



*Research article*

## Gauss-Bonnet theorems in the generalized affine group and the generalized BCV spaces

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**Abstract:** In this paper, we compute sub-Riemannian limits of Gaussian curvature for a Euclidean  $C^2$ -smooth surface in the generalized affine group and the generalized BCV spaces away from characteristic points and signed geodesic curvature for Euclidean  $C^2$ -smooth curves on surfaces. We get Gauss-Bonnet theorems in the generalized affine group and the generalized BCV spaces.

**Keywords:** the generalized affine group; the generalized BCV spaces; Gauss-Bonnet theorem; sub-Riemannian limit

**Mathematics Subject Classification:** 53C40, 53C42

### 1. Introduction

In [4], Diniz and Veloso gave the definition of Gaussian curvature for non-horizontal surfaces in sub-Riemannian Heisenberg space  $\mathbb{H}^1$  and the proof of the Gauss-Bonnet theorem. In [1], intrinsic Gaussian curvature for a Euclidean  $C^2$ -smooth surface in the Heisenberg group  $\mathbb{H}^1$  away from characteristic points and intrinsic signed geodesic curvature for Euclidean  $C^2$ -smooth curves on surfaces are defined by using a Riemannian approximation scheme. These results were then used to prove a Heisenberg version of the Gauss-Bonnet theorem. In [5], Veloso verified that Gaussian curvature of surfaces and normal curvature of curves in surfaces introduced by [4] and by [1] to prove Gauss-Bonnet theorems in Heisenberg space  $\mathbb{H}^1$  were unequal and he applied the same formalism of [4] to get the curvatures of [1]. With the obtained formulas, the Gauss-Bonnet theorem can be proved as a straightforward application of Stokes theorem in [5].

In [1] and [2], Balogh-Tyson-Vecchi used that the Riemannian approximation scheme may depend upon the choice of the complement to the horizontal distribution in general. In the context of  $\mathbb{H}^1$  the choice which they have adopted is rather natural. The existence of the limit defining the intrinsic curvature of a surface depends crucially on the cancellation of certain divergent quantities in the limit. Such cancellation stems from the specific choice of the adapted frame bundle on the surface, and

on symmetries of the underlying left-invariant group structure on the Heisenberg group. In [1], they proposed an interesting question to understand to what extent similar phenomena hold in other sub-Riemannian geometric structures. In [6], Wang and Wei gave sub-Riemannian limits of Gaussian curvature for a Euclidean  $C^2$ -smooth surface in the affine group and the group of rigid motions of the Minkowski plane away from characteristic points and signed geodesic curvature for Euclidean  $C^2$ -smooth curves on surfaces. And they got Gauss-Bonnet theorems in the affine group and the group of rigid motions of the Minkowski plane. In [7], Wang and Wei gave sub-Riemannian limits of Gaussian curvature for a Euclidean  $C^2$ -smooth surface in the BCV spaces and the twisted Heisenberg group away from characteristic points and signed geodesic curvature for Euclidean  $C^2$ -smooth curves on surfaces. And they got Gauss-Bonnet theorems in the BCV spaces and the twisted Heisenberg group.

In this paper, we solve this problem for the generalized affine group and the generalized BCV spaces. In the case of the generalized affine group, the cancellation of certain divergent quantities in the limit happens and the limit of the Riemannian Gaussian curvature exists. In the case of the generalized BCV spaces, the result is the same as the generalized affine group. We also get Gauss-Bonnet theorems in the generalized affine group and the generalized BCV spaces.

In Section 2, we compute the sub-Riemannian limit of curvature of curves in the generalized affine group. In Section 3, we compute sub-Riemannian limits of geodesic curvature of curves on surfaces and the Riemannian Gaussian curvature of surfaces in the generalized affine group. In Section 4, we prove the Gauss-Bonnet theorem in the generalized affine group. In Section 5, we compute the sub-Riemannian limit of curvature of curves in the generalized BCV spaces. In Section 6, we compute sub-Riemannian limits of geodesic curvature of curves on surfaces and the Riemannian Gaussian curvature of surfaces in the generalized BCV spaces and get a Gauss-Bonnet theorem in the generalized BCV spaces.

## 2. The sub-Riemannian limit of curvature of curves in the generalized affine group

When  $TM = H \oplus H^\perp$  and  $g^{TM} = g^H \oplus g^{H^\perp}$ , we may consider the rescaled metric  $g_L = g^H \oplus Lg^{H^\perp}$ , then we may consider the sub-Riemannian limit of some geometric objects like the Gauss curvature and the mean curvature  $\dots$ , when  $L$  goes to the infinity. In this case, we call the  $(M, g^{TM})$  as the manifold with the splitting tangent bundle. In this paper, our main objects: the generalized affine group and the generalized BCV spaces are not sub-Riemannian manifolds (groups) in general. But they are manifolds with the splitting tangent bundle. So we can use the Riemannian approximation scheme to get the Gauss-Bonnet theorems in these spaces.

Firstly we give some notations on the generalized affine group. Let  $\mathbb{G}$  be the generalized affine group and choose the underlying manifold  $\mathbb{G} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid f(x_1, x_2, x_3) > 0\}$ . On  $\mathbb{G}$ , we let

$$X_1 = f\partial_{x_1}, \quad X_2 = f\partial_{x_2} + \partial_{x_3}, \quad X_3 = f\partial_{x_2}. \quad (2.1)$$

where  $f$  be a smooth function with respect to  $x_1, x_2, x_3$ . Then

$$\partial_{x_1} = \frac{1}{f}X_1, \quad \partial_{x_2} = \frac{1}{f}X_3, \quad \partial_{x_3} = X_2 - X_3, \quad (2.2)$$

and  $\text{span}\{X_1, X_2, X_3\} = T\mathbb{G}$ . Let  $H = \text{span}\{X_1, X_2\}$  be the horizontal distribution on  $\mathbb{G}$ . Let  $\omega_1 = \frac{1}{f}dx_1$ ,  $\omega_2 = dx_3$ ,  $\omega = \frac{1}{f}dx_2 - dx_3$ . Then  $H = \text{Ker}\omega$ . For the constant  $L > 0$ , let  $g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2$

$\omega_2 + L\omega \otimes \omega$ ,  $g = g_1$  be the Riemannian metric on  $\mathbb{G}$ . Then  $X_1, X_2, \widetilde{X}_3 := L^{-\frac{1}{2}}X_3$  are orthonormal basis on  $T\mathbb{G}$  with respect to  $g_L$ . We have

$$[X_1, X_2] = -(f_2 + \frac{f_3}{f})X_1 + f_1X_3, \quad [X_1, X_3] = -f_2X_1 + f_1X_3, \quad [X_2, X_3] = \frac{f_3}{f}X_3. \quad (2.3)$$

where  $f_i = \frac{\partial f}{\partial x_i}$ , for  $1 \leq i \leq 3$ .

Let  $\nabla^L$  be the Levi-Civita connection on  $\mathbb{G}$  with respect to  $g_L$ . Then we have the following lemma,

**Lemma 2.1.** *Let  $\mathbb{G}$  be the generalized affine group, then*

$$\begin{aligned} \nabla_{X_1}^L X_1 &= (f_2 + \frac{f_3}{f})X_2 + \frac{f_3}{L}, & \nabla_{X_1}^L X_2 &= -(f_2 + \frac{f_3}{f})X_1 + \frac{f_1}{2}X_3, & \nabla_{X_2}^L X_1 &= -\frac{f_1}{2}X_3, \\ \nabla_{X_2}^L X_2 &= 0, & \nabla_{X_1}^L X_3 &= -f_2X_1 - \frac{f_1L}{2}X_2, & \nabla_{X_3}^L X_1 &= -\frac{f_1L}{2}X_2 - f_1X_3, \\ \nabla_{X_2}^L X_3 &= \frac{f_1L}{2}X_1, & \nabla_{X_3}^L X_2 &= \frac{f_1L}{2}X_1 - \frac{f_3}{f}X_3, & \nabla_{X_3}^L X_3 &= f_1LX_1 + \frac{f_3L}{f}X_2. \end{aligned} \quad (2.4)$$

*Proof.* By the Koszul formula, we have

$$2\langle \nabla_{X_i}^L X_j, X_k \rangle_L = \langle [X_i, X_j], X_k \rangle_L - \langle [X_j, X_k], X_i \rangle_L + \langle [X_k, X_i], X_j \rangle_L, \quad (2.5)$$

where  $i, j, k = 1, 2, 3$ . So lemma 2.1 holds.  $\square$

**Definition 2.2.** *Let  $\gamma : [a, b] \rightarrow (\mathbb{G}, g_L)$  be a Euclidean  $C^1$ -smooth curve. We say that  $\gamma$  is regular if  $\dot{\gamma} \neq 0$  for every  $t \in [a, b]$ . Moreover we say that  $\gamma(t)$  is a horizontal point of  $\gamma$  if*

$$\omega(\dot{\gamma}(t)) = \frac{\dot{\gamma}_2(t)}{f} - \dot{\gamma}_3(t) = 0,$$

where  $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$  and  $\dot{\gamma}_i(t) = \frac{\partial \gamma_i(t)}{\partial t}$ .

