$.*

Proof. First, we show that $a \wedge (p \oplus q) \leq (a \wedge p) \oplus (a \wedge q)$ for each $a, p, q \in E$. Now, $(a \wedge p) \oplus (a \wedge q) = (a \wedge p)^- \to (a \wedge q) = (a^- \vee p^-) \to (a \wedge q) = (a^- \to a \wedge q) \wedge (p^- \to a \wedge q) = ((a^- \to a) \wedge (a^- \to q)) \wedge (p^- \to a) \wedge (p^- \to q) = (a \oplus a) \wedge (a \oplus q) \wedge (p \oplus a) \wedge (p \oplus q)$. Since $a \leq (a \oplus a) \wedge (a \oplus q) \wedge (p \oplus a)$, then $a \wedge (p \oplus q) \leq (a \oplus a) \wedge (a \oplus q) \wedge (p \oplus a) \wedge (p \oplus q) = (a \wedge p) \oplus (a \wedge q)$.

Assume that the inequation holds for n. Thus, $a \wedge (p_1 \oplus p_2 \oplus \cdots \oplus p_n \oplus p_{n+1}) \leq (a \wedge (p_1 \oplus p_2 \oplus \cdots \oplus p_n)) \oplus (a \wedge p_{n+1}) \leq (a \wedge p_1) \oplus (a \wedge p_2) \oplus \cdots \oplus (a \wedge p_n) \oplus (a \wedge p_{n+1})$. Therefore, the conclusion holds. □

Definition 3.4. *Suppose that I is a nonempty subset of an EQ-algebra. Then I is called an ideal if:*
($I1$) For any $p, q \in E$, if $p \leq q$ and $q \in I$, then $p \in I$.
($I2$) For any $p, q \in I$, $p \oplus q \in I$.

Obviously, there are two trivial ideals, E and $\{0\}$. Let's denote the set of all ideals of ε by $I(\varepsilon)$. If $I \neq E$ and $I \in I(\varepsilon)$, I is said to be a proper ideal.

Example 3.5. *[17] Suppose that $E = \{0, p, q, x, y, 1\}$, with $0 < p < q < x < y < 1$. Define $\odot$ and $\sim$ in the following:*

$\odot$	0	p	q	x	y	1
0	0	0	0	0	0	0
p	0	0	0	0	0	p
q	0	0	0	0	p	q
x	0	0	0	p	p	x
y	0	0	p	p	p	y
1	0	p	q	x	y	1

$\sim$	0	p	q	x	y	1
0	1	x	q	p	0	0
p	x	1	q	p	p	p
q	q	q	1	q	q	q
x	p	p	q	1	x	x
y	0	p	q	x	1	y
1	0	p	q	x	y	1

Then $(E, \wedge, \odot, \sim, 1)$ is an EQ-algebra and $I_1 = \{0, p\}$ and $I_2 = \{0, p, q\}$ are the ideals.

Let $\{I_\lambda : \lambda \in \Lambda\}$ be a family of ideals, then $\cap_{\lambda \in \Lambda} I_\lambda$ is an ideal, but $\cup_{\lambda \in \Lambda} I_\lambda$ is not an ideal generally.

Example 3.6. *[17] Suppose $E = \{0, p, q, x, y, e, f, 1\}$, with $0 < p < x < y < e < 1$ and $0 < q < x < y < f < 1$. We can check that $(E, \wedge, \odot, \sim, 1)$ is an EQ-algebra, where $\odot$ and $\sim$ are given by the following table:*

$\odot$	0	p	q	x	y	e	f	1
0	0	0	0	0	0	0	0	0
p	0	0	0	0	0	0	0	p
q	0	0	0	0	0	0	0	q
x	0	0	0	0	0	0	0	x
y	0	0	0	0	y	y	y	y
e	0	0	0	0	y	e	y	e
f	0	0	0	0	y	y	y	f
1	0	p	q	c	y	y	f	1

$\sim$	0	p	q	x	y	e	f	1
0	1	e	f	y	x	p	q	0
p	e	1	y	f	x	p	x	p
q	f	y	1	e	x	x	q	q
x	y	f	e	1	x	x	x	x
y	x	x	x	x	1	f	e	y
e	p	p	x	x	f	1	y	e
f	q	x	q	x	e	y	1	f
1	0	p	q	x	y	e	f	1

Also, $I_1 = \{0, p\}$ *and* $I_2 = \{0, q\}$ *are two ideals, but* $I_1 \cup I_2 = \{0, p, q\}$ *is not an ideal as* $p \oplus q = x \notin I_1 \cup I_2$.

Theorem 3.7. *Suppose that* ε *is residuated and* $\emptyset \neq I \subseteq E$. *Then* $I \in I(\varepsilon)$ *if and only if* $0 \in I$ *and* $p, p^- \odot q \in I$ *imply* $q \in I$ *for each* $p, q \in E$.

Proof. ($\Rightarrow$) Let $I \in I(\varepsilon)$. Then $0 \in I$ by $I \neq \emptyset$ and ($I1$). Assume that p, $p^- \odot q \in I$, then $q \to (p \oplus (p^- \odot q)) = q \to (p^- \to (p^- \odot q)) = (q \odot p^-) \to (p^- \odot q) = 1$. Hence, $q \leq p \oplus (p^- \odot q) \in I$, and so $q \in I$.
($\Leftarrow$) Suppose that $p, q \in E$ satisfying $q \in I$ and $p \leq q$. Then $q^- \leq p^-$ and $q^- \odot p \leq p^- \odot p = 0 \in I$. By assumption, we have $p \in I$. Now, let $p, q \in I$. Since $p^- \odot (p \oplus q) = p^- \odot (p^- \to q) \leq q \in I$, then $p^- \odot (p \oplus q) \in I$. Hence, $p \oplus q \in I$, and so $I \in I(\varepsilon)$. □

Theorem 3.8. *Suppose that* ε *is residuated and* $\emptyset \neq I \subseteq E$. *Then* $I \in I(\varepsilon)$ *if and only if* $0 \in I$ *and* $p, (p^- \to q^-)^- \in I$ *imply* $q \in I$ *for each* $p, q \in E$.

Proof. ($\Rightarrow$) Let $I \in I(\varepsilon)$. It follows from ($I1$) and $I \neq \emptyset$ that $0 \in I$. Suppose that $p, (p^- \to q^-)^- \in I$, then $p^- \odot q^{--} \in I$ as $p^- \odot q^{--} \leq (p^- \odot q^{--})^{--} = ((p^- \odot q^{--})^-)^- = (p^- \to q^{---})^- = (p^- \to q^-)^-$ and ($I1$). Since $p^- \odot q \leq p^- \odot q^{--}$, then $p^- \odot q \in I$. Hence, $q \in I$ by Theorem 3.7.
($\Leftarrow$) By the Theorem 3.7, we only need to show that $p^- \odot q \in I$ and $p \in I$ imply $q \in I$ for every $p, q \in E$. Setting $q = p^{--}$, then $p^{--} \in I$. Let $p, q \in E$ satisfying $p^- \odot q \in I$ and $p \in I$, then $(p^- \odot q)^{--} \in I$. Since $(p^- \odot q)^{--} = ((p^- \odot q)^-)^- = (p^- \to q^-)^-$, then $(p^- \to q^-)^- \in I$, thus $q \in I$ by assumption. □

Corollary 3.9. *Suppose that* ε *is residuated and* $I \in I(\varepsilon)$. *Then*
(1) $p, q \in I$ *if and only if* $p \vee_1 q \in I$;
(2) $p \in I$ *if and only if* $p^{--} \in I$ *for each* $p \in E$.

Proof. (1) ($\Rightarrow$) Suppose that $p, q \in I$, since $p^- \odot (p \vee_1 q) = (p^- \odot p) \vee_1 (p^- \odot q) = p^- \odot q \leq q$, then $p^- \odot (p \vee_1 q) \in I$. Thus, $(p \vee_1 q) \in I$ by the Theorem 3.7.
($\Leftarrow$) Clearly.
(2) Assume that $p \in I$, then $p^{--} \in I$ by taking $q = p^{--}$ in the Theorem 3.8. Conversely, since $p \leq p^{--} \in I$ and $I \in I(\varepsilon)$, then $p \in I$. □

Proposition 3.10. *Assume that* ε *is residuated and* $0 \in I \in I(\varepsilon)$. *Then the following assertions are equivalent:*
(1) $I \in I(\varepsilon)$;
(2) $L(p, q) = \{u \in E : u \odot p^- \leq q\} \subseteq I$ *for each* $p, q \in I$;
(3) if $(u \odot p^-) \odot q^- = 0$, *then* $u \in I$ *for every* $u \in E$ *and* $p, q \in I$.

Proof. (1) $\Rightarrow$ (2): For any $a \in L(p,q)$, $a \odot p^- \le q$ for every $p, q \in I$. Since $q \in I$ and $I \in I(\varepsilon)$, then $a \odot p^- \in I$. Thus, $a \in I$ by the Theorem 3.7. Hence, $L(p,q) \subseteq I$.
(2) $\Rightarrow$ (3) Suppose that $p, q \in I$ satisfying $(u \odot p^-) \odot q^- = 0$. Since $(u \odot p^-) \odot q^- \le 0$ and $0, q \in I$, then $u \odot p^- \in L(q, 0) \subseteq I$ by (2). Thus, $u \odot p^- \in I$. Also, since $u \odot p^- \le u \odot p^-$ and $u \odot p^-, p \in I$, then, $u \in L(p, u \odot p^-) \subseteq I$ by (2), and so $u \in I$.
(3) $\Rightarrow$ (1) Now, let $p, p^- \odot q \in I$. Since $(q \odot p^-) \odot (p^- \odot q)^- = 0$, then $q \in I$ by (3). Thus, $I \in I(\varepsilon)$ by Theorem 3.7. □

Next, we learn the relationship between ideals and filters on EQ-algebras. For every $\emptyset \ne X \subseteq E$, define $X^- = \{x \in E \mid x^- \in X\}$.

Proposition 3.11. *Suppose that F is a filter of an EQ-algebra ε. Then F' is an ideal, where $F' = \{p \in E|$ there exists $q \in F$ satisfying $p^{--} \le q^-\}$.*

Proof. $(I1)$ Assume that $p, q \in E$ satisfying $p \le q$ and $q \in F'$, then there is $q_0 \in F$ satisfying $q^{--} \le q_0^-$. Since $p^{--} \le q^{--} \le q_0^-$, then $p \in F'$.
$(I2)$ Assume that $p, q \in F$, then there are $p_0, q_0 \in F$ satisfying $p^{--} \le p_0^-$ and $q^{--} \le q_0^-$. Then $p_0 \odot q_0 \in F$ as F is a filter. Since $(p \oplus q)^{--} = (p^- \odot q^-)^- = p^- \to q^{--} \le p^- \to q_0^- \le p_0 \to q_0^- = (p_0 \odot q_0)^-$, thus $p \oplus q \in F'$. □

Proposition 3.12. *Assume that ε is residuated and $I \in I(\varepsilon)$. Then I^* is a filter, where $I^* = \{p \in E|$ there exists $q \in I$ satisfying $p^{--} \ge q^-\}$.*

Proof. $(F1)$ Since $1^{--} = 1$, then for each $q \in I$, $q^- \le 1 = 1^{--}$, thus $1 \in I^*$.
$(F2)$ Assume $p, q \in I^*$, then there exist $p_0, q_0 \in I$ satisfying $p^{--} \ge p_0^-$ and $q^{--} \ge q_0^-$. Since $p \to q^- \le q^{--} \to p^- \le q_0^- \to p^- \le p^{--} \to q_0^{--} \le p_0^- \to q_0^{--} = p_0 \oplus q_0^{--}$, then $(p \odot q)^{--} = (p \to q^-)^- \ge (p_0 \oplus q_0^{--})^-$. Since $p_0, q_0 \in I$ and $I \in I(\varepsilon)$, then $p_0, q_0^{--} \in I$ and $p_0 \oplus q^{--} \in I$, hence $p \odot q \in I^*$.
$(F3)$ First we prove that if $p, q \in E$ with $p \le q$ and $p \in I^*$, then $q \in I^*$. Since $p \in I^*$, then there exists p_0 satisfying $p^{--} \ge p_0^-$, thus $q^{--} \ge p^{--} \ge p_0^-$, i.e. $q \in I^*$. Now, assume $p, p \to q \in I^*$, then $p \odot (p \to q) \in I^*$. Thus, $q \in I^*$ as $p \odot (p \to q) \le q$.
$(F4)$ Since ε is residuated, then for every $p, q, u \in E$, $p \to q \le (p \odot u) \to (q \odot u)$. If $p \to q \in I^*$, then $(p \odot u) \to (q \odot u) \in I^*$.
To sum up, I^* is a filter. □