**Definition 2.3.** *Let  $\gamma : [a, b] \rightarrow (\mathbb{G}, g_L)$  be a Euclidean  $C^2$ -smooth regular curve in the Riemannian manifold  $(\mathbb{G}, g_L)$ . The curvature  $k_\gamma^L$  of  $\gamma$  at  $\gamma(t)$  is defined as*

$$k_\gamma^L := \sqrt{\frac{\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\|_L^2}{\|\dot{\gamma}\|_L^4} - \frac{\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma} \rangle_L^2}{\|\dot{\gamma}\|_L^6}}. \quad (2.6)$$

**Lemma 2.4.** *Let  $\gamma : [a, b] \rightarrow (\mathbb{G}, g_L)$  be a Euclidean  $C^2$ -smooth regular curve in the Riemannian manifold  $(\mathbb{G}, g_L)$ . Then,*

$$\begin{aligned} k_\gamma^L &= \left\{ \left\{ \left[ \frac{\ddot{\gamma}_1 - f_2\dot{\gamma}_1\dot{\gamma}_3}{f} - \frac{f'\dot{\gamma}_1 + f_3\dot{\gamma}_1\dot{\gamma}_3}{f^2} \right] + \left[ f_1L\dot{\gamma}_3 - \frac{f_2\dot{\gamma}_1}{f} + f_1L\omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \right\}^2 \right. \\ &+ \left\{ \left[ \ddot{\gamma}_3 + \frac{f_2\dot{\gamma}_1^2}{f^2} + \frac{f_3\dot{\gamma}_1^2}{f^3} \right] + \left[ \frac{f_3L\omega(\dot{\gamma}(t))}{f} - \frac{f_1L\dot{\gamma}_1}{f} \right] \omega(\dot{\gamma}(t)) \right\}^2 \\ &\left. + L \left\{ \frac{f_2\dot{\gamma}_1^2}{f^2L} - \left[ \frac{f_1\dot{\gamma}_1}{f} + \frac{f_3\dot{\gamma}_3}{f} \right] \omega(\dot{\gamma}(t)) + \frac{d}{dt} \omega(\dot{\gamma}(t)) \right\}^2 \right\} \cdot \left[ \left( \frac{\dot{\gamma}_1}{f} \right)^2 + \dot{\gamma}_3^2 + L(\omega(\dot{\gamma}(t)))^2 \right]^{-2} \end{aligned} \quad (2.7)$$

$$- \left\{ \left[ \frac{\dot{\gamma}_1 \ddot{\gamma}_1}{f^2} - \frac{f' \dot{\gamma}_1^2}{f^3} + \dot{\gamma}_3 \ddot{\gamma}_3 \right] + \frac{L d\omega(\dot{\gamma}(t))}{dt} \omega(\dot{\gamma}(t)) \right\}^2 \cdot \left[ \left( \frac{\dot{\gamma}_1}{f} \right)^2 + \dot{\gamma}_3^2 + L(\omega(\dot{\gamma}(t)))^2 \right]^{-3} \right\}^{\frac{1}{2}}$$

In particular, if  $\gamma(t)$  is a horizontal point of  $\gamma$ ,

$$\begin{aligned} k_\gamma^L = & \left\{ \left\{ \left[ \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right]^2 + \left[ \dot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right]^2 \right. \right. \\ & \left. \left. + L \left[ \frac{f_2 \dot{\gamma}_1^2}{f^2 L} + \frac{d}{dt} \omega(\dot{\gamma}(t)) \right]^2 \right\} \cdot \left[ \left( \frac{\dot{\gamma}_1}{f} \right)^2 + \dot{\gamma}_3^2 \right]^{-2} \right. \\ & \left. - \left[ \frac{\dot{\gamma}_1 \ddot{\gamma}_1}{f^2} - \frac{f' \dot{\gamma}_1^2}{f^3} + \dot{\gamma}_3 \ddot{\gamma}_3 \right]^2 \cdot \left[ \left( \frac{\dot{\gamma}_1}{f} \right)^2 + \dot{\gamma}_3^2 \right]^{-3} \right\}^{\frac{1}{2}} \end{aligned} \quad (2.8)$$

where  $f' = \dot{\gamma}(f) = \frac{d}{dt} f(\gamma(t))$ .

*Proof.* By (2.2), we have

$$\dot{\gamma}(t) = \frac{\dot{\gamma}_1}{f} X_1 + \dot{\gamma}_3 X_2 + \omega(\dot{\gamma}(t)) X_3. \quad (2.9)$$

By Lemma 2.1 and (2.9), we have

$$\begin{aligned} \nabla_{\dot{\gamma}}^L X_1 &= \left[ \frac{f f_2 \dot{\gamma}_1(t) + f_3 \dot{\gamma}_1(t)}{f^2} - \frac{f_1 L \omega(\dot{\gamma}(t))}{2} \right] X_2 + \left[ \frac{f_2 \dot{\gamma}_1(t)}{f L} - \frac{f_1 \dot{\gamma}_3(t)}{2} - f_1 \omega(\dot{\gamma}(t)) \right] X_3, \\ \nabla_{\dot{\gamma}}^L X_2 &= \left[ -\frac{f f_2 \dot{\gamma}_1(t) + f_3 \dot{\gamma}_1(t)}{f^2} + \frac{f_1 L \omega(\dot{\gamma}(t))}{2} \right] X_1 + \left[ -\frac{f_3 \omega(\dot{\gamma}(t))}{f} + \frac{f_1 \dot{\gamma}(t)}{2f} \right] X_3, \\ \nabla_{\dot{\gamma}}^L X_3 &= \left[ -\frac{f_2 \dot{\gamma}_1(t)}{f} + \frac{L f_1 \dot{\gamma}_3(t)}{2} + f_1 L \omega(\dot{\gamma}(t)) \right] X_1 + \left[ -\frac{f_1 L \dot{\gamma}(t)}{2f} + \frac{f_3 L \omega(\dot{\gamma}(t))}{f} \right] X_2. \end{aligned} \quad (2.10)$$

By (2.9) and (2.10), we have

$$\begin{aligned} \nabla_{\dot{\gamma}}^L \dot{\gamma} &= \left\{ \left[ \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right] + \left[ f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \right\} X_1 \\ &+ \left\{ \left[ \dot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right] + \left[ \frac{f_3 L \omega(\dot{\gamma}(t))}{f} - \frac{f_1 L \dot{\gamma}_1}{f} \right] \omega(\dot{\gamma}(t)) \right\} X_2 \\ &+ \left\{ \frac{f_2 \dot{\gamma}_1^2}{f^2 L} - \left( \frac{f_1 \dot{\gamma}_1}{f} + \frac{f_3 \dot{\gamma}_3}{f} \right) \omega(\dot{\gamma}(t)) + \frac{d\omega(\dot{\gamma}(t))}{dt} \right\} X_3 \end{aligned} \quad (2.11)$$

By (2.6), (2.9) and (2.11), we get Lemma 2.4.  $\square$

**Definition 2.5.** Let  $\gamma : [a, b] \rightarrow (\mathbb{G}, g_L)$  be a Euclidean  $C^2$ -smooth regular curve in the Riemannian manifold  $(\mathbb{G}, g_L)$ . We define the intrinsic curvature  $k_\gamma^\infty$  of  $\gamma$  at  $\gamma(t)$  to be

$$k_\gamma^\infty := \lim_{L \rightarrow +\infty} k_\gamma^L,$$

if the limit exists.

We introduce the following notation: for continuous functions  $f_1, f_2 : (0, +\infty) \rightarrow \mathbb{R}$ ,

$$f_1(L) \sim f_2(L), \text{ as } L \rightarrow +\infty \Leftrightarrow \lim_{L \rightarrow +\infty} \frac{f_1(L)}{f_2(L)} = 1. \quad (2.12)$$

**Lemma 2.6.** *Let  $\gamma : [a, b] \rightarrow (\mathbb{G}, g_L)$  be a Euclidean  $C^2$ -smooth regular curve in the Riemannian manifold  $(\mathbb{G}, g_L)$ . Then*

$$k_\gamma^\infty = \frac{\sqrt{[f_1\dot{\gamma}_1 - \frac{f_3\dot{\gamma}_2}{f} + f_3\dot{\gamma}_3]^2 + (f_1\dot{\gamma}_2)^2}}{|f|\omega(\dot{\gamma}(t))|}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0, \quad (2.13)$$

$$\begin{aligned} k_\gamma^\infty = & \left\{ \left\{ \left[ \frac{\ddot{\gamma}_1 - f_2\dot{\gamma}_1\dot{\gamma}_3}{f} - \frac{f'\dot{\gamma}_1 + f_3\dot{\gamma}_1\dot{\gamma}_3}{f^2} \right]^2 + \left[ \ddot{\gamma}_3 + \frac{f_2\dot{\gamma}_1^2}{f^2} + \frac{f_3\dot{\gamma}_1^2}{f^3} \right]^2 \right. \right. \\ & \left. \left. + \left( \frac{f_2\dot{\gamma}_1^2}{f^2} \right)^2 \right\} \cdot \left[ \left( \frac{\dot{\gamma}_1}{f} \right)^2 + \dot{\gamma}_3^2 \right]^{-2} \right. \\ & \left. - \left[ \frac{\dot{\gamma}_1\ddot{\gamma}_1}{f^2} - \frac{f'\dot{\gamma}_1^2}{f^3} + \dot{\gamma}_3\ddot{\gamma}_3 \right]^2 \cdot \left[ \left( \frac{\dot{\gamma}_1}{f} \right)^2 + \dot{\gamma}_3^2 \right]^{-3} \right\}^{\frac{1}{2}} \\ & \text{if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0, \end{aligned} \quad (2.14)$$

$$\lim_{L \rightarrow +\infty} \frac{k_\gamma^L}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot{\gamma}(t)))|}{\left(\frac{\dot{\gamma}_1}{f}\right)^2 + \dot{\gamma}_3^2}, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \quad (2.15)$$

*Proof.* Using the notation introduced in (2.12), when  $\omega(\dot{\gamma}(t)) \neq 0$ , we have

$$\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\|_L^2 \sim \left( \frac{\omega(\dot{\gamma}(t))}{f} \right)^2 \left\{ \left[ f_1\dot{\gamma}_1 - \frac{f_3\dot{\gamma}_2}{f} + f_3\dot{\gamma}_3 \right]^2 + (f_1\dot{\gamma}_2)^2 \right\} L^2, \text{ as } L \rightarrow +\infty,$$

$$\|\dot{\gamma}\|_L^2 \sim L\omega(\dot{\gamma}(t))^2, \text{ as } L \rightarrow +\infty,$$

$$\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma} \rangle_L^2 \sim O(L^2) \text{ as } L \rightarrow +\infty.$$

Therefore

$$\frac{\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\|_L^2}{\|\dot{\gamma}\|_L^4} \rightarrow \frac{\left\{ \left[ f_1\dot{\gamma}_1 - \frac{f_3\dot{\gamma}_2}{f} + f_3\dot{\gamma}_3 \right]^2 + (f_1\dot{\gamma}_2)^2 \right\}}{f^2\omega(\dot{\gamma}(t))^2}, \text{ as } L \rightarrow +\infty,$$

$$\frac{\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma} \rangle_L^2}{\|\dot{\gamma}\|_L^6} \rightarrow 0, \text{ as } L \rightarrow +\infty.$$

So by (2.6), we have (2.13). (2.14) comes from (2.8) and

$$\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0.$$

When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$ , we have

$$\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\|_L^2 \sim L \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right]^2, \text{ as } L \rightarrow +\infty,$$

$$\|\dot{\gamma}\|_L^2 = \left( \frac{\dot{\gamma}_1}{f} \right)^2 + \dot{\gamma}_3^2,$$

$$\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma} \rangle_L = O(1) \text{ as } L \rightarrow +\infty.$$

By (2.6), we get (2.15). □

### 3. The sub-Riemannian limit of geodesic curvature of curves on surfaces in the generalized affine group

We will say that a surface  $\Sigma \subset (\mathbb{G}, g_L)$  is regular if  $\Sigma$  is a Euclidean  $C^2$ -smooth compact and oriented surface. In particular we will assume that there exists a Euclidean  $C^2$ -smooth function  $u : \mathbb{G} \rightarrow \mathbb{R}$  such that

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{G} : u(x_1, x_2, x_3) = 0\}$$

and  $u_{x_1} \partial_{x_1} + u_{x_2} \partial_{x_2} + u_{x_3} \partial_{x_3} \neq 0$ . Let  $\nabla_H u = X_1(u)X_1 + X_2(u)X_2$ . A point  $x \in \Sigma$  is called *characteristic* if  $\nabla_H u(x) = 0$ . We define the characteristic set  $C(\Sigma) := \{x \in \Sigma | \nabla_H u(x) = 0\}$ . Our computations will be local and away from characteristic points of  $\Sigma$ . Let us define first

$$p := X_1 u, \quad q := X_2 u, \quad \text{and } r := \widetilde{X}_3 u.$$

We then define

$$\begin{aligned} l &:= \sqrt{p^2 + q^2}, & l_L &:= \sqrt{p^2 + q^2 + r^2}, & \bar{p} &:= \frac{p}{l}, \\ \bar{q} &:= \frac{q}{l}, & \bar{p}_L &:= \frac{p}{l_L}, & \bar{q}_L &:= \frac{q}{l_L}, & \bar{r}_L &:= \frac{r}{l_L}. \end{aligned} \quad (3.1)$$

In particular,  $\bar{p}^2 + \bar{q}^2 = 1$ . These functions are well defined at every non-characteristic point. Let

$$v_L = \bar{p}_L X_1 + \bar{q}_L X_2 + \bar{r}_L \widetilde{X}_3, \quad e_1 = \bar{q}_L X_1 - \bar{p}_L X_2, \quad e_2 = \bar{r}_L \bar{p}_L X_1 + \bar{r}_L \bar{q}_L X_2 - \frac{l}{l_L} \widetilde{X}_3, \quad (3.2)$$

then  $v_L$  is the Riemannian unit normal vector to  $\Sigma$  and  $e_1, e_2$  are the orthonormal basis of  $\Sigma$ . On  $T\Sigma$  we define a linear transformation  $J_L : T\Sigma \rightarrow T\Sigma$  such that

$$J_L(e_1) := e_2; \quad J_L(e_2) := -e_1. \quad (3.3)$$

For every  $U, V \in T\Sigma$ , we define  $\nabla_U^{\Sigma, L} V = \pi \nabla_U^L V$  where  $\pi : T\mathbb{G} \rightarrow T\Sigma$  is the projection. Then  $\nabla^{\Sigma, L}$  is the Levi-Civita connection on  $\Sigma$  with respect to the metric  $g_L$ . By (2.11), (3.2) and

$$\nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma} = \langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, e_1 \rangle_L e_1 + \langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, e_2 \rangle_L e_2, \quad (3.4)$$

we have

$$\begin{aligned} \nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma} &= \left\{ \bar{q} \left[ \left( \frac{\dot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \left( f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right) \omega(\dot{\gamma}(t)) \right] \right. \\ &\quad \left. - \bar{p} \left[ \left( \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \left( \frac{f_1 L \dot{\gamma}_1 - f_3 L \omega(\dot{\gamma}(t))}{f} \right) \omega(\dot{\gamma}(t)) \right] \right\} e_1 \\ &\quad + \left\{ \bar{r}_L \bar{p} \left[ \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \left( f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right) \omega(\dot{\gamma}(t)) \right] \right. \\ &\quad \left. + \bar{r}_L \bar{q} \left[ \left( \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \left( \frac{f_1 L \dot{\gamma}_1 - f_3 L \omega(\dot{\gamma}(t))}{f} \right) \omega(\dot{\gamma}(t)) \right] \right. \\ &\quad \left. - \frac{l}{l_L} L^{\frac{1}{2}} \left[ \frac{f_2 \dot{\gamma}_1^2}{f^2 L} - \left( \frac{f_1 \dot{\gamma}_1}{f} + \frac{f_3 \dot{\gamma}_3}{f} \right) \omega(\dot{\gamma}(t)) + \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right] \right\} e_2. \end{aligned} \quad (3.5)$$