Proposition 3.13. *Suppose that ε is residuated with (DNP), then $I \in I(\varepsilon)$ if and only if I^- is a filter.*

Proof. $(\Rightarrow)$ $(F1)$ $1 \in I^-$ as $0 \in I$.
$(F2)$ Suppose $p, q \in I^-$, then $p^-, q^- \in I$, and so $(p \odot q)^- = p^- \oplus q^- \in I$. Thus, $(p \odot q) \in I^-$.
$(F3)$ Now, suppose $p \le q$ and $p \in I^-$, then $p^- \in I$ and $q^- \le p^-$, hence $q^- \in I$. That is, $q \in I^-$. If $p, p \to q \in I^-$, then $p \odot (p \to q) \in I^-$. It follows from $p \odot (p \to q) \le q$ that $q \in I^-$.
$(F4)$ Suppose $p \to q \in I^-$, since $p \to q \le (p \odot u) \to (q \odot u)$, then $(p \odot u) \to (q \odot u) \in I^-$.
$(\Leftarrow)$ $(I1)$ Since $1 \in I^-$, then $0 \in I$. Let $p, q \in E$ with $p \le q$ and $q \in I$. Then $q^- \in I^-$ and $q^- \le p^-$, and thus $p^- \in I^-$. That is, $p = p^{--} \in I$.
$(I2)$ Note that $p^-, q^- \in I^-$ and $(p^- \odot q^-) \in I^-$ for each $p, q \in I$. Then $(p^- \odot q^-) \to (p^- \to q)^- = (p^- \to q) \to ((p^- \odot q^-) \to 0) = (p^- \to q) \to ((p^- \to q^{--}) = 1$ and $(p^- \odot q^-) \le (p^- \to q)^-$. Thus, $(p^- \to q)^- \in I^-$ and $(p^- \to q)^{--} \in I$. Hence, $p \oplus q = (p^- \to q) \in I$. □

Let A be a nonempty set of an EQ-algebra ε. The smallest ideal of ε including A is said to be the ideal generated by A and is marked by $\langle A\rangle$.

Theorem 3.14. *Let X be a nonempty set of an EQ-algebra ε. Then $\langle X\rangle = \{a \in E \mid a \leq (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n), p_i \in X, i = 1, 2, \cdots, n\}$.*

Proof. Let $A = \{a \in E \mid a \leq (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n), p_i \in X, i = 1, 2, \cdots, n\}$. Firstly, we check that $A \in I(\varepsilon)$. Since $0 \in A$, then A is nonempty. Assume $a, b \in E$ with $a \leq b$ and $b \in A$. Since $b \in A$, then there exist $p_i \in X, i = 1, 2, \cdots, n$ with $b \leq (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n)$. It follows from $a \leq b$ that $a \leq (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n)$, and thus $a \in A$. Now, assume $a, b \in A$, then there are $p_i, q_i \in X, i = 1, 2, \cdots, n$ satisfying $a \leq (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n)$ and $b \leq (\cdots((q_1 \oplus q_2) \oplus q_3) \cdots \oplus q_n)$, thus $(\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n)^- \leq a^-$. Hence $a \oplus b = a^- \to b \leq (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n)^- \to b \leq (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n)^- \to (\cdots((q_1 \oplus q_2) \oplus q_3) \cdots \oplus q_n) = (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n) \oplus (\cdots((q_1 \oplus q_2) \oplus q_3) \cdots \oplus q_n) \leq (\cdots(\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n) \oplus q_1) \oplus \cdots \oplus q_n)$ by Proposition 3.2 (5), i.e. $(a \oplus b) \in A$. Therefore, $A \in I(\varepsilon)$.

Obviously, $X \subseteq A$. Hence we only need to show that for any ideal C satisfying $X \subseteq C$, we can obtain $A \subseteq C$. For any $a \in A$, then there are $p_i \in X, i = 1, 2, \cdots, n$ with $a \leq (\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n)$. Since $X \subseteq C$ and $C \in I(\varepsilon)$, then $(\cdots((p_1 \oplus p_2) \oplus p_3) \cdots \oplus p_n) \in C$ and $a \in C$. Hence, A is the smallest ideal containing X. □

Proposition 3.15. *Suppose that ε is residuated and $I \in I(\varepsilon)$, $p \in E$. Then $\langle I \cup \{p\}\rangle = \{a \in E \mid a \leq q_1 \oplus (q_2 \cdots \oplus (q_n \oplus np)), q_i \in I, i = 1, 2, \cdots, n\} \cup \{a \in E \mid a \leq p \oplus (p \oplus \cdots \oplus (p \oplus q)), q \in I\}$.*

Proof. Let $A = \{a \in E \mid a \leq q_1 \oplus (q_2 \cdots \oplus (q_n \oplus np)), q_i \in I, i = 1, 2, \cdots, n\} \cup \{a \in E \mid a \leq p \oplus (p \oplus \cdots \oplus (p \oplus q)), q \in I\}$. Obviously, $(I \cup \{p\}) \subseteq A$ and $0 \in A$. Firstly, we prove that A is an ideal. Suppose that $b, c \in E$ with $b \leq c$ and $c \in A$, then there are $q_i \in I, i = 1, 2, \cdots, n$ and $q \in I$ such that $c \leq q_1 \oplus (q_2 \cdots \oplus (q_n \oplus np))$ or $c \leq p \oplus (p \oplus \cdots \oplus (p \oplus q))$, thus $b \leq c \leq q_1 \oplus (q_2 \cdots \oplus (q_n \oplus np))$ or $b \leq c \leq p \oplus (p \oplus \cdots \oplus (p \oplus q))$, and so $b \in A$, i.e. $(I1)$ holds. Assume $b, c \in A$, then there are $q_i, u_j \in I, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$ and $q, u \in I$ satisfying $b \leq q_1 \oplus (q_2 \cdots \oplus (q_n \oplus np))$ or $b \leq p \oplus (p \oplus \cdots \oplus (p \oplus q))$ and $c \leq u_1 \oplus (u_2 \cdots \oplus (u_n \oplus mp))$ or $c \leq p \oplus (p \oplus \cdots \oplus (p \oplus u))$.

Case 1: If $b \leq q_1 \oplus (q_2 \cdots \oplus (q_n \oplus np))$ and $c \leq u_1 \oplus (u_2 \cdots \oplus (u_n \oplus mp))$, then, by the Proposition 3.2 (5), we have

$$
\begin{aligned}
& b \oplus c \\
& \leq [q_1 \oplus (q_2 \oplus \cdots \oplus (q_n \oplus np))] \oplus [u_1 \oplus (u_2 \oplus \cdots \oplus (u_n \oplus mp))] \\
& \leq q_1 \oplus \{[q_2 \oplus \cdots \oplus (q_n \oplus np)] \oplus [u_1 \oplus (u_2 \oplus \cdots \oplus (u_n \oplus mp))]\} \\
& \leq q_1 \oplus (q_2 \oplus \{[q_3 \oplus \cdots \oplus (q_n \oplus np)] \oplus [u_1 \oplus (u_2 \cdots \oplus (u_n \oplus mp))]\} \\
& \leq \cdots \leq q_1 \oplus (q_2 \oplus (\cdots (q_n \oplus (np \oplus (u_1 \oplus (\cdots \oplus (u_n \oplus mp) \cdots)))) \cdots))
\end{aligned}
$$

Hence, $b \oplus c \in A$.

Other cases are analogous to the Case 1, then we also get $b \oplus c \in A$. Hence, A is an ideal.

Now, let B be any ideal with $(I \cup \{p\}) \subseteq B$, then for each $b \in A$, there are $q_i \in I, i = 1, 2, \cdots, n$ and $q \in I$ such that $b \leq q_1 \oplus (q_2 \cdots \oplus (q_n \oplus np)$ or $b \leq p \oplus (p \oplus \cdots \oplus (p \oplus q))$. Since B is an ideal and $(I \cup \{p\}) \subseteq B$, then we have $q_i \in B (i = 1, 2, \cdots, n)$, $q, q_1 \oplus (q_2 \cdots \oplus (q_n \oplus np), p \oplus (p \oplus \cdots \oplus (p \oplus q)) \in B$. Hence $b \in B$, i.e. $A \subseteq B$. Therefore, the conclusion holds. □

By Proposition 3.2 (8), if ε satisfies the condition (DNP), then the operation $\oplus$ is commutative and associative. So we have the following statement:

Remark 3.16. *Note that* $nc \triangleq c \oplus (c \oplus \cdots \oplus (c \oplus c) \cdots) = (c^-)^{n-1} \to c$. *If* ε *satisfies the condition* (DNP)*, then* $nc = ((c^-)^n)^-$.

Proposition 3.17. *Suppose that* $I \in I(\varepsilon)$ *and* $c \in E$. *Then*
(1) $\langle c \rangle = \{p \in E : \exists n \in N, s.t. p \leq nc\}$;
if ε *satisfies the condition* (DNP),
(2) $\langle I \cup \{c\} \rangle = \{p \in E : \exists n \in N, q \in I, s.t.\ p \leq (nc) \oplus q\} = \{p \in E : \exists n \in N, s.t.\ p \odot (nc)^- \in I\}$;
(3) $\langle I \cup \{p\} \rangle \cap \langle I \cup \{q\} \rangle = \langle I \cup \{p \wedge q\} \rangle$;
(4) $\langle \langle p \rangle \cup \langle q \rangle \rangle = \langle p \oplus q \rangle$;
(5) $\langle p \rangle \cap \langle q \rangle = \langle p^{--} \wedge q^{--} \rangle = \langle p \wedge q \rangle$.