Moreover if  $\omega(\dot{\gamma}(t)) = 0$ , then

$$\begin{aligned} \nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma} &= \left\{ \bar{q} \left[ \frac{\dot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right] - \bar{p} \left[ \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right] \right\} e_1 \\ &\quad + \left\{ \bar{r}_L \bar{p} \left[ \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right] + \bar{r}_L \bar{q} \left[ \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right] \right. \\ &\quad \left. - \frac{l}{l_L} L^{\frac{1}{2}} \left[ \frac{f_2 \dot{\gamma}_1^2}{f^2 L} - \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right] \right\} e_2. \end{aligned} \quad (3.6)$$

**Definition 3.1.** Let  $\Sigma \subset (\mathbb{G}, g_L)$  be a regular surface. Let  $\gamma : [a, b] \rightarrow \Sigma$  be a Euclidean  $C^2$ -smooth regular curve. The geodesic curvature  $k_{\gamma,\Sigma}^L$  of  $\gamma$  at  $\gamma(t)$  is defined as

$$k_{\gamma,\Sigma}^L := \sqrt{\frac{\|\nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma}\|_{\Sigma,L}^2 - \langle \nabla_{\dot{\gamma}}^{\Sigma,L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma,L}^2}{\|\dot{\gamma}\|_{\Sigma,L}^4 - \|\dot{\gamma}\|_{\Sigma,L}^6}}. \quad (3.7)$$

**Definition 3.2.** Let  $\Sigma \subset (\mathbb{G}, g_L)$  be a regular surface. Let  $\gamma : [a, b] \rightarrow \Sigma$  be a Euclidean  $C^2$ -smooth regular curve. We define the intrinsic geodesic curvature  $k_{\gamma,\Sigma}^\infty$  of  $\gamma$  at  $\gamma(t)$  to be

$$k_{\gamma,\Sigma}^\infty := \lim_{L \rightarrow +\infty} k_{\gamma,\Sigma}^L,$$

if the limit exists.

**Lemma 3.3.** Let  $\Sigma \subset (\mathbb{G}, g_L)$  be a regular surface. Let  $\gamma : [a, b] \rightarrow \Sigma$  be a Euclidean  $C^2$ -smooth regular curve. Then

$$k_{\gamma,\Sigma}^\infty = \frac{|\bar{p} \left( f_1 \dot{\gamma}_1 - \frac{f_3 \dot{\gamma}_2}{f} + f_3 \dot{\gamma}_3 \right) + \bar{q} f_1 \dot{\gamma}_2|}{|f| |\omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0, \quad (3.8)$$

$$k_{\gamma,\Sigma}^\infty = 0, \quad \text{if } \omega(\dot{\gamma}(t)) = 0, \quad \text{and} \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,$$

$$\lim_{L \rightarrow +\infty} \frac{k_{\gamma,\Sigma}^L}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot{\gamma}(t)))|}{\left( \bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3 \right)^2}, \quad \text{if } \omega(\dot{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \quad (3.9)$$

*Proof.* we know  $\dot{\gamma}(t) = \dot{\gamma}_1(t)\partial x_1 + \dot{\gamma}_2(t)\partial x_2 + \dot{\gamma}_3(t)\partial x_3$ , then by (2.2),  $\dot{\gamma}(t) = \frac{\dot{\gamma}_1(t)}{\gamma_1(t)}X_1 + \gamma_3(t)X_2 + \omega(\dot{\gamma}(t))X_3$ .  
Let

$$\dot{\gamma}(t) = \lambda_1 e_1 + \lambda_2 e_2.$$

Then

$$\begin{cases} \frac{\dot{\gamma}_1(t)}{\gamma_1(t)} = \lambda_1 \bar{q} + \lambda_2 \bar{r}_L \bar{p} \\ \dot{\gamma}_3(t) = -\lambda_1 \bar{p} + \lambda_2 \bar{r}_L \bar{q} \\ \omega(\dot{\gamma}(t)) = -\lambda_2 \frac{l}{L} L^{-\frac{1}{2}} \end{cases} \quad (3.10)$$

We have

$$\begin{cases} \lambda_1 = \bar{q} \frac{\dot{\gamma}_1(t)}{\gamma_1(t)} - \bar{p} \dot{\gamma}_3(t) \\ \lambda_2 = -\lambda_2 \frac{l}{L} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) \end{cases} \quad (3.11)$$

Thus  $\dot{\gamma} \in T\Sigma$ , we have

$$\dot{\gamma} = (\bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3) e_1 - \frac{l}{L} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) e_2. \quad (3.12)$$

By (3.6), we have

$$\begin{aligned} \|\nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}\|_{L, \Sigma}^2 &= \left\{ \bar{q} \left[ \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \left( f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right) \omega(\dot{\gamma}(t)) \right] \right. \\ &\quad \left. - \bar{p} \left[ \left( \dot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \left( \frac{f_1 L \dot{\gamma}_1 - f_3 L \omega(\dot{\gamma}(t))}{f} \right) \omega(\dot{\gamma}(t)) \right] \right\}^2 \\ &\quad + \left\{ \bar{r}_L \bar{p} \left[ \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \left( f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right) \omega(\dot{\gamma}(t)) \right] \right. \\ &\quad \left. + \bar{r}_L \bar{q} \left[ \left( \dot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \left( \frac{f_1 L \dot{\gamma}_1}{f} - \frac{f_3 L \omega(\dot{\gamma}(t))}{f} \right) \omega(\dot{\gamma}(t)) \right] \right. \\ &\quad \left. - \frac{l}{L} L^{\frac{1}{2}} \left[ \frac{f_2 \dot{\gamma}_1^2}{f^2 L} - \left( \frac{f_1 \dot{\gamma}_1}{f} + \frac{f_3 \dot{\gamma}_3}{f} \right) \omega(\dot{\gamma}(t)) + \frac{d}{dt} (\omega(\dot{\gamma}(t))) \right] \right\}^2 \\ &\sim L^2 \frac{\left[ \bar{p} \left( f_1 \dot{\gamma}_1 - \frac{f_3 \dot{\gamma}_2}{f} + f_3 \dot{\gamma}_3 \right) + \bar{q} f_1 \dot{\gamma}_2 \right]^2 \omega(\dot{\gamma}(t))^2}{f^2}, \quad \text{as } L \rightarrow +\infty. \end{aligned} \quad (3.13)$$

Similarly, we have that when  $\omega(\dot{\gamma}(t)) \neq 0$ ,

$$\|\dot{\gamma}\|_{\Sigma, L} = \sqrt{(\bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3)^2 + (\frac{l}{L})^2 L \omega(\dot{\gamma}(t))^2} \sim L^{\frac{1}{2}} |\omega(\dot{\gamma}(t))|, \quad \text{as } L \rightarrow +\infty. \quad (3.14)$$