Proof. (1) By the Theorem 3.14.
(2) By the Theorem 3.14, we know that $\langle I \cup \{c\} \rangle = \{p \in E : \exists n \in N, q \in I, s.t.\ p \leq (nc) \oplus q\}$. The problem turns into proving that $\{p \in E : \exists n \in N, q \in I, s.t.\ p \leq (nc) \oplus q\} = \{p \in E : \exists n \in N, s.t.\ p \odot (nc)^- \in I\}$. Since $p \leq (nc) \oplus q \Leftrightarrow p \odot (nc)^- \leq q \in I$, it follows that $p \odot (nc)^- \in I$, which implies $\{p \in E : \exists n \in N, q \in I, s.t.\ p \leq (nc) \oplus q\} \subseteq \{p \in E : \exists n \in N, s.t.\ p \odot (nc)^- \in I\}$. Conversely, we just take $q = p \odot (nc)^-$. Hence, the conclusion holds.
(3) For each $c \in \langle I \cup \{p \wedge q\} \rangle$, there is $n \in N$ satisfying $c \odot (n(p \wedge q))^- \in I$. Since $c \odot (np)^- = c \odot (p^-)^n$, then $c \odot (n(p \wedge q))^- = c \odot ((p \wedge q)^-)^n \geq c \odot (p^-)^n$, thus $c \odot (np)^- \in I$. Similarly, we have $c \odot (mq)^- \in I$. Thus $c \in \langle I \cup \{p\} \rangle \cap \langle I \cup \{q\} \rangle$, and so $\langle I \cup \{p \wedge q\} \rangle \subseteq \langle I \cup p \rangle \cap \langle I \cup \{q\} \rangle$. Conversely, let $c \in \langle I \cup p \rangle \cap \langle I \cup \{q\} \rangle$. Then there are $n, m \in N$ satisfying $c \odot (np)^-$ and $c \odot (mq)^- \in I$, and so $c \odot (p^-)^n$ and $c \odot (q^-)^m \in I$. Let $Y_1 = c \odot (p^-)^n$ and $Y_2 = c \odot (q^-)^m$. Then $Y_1^- = (c \odot (p^-)^n)^- = (p^-)^n \to c^-$ and $Y_2^- = (c \odot (q^-)^m)^- = (q^-)^m \to c^-$. Thus, $(p^-)^n \to (Y_2^- \to (Y_1^- \to c^-)) = 1$ and $(q^-)^m \to (Y_2^- \to (Y_1^- \to c^-)) = 1$. By Proposition 3.1 (3), there is $p \in N$ satisfying $[(p \wedge q)^-]^p \to (Y_2^- \to (Y_1^- \to c^-)) = (p^- \vee q^-)^p \to (Y_2^- \to (Y_1^- \to c^-)) = \bigwedge\{(c_1 \odot ... \odot c_p) \to (Y_2^- \to (Y_1^- \to c^-)) : c_i \in \{p^-, q^-\}\} = 1$. Hence, $c \in \langle I \cup \{p \wedge q\} \rangle$.
(4) Since $\langle p \rangle, \langle q \rangle \subseteq \langle p \oplus q \rangle$, then $\langle p \rangle \cup \langle q \rangle \subseteq \langle p \oplus q \rangle$, and so $\langle \langle p \rangle \cup \langle q \rangle \rangle \subseteq \langle p \oplus q \rangle$. Now, let $w \in \langle p \oplus q \rangle$, then $w \leq n(p \oplus q) \leq (np) \oplus (nq)$for some $n \in N^+$. It follows that from $np \in \langle p \rangle$ and $nq \in \langle q \rangle$ that $w \in \langle \langle p \rangle \cup \langle q \rangle \rangle$. Hence, $\langle \langle p \rangle \cup \langle q \rangle \rangle = \langle p \oplus q \rangle$.
(5) By $p \in \langle p \rangle$ and $q \in \langle q \rangle$, we know that $p^{--} \in \langle p \rangle$ and $q^{--} \in \langle q \rangle$ by Corollary 3.9 (2). Then $p^{--} \wedge q^{--} \in \langle p \rangle \cap \langle q \rangle$, and so $\langle p^{--} \wedge q^{--} \rangle \subseteq \langle p \rangle \cap \langle q \rangle$. Conversely, let $w \in \langle p \rangle \cap \langle q \rangle$, then $w \leq np$ and $w \leq mq$ for some $n, m \in N^+$. Thus, $w \leq (np) \wedge (mq) \leq nm(p^{--} \wedge q^{--})$, that is, $w \in \langle p^{--} \wedge q^{--} \rangle$. Since ε satisfies the condition (DNP), then, $\langle p \rangle \cap \langle q \rangle = \langle p^{--} \wedge q^{--} \rangle = \langle p \wedge q \rangle$ holds.

□

Proposition 3.18. *Let* ε *be an EQ-algebra. Then* $(I(\varepsilon), \vee, \wedge)$ *is a frame, where* $\vee_{i \in I} I_i = \langle \cup_{i \in I} I_i \rangle$, $\wedge_{i \in I} I_i = \cap_{i \in I} I_i$ *for each* $I_i \in I(\varepsilon)$, $i \in I$.

Proof. It's obvious that $(I(\varepsilon), \vee, \wedge)$ is a complete lattice. Next, we show that $I \wedge (\vee_{i \in I} I_i) = \vee_{i \in I}(I \wedge I_i)$, that is, $I \cap \langle \cup_{i \in I} I_i \rangle = \langle \cup_{i \in I}(I \cap I_i) \rangle$. Obviously, $\langle \cup_{i \in I}(I \cap I_i) \rangle \subseteq I \cap \langle \cup_{i \in I} I_i \rangle$ holds. On the other hand, for each $p \in I \cap \langle \cup_{i \in I} I_i \rangle$. Then $p \in I$ and $p \leq p_{i_1} \oplus p_{i_2} \oplus \cdots \oplus p_{i_n}$ for some $i_1, i_2, \cdots, i_n \in I$ and $p_{i_j} \in I_{i_j} (1 \leq j \leq n)$. Hence, $p = p \wedge (p_{i_1} \oplus p_{i_2} \oplus \cdots \oplus p_{i_n}) \leq (p \wedge p_{i_1}) \oplus (p \wedge p_{i_2}) \oplus \cdots \oplus (p \wedge p_{i_n})$ by Proposition 3.3. Since $p \wedge p_{i_j} \in I \wedge I_{i_j}$ for each $1 \leq j \leq n$, then $p \in \langle \cup_{i \in I}(I \cap I_i) \rangle$. It follows that $I \cap \langle \cup_{i \in I} I_i \rangle = \langle \cup_{i \in I}(I \cap I_i) \rangle$, and so the conclusion holds. □

Theorem 3.19. *Given a residuated EQ-algebra with* (DNP). *Define* $I_1 \to I_2 := \{p \in E \mid \langle p \rangle \cap I_1 \subseteq I_2\}$ *for each* $I_1, I_2 \in I(\varepsilon)$. *Then*

(1) $I_1 \to I_2 \in I(\varepsilon)$;
(2) $I_1 \cap I \subseteq I_2$ *if and only if* $I \subseteq I_1 \to I_2$, *where* $I \in I(\varepsilon)$.

Proof. (1) Since $\langle 0\rangle \cap I_1 \subseteq I_2$, then $0 \in I_1 \to I_2$. Let $p, q \in E$ with $p \leq q$ and $q \in I_1 \to I_2$, then $\langle q\rangle \cap I_1 \subseteq I_2$. It follows from $\langle p\rangle \cap I_1 \subseteq \langle q\rangle \cap I_1 \subseteq I_2$ that $p \in I_1 \to I_2$, that is $(I1)$ holds.

As for $(I2)$, we first show that $\langle p\oplus q\rangle \subseteq \langle p\rangle\vee\langle q\rangle$. For each $w \in \langle p\oplus q\rangle$, then $w \leq n(p\oplus q) = (np)\oplus(nq)$ for some $n \in N$ by Proposition 3.17 (1). Since $(np) \in \langle p\rangle$ and $(nq) \in \langle q\rangle$, then $w \in \langle p\rangle \vee \langle q\rangle$. Now, let $p, q \in I_1 \to I_2$. Then $\langle p\rangle \cap I_1 \subseteq I_2$ and $\langle q\rangle \cap I_1 \subseteq I_2$, and so $((\langle p\rangle \cap I_1) \vee (\langle q\rangle \cap I_1)) \subseteq I_2$. Hence, $\langle p \oplus q\rangle \cap I_1 \subseteq (\langle p\rangle \vee \langle q\rangle) \cap I_1 \subseteq I_2$, that is, $p \oplus q \in I_1 \to I_2$. Therefore, $I_1 \to I_2 \in I(\varepsilon)$.
(2) Assume that $I_1 \cap I \subseteq I_2$ and $p \in I$, then $\langle p\rangle \cap I_1 \subseteq I_2$. Hence $p \in I_1 \to I_2$, and so $I \subseteq I_1 \to I_2$. Conversely, if $I \subseteq I_1 \to I_2$ and $p \in I_1 \cap I$, then $p \in I \subseteq I_1 \to I_2$. Thus, $p \in \langle p\rangle \cap I_1 \subseteq I_2$, that is, $I_1 \cap I \subseteq I_2$. □

Corollary 3.20. *Suppose that* ε *is residuated with* (*DNP*)*, then* $(I(\varepsilon), \vee, \wedge, \to, \{0\})$ *is a Heyting-algebra, where* $I^- = I \to \{0\} = \{p \in E \mid \langle p\rangle \cap I = \{0\}\}$ *for each* $I \in I(\varepsilon)$.

Proposition 3.21. *Given a residuated EQ-algebra with* (*DNP*)*. Then*
(1) *For each* $I \in I(\varepsilon)$, $I^- = \{p \in E \mid p \wedge a = 0$ *for each* $a \in I\}$;
(2) *For each* $p \in E$, $\langle p\rangle^- = \{q \in E \mid p \wedge q = 0\}$.

Proof. (1) For each $p \in I^-$, then $\langle p\rangle\cap I = \{0\}$. It follows from Proposition 3.17 (5) that $p\wedge a \in \langle p\rangle\cap I$ for each $a \in I$, and so $p\wedge a = 0$ for each $a \in I$. Conversely, assume $p \in E$ with $p\wedge a = 0$ for each $a \in I$. Let $w \in \langle p\rangle \cap I$, then $w \leq np$ for some $n \in N^+$ and $w \in I$. Thus, $p \wedge w = 0$ and $w = w \wedge np \leq n(p \wedge w) = 0$. Thus, $\langle p\rangle \cap I = \{0\}$. Hence, $I^- = I \to \{0\} = \{p \in E \mid \langle p\rangle \cap I = \{0\}\} = \{p \in E \mid p \wedge a = 0$ for each $a \in I\}$.
(2) By (1), we get that $\langle p\rangle^- = \{q \in E \mid q \wedge a = 0$ for each $a \in \langle p\rangle\}$. For each $q \in \langle p\rangle^-$, then $q \wedge a = 0$ for each $a \in \langle p\rangle$. Since $p \in \langle p\rangle$, then $q \wedge p = 0$, that is, $\langle p\rangle^- \subseteq \{q \in E \mid p \wedge q = 0\}$. On the other hand, let $q \in \{q \in E \mid p\wedge q = 0\}$ and $a \in \langle p\rangle$. Then $a \leq np$ for some $n \in N^+$. Now, $q\wedge a \leq q\wedge(np) \leq n(p\wedge q) = 0$, thus $q \wedge a = 0$, that is, $\{q \in E \mid p \wedge q = 0\} \subseteq \langle p\rangle^-$. Hence, the conclusion holds. □

Theorem 3.22. *Given a residuated EQ-algebra with* (*DNP*)*. Then the following statements are equivalent:*
(1) $(I(\varepsilon), \vee, \wedge, ^-, \{0\}, E)$ *is a Boolean algebra;*
(2) *Each ideal of* E *is principal and for each* $p \in E$*, there is* $n \in N^+$ *satisfying* $p \wedge (np)^- = 0$.

Proof. (1) ⇒ (2) By assumption, $I \vee I^- = E$ for each $I \in I(\varepsilon)$, thus $1 \in I \vee I^-$. Hence, there are $u \in I$ and $v \in I^-$ satisfying $u \oplus v = 1$. By Proposition 3.21 (1), we have $p \wedge v = 0$ for each $p \in I$. Now $p \odot v \leq p \wedge v = 0$, then $p \odot v = 0$, and so $p \to v^- = (p \odot v)^- = 1$. Hence, $p \leq v^-$. Since $u \oplus v = 1$, then $(nu)^- \to v = (nu) \oplus v = 1$ for each $n \in N^+$, thus $(nu)^- \leq v$, and so $v^- \leq (nu)^{--} = (nu)$. Hence, $p \leq v^- \leq nu$ for each $p \in I$, that is, $I = \langle u\rangle$.