By (3.6) and (3.12), we have

$$\begin{aligned} \langle \nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma, L} & \quad (3.15) \\ &= \left( \bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3 \right) \cdot \left\{ \bar{q} \left[ \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \left( f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right) \omega(\dot{\gamma}(t)) \right] \right. \end{aligned}$$



$$\begin{aligned}
& -\bar{p} \left[ \left( \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \left( \frac{f_1 L \dot{\gamma}_1 - f_3 L \omega(\dot{\gamma}(t))}{f} \right) \omega(\dot{\gamma}(t)) \right] \Big\} \\
& - \frac{l}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) \cdot \left\{ \bar{r}_L \bar{p} \left[ \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \left( f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right) \omega(\dot{\gamma}(t)) \right] \right. \\
& + \bar{r}_L \bar{q} \left[ \left( \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \left( \frac{f_1 L \dot{\gamma}_1}{f} - \frac{f_3 L \omega(\dot{\gamma}(t))}{f} \right) \omega(\dot{\gamma}(t)) \right] \\
& \left. - \frac{l}{l} L^{\frac{1}{2}} \left[ \frac{f_2 \dot{\gamma}_1^2}{f^2 L} - \left( \frac{f_1 \dot{\gamma}_1}{f} + \frac{f_3 \dot{\gamma}_3}{f} \right) \omega(\dot{\gamma}(t)) + \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right] \right\} \\
& \sim M_0 L,
\end{aligned}$$

where  $M_0$  does not depend on  $L$ . By (3.7), (3.13)–(3.15), we get (3.8). When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$ , we have

$$\begin{aligned}
\|\nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}\|_{L, \Sigma}^2 &= \left[ \bar{q} \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) - \bar{p} \left( \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) \right]^2 \\
&+ \left[ \bar{r}_L \bar{p} \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \bar{r}_L \bar{q} \left( \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \frac{l}{l} L^{\frac{1}{2}} \frac{f_2 \dot{\gamma}_1^2}{f^2 L} \right]^2 \\
&\sim \left[ \bar{q} \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) - \bar{p} \left( \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) \right]^2
\end{aligned} \tag{3.16}$$

and

$$\|\dot{\gamma}\|_{\Sigma, L} = \left| \bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3 \right|, \tag{3.17}$$

$$\langle \nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma, L} = \left( \bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3 \right) \cdot \left[ \bar{q} \left( \frac{\ddot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) - \bar{p} \left( \ddot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) \right] \tag{3.18}$$

By (3.16)–(3.18) and (3.7), we get  $k_{\gamma, \Sigma}^{\infty} = 0$ . When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$ , we have

$$\begin{aligned}
\|\nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}\|_{L, \Sigma}^2 &\sim L \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right]^2, \\
\langle \nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma, L} &= O(1),
\end{aligned}$$

so we get (3.9).  $\square$

**Definition 3.4.** Let  $\Sigma \subset (\mathbb{G}, g_L)$  be a regular surface. Let  $\gamma : [a, b] \rightarrow \Sigma$  be a Euclidean  $C^2$ -smooth regular curve. The signed geodesic curvature  $k_{\gamma, \Sigma}^{L, s}$  of  $\gamma$  at  $\gamma(t)$  is defined as

$$k_{\gamma, \Sigma}^{L, s} := \frac{\langle \nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{\Sigma, L}}{\|\dot{\gamma}\|_{\Sigma, L}^3}, \tag{3.19}$$

where  $J_L$  is defined by (3.3).

**Definition 3.5.** Let  $\Sigma \subset (\mathbb{G}, g_L)$  be a regular surface. Let  $\gamma : [a, b] \rightarrow \Sigma$  be a Euclidean  $C^2$ -smooth regular curve. We define the intrinsic geodesic curvature  $k_{\gamma, \Sigma}^\infty$  of  $\gamma$  at the non-characteristic point  $\gamma(t)$  to be

$$k_{\gamma, \Sigma}^{\infty, s} := \lim_{L \rightarrow +\infty} k_{\gamma, \Sigma}^{L, s},$$

if the limit exists.

**Lemma 3.6.** Let  $\Sigma \subset (\mathbb{G}, g_L)$  be a regular surface. Let  $\gamma : [a, b] \rightarrow \Sigma$  be a Euclidean  $C^2$ -smooth regular curve. Then

$$k_{\gamma, \Sigma}^{\infty, s} = \frac{\bar{p} \left( f_1 \dot{\gamma}_1 - \frac{f_3 \dot{\gamma}_2}{f} + f_3 \dot{\gamma}_3 \right) + \bar{q} f_1 \dot{\gamma}_2}{|f \omega(\dot{\gamma}(t))|}, \quad \text{if } \omega(\dot{\gamma}(t)) \neq 0, \quad (3.20)$$

$$k_{\gamma, \Sigma}^{\infty, s} = 0, \quad \text{if } \omega(\dot{\gamma}(t)) = 0, \quad \text{and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,$$

$$\lim_{L \rightarrow +\infty} \frac{k_{\gamma, \Sigma}^{L, s}}{\sqrt{L}} = \frac{(-\bar{q} \frac{\dot{\gamma}_1}{f} + \bar{p} \dot{\gamma}_3) \frac{d}{dt}(\omega(\dot{\gamma}(t)))}{|\bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3|^3}, \quad \text{if } \omega(\dot{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \quad (3.21)$$

*Proof.* By (3.3) and (3.12), we have

$$J_L(\dot{\gamma}) = \frac{l_L}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) e_1 + \left( \bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3 \right) e_2. \quad (3.22)$$

By (3.5) and (3.22), we have

$$\begin{aligned} & \langle \nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L\Sigma} \quad (3.23) \\ &= \frac{l_L}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) \left\{ \bar{q} \left[ \left( \frac{\dot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \left( f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right) \omega(\dot{\gamma}(t)) \right] \right. \\ & \quad \left. - \bar{p} \left[ \left( \dot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \left( \frac{f_1 L \dot{\gamma}_1 - f_3 L \omega(\dot{\gamma}(t))}{f} \right) \omega(\dot{\gamma}(t)) \right] \right\} \\ & \quad + \left( \bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3 \right) \cdot \left\{ \bar{r}_L \bar{p} \left[ \left( \frac{\dot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \left( f_1 L \dot{\gamma}_3 - \frac{f_2 \dot{\gamma}_1}{f} + f_1 L \omega(\dot{\gamma}(t)) \right) \omega(\dot{\gamma}(t)) \right] \right. \\ & \quad \left. + \bar{r}_L \bar{q} \left[ \left( \dot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \left( \frac{f_1 L \dot{\gamma}_1 - f_3 L \omega(\dot{\gamma}(t))}{f} \right) \omega(\dot{\gamma}(t)) \right] \right. \\ & \quad \left. - \frac{l}{l_L} L^{\frac{1}{2}} \left[ \frac{f_2 \dot{\gamma}_1^2}{f^2 L} - \left( \frac{f_1 \dot{\gamma}_1}{f} + \frac{f_3 \dot{\gamma}_3}{f} \right) \omega(\dot{\gamma}(t)) + \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right] \right\}, \\ & \sim L^{\frac{3}{2}} \omega(\dot{\gamma}(t))^2 \frac{\bar{p} \left( f_1 \dot{\gamma}_1 - \frac{f_3 \dot{\gamma}_2}{f} + f_3 \dot{\gamma}_3 \right) + \bar{q} f_1 \dot{\gamma}_2}{f} \quad \text{as } L \rightarrow +\infty. \end{aligned}$$

So we get (3.20). When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$ , we get

$$\begin{aligned} & \langle \nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L\Sigma} \quad (3.24) \\ &= \left( \bar{q} \frac{\dot{\gamma}_1}{f} - \bar{p} \dot{\gamma}_3 \right) \cdot \left[ \bar{r}_L \bar{p} \left( \frac{\dot{\gamma}_1 - f_2 \dot{\gamma}_1 \dot{\gamma}_3}{f} - \frac{f' \dot{\gamma}_1 + f_3 \dot{\gamma}_1 \dot{\gamma}_3}{f^2} \right) + \bar{r}_L \bar{q} \left( \dot{\gamma}_3 + \frac{f_2 \dot{\gamma}_1^2}{f^2} + \frac{f_3 \dot{\gamma}_1^2}{f^3} \right) - \frac{l}{l_L} L^{-\frac{1}{2}} \frac{f_2 \dot{\gamma}_1^2}{f^2} \right] \end{aligned}$$

$\sim M_0 L^{-\frac{1}{2}}$  as  $L \rightarrow +\infty$ .