For each $p \in E$, it follows from $I(\varepsilon)$ is a Boolean algebra that $E = \langle p\rangle \vee \langle p\rangle^- = \langle\langle p\rangle^- \cup \{p\}\rangle = \{a \in E \mid \exists n \in N^+, q \in \langle p\rangle^-,\ s.t.\ a \leq np \oplus q\}$. Since $1 \in E$, then there are $n \in N^+$ and $q \in \langle p\rangle^-$ satisfying $np \oplus q = 1$. By $q \in \langle p\rangle^-$, we have $q \wedge p = 0$. Since $(np)^- \to q = np \oplus q = 1$, then $(np)^- \leq q$, thus $(np)^- \wedge p \leq q \wedge p = 0$. Hence, $(np)^- \wedge p = 0$.
(2) ⇒ (1) By Corollary 3.20, we obtain that $I(\varepsilon)$ is a Heyting algebra, then it suffices to show that $I^- = \{0\}$ only for $I = E$. By assumption, $I = \langle p\rangle$ for some $p \in E$ and $p \wedge (np)^- = 0$. By Proposition 3.21 (2), we have $(np)^- \in \langle p\rangle^- = \{0\}$, thus $(np)^- = 0$, and so $(np)^{--} = 1$. It follows that $1 \in \langle p\rangle = I$, and so $I = E$. □

Proposition 3.23. *Let $\xi : A \to B$ be a homomorphism of EQ-algebras. Then*
(1) if $I \in I(B)$, then $\xi^{-1}(I) \in I(A)$;
(2) if ξ is surjective and $I \in I(A)$, then $\xi(I) \in I(B)$;
(3) if $ker(\xi) = \{x \in A \mid \xi(x) = 0\}$, then $ker(\xi) \in I(A)$.

Proof. (1) Since $\xi(0) = 0 \in I$, then $0 \in \xi^{-1}(I)$. It follows from $p \leq q$ and $q \in \xi^{-1}(I)$ that $\xi(q) \in I$. Since $p \to q = 1$ and ξ is a homomorphism, then $\xi(p) \leq \xi(q)$, and so $\xi(p) \in I$ and $p \in \xi^{-1}(I)$. Assume that $p, q \in \xi^{-1}(I)$. Then $\xi(p), \xi(q) \in I$ and $\xi(p^- \to q) = \xi(p)^- \to \xi(q) \in I$ as $I \in I(B)$. Thus, $p^- \to q \in \xi^{-1}(I)$ and $\xi^{-1}(I) \in I(A)$.
(2) Obviously, $0 \in \xi(I)$. Assume that $p \leq q$ and $q \in \xi(I)$. Then, there is $a \in I$ satisfying $\xi(a) = q$. Since $p = \xi(b) \leq \xi(a) = q$, where $b \in A$, then $\xi(1) = 1 = \xi(b) \to \xi(a) = \xi(b \to a)$, thus $b \leq a$. Moreover, because $a \in I$ and $I \in I(A)$, so $b \in I$ and $p = \xi(b) \in \xi(I)$. Now, let $p, q \in \xi(I)$. Then there are $a, b \in I$ with $\xi(a) = p$ and $\xi(b) = q$. Since $I \in I(A)$, $a^- \to b \in I$, then $p^- \to q = \xi(a)^- \to \xi(b) \in \xi(I)$. Hence, $\xi(I) \in I(B)$.
(3) $ker(\xi) \neq \emptyset$ as $\xi(0) = 0$. Let $p, q \in ker(\xi)$. Then $\xi(p) = \xi(q) = 0$ and $\xi(p^- \to q) = \xi(p)^- \to \xi(q) = 0^- \to 0 = 0$. Hence, $p^- \to q \in ker\xi$. Assume that $p \leq q$ and $q \in ker(\xi)$. Since ξ is monotone, $\xi(p) \leq \xi(q) = 0$. Then $\xi(p) = 0$ and so $p \in ker(\xi)$. Hence, $ker(\xi) \in I(A)$. □

Proposition 3.24. *Suppose that ε is residuated with (DNP) and $I \in I(\varepsilon)$. Define a binary operation $\approx_I$ in the following:*

$$p \approx_I q \text{ iff } (p \sim q)^- \in I$$

Then $\approx_I$ is a congruence relation on ε.

Proof. Firstly, we prove that $\approx_I$ is an equivalence relation. Since $(p \sim p)^- = 0 \in I$, then $p \approx_I p$, i.e. $\approx_I$ is reflexivity. Assume $p \approx_I q$, then $(p \sim q)^- \in I$, and so $(q \sim p)^- \in I$, i.e. $q \approx_I p$, hence $\approx_I$ is symmetry. If $p \approx_I q$ and $q \approx_I u$, then $(p \sim q)^- \in I$ and $(q \sim u)^- \in I$, thus $p \sim q, q \sim u \in I^-$ and $(p \sim q) \odot (q \sim u) \in I$. Since $(p \sim q) \odot (q \sim u) \leq p \sim u$, then $p \sim u \in I$, and so $(p \sim u)^- \in I$, i.e. $\approx_I$ is transitive.

Now, assume $p \approx_I q$, then $(p \sim q)^- \in I$. Since $p \sim q \leq (p \wedge u) \sim (q \wedge u)$, then $((p \wedge u) \sim (q \wedge u))^- \leq (p \sim q)^-$, hence $((p \wedge u) \sim (q \wedge u))^- \in I$, i.e. $(p \wedge u) \approx_I (q \wedge u)$.

Assume $p \approx_I q$, then $(p \sim q)^- \in I$, and so $p \sim q \in I^-$. Since $p \sim q \leq p \to q, q \to p$, then $p \to q, q \to p \in I^-$. Hence $(p \odot u) \to (q \odot u), (q \odot u) \to (p \odot u) \in I^-$ and $((p \odot u) \to (q \odot u)) \odot ((q \odot u) \to (p \odot u)) \in I^-$. Since $((p \odot u) \to (q \odot u)) \odot ((q \odot u) \to (p \odot u)) \leq (p \odot u) \sim (q \odot u)$, then $(p \odot u) \sim (q \odot u) \in I^-$. Thus, $((p \odot u) \sim (q \odot u))^- \in I$, i.e. $(p \odot u) \approx_I (q \odot u)$.

Assume $p \approx_I q$, $z \approx_I w$, then $(p \sim q)^- \in I$ and $(z \sim w)^- \in I$, and so $(p \sim q) \in I^-$ and $(z \sim w) \in I^-$. Thus, $(p \sim q) \odot (z \sim w) \in I^-$. Since $(p \sim q) \odot (z \sim w) \leq (p \sim z) \sim (q \sim w)$, then $((p \sim z) \sim (q \sim w))^- \leq ((p \sim q) \odot (z \sim w))^- \in I$, hence $((p \sim z) \sim (q \sim w))^- \in I$, i.e. $(p \sim z) \approx_I (q \sim w)$.
Therefore, the conclusion holds. □

Define $\varepsilon/I = \{[v] \mid v \in E\}$, where $[v] = \{y \in E \mid v \approx_I y\}$. Define binary operations $\wedge_I$, $\odot_I$, $\sim_I$ as follows: For all $v, y \in E$

$$[v] \wedge_I [y] = [v \wedge y], [v] \odot_I [y] = [v \odot y], [v] \sim_I [y] = [v \sim y].$$

Then, $(E/I, \wedge_I, \odot_I, \sim_I, 1)$ is an EQ-algebra.

Theorem 3.25. *Let $I \in I(\varepsilon)$, then $I = [0]$.*

Proof. At first, we show that $[0]$ is an ideal. Obviously, $0 \in [0]$. Assume that $p, q \in E$ with $p, (p^- \odot q) \in [0]$, then $p \approx_I 0$ and $(p^- \odot q) \approx_I 0$. Since $p^- = (p \to 0) \approx_I (0 \to 0) = 1$, then $(p^- \odot q) \approx_I (1 \odot q) = q$. Thus, $q \approx_I 0$ and $q \in [0]$ by symmetry and transitivity. Hence, by the Theorem 3.7, we know that $[0]$ is an ideal.

Now we show that $I = [0]$. For each $p \in I$, we have $(p \sim 0)^- = (p \to 0)^- = p^{--} \in I$ and $(0 \sim p)^- = 0 \in I$, then $p \approx_I 0$ and $p \in [0]$. Conversely, for any $p \in [0]$, then $p \approx_I 0$, and so $p^{--} \in I$. Thus, $p \in I$ as $p \leq p^{--}$. Hence $I = [0]$. □

4. Special ideals on EQ-algebras

In this part, we will introduce some special ideals, such as implicative ideals, primary ideals, prime ideals and maximal ideals. Also, We discuss some related properties and their relations.

Definition 4.1. *Assume that I is a non-void subset of an EQ-algebra, then I is called an implicative ideal if the following hold for each $p, q, u \in E$:*
$(IM1)$ *$0 \in I$;*
$(IM2)$ *If $p, q \in I$, then $p \oplus q \in I$;*
$(IM3)$ *If $p \odot q^- \odot u^- \in I$ and $q \odot u^- \in I$, then $p \odot u^- \in I$.*

Let's denote the set of all implicative ideals of ε by $IM(\varepsilon)$.

Example 4.2. *In Example 3.5, we can show that $I = \{0, p, q\} \in IM(\varepsilon)$.*

Proposition 4.3. *If ε is residuated with (DNP), then $I \in IM(\varepsilon)$ if and only if $I^- \triangleq F$ is a positive implicative filter of ε.*

Proof. By Proposition 3.13, we obtain that $I \in I(\varepsilon)$ if and only if I^- is a filter.
$(\Rightarrow)$ Assume that $(p \odot q) \to u \in I^-$ and $p \to q \in I^-$. Then, there are $t_1, t_2 \in I$ satisfying $((p \odot q) \to u)^- = t_1 \in I$ and $(p \to q)^- = t_2 \in I$. That is, $p \odot q \odot u^- = t_1 \in I$ and $p \odot q^- = t_2 \in I$. By the commutativity and (DNP), we have $u^- \odot (q^-)^- \odot (p^-)^- = t_1 \in I$ and $q^- \odot (p^-)^- = t_2 \in I$. By $(IM3)$, we get that $(p \to u)^- = u^- \odot p = u^- \odot (p^-)^-$. Hence, $p \to u \in I^-$.
$(\Leftarrow)$ Similarly, suppose that $p \odot q^- \odot u^- \in I$ and $q \odot u^- \in I$, then $u^- \odot q^- \odot (p^-)^- \in I$ and $u^- \odot (q^-)^- \in I$. That is, $(u^- \odot q^- \to p^-)^- \in I$ and $(u^- \to q^-)^- \in I$. It follows that $u^- \odot q^- \to p^- \in I^-$ and $u^- \to q^- \in I^-$. By Definition 2.6, we have $u^- \to p^- \in I^-$. Hence, $(u^- \to p^-)^- \in I$, and so $u^- \odot p \in I$. □

Proposition 4.4. *Each implicative ideal of ε is an ideal.*

Proof. Assume that $I \in IM(\varepsilon)$ and $p, q \in E$ satisfying $p \leq q$ and $q \in I$, then $p \to q = 1$. Since $p \to q \leq p \to q^{--} = (p \odot q^-)^-$, then $(p \odot q^-)^- = 1$ and $(p \odot q^-)^{--} = 0 \in I$. Thus, $p \odot q^- = 0 \in I$ as $p \odot q^- \leq (p \odot q^-)^{--}$. Taking $u = 0$, then $p \odot q^- \odot u^- = p \odot q^- = 0 \in I$ and $q \odot u^- = q \in I$. Hence, $p = p \odot u^- \in I$ as $I \in IM(\varepsilon)$. Therefore, $I \in I(\varepsilon)$. □

Example 4.5. *In Example 3.5, we know that $I = \{0, p\} \in I(\varepsilon)$, but $I \notin IM(\varepsilon)$ as $1 \odot x^- \odot q^- = 0 \in I$ and $x \odot q^- = 0 \in I$, but $1 \odot q^- = q \notin I$. Therefore, this example implies the converse of the Proposition 4.4 is not true generally.*

Proposition 4.6. *Suppose that ε is residuated with (DNP) and $I \in I(\varepsilon)$. Then the following assertions are equivalent:*
(1) *$I \in IM(\varepsilon)$;*
(2) *If $p \odot v \odot v \in I$, then $p \odot v \in I$ for each $p, v \in E$;*
(3) *If $p^2 \in I$, then $p \in I$ for each $p \in E$;*
(4) *$\{p \in E \mid p^2 = 0\} \subseteq I$ for each $p \in E$.*