So  $k_{\gamma, \Sigma}^{\infty, s} = 0$ . When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$ , we have

$$\langle \nabla_{\dot{\gamma}}^{\Sigma, L} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L, \Sigma} \sim L^{\frac{1}{2}} \left( -\bar{q} \frac{\dot{\gamma}_1}{f} + \bar{p} \dot{\gamma}_3 \right) \frac{d}{dt}(\omega(\dot{\gamma}(t))) \text{ as } L \rightarrow +\infty. \quad (3.25)$$

So we get (3.21).  $\square$

In the following, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in the generalized affine group. We define the *second fundamental form*  $II^L$  of the embedding of  $\Sigma$  into  $(\mathbb{G}, g_L)$ :

$$II^L = \begin{pmatrix} \langle \nabla_{e_1}^L v_L, e_1 \rangle_L, & \langle \nabla_{e_1}^L v_L, e_2 \rangle_L \\ \langle \nabla_{e_2}^L v_L, e_1 \rangle_L, & \langle \nabla_{e_2}^L v_L, e_2 \rangle_L \end{pmatrix}. \quad (3.26)$$

Similarly to Theorem 4.3 in [3], we have

**Theorem 3.7.** *The second fundamental form  $II^L$  of the embedding of  $\Sigma$  into  $(\mathbb{G}, g_L)$  is given by*

$$II^L = \begin{pmatrix} h_{11}, & h_{12} \\ h_{21}, & h_{22} \end{pmatrix}. \quad (3.27)$$

where

$$\begin{aligned} h_{11} &= \frac{l}{l_L} [X_1(\bar{p}) + X_2(\bar{q})] - \bar{q}_L \left( f_2 + \frac{f_3}{f} \right) - \bar{q}^2 f_2 \bar{r}_L L^{-\frac{1}{2}}, \\ h_{12} = h_{21} &= -\frac{l_L}{l} \langle e_1, \nabla_H(\bar{r}_L) \rangle_L - \frac{f_1 \sqrt{L}}{2} - \bar{p} \bar{q} f_2 L^{-\frac{1}{2}}, \\ h_{22} &= -\frac{l^2}{l_L^2} \langle e_2, \nabla_H(\frac{r}{l}) \rangle_L + \widetilde{X}_3(\bar{r}_L) - f_1 \bar{p}_L - \bar{q}_L \frac{f_3}{f} - \bar{r}_L \bar{P}^2 f_2 L^{-\frac{1}{2}}. \end{aligned}$$

*Proof.* By  $e_i \langle V_L, e_j \rangle_L - \langle \nabla_{e_i}^L V_L, e_j \rangle_L - \langle \nabla_{e_i}^L e_j, V_L \rangle_L = 0$  and  $e_i \langle V_L, e_j \rangle_L = 0$ , we have  $\langle \nabla_{e_i}^L V_L, e_j \rangle_L = -\langle \nabla_{e_i}^L e_j, V_L \rangle_L$ ,  $i, j = 1, 2$ .

By lemma 2.1 and (3.2),

$$\begin{aligned} \nabla_{e_1}^L e_1 &= \nabla_{(\bar{q}X_1 - \bar{p}X_2)}^L (\bar{q}X_1 - \bar{p}X_2) \\ &= [\bar{q}X_1(\bar{q}) - \bar{p}X_2(\bar{p}) + \bar{p}\bar{q}(f_2 + \frac{f_3}{f})]X_1 \\ &\quad - [\bar{q}X_1(\bar{p}) - \bar{p}X_2(\bar{p}) - \bar{q}^2(f_2 + \frac{f_3}{f})]X_2 + \bar{q}^2 f_3 L^{-\frac{1}{2}} \widetilde{X}_3. \end{aligned} \quad (3.28)$$

Then

$$\begin{aligned} h_{11} &= -\langle \nabla_{e_1}^L e_1, V_L \rangle_L \\ &= -\bar{p}_L [\bar{q}X_1(\bar{q}) - \bar{p}X_2(\bar{p})] + \bar{q}_L [\bar{q}X_1(\bar{p}) - \bar{p}X_2(\bar{p})] \end{aligned}$$

$$\begin{aligned}
& -\bar{q}_L \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \bar{p}_L \left( F_1 + \frac{F_3 f}{F} \right) + \bar{r}_L L^{-\frac{1}{2}} \frac{F_3}{F} \\
& = \frac{l}{l_L} [X_1(\bar{p}) + X_2(\bar{q})] - \bar{q}_L \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \bar{p}_L \left( F_1 + \frac{F_3 f}{F} \right) + \bar{r}_L L^{-\frac{1}{2}} \frac{F_3}{F}.
\end{aligned} \tag{3.29}$$

Similarly,

$$\begin{aligned}
\nabla_{e_1}^L e_2 & = \nabla_{(\bar{q}X_1 - \bar{p}X_2)}^L (\bar{r}_L \bar{p}X_1 + \bar{r}_L \bar{q}X_2 - \frac{l}{l_L} \widetilde{X}_3) \\
& = [\bar{q}X_1(\bar{r}_L \bar{p}) - \bar{p}X_2(\bar{r}_L \bar{p}) + \frac{f_1 \bar{p}_L \sqrt{L}}{2} \\
& \quad - \bar{r}_L \bar{q}^2 (f_2 + f_3 f) + \bar{q}_L f_2 L^{-\frac{1}{2}}] X_1 \\
& \quad + [\bar{q}X_1(\bar{q}) - \bar{p}X_2(\bar{q}) + \frac{f_1 \bar{q}_L \sqrt{L}}{2} \\
& \quad + \bar{r}_L \bar{p} \bar{q} (f_2 + \frac{f_3}{f})] X_2 \\
& \quad + [\bar{p}X_2(\frac{l}{l_L}) - \bar{q}X_1(\frac{l}{l_L}) + \frac{f_1 \bar{r}_L \sqrt{L}}{2} + \bar{r}_L \bar{p} \bar{q} f_3 L^{-\frac{1}{2}}] \widetilde{X}_3.
\end{aligned} \tag{3.30}$$

Then

$$\begin{aligned}
h_{12} & = -\langle \nabla_{e_1}^L e_2, V_L \rangle_L \\
& = -\frac{l}{l_L} [\bar{q}X_1(\bar{r}_L) - \bar{p}X_2(\bar{r}_L)] + \bar{r}_L [\bar{q}X_1(\frac{l}{l_L}) - \bar{p}X_2(\frac{l}{l_L})] - \frac{1}{2} L^{\frac{1}{2}} (fF_2 - F_1 \bar{f} + \bar{f}' F - F f') \\
& = -\frac{l_L}{l} \langle e_1, \nabla_H(\bar{r}_L) \rangle_L - \frac{1}{2} L^{\frac{1}{2}} (fF_2 - F_1 \bar{f} + \bar{f}' F - F f').
\end{aligned} \tag{3.31}$$

Since

$$\begin{aligned}
\langle \nabla_{e_2}^L V_L, e_1 \rangle_L & = -\langle \nabla_{e_2}^L e_1, V_L \rangle_L = -\langle \nabla_{e_1}^L e_2 + [e_2, e_1], V_L \rangle_L \\
& = -\langle \nabla_{e_1}^L e_2, V_L \rangle = \langle \nabla_{e_1}^L V_L, e_2 \rangle_L.
\end{aligned} \tag{3.32}$$

Then,

$$h_{21} = h_{12} = -\frac{l_L}{l} \langle e_1, \nabla_H(\bar{r}_L) \rangle_L - \frac{1}{2} L^{\frac{1}{2}} (fF_2 - F_1 \bar{f} + \bar{f}' F - F f'). \tag{3.33}$$

Since

$$\nabla_{e_2}^L e_2 = \nabla_{(\bar{r}_L \bar{p}X_1 + \bar{r}_L \bar{q}X_2 - \frac{l}{l_L} \widetilde{X}_3)}^L (\bar{r}_L \bar{p}X_1 + \bar{r}_L \bar{q}X_2 - \frac{l}{l_L} \widetilde{X}_3)$$

$$\begin{aligned}
&= [\bar{q}X_1(\bar{r}_L\bar{p}) - \bar{p}X_2(\bar{r}_L\bar{p}) + f_1\bar{q}_L\sqrt{L} - \bar{r}_L^2\bar{p}\bar{q}(f_2 + \frac{f_3}{f}) \\
&\quad + \bar{r}_L\bar{p}_L f_2 L^{-\frac{1}{2}} + (\frac{l}{l_L})^2 f_1] X_1 \\
&\quad + [\bar{q}X_1(\bar{q}) - \bar{p}X_2(\bar{q}) + f_1\bar{p}_L\sqrt{L} + \bar{r}_L^2\bar{p}\bar{q}(f_2 + f_3 f) + (\frac{l}{l_L})^2 \frac{f_3}{f} X_2 \\
&\quad + [\bar{p}X_2(\frac{l}{l_L}) - \bar{q}X_1(\frac{l}{l_L}) + \bar{r}_L^2\bar{p}\bar{q}\frac{f_3}{f} L^{-\frac{1}{2}} + \bar{r}_L\bar{p}_L f_1 + \bar{r}_L\bar{q}_L \frac{f_3}{f}] \widetilde{X}_3.
\end{aligned} \tag{3.34}$$