Proof. (1) $\Rightarrow$ (2) Let $p \odot v \odot v \in I$. Then, $p \odot v^{--} \odot v^{--} \in I$. Since $v^{-} \odot v^{--} = 0$ and $I \in IM(\varepsilon)$, then we have $p \odot v^{--} \in I$. It follows that $p \odot v \in I$.
(2) $\Rightarrow$ (1) Assume that $p \odot v^{-} \odot u^{-} \in I$ and $v \odot u^{-} \in I$. Since $(p \odot u^{-} \odot u^{-}) \odot (v \odot u^{-})^{-} = (p \odot u^{-} \odot u^{-}) \odot (u^{-} \to v^{-}) \leq p \odot u^{-} \odot v^{-}$ and $I \in I(\varepsilon)$, then $(p \odot u^{-} \odot u^{-}) \odot (v \odot u^{-})^{-} \in I$, thus $p \odot u^{-} \odot u^{-} \in I$ by the Theorem 3.7. Hence, $p \odot u^{-} \in I$ by (2). Therefore, $I \in IM(\varepsilon)$.
(2) $\Rightarrow$ (3) Since $I \in I(\varepsilon)$, then $(p^2)^{--} \in I$ by the Corollary 3.9. By the Proposition 2.3 (6), Proposition 2.4, Proposition 3.2 (4),
$(p^{--} \odot p^{--}) \to (p^2)^{--}$
$= (p^{--} \odot p^{--}) \to ((p^2)^{-} \to 0)$
$= p^{--} \to (p^{--} \to ((p^2)^{-} \to 0))$
$= p^{--} \to ((p^2)^{-} \to p^{---})$
$= p^{--} \to ((p^2)^{-} \to p^{-})$
$= (p^2)^{-} \to (p^{--} \to p^{-})$
$= (p^2)^{-} \to (p \to p^{-})$
$= p \to ((p^2)^{-} \to p^{-})$
$= p \to (p \to (p^2)^{--})$
$= p^2 \to (p^2)^{--} = 1$
Hence, $p^{--} \odot p^{--} \leq (p^2)^{--}$. Since $(p^2)^{--} \in I$ and $I \in I(\varepsilon)$. Then $1 \odot p^{--} \odot p^{--} = p^{--} \odot p^{--} \in I$. Hence, $1 \odot p = p \in I$ by (2).
(3) $\Rightarrow$ (2) Let $p \odot v \odot v \in I$ for each $p, v \in E$. Since $p^2 \leq p$, then $(p \odot v)^2 = p^2 \odot v^2 \leq p \odot v \odot v$. Thus, $p \odot v \in I$ by (3).
(3) $\Rightarrow$ (4) Let $a \in \{p \in E \mid p^2 = 0\}$. Then $a^2 = 0$, and so $a \in I$. Hence, $\{p \in E \mid p^2 = 0\} \subseteq I$.
(4) $\Rightarrow$ (3) Let $p^2 \in I$ for every $p \in E$. Then $p^2/I = 0/I$. Since $\{0/I\} \in I(\varepsilon/I)$, then $p/I \in \{0/I\}$. Hence, $p \in I$. □

Corollary 4.7. *Assume that $I \in IM(\varepsilon)$ and $J \in I(\varepsilon)$ with $I \subseteq J$, then $J \in IM(\varepsilon)$.*

Proof. If $I \subseteq J$ and $I \in IM(\varepsilon)$, then $\{p \in E \mid p^2 = 0\} \subseteq I \subseteq J$ by the Proposition 4.6. Since $J \in I(\varepsilon)$, then $J \in IM(\varepsilon)$. □

Corollary 4.8. *If $I \in IM(\varepsilon)$, then for each $p \in E$, $p^{-} \wedge p \in I$.*

Proof. Since $p^{-} \wedge p \leq p, p^{-}$, then $(p^{-} \wedge p)^2 \leq p \odot p^{-} = 0$ and $(p^{-} \wedge p)^2 = 0$. Hence $p^{-} \wedge p \in I$ by the Proposition 4.6. □

Proposition 4.9. *Suppose that ε is residuated with $p^{--} \odot q \odot t^{-} \in I$, $p \odot q^{-} \in I$ and $I \in IM(\varepsilon)$, then $p \odot t^{-} \in I$.*

Proof. Let $p^{--} \odot q \odot t^{-} \in I$ and $p \odot q^{-} \in I$ for each $p, q, t \in E$. Then by the Proposition 3.2 (4),

$(p^{--} \odot q \odot t^-)^- \odot (p^{--} \odot q^{--} \odot t^-)$
$= ((q \odot t^-) \to p^{---}) \odot (p^{--} \odot q^{--} \odot t^-)$
$= ((q \odot t^-) \to p^-) \odot (p^{--} \odot q^{--} \odot t^-)$
$= t^- \odot (t^- \to (q \to p^-)) \odot p^{--} \odot q^{--}$
$\leq (q \to p^-) \odot p^{--} \odot q^{--}$
$= (q \to p^-) \odot (p^- \to 0) \odot q^{--}$
$\leq q^- \odot q^{--} = 0 \in I.$
Since $I \in I(\varepsilon)$, $(p^{--} \odot q \odot t^-)^- \odot (p^{--} \odot q^{--} \odot t^-) \in I$, then, by the Theorem 3.7, $p^{--} \odot q^{--} \odot t^- \in I$. From $(p \odot q^-)^- \odot (q^- \odot p^{--}) = q^- \odot (q^- \to p^-) \odot p^{--} \leq p^- \odot p^{--} = 0 \in I$ and $I \in I(\varepsilon)$, $(p \odot q^-)^- \odot (q^- \odot p^{--}) \in I$. Hence, $q^- \odot p^{--} \in I$ as $p \odot q^- \in I$. It follows that $t^- \odot p^{--} \in I$ as $I \in IM(\varepsilon)$. Since $t^- \odot p \leq t^- \odot p^{--}$, then $p \odot t^- \in I$. □

Proposition 4.10. *Assume that $\emptyset \neq I \subseteq E$. Then the following assertions are equivalent:*
(1) *$I \in IM(\varepsilon)$;*
(2) *$I \in I(\varepsilon)$ and if $p \odot (q^-)^2 \in I$, then $p \odot q^- \in I$ for each $p, q \in E$;*
(3) *$I \in I(\varepsilon)$ and if $(p \odot q^-) \odot t^- \in I$, then $(p \odot t^-) \odot (q \odot t^-)^- \in I$ for each $p, q, t \in E$;*
(4) *$0 \in I$ and if $(p \odot (q^-)^2) \odot t^- \in I$ and $t \in I$, then $p \odot q^- \in I$ for each $p, q, t \in E$.*

Proof. (1) ⇒ (2): By Proposition 4.4, $I \in I(\varepsilon)$. Suppose that $(p \odot q^-) \odot q^- = p \odot (q^-)^2 \in I$. Since $q \odot q^- = 0 \in I$, then $p \odot q^- \in I$ by $(IM3)$.
(2) ⇒ (3) Let $(p \odot q^-) \odot t^- \in I$. Since $p \odot (q \odot t^-)^- \odot t^- \odot t^- = p \odot t^- \odot t^- \odot (t^- \to q^-) \leq (p \odot q^-) \odot t^- \in I$ and $I \in I(\varepsilon)$, thus, $p \odot (q \odot t^-)^- \odot t^- \odot t^- \in I$. By (2), we get that $(p \odot t^-) \odot (q \odot t^-)^- = p \odot (q \odot t^-)^- \odot t^- \in I$.
(3) ⇒ (1) Assume that $(p \odot q^-) \odot t^- \in I$ and $q \odot t^- \in I$. Then, $(p \odot t^-) \odot (q \odot t^-)^- \in I$ by (3). Since $q \odot t^- \in I$ and $I \in I(\varepsilon)$, then $p \odot t^- \in I$ by the Theorem 3.7. Hence, $I \in IM(\varepsilon)$.
(2) ⇒ (4) Let $(p \odot (q^-)^2) \odot t^- \in I$ and $t \in I$. Then, $0 \in I$ and $p \odot (q^-)^2 \in I$ as $I \in I(\varepsilon)$. Hence, $p \odot q^- \in I$ by (2).
(4) ⇒ (2) It's easy by taking $t = 0$ in (4). □

Proposition 4.11. *Let $\xi : A \to B$ be a homomorphism of EQ-algebras. Then*
(1) *If $I \in IM(B)$, then $\xi^{-1}(I) \in IM(A)$.*
(2) *If ξ is surjective and $I \in IM(A)$, then $\xi(I) \in IM(B)$.*

Proof. The proof is analogous to that of the Proposition 3.23. □

Proposition 4.12. *Let $I \in I(\varepsilon)$. Then $I \in IM(\varepsilon)$ iff $\langle I \cup \{u\} \rangle = I_u$, where $I_u = \{p \in E : p \odot u^- \in I\}$ and $u \in E$.*

Proof. (⇒) For any $u \in E$, $I_u \neq \emptyset$ as $0 \in I_u$. Let $p^- \odot q, p \in I_u$. Then $(p^- \odot q) \odot u^- \in I$ and $p \odot u^- \in I$. Since $I \in IM(\varepsilon)$, then $q \odot u^- \in I$, and so $q \in I_u$. Hence, $I_u \in I(\varepsilon)$ by the Theorem 3.7. Also, $u \in I_u$ as $u \odot u^- = 0 \in I$. Let $p \in I$. Since $p \odot u^- \leq p$, then $p \odot u^- \in I$, and so $p \in I_u$. Hence, $I \subseteq I_u$. Now, assume there is $J \in I(\varepsilon)$ satisfying $I \cup \{u\} \subseteq J$. For any $p \in I_u$, then $p \odot u^- \in I \subseteq J$, and so $p \odot u^- \in J$. Since $u \in J$ and $J \in I(\varepsilon)$, then $p \in J$, and so $I_u \subseteq J$. Hence, the conclusion holds.
(⇐) Let $I \in I(\varepsilon)$, $p \odot q^- \odot t^- \in I$ and $q \odot t^- \in I$ for every $p, q, t \in E$. Then, $p \odot q^- \in I_t$ and $q \in I_t$ by definition. Since $I_t \in I(\varepsilon)$, then $p \in I_t$, and so $p \odot t^- \in I$. Hence, $I \in IM(\varepsilon)$. □

Proposition 4.13. *Let $I, J \in I(\varepsilon)$. Then*
(1) $I_{u_1} = I$ *iff* $u_1 \in I$;
(2) *if* $u_1 \leq u_2$, *then* $I_{u_1} \subseteq I_{u_2}$;
(3) *if* $I \subseteq J$, *then* $I_{u_1} \subseteq J_{u_1}$;
(4) $(I \cap J)_{u_1} = I_{u_1} \cap J_{u_1}$ *and* $(I \cup J)_{u_1} = I_{u_1} \cup J_{u_1}$;
(5) $I_{u_1 \oplus u_2} \subseteq (I_{u_1})_{u_2}$.

Proof. (1) Since $I \cup \{u_1\} \subseteq I_{u_1} = I$, then $u_1 \in I$. Conversely, if $u_1 \in I$, since $I_{u_1} = \langle I \cup \{u_1\} \rangle$ and $I \in I(\varepsilon)$, then $I_{u_1} = I$.
(2) Suppose that $p \in I_{u_1}$ and $u_1 \leq u_2$, then $p \odot u_1^- \in I$. Since $u_1 \leq u_2$, then $u_2^- \leq u_1^-$, and so $p \odot u_2^- \leq p \odot u_1^-$, hence $p \odot u_2^- \in I$ and $p \in I_{u_2}$. Thus, $I_{u_1} \subseteq I_{u_2}$.
(3) Let $I, J \in I(\varepsilon)$ and $I \subseteq J$. If $p \in I_{u_1}$, then $p \odot u_1^- \in I$, and so $p \odot u_1^- \in J$. Hence, $p \in J_{u_1}$ and $I_{u_1} \subseteq J_{u_1}$.
(4) Since $I \cap J \subseteq I, J$, then, $(I \cap J)_{u_1} \subseteq I_{u_1} \cap J_{u_1}$ by (3). Suppose that $p \in I_{u_1} \cap J_{u_1}$. Then $p \odot u_1^- \in I$ and $p \odot u_1^- \in J$, and so $p \odot u_1^- \in I \cap J$. Hence, $p \in (I \cap J)_{u_1}$. Analogously, we can prove that $(I \cup J)_{u_1} = I_{u_1} \cup J_{u_1}$.
(5) Let $p \in I_{u_1 \oplus u_2}$, then $p \odot (u_1 \oplus u_2)^- = p \odot (u_1^- \to u_2)^- \in I$. Since $(u_1^- \odot u_2^-) \to (u_1^- \to u_2)^- = (u_1^- \to u_2) \to ((u_1^- \odot u_2^-) \to 0) = (u_1^- \to u_2) \to (u_1^- \to u_2^{--}) = 1$. Thus, $(u_1^- \odot u_2^-) \leq (u_1^- \to u_2)^-$, and so $p \odot (u_1^- \odot u_2^-) \leq p \odot (u_1^- \to u_2)^- \in I$. Since $I \in I(\varepsilon)$, $p \odot (u_1^- \odot u_2^-) \in I$. Hence, $p \in (I_{u_1})_{u_2}$. □

Proposition 4.14. *Let ε be an EQ-algebra with (DNP), then the assertions are equivalent:*
(1) *If* $I \in I(\varepsilon)$, *then* $I \in IM(\varepsilon)$;
(2) $\{0\} \in IM(\varepsilon)$;
(3) *For each* $a \in E$, *the set* $E(a) = \{p \in E \mid p \odot a^- = 0\} \in I(\varepsilon)$.

Proof. (1) ⇒ (2) Since $\{0\} \in I(\varepsilon)$, then $\{0\} \in IM(\varepsilon)$.
(2) ⇒ (3) ($I1$) $E(a) \neq \emptyset$ as $0 \in E(a)$. Assume that $p \leq q$ and $q \in E(a)$, then $p \odot a^- \leq q \odot a^- = 0$, thus $p \odot a^- = 0$, hence $p \in E(a)$.
($I2$) Assume $p, q \in E(a)$, then $p \odot a^- = 0$, $q \odot a^- = 0$. Since $(p^- \to q) \odot q^- \odot a^- = (p^- \to q) \odot (q \to 0) \odot a^- \leq p^{--} \odot a^- = p \odot a^- = 0$ by the Proposition 2.3 (5), then $(p^- \to q) \odot q^- \odot a^- = 0 \in \{0\}$. Since $q \odot a^- = 0 \in \{0\}$ and $\{0\} \in IM(\varepsilon)$, then $(p^- \to q) \odot a^- = 0$, and so $(p^- \to q) \in E(a)$.
(3) ⇒ (1) Let $I \in I(\varepsilon)$ with $p \odot q^- \odot z^- \in I$ and $q \odot z^- \in I$ for each $p, q, z \in E$. For any $a/I \in \varepsilon/I$, $(\varepsilon/I)(a/I) \in I(\varepsilon/I)$. Then $(p \odot q^-)/I \odot z^-/I = 0$ and $q/I \odot z^-/I = 0$. Thus, $(p \odot q^-)/I \in (\varepsilon/I)(z/I)$ and $q/I \in (\varepsilon/I)(z/I)$, and so $q/I \in (\varepsilon/I)(z/I)$ as $(\varepsilon/I)(z/I) \in I(\varepsilon/I)$. Hence $p/I \odot z^-/I = 0$, and then $p \odot z^- \in I$. Therefore, $I \in IM(\varepsilon)$. □

Definition 4.15. *Assume that P is a proper ideal of an EQ-algebra ε. If $p \odot q \in P$ implies $p^n \in P$ or $q^n \in P$ for some $n \in N$ and each $p, q \in E$, then P is said to be a primary ideal.*

Example 4.16. *In Example 3.5, we can check that $I = \{0, p\}$ is a primary ideal.*

Proposition 4.17. *Suppose that P is a proper ideal of ε with (DNP). Then P is a primary ideal iff* $[u] \odot_p [v] = [0]$ *implies* $[u]^n = [0]$ *or* $[v]^n = [0]$ *for some $n \in N$ and each* $[u], [v] \in \varepsilon/P$.

Proof. (⇒) Assume P is a primary ideal and $[u] \odot_p [v] = [0]$, then $[u \odot v] = [u] \odot_p [v] = [0]$, thus $(u \odot v)^{--} = ((u \odot v) \to 0)^- \in P$, and so $(u \odot v) \in P$. Hence, $u^n \in P$ or $v^n \in P$ for some $n \in N$ by definition. Assume $u^n \in P$, then $(u^n \to 0)^- = (u^n)^{--} = u^n \in P$. Also $(0 \to u^n)^- = 0 \in P$, i.e. $[u]^n = [0]$. If we assume $v^n \in P$, we can get $[v]^n = [0]$.

(⇐) Assume $u \odot v \in P$, then $((u \odot v) \to 0)^- = (u \odot v)^{--} \in P$. Since $(0 \to (u \odot v))^- = 0 \in P$, then $[u] \odot_p [v] = [u \odot v] = [0]$. Hence, $[u]^n = [0]$ or $[v]^n = [0]$ for some $n \in N$. If $[u]^n = [0]$, then $u^n = (u^n)^{--} = (u^n \to 0)^- \in P$. If $[v]^n = [0]$, we can obtain $(v^n) \in P$. Hence, the conclusion holds. □

Definition 4.18. *Let P be a proper ideal of an EQ-algebra ε. P is said to be a prime ideal of ε if $p \wedge y \in P$ implies $p \in P$ or $y \in P$ for each $p, y \in E$. Let's denote the set of all prime ideals of ε by $I_p(\varepsilon)$.*

Example 4.19. *In Example 3.5, we can check that both $I_1 = \{0, p\} \in I_p(\varepsilon)$ and $I_2 = \{0, p, q\} \in I_p(\varepsilon)$.*

Proposition 4.20. *If ε is residuated with (DNP), then $I \in I_p(\varepsilon)$ if and only if $I^- = F$ is a prime filter of ε.*

Proof. The proof is analogous to that of Proposition 3.13. □

Proposition 4.21. *Assume that ε satisfies (DNP) and P is a proper ideal of ε. Then $P \in I_p(\varepsilon)$ iff $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for each $I, J \in I(\varepsilon)$.*

Proof. (⇒) Let $I, J \in I(\varepsilon)$ with $I \cap J \subseteq P$, but $I \nsubseteq P$ and $J \nsubseteq P$, then there are $p \in I - P$ and $q \in J - P$. Since $p \wedge q \leq p, q$ and $I, J \in I(\varepsilon)$, then $p \wedge q \in I \cap J \subseteq P$ and $p \wedge q \in P$. Thus, $p \in P$ or $q \in P$ as $P \in I_p(\varepsilon)$, which generates a contradiction. Hence, $I \subseteq P$ or $J \subseteq P$.
(⇐) Let $P \in I(\varepsilon)$ with $p \wedge q \in P$ for each $p, q \in E$. If $p, q \notin P$, then $\langle P \cup \{p\} \rangle \cap \langle P \cup \{q\} \rangle = \langle P \cup \{p \wedge q\} \rangle = P$ by Proposition 3.17. Thus, $\langle P \cup \{p\} \rangle \subseteq P$ or $\langle P \cup \{q\} \rangle \subseteq P$, and so $p \in P$ or $q \in P$, which generates a contradiction. Hence, $P \in I_p(\varepsilon)$. □

Proposition 4.22. *Let $\xi : E_1 \to E_2$ be a homomorphism of EQ-algebras. Then*
(1) *if $P \in I_p(E_2)$, then $\xi^{-1}(P) \in I_p(E_1)$;*
(2) *if ξ is surjective and $P \in I_p(E_1)$ with $P \neq E_2$, then $\xi(P) \in I_p(E_2)$.*

Proof. The proof is similar to the Proposition 3.23. □

Proposition 4.23. *Assume that ε is an EQ-algebra. If $p^2 = p$ for each $p \in E$, then every primary ideal is prime.*

Proof. Assume P is a primary ideal and $p \wedge q \in P$, since $(p \odot q) \leq (p \wedge q)$, then $p \odot q \in P$, and so $p^n \in P$ or $q^n \in P$ for some $n \in N$ by definition. If $p^n \in P$, then $p^n = p \in P$ as $p^2 = p$. If $q^n \in P$, we get that $q \in P$. Hence P is prime. □

Definition 4.24. *Let M be a proper ideal of ε. If there is no proper ideal strictly contains M, then M is called a maximal ideal. Let us denote the set of all maximal ideals by $I_M(\varepsilon)$.*

Proposition 4.25. *Let I be a proper ideal of an EQ-algebra ε. Then $I \in I_M(\varepsilon)$ if and only if $\langle I \cup \{p\} \rangle = E$ for each $p \in E \setminus I$.*

Proof. (⇒) If $p \notin I$, then $I \subsetneq \langle I \cup \{p\} \rangle \subseteq E$. Hence, $\langle I \cup \{p\} \rangle = E$ by definition.
(⇐) Assume G is an ideal with $I \subsetneq G \subseteq E$. Then there exists $p \in G$ but $p \notin I$. Thus $E = \langle I \cup \{p\} \rangle \subseteq G$, and so $E = G$. Therefore, $I \in I_M(\varepsilon)$. □

Proposition 4.26. *Suppose that ε is residuated with (DNP) and $I \in I_M(\varepsilon)$. Then $p \in E \setminus I$ if and only if $(np)^- \in I$ for some $n \in N$.*

Proof. (⇒) If $p \notin I$, then $\langle I \cup \{p\}\rangle = E$ by the Proposition 4.25, and so $1 \in \langle I \cup \{p\}\rangle$. By the Proposition 3.17 (2), there are $n \in N$ and $q \in I$ satisfying $1 \le q \oplus (np) = q^- \to (np)$, hence $q^- \le (np)$, and so $(np)^- \le q^{--} = q \in I$. Therefore $(np)^- \in I$.
(⇐) Let $p \in I$ and $(np)^- \in I$. Then $(np) \in I$ and $(np) \oplus (np)^- = 1 \in I$. But I is a proper ideal, which generates a contradiction. Hence, $p \notin I$. □

Proposition 4.27. *Given an EQ-algebra ε with (DNP), then every maximal ideal is prime.*

Proof. Let $M \in Max(\varepsilon)$ and $p \wedge q \in M$ for any $p, q \in E$. If $p \notin M$, then $M \subsetneq \langle M \cup \{p\}\rangle$. Since $M \in Max(\varepsilon)$, we get $\langle M \cup \{p\}\rangle = E$. In a similar way, if $q \notin M$, then $\langle M \cup \{q\}\rangle = E$. By the Proposition 3.17, we have $E = \langle M \cup \{p\}\rangle \cap \langle M \cup \{q\}\rangle = \langle M \cup \{p \wedge q\}\rangle = M$, which is a contradiction. Therefore, $M \in I_p(\varepsilon)$. □

Proposition 4.28. *Let $\xi : E_1 \to E_2$ be a homomorphism of EQ-algebras. Then*
(1) *if $M \in I_M(E_2)$, then $\xi^{-1}(M) \in I_M(E_1)$;*
(2) *if ξ is surjective and $M \in I_M(E_1)$ such that $\xi(M) \neq E_2$, then $\xi(M) \in I_M(E_2)$.*

Proof. The proof is similar to the Proposition 3.23. □

Theorem 4.29. *Let I be a prime ideal of an EQ-algebra with (DNP). If I is an implicative ideal, then it is maximal.*

Proof. Assume that $I \notin I_M(\varepsilon)$. Then there is $J \in I(\varepsilon)$ satisfying $I \subsetneq J \subsetneq E$, and so there is an element $p \in J - I$. Since $(p^- \wedge p) \odot p^- \le p^- \odot p = 0 \in I$, then $(p^- \wedge p) \odot p^- \in \langle I \cup \{p\}\rangle$, thus $p \wedge p^- \in \langle I \cup \{p\}\rangle \subseteq J$. If $p \wedge p^- \in I$, then $p^- \in I$ as $I \in I_p(\varepsilon)$, and so $p^- \in J$. Since $J \in I(\varepsilon)$, $p \oplus p^- = 1 \in J$, which generates a contradiction. Hence $p \wedge p^- = p$, then $p \le p^-$, and so $p^2 = 0 \in I$. Because I is an implicative ideal, so $p \in I$ by Proposition 4.6, which is a contradiction. Hence, $I \in I_M(\varepsilon)$. □

5. Topological space of prime ideals

In this part, we give the topological properties of prime ideals.