Then,

$$\begin{aligned}
h_{22} &= -\langle \nabla_{e_2}^L e_2, V_L \rangle_L \\
&= -\bar{p} \frac{r}{l} X_1(\bar{r}_L) - \bar{q} \frac{r}{l} X_2(\bar{r}_L) + \widetilde{X}_3(\bar{r}_L) - \bar{p}_L \frac{fF_3}{F} - \bar{q}_L \frac{\bar{f}F_3}{F} - \bar{r}_L \frac{F_3}{F} L^{-\frac{1}{2}} \\
&= -\frac{l^2}{l_L^2} \langle e_2, \nabla_H(\frac{r}{l}) \rangle_L + \widetilde{X}_3(\bar{r}_L) - \bar{p}_L \frac{fF_3}{F} - \bar{q}_L \frac{\bar{f}F_3}{F} - \bar{r}_L \frac{F_3}{F} L^{-\frac{1}{2}}.
\end{aligned} \tag{3.35}$$

□

The Riemannian mean curvature  $\mathcal{H}_L$  of  $\Sigma$  is defined by

$$\mathcal{H}_L := \text{tr}(II^L).$$

Let

$$\mathcal{K}^{\Sigma,L}(e_1, e_2) = -\langle R^{\Sigma,L}(e_1, e_2)e_1, e_2 \rangle_{\Sigma,L}, \quad \mathcal{K}^L(e_1, e_2) = -\langle R^L(e_1, e_2)e_1, e_2 \rangle_L. \tag{3.36}$$

By the Gauss equation, we have

$$\mathcal{K}^{\Sigma,L}(e_1, e_2) = \mathcal{K}^L(e_1, e_2) + \det(II^L). \tag{3.37}$$

**Proposition 3.8.** *Away from characteristic points, the horizontal mean curvature  $\mathcal{H}_\infty$  of  $\Sigma \subset \mathbb{G}$  is given by*

$$\mathcal{H}_\infty = \lim_{L \rightarrow +\infty} \mathcal{H}_L = X_1(\bar{p}) + X_2(\bar{q}) - f_1\bar{p} - \bar{q}f_2 - 2\bar{q}\frac{f_3}{f}. \tag{3.38}$$

*Proof.* By

$$\begin{aligned}
\frac{l^2}{l_L^2} \langle e_2, \nabla_H(\frac{r}{l}) \rangle_L &= \frac{\bar{p}r}{l} X_1(\bar{r}_L) + \frac{\bar{q}r}{l} X_2(\bar{r}_L) = O(L^{-1}) \\
\frac{l}{l_L} [X_1(\bar{p}) + X_2(\bar{q})] &\rightarrow X_1(\bar{p}) + X_2(\bar{q}), \quad \widetilde{X}_3(\bar{r}_L) \rightarrow 0, \\
\bar{q}^2 f_2 \bar{r}_L L^{-\frac{1}{2}} &\rightarrow O(L^{-1}), \quad \bar{q}_L \rightarrow \bar{q}, \\
\bar{r}_L \bar{p}^2 f_2 L^{-\frac{1}{2}} &\rightarrow O(L^{-1}), \quad \bar{p}_L \rightarrow \bar{p},
\end{aligned}$$

we get (3.38). □

Define the curvature of a connection  $\nabla$  by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (3.39)$$

Then by Lemma 2.1 and (3.39), we have the following lemma,

**Lemma 3.9.** *Let  $\mathbb{G}$  be the affine group, then*

$$\begin{aligned} R^L(X_1, X_2)X_1 &= \left[ \frac{3Lf_1^2}{4} + \left( f_2 + \frac{f_3}{f} \right)^2 \right] X_2 + \left[ X_1 \left( -\frac{f_1}{2} \right) - X_2 \left( \frac{f_2}{L} \right) + f_1^2 + \frac{f_2^2}{L} + \frac{f_2 f_3}{Lf} \right] X_3, \\ R^L(X_1, X_2)X_2 &= \left[ -\frac{3Lf_1^2}{4} + X_2 \left( f_2 + \frac{f_3}{f} \right) - \left( f_2 + \frac{f_3}{f} \right)^2 \right] X_1 + \frac{f_1 f_3}{f} X_3, \\ R^L(X_1, X_2)X_3 &= \left[ X_1 \left( \frac{f_1 L}{2} \right) + X_2 (f_2) - f_1^2 L - f_2 \left( f_2 + \frac{f_3}{f} \right) \right] X_1 + \left[ X_2 \left( \frac{f_1 L}{f} \right) - \frac{f_1 f_3 L}{f} \right] X_2, \\ R^L(X_1, X_3)X_1 &= - \left[ X_1 \left( \frac{f_1 L}{2} \right) + X_3 \left( f_2 + \frac{f_3}{f} \right) + \frac{f_2 f_3}{f} - f_1^2 L - f_2 \left( f_2 + \frac{f_3}{f} \right) \right] X_2 \\ &+ \left[ -\frac{f_1^2 L}{4} + f_1^2 + \frac{f_2^2}{L} - X_3 \left( \frac{f_2}{L} \right) - X_1 (f_1) + \frac{f_3}{f} \left( f_2 + \frac{f_3}{f} \right) \right] X_3, \\ R^L(X_1, X_3)X_2 &= \left[ X_1 \left( \frac{f_1 L}{2} \right) - f_1^2 L - f_2 \left( f_2 + \frac{f_3}{f} \right) + X_3 \left( f_2 + \frac{f_3}{f} \right) + \frac{f_2 f_3}{f} \right] X_1 \\ &+ \left[ -X_1 \left( \frac{f_3}{f} \right) + f_1 f_2 + \frac{f_1 f_3}{f} - f_1 \left( f_2 + \frac{f_3}{f} \right) - X_3 \left( \frac{f_1}{2} \right) \right] X_3, \\ R^L(X_1, X_3)X_3 &= \left[ X_1 (f_1 L) - \frac{f_3 L}{f} \left( f_2 + \frac{f_3}{f} \right) + X_3 (f_2) + \frac{f_1^2 L^2}{4} - f_1^2 L - f_2^2 \right] X_1 \\ &+ \left[ f_1 L \left( f_2 + \frac{f_3}{f} \right) + X_1 \left( \frac{f_3 L}{f} \right) - f_1 f_2 L - \frac{f_1 f_3 L}{f} + X_3 \left( \frac{f_1 L}{2} \right) \right] X_2, \\ R^L(X_2, X_3)X_1 &= \left[ -X_2 \left( \frac{f_1 L}{2} \right) + \frac{f_1 f_3 L}{f} \right] X_2 + \left[ X_3 \left( \frac{f_1}{2} \right) - X_2 (f_1) + \frac{f_1 f_3}{f} \right] X_3, \\ R^L(X_2, X_3)X_2 &= \left[ X_2 \left( \frac{f_1 L}{2} \right) - \frac{f_1 f_3 L}{f} \right] X_1 + \left[ \frac{f_3^2}{f^2} - \frac{f_1^2 L}{4} - X_2 \left( \frac{f_3}{f} \right) \right] X_3, \\ R^L(X_2, X_3)X_3 &= \left[ X_2 (f_1 L) - X_3 \left( \frac{f_1 L}{2} \right) - \frac{f_1 f_3 L}{f} \right] X_1 + \left[ X_2 \left( \frac{f_3 L}{f} \right) - \frac{f_3^2 L}{f^2} + \frac{f_1^2 L^2}{4} \right] X_2. \end{aligned} \quad (3.40)$$