Proposition 5.1. *Let $P \in I_p(\varepsilon)$. Then*
(1) *if P_1 is a proper ideal and $P \subseteq P_1$, then $P_1 \in I_p(\varepsilon)$;*
(2) *if $\{P_i\}_{i \in I}$ is a family of ideals and $P \subseteq \cap_{i \in I} P_i$, then $\{P_i\}_{i \in I}$ is a chain.*

Proof. (1) For each $x, y \in E$ with $x \wedge y \in P \subseteq P_1$, since P is prime, then either $x \in P \subseteq P_1$ or $y \in P \subseteq P_1$, hence $P_1 \in I_p(\varepsilon)$.
(2) For each $P_1, P_2 \in \{P_i\}_{i \in I}$. If $P_1 = E$ or $P_2 = E$, then the conclusion holds. Now, we suppose that $P_1 \neq E$, $P_2 \neq E$ and $P_1 \nsubseteq P_2$, $P_2 \nsubseteq P_1$, then there are $x, y \in E$ satisfying $x \in P_1 - P_2$ and $y \in P_2 - P_1$. Since $P \subseteq \cap_{i \in I} P_i \subseteq P_1 \cap P_2$, then $P_1 \cap P_2$ is prime by (1). Also since $\langle x\rangle \in P_1$ and $\langle y\rangle \in P_2$, then $\langle x\rangle \cap \langle y\rangle \subseteq P_1 \cap P_2$. Thus $x \in \langle x\rangle \subseteq P_1 \cap P_2$ or $y \in \langle y\rangle \subseteq P_1 \cap P_2$, which is a contradiction. It follows that $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$, hence $\{P_i\}_{i \in I}$ is a chain. □

Proposition 5.2. *Assume that I is a proper ideal of an EQ-algebra with (DNP) and $\emptyset \neq S \subseteq E$ satisfying $I \cap S = \emptyset$. Then there exists $P \in I_p(\varepsilon)$ satisfying $I \subseteq P$ and $P \cap S = \emptyset$.*

Proof. Let $\Sigma = \{J \in I(\varepsilon) \mid I \subseteq J \text{ and } J \cap S = \emptyset\}$. Then $\Sigma \neq \emptyset$ as $I \in \Sigma$. If $\{J_\lambda\}_{\lambda\in\Delta}$ is a chain of ideals that are in Σ, then by *Zorn's* lemma we have $P = \cup_{\lambda\in\Delta} J_\lambda$ is a maximal element of Σ. So, it only need to prove that P is an ideal. Then, we know that $P \in I_p(\varepsilon)$ by Proposition 4.27. Assume that $x \leq y$ and $y \in P$. Then there exists $\lambda_0 \in \Delta$ with $y \in J_{\lambda_0}$. Since J_{λ_0} is an ideal, then $x \in J_{\lambda_0} \subseteq P$, i.e. $(I1)$ holds. Let $x, y \in P$. Then there are $\lambda \in \Delta$ and $\mu \in \Delta$ satisfying $x \in J_\lambda$ and $y \in J_\mu$. Since $\{J_\lambda\}_{\lambda\in\Delta}$ is a chain, then either $J_\lambda \subseteq J_\mu$ or $J_\mu \subseteq J_\lambda$. Assume $J_\lambda \subseteq J_\mu$, then $x, y \in J_\mu$. Hence $x \oplus y \in J_\mu \subseteq P$, i.e. $(I2)$ holds. Therefore P is an ideal. Hence, the conclusion holds. □

Corollary 5.3. *Assume that ε is an EQ-algebra with (DNP). Then for any proper ideal I of ε with $x \notin I$, there is $P \in I_p(\varepsilon)$ satisfying $I \subseteq P$ and $x \notin P$.*

Proof. Since I is proper, then there is $x \in E - I$. Let $S = \{x\}$ in the Proposition 5.2, the proof is clear. □

Next we discuss the topological properties of prime ideals. For each $X \subseteq E$, we denote that $S(X) = \{P \in I_P(\varepsilon) \mid X \nsubseteq P\}$.

Proposition 5.4. *Suppose that ε is an EQ-algebra, then*
(1) *$X_1 \subseteq X_2 \subseteq E$ implies $S(X_1) \subseteq S(X_2)$;*
(2) *$S(X_1) = \emptyset$ if and only if $X_1 = \emptyset$ or $X_1 = \{0\}$;*
(3) *$S(X_1) = I_P(\varepsilon)$ if and only if $\langle X_1\rangle = E$;*
(4) *$S(X_1) = S(\langle X_1\rangle)$;*
(5) *If $X_1, X_2 \subseteq E$, then $\langle X_1\rangle = \langle X_2\rangle$ if and only if $S(X_1) = S(X_2)$;*
(6) *If $S(X_1) \subseteq S(X_2)$, then $\langle X_1\rangle \subseteq \langle X_2\rangle$.*

Proof. (1) For any $P \in S(X_1)$, then $X_1 \nsubseteq P$. Since $X_1 \subseteq X_2$, then $X_2 \nsubseteq P$ and $X_2 \in S(X_2)$. Hence, $S(X_1) \subseteq S(X_2)$.
(2) $(\Rightarrow)$ Assume that $S(X_1) = \emptyset$ but $X_1 \neq \emptyset$ and $X_1 \neq \{0\}$, then there are $0 \neq a \in X_1$ and $P \in I_p(\varepsilon)$ such that $a \notin P$ by Corollary 5.3, i.e. $X_1 \nsubseteq P$, and so $P \in S(X_1)$, which is contradiction. Hence, either $X_1 = \emptyset$ or $X_1 = \{0\}$.
$(\Leftarrow)$ If $X_1 = \emptyset$, then for each $P \in I_P(\varepsilon)$, we have $\emptyset \subseteq P$, i.e. $P \notin S(\emptyset)$, hence $S(\emptyset) = \emptyset$. Also, If $X_1 = \{0\}$, we can obtain $S(\{0\}) = \emptyset$.
(3) $(\Rightarrow)$ Assume $S(X_1) = I_P(\varepsilon)$ but $\langle X_1\rangle \neq E$. Then $\langle X_1\rangle$ is proper. By Corollary 5.3, there is $P \in I_p(\varepsilon)$ satisfying $\langle X_1\rangle \subseteq P$. Then, $X_1 \subseteq \langle X_1\rangle \subseteq P$, i.e. $P \notin S(X_1)$, which generates a contradiction. Hence $\langle X_1\rangle = E$.
$(\Leftarrow)$ If $\langle X_1\rangle = E$, then E is the smallest ideal including X_1. Hence $X_1 \nsubseteq P$ for any $P \in I_P(\varepsilon)$, and so $I_P(\varepsilon) \subseteq S(X_1)$. Therefore, $S(X_1) = I_P(\varepsilon)$.
(4) Since $X_1 \subseteq \langle X_1\rangle$, then $S(X_1) \subseteq S(\langle X_1\rangle)$ by (1). Conversely, for any $P \in S(\langle X_1\rangle)$, we have $\langle X_1\rangle \nsubseteq P$. Hence $X_1 \nsubseteq P$, i.e. $P \in S(X_1)$. Therefore $S(X_1) = S(\langle X_1\rangle)$.
(5) If $\langle X_1\rangle = \langle X_2\rangle$, then $S(X_1) = S(\langle X_1\rangle) = S(\langle X_2\rangle) = S(X_2)$. Conversely, assume $S(X_1) = S(X_2)$, but $\langle X_1\rangle \neq \langle X_2\rangle$, then there is $P \in I_p(\varepsilon)$ satisfying $\langle X_1\rangle \subseteq P$, $\langle X_2\rangle \nsubseteq P$ by Proposition 5.2. Thus $P \notin S(X_1)$, $P \in S(X_2)$, which is a contradiction. Hence $\langle X_1\rangle = \langle X_2\rangle$.
(6) For any $P \in I_P(\varepsilon)$, if $X_2 \subseteq P$, then $P \notin S(X_2)$, and so $P \notin S(X_1)$. Hence $X_1 \subseteq P$. Therefore $\langle X_1\rangle \subseteq \langle X_2\rangle$. □

Proposition 5.5. *Suppose that ε is an EQ-algebra, then*
(1) *If $\{X_i\}_{i\in I}$ is a family of subset of E, then $S(\cup_{i\in I}X_i) = \cup_{i\in I}S(X_i)$;*
(2) *For each $Z_1, Z_2 \in E$, $S(Z_1) \cap S(Z_2) = S(\langle Z_1\rangle \cap \langle Z_2\rangle)$.*

Proof. (1) For each $i \in I$, $X_i \subseteq \cup_{i\in I}X_i$, then $S(X_i) \subseteq S(\cup_{i\in I}X_i)$ for any $i \in I$. Thus $\cup_{i\in I}S(X_i) \subseteq S(\cup_{i\in I}X_i)$. Conversely, for any $P \in S(\cup_{i\in I}X_i)$, then $\cup_{i\in I}X_i \nsubseteq P$. Hence, there exists $i_0 \in I$ satisfying $X_{i_0} \nsubseteq P$, hence $P \in S(X_{i_0}) \subseteq \cup_{i\in I}S(X_i)$, and so $S(\cup_{i\in I}X_i) \subseteq \cup_{i\in I}S(X_i)$. Hence, $S(\cup_{i\in I}X_i) = \cup_{i\in I}S(X_i)$.
(2) On one hand, if $P \in S(Z_1) \cap S(Z_2)$, then $Z_1 \nsubseteq P$ and $Z_2 \nsubseteq P$. Hence $\langle Z_1\rangle \nsubseteq P$ and $\langle Z_2\rangle \nsubseteq P$. If $\langle Z_1\rangle \cap \langle Z_2\rangle \subseteq P$, then $\langle Z_1\rangle \subseteq P$ or $\langle Z_2\rangle \subseteq P$, which generates a contradiction, hence $\langle Z_1\rangle \cap \langle Z_2\rangle \nsubseteq P$, i.e. $P \in S(\langle Z_1\rangle \cap \langle Z_2\rangle)$, and so $S(Z_1) \cap S(Z_2) \subseteq S(\langle Z_1\rangle \cap \langle Z_2\rangle)$. On the other hand, assume $P \in S(\langle Z_1\rangle \cap \langle Z_2\rangle)$, then $\langle Z_1\rangle \nsubseteq P$, $\langle Z_2\rangle \nsubseteq P$. Thus $Z_1 \nsubseteq P$ and $Z_2 \nsubseteq P$. Hence $P \in S(Z_1) \cap S(Z_2)$, i.e. $S(\langle Z_1\rangle \cap \langle Z_2\rangle) \subseteq S(Z_1) \cap S(Z_2)$. Hence, $S(Z_1) \cap S(Z_2) = S(\langle Z_1\rangle \cap \langle Z_2\rangle)$. □

Let ε be an EQ-algebra, if $X = \{a\}$, then we denote that $S(a) = \{P \in I_P(\varepsilon) \mid a \notin P\}$.

Proposition 5.6. *Assume that ε is an EQ-algebra, then*
(1) *$S(0) = \emptyset$ and $S(1) = I_P(\varepsilon)$;*
(2) *if $a \le b$, then $S(a) \subseteq S(b)$;*
(3) *$S(a) = S(b)$ if and only if $\langle a\rangle = \langle b\rangle$;*
(4) *$S(a) \cap S(b) = S(a \wedge b)$;*
(5) *$S(a) \cup S(b) = S(a \oplus b)$.*

Proof. (1) For any $P \in I_P(\varepsilon)$, $0 \in P$, then $S(0) = \emptyset$. Since P is proper, then $1 \notin P$. Thus $P \in S(1)$, and so $S(1) = I_P(\varepsilon)$.
(2) − (3) The proof is analogous to that of the Proposition 5.4 (1) and (5).
(4) For any $P \in I_P(\varepsilon)$, $a \wedge b \notin P$ iff $a \notin P$ and $b \notin P$, hence $P \in S(a \wedge b)$ iff $a \wedge b \notin P$ iff $a \notin P$ and $b \notin P$ iff $P \in S(a)$ and $P \in S(b)$ iff $P \in S(a) \cap S(b)$.
(5) We can check that for each ideal I of ε, $a \notin I$ or $b \notin I$ iff $a\oplus b \notin I$ by using the proof by contradiction. Then for any $P \in I_P(\varepsilon)$, we have $a \notin P$ or $b \notin P$ iff $a \oplus b \notin P$, it follows that $P \in S(a) \cup S(b)$ iff $P \in S(a \oplus b)$. □

Definition 5.7. *Let ε be an EQ-algebra and $\mathfrak{I} = \{S(X) \mid X \subseteq E\}$, then by the Proposition 5.4 and 5.5 we have*
(1) *$\emptyset$, $I_P(\varepsilon) \in \mathfrak{I}$;*
(2) *if $S(X)$, $S(Y) \in \mathfrak{I}$, then $S(X) \cap S(Y) \in \mathfrak{I}$;*
(3) *if $S(X_i)_{i\in I} \in \mathfrak{I}$, then $\cup_{i\in I}S(X_i) \in \mathfrak{I}$.*
Hence, $\mathfrak{I}$ is a topology on $I_P(\varepsilon)$ and $(I_P(\varepsilon), \mathfrak{I})$ is a topological space of prime ideals.

Proposition 5.8. *Suppose that ε is an EQ-algebra, then $\{S(a)\}_{a\in E}$ is a topological base of $\mathfrak{I}$.*

Proof. For any $S(X) \in \mathfrak{I}$, then, by the Proposition 5.5, we have $S(X) = S(\cup_{a\in X}\{a\}) = \cup_{a\in X}S(\{a\})$. It follows that every member in $\mathfrak{I}$ can be expressed as the union of elements in subset of $\{S(a)\}_{a\in E}$. Hence, the conclusion holds. □

Proposition 5.9. *Let ε be an EQ-algebra, then $S(a)$ is compact in $(I_P(\varepsilon), \mathfrak{I})$ for any $a \in E$.*

Proof. We only need to prove that each open covering of $S(a)$ has a finite open covering. Assume $S(a) = \cup_{i\in I}S(a_i) = S(\cup_{i\in I}\{a_i\})$, then $\langle a\rangle = \langle \cup_{i\in I}\{a_i\}\rangle$ by the Proposition 5.4 (5), thus $a \in \langle \cup_{i\in I}\{a_i\}\rangle$ and there are $n \in N$ and $i_1, i_2, \cdots, i_n \in I$ satisfying $a \leq a_{i_1} \oplus a_{i_2} \oplus \cdots \oplus a_{i_n}$. By the Proposition 5.6 (2) and (5), we have $S(a) \subseteq S(a_{i_1} \oplus a_{i_2} \oplus \cdots \oplus a_{i_n}) = S(a_{i_1}) \cup \cdots \cup S(a_{i_n})$. Conversely, $S(a_{i_1}) \cup \cdots \cup S(a_{i_n}) \subseteq \cup_{i \in I} S(a_i) = S(a)$. Hence, $S(a) = S(a_{i_1}) \cup \cdots \cup S(a_{i_n})$. □

Proposition 5.10. *Suppose that ε is an EQ-algebra, Then $(I_P(\varepsilon), \mathfrak{I})$ is a compact T_0 topological space.*

Proof. Since $I_P(\varepsilon) = S(1)$, then $I_P(\varepsilon)$ is compact by the Proposition 5.9. Now we show that $I_P(\varepsilon)$ is a T_0 topological space. For every $P, Q \in I_P(\varepsilon)$ and $P \neq Q$, then $P \nsubseteq Q$ or $Q \nsubseteq P$. If $P \nsubseteq Q$, then there exists a satisfying $a \in P$ but $a \notin Q$. Let $U = S(a)$, then $Q \in U$, $P \notin U$. If $Q \nsubseteq P$, the proof is similar. Hence, the conclusion holds. □

From the above discussions, we know that the topological space $I_p(E)$ is a prime spectrum [16]. Next, we study the reticulation of EQ-algebras.

Definition 5.11. *Given an EQ-algebra ε, a bounded distributive lattice L, and a map $\lambda : E \to L$. We call the pair (L, λ) an ideal reticulation, if it satisfies the following properties:*
$(R1)$ *$\lambda(a \oplus b) = \lambda(a) \vee \lambda(b)$;*
$(R2)$ *$\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$;*
$(R3)$ *$\lambda(0) = 0, \lambda(1) = 1$;*
$(R4)$ *λ is surjective;*
$(R5)$ *$\lambda(a) \leq \lambda(b)$ iff $na \leq b$ for some $n \in N^+$.*

Example 5.12. *Let L be the set of all principal ideals and $\lambda : a \to \langle a \rangle$, where $a \in E$. Then, we can check that (L, λ) be an ideal reticulation by Proposition 3.17.*

Proposition 5.13. *Given an ideal I of EQ-algebra ε and $a, b \in E$. Then*
(1) *If $\lambda(a) = \lambda(b)$, then $a \in I$ if and only if $b \in I$.*
(2) *$\lambda(a) = \lambda(b)$ if and only if $\langle a \rangle = \langle b \rangle$.*
(3) *$\lambda(a) \in \lambda(I)$ if and only if $a \in I$.*
(4) *If I is prime of E, then $\lambda(I)$ is a prime ideal of L.*

Proof. (1) Assume that $a \in I$, since $\lambda(b) \leq \lambda(a)$, then $nb \leq a$ for some $n \in N^+$. Thus, $b \in I$. Similarly, we can prove that $b \in I$ implies that $a \in I$.
(2) Let $\lambda(a) = \lambda(b)$. Now, $\lambda(a) \leq \lambda(b) \Leftrightarrow a \leq nb \Leftrightarrow a \in \langle b \rangle$, it follows that $\langle a \rangle \subseteq \langle b \rangle$. Analogously, we have $\langle b \rangle \subseteq \langle a \rangle$.
(3) If $\lambda(a) \in \lambda(I)$, then there is $b \in I$ such that $\lambda(a) = \lambda(b)$. It follows that $a \in I$ by (1).
(4) Since $0 \in I \neq \emptyset$, then $\lambda(0) \in \lambda(I) \neq \emptyset$. Assume that $\lambda(a), \lambda(b) \in L$ with $\lambda(a) \leq \lambda(b)$, then $na \leq b$. Thus, $na \in I$ and so $a \in I$. Hence, $\lambda(a) \in \lambda(I)$, that is, $(I1)$ holds. Let $\lambda(a), \lambda(b) \in \lambda(I)$. Then, $a, b \in I$ by (3). Hence, $a \oplus b \in I$, and so $\lambda(a \oplus b) \in \lambda(I)$. That is, $\lambda(a) \vee \lambda(b) \in \lambda(I)$. Hence, $\lambda(I)$ is an ideal.

Since $I \neq E$, then there is $a \in E$ but $a \notin I$. Thus,$\lambda(a) \notin \lambda(I)$ by (3). Hence, $\lambda(I)$ is proper. Assume that $\lambda(a) \wedge \lambda(b) \in \lambda(I)$, then $\lambda(a \wedge b) \in \lambda(I)$. Thus, $a \in I$ or $b \in I$, it follows that $\lambda(a) \in \lambda(I)$ or $\lambda(b) \in \lambda(I)$. Hence, $\lambda(I)$ is prime. □

Proposition 5.14. *The map $\lambda : E \to L$ is surjective if and only if $\lambda^* : \mathcal{P}(L) \to \mathcal{P}(E)$ is injective.*

Proof. Let λ be surjective. If $\lambda^*(X) = \lambda^*(Y)$ for some $X, Y \in \mathcal{P}(L)$, then $\lambda\lambda^*(X) = \lambda\lambda^*(Y)$. It follows that $X = \lambda\lambda^*(X) = \lambda\lambda^*(Y) = Y$, that is, λ^* is injective. Conversely, for each $y \in Y$, $\lambda^*(\{y\}) \neq \emptyset$. Otherwise, if $\lambda^*(\{y\}) = \emptyset$ for some $y_0 \in Y$. Then,$\lambda^*(Y) = \lambda^*(Y \setminus \{y_0\})$, but $Y \neq Y \setminus \{y_0\}$, which is a contradiction. □

Now, we define the restriction $\lambda^* : I_p(L) \to I_p(E)$ by $\lambda^*(I) = \lambda^{-1}(I)$ for each $I \in I_p(L)$.

Proposition 5.15. *If $I \in I_p(L)$, then $\lambda^*(I) \in I_p(E)$.*

Proof. First, we show that $\lambda^*(I)$ is an ideal of E. Let $a, b \in E$ such that $a \leq b$ and $b \in \lambda^*$. Then $\lambda(a) \leq \lambda(b) \in I$. Thus, $\lambda(a) \in I$, it follows that $a \in \lambda^*(I)$. That is, $(I1)$ holds. Now, assume $a, b \in \lambda^*$, then $\lambda(a), \lambda(b) \in I$, and so $\lambda(a) \vee \lambda(b) \in I$. That is $\lambda(a \oplus b) \in I$, it follows that $a \oplus b \in \lambda^*(E)$.

Since $I \neq L$ and λ^* is injective, then $\lambda^*(I) \neq E$. Assume that $a \wedge b \in \lambda^*(I)$, then $\lambda(a) \wedge \lambda(b) = \lambda(a \wedge b) \in I$. Thus, $\lambda(a) \in I$ or $\lambda(b) \in I$, it follows that $a \in \lambda^*(I)$ or $b \in \lambda^*(I)$. Hence, $\lambda^*(I) \in I_p(E)$. □

Theorem 5.16. *Given an ideal I of EQ-algebra ε. Then*
(1) $\lambda^*\lambda(I) = I$.
(2) *The map λ^* is surjective.*

Proof. (1) By Proposition 5.13 (3), we have $a \in I \Leftrightarrow \lambda(a) \in \lambda(I)$ iff $a \in \lambda^*\lambda(I)$.
(2) Let $I \in I_p(E)$, $\lambda(I) \in I_p(L)$ by Proposition 5.13 (4). By (1), we know that $\lambda^*\lambda(I) = I$, then λ^* is surjective. □

6. Conclusions

In this article, we firstly introduced ideals and other special ideals in EQ-algebras, such as implicative ideals, primary ideals, prime ideals and maximal ideals. We discuss their properties and the relationships among them, for example, we proved that every maximal ideal is prime and if prime ideals are implicative, then they are maximal in the EQ-algebras with the condition (DNP). And each primary ideal is prime under certain condition. Also, we gave the generation formula of ideals and their equivalent characterizations. By studying the relationship between ideals and filters, we obtain that filters and ideals are not dual notions in EQ-algebra. But we prove that they are dual in a residuated EQ-algebra with the condition (DNP). Also, We gave an equivalent condition for $I(\varepsilon)$ to become a Boolean algebra. Finally, we introduced the topological properties of prime ideals. We got that the set of all prime ideals is a compact T_0 topological space. Also, we transferred the spectrum of EQ-algebras to bounded distributive lattices and given the ideal reticulation of EQ-algebras.

As we all know, fuzzy ideals are the generalization of ideals. An ideal can uniquely determine a fuzzy ideal, but given a fuzzy ideal, we can get more than one ideals. So the study of fuzzy ideals is more extensive than that of ideals, and more algebraic structure information can be obtained. Therefore, we will devote ourselves to the study of fuzzy ideals on EQ-algebras in the future work.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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