**Proposition 3.10.** *Away from characteristic points, we have*

$$\mathcal{K}^{\Sigma, L}(e_1, e_2) \rightarrow A_0 + O(L^{-\frac{1}{2}}), \text{ as } L \rightarrow +\infty, \quad (3.41)$$

where

$$\begin{aligned} A_0 &:= -f_1 \langle e_1, \nabla_H \left( \frac{X_3 u}{|\nabla_H u|} \right) \rangle - f_1 f_2 \bar{p} \bar{q} - \bar{q}^2 f_1^2 - \frac{3}{4} \frac{(X_3 u)^2}{\bar{p}^2} f_1^2 + \bar{p}^2 X_2 \left( \frac{f_3}{f} \right) \\ &- \bar{p} \left[ X_1(\bar{p}) + X_2(\bar{q}) - \bar{q} \left( f_2 + \frac{f_3}{f} \right) \right] \left( f_1 \bar{p} + \bar{q} \frac{f_3}{f} \right) \end{aligned} \quad (3.42)$$

$$\begin{aligned}
& + 2\bar{q}\frac{X_3u}{l}f_1^2 + 2\bar{q}\frac{X_3u}{X_1}\left(-\frac{f_1}{2}\right) - 2\bar{p}\frac{X_3u}{l}\frac{f_1f_3}{f} + \bar{q}^2X_1(f_1) \\
& - \bar{q}^2\left(f_2 + \frac{f_3}{f}\right) - 2\bar{q}\bar{p}X_1\left(\frac{f_3}{f}\right) - 2\bar{q}\bar{p}X_3\left(\frac{f_1}{2}\right) - \bar{p}^2\frac{f_3^2}{f^2}.
\end{aligned}$$

*Proof.* By (3.2), we have

$$\begin{aligned}
& \langle R^L(e_1, e_2)e_1, e_2 \rangle_L \tag{3.43} \\
& = \bar{r}_L^2 \langle R^L(X_1, X_2)X_1, X_2 \rangle_L - 2\frac{l}{l_L}\bar{q}L^{-\frac{1}{2}}\bar{r}_L \langle R^L(X_1, X_2)X_1, X_3 \rangle_L \\
& + 2\frac{l}{l_L}\bar{p}L^{-\frac{1}{2}}\bar{r}_L \langle R^L(X_1, X_2)X_2, X_3 \rangle_L + \left(\frac{l}{l_L}\bar{q}\right)^2 L^{-1} \langle R^L(X_1, X_3)X_1, X_3 \rangle_L \\
& - 2\left(\frac{l}{l_L}\right)^2 \bar{p}\bar{q}L^{-1} \langle R^L(X_1, X_3)X_2, X_3 \rangle_L + \left(\bar{p}\frac{l}{l_L}\right)^2 L^{-1} \langle R^L(X_2, X_3)X_2, X_3 \rangle_L.
\end{aligned}$$

By Lemma 3.9, we have

$$\begin{aligned}
\mathcal{K}^L(e_1, e_2) & = \frac{1}{4}\frac{l^2}{l_L^2}f_1^2L - \frac{3}{4}Lf_1^2\bar{r}_L^2 + 2\frac{l}{l_L}\bar{q}L^{\frac{1}{2}}\bar{r}_Lf_1^2 - \bar{q}^2\frac{l^2}{l_L^2}f_1^2 \tag{3.44} \\
& - \bar{r}_L^2\left(f_2 + \frac{f_3}{f}\right)^2 + 2\frac{l}{l_L}\bar{q}L^{\frac{1}{2}}\bar{r}_LX_1\left(-\frac{f_1}{2}\right) - 2\frac{l}{l_L}\bar{q}L^{\frac{1}{2}}\bar{r}_LX_2\left(-\frac{f_2}{L}\right) \\
& + 2\frac{l}{l_L}\bar{q}L^{-\frac{1}{2}}\bar{r}_Lf_2^2 + 2\frac{l}{l_L}\bar{q}L^{-\frac{1}{2}}\bar{r}_L\frac{f_2f_3}{f} - 2\bar{p}L^{\frac{1}{2}}\bar{r}_L\frac{f_1f_3}{f} \\
& - \bar{q}^2\frac{l^2}{l_L^2}\frac{f_2^2}{L} + \bar{q}^2\frac{l^2}{l_L^2}X_3\left(\frac{f_2}{L}\right) + \bar{q}^2\frac{l^2}{l_L^2}X_1(f_1) - \bar{q}^2\frac{l^2}{l_L^2}\left(f_2 + \frac{f_3}{f}\right) \\
& - 2\bar{q}\bar{p}\frac{l^2}{l_L^2}X_1\left(\frac{f_3}{f}\right) + 2\bar{q}\bar{p}\frac{l^2}{l_L^2}f_1f_2 + 2\bar{q}\bar{p}\frac{l^2}{l_L^2}\frac{f_1f_3}{f} - 2\bar{q}\bar{p}\frac{l^2}{l_L^2}f_1\left(f_2 + \frac{f_3}{f}\right) \\
& - 2\bar{q}\bar{p}\frac{l^2}{l_L^2}X_3\left(\frac{f_1}{2}\right) - \bar{p}^2\frac{l^2}{l_L^2}\frac{f_3^2}{f^2} + \bar{p}^2\frac{l^2}{l_L^2}X_2\left(\frac{f_3}{f}\right).
\end{aligned}$$

By (3.35) and

$$\nabla_H(\bar{r}_L) = L^{-\frac{1}{2}}\nabla_H\left(\frac{X_3u}{|\nabla_Hu|}\right) + O(L^{-1}) \text{ as } L \rightarrow +\infty$$

we get

$$\begin{aligned}
\det(II^L) & = h_{11}h_{22} - h_{12}h_{21} \\
& = -\frac{f_1^2L}{4} - f_1\langle e_1, \nabla_H\left(\frac{X_3u}{|\nabla_Hu|}\right) \rangle - f_1f_2\bar{p}\bar{q} \tag{3.45} \\
& - \bar{p}\left[X_1(\bar{p}) + X_2(\bar{q}) - \bar{q}\left(f_2 + \frac{f_3}{f}\right)\right]\left(f_1\bar{p} + \bar{q}\frac{f_3}{f}\right) + O(L^{-\frac{1}{2}}).
\end{aligned}$$

By (3.38), (3.44), (3.45) we get (3.41).  $\square$

#### 4. A Gauss-Bonnet theorem in the generalized affine group

Let us first consider the case of a regular curve  $\gamma : [a, b] \rightarrow (\mathbb{G}, g_L)$ . We define the Riemannian length measure

$$ds_L = \|\dot{\gamma}\|_L dt.$$

**Lemma 4.1.** *Let  $\gamma : [a, b] \rightarrow (\mathbb{G}, g_L)$  be a Euclidean  $C^2$ -smooth and regular curve. Let*

$$ds := |\omega(\dot{\gamma}(t))| dt, \quad d\bar{s} := \frac{1}{2} \frac{1}{|\omega(\dot{\gamma}(t))|} \left( \frac{\dot{\gamma}_1^2}{f^2} + \dot{\gamma}_3^2 \right) dt. \quad (4.1)$$

Then

$$\lim_{L \rightarrow +\infty} \frac{1}{\sqrt{L}} \int_{\gamma} ds_L = \int_a^b ds. \quad (4.2)$$

When  $\omega(\dot{\gamma}(t)) \neq 0$ , we have

$$\frac{1}{\sqrt{L}} ds_L = ds + d\bar{s}L^{-1} + O(L^{-2}) \text{ as } L \rightarrow +\infty. \quad (4.3)$$

When  $\omega(\dot{\gamma}(t)) = 0$ , we have

$$\frac{1}{\sqrt{L}} ds_L = \frac{1}{\sqrt{L}} \sqrt{\frac{\dot{\gamma}_1^2}{f^2} + \dot{\gamma}_3^2} dt. \quad (4.4)$$

*Proof.* We know that

$$\|\dot{\gamma}(t)\|_L = \sqrt{\left(\frac{\dot{\gamma}_1}{f}\right)^2 + \dot{\gamma}_3^2 + L\omega(\dot{\gamma}(t))^2},$$

similar to the proof of Lemma 6.1 in [1], we can prove (4.2). When  $\omega(\dot{\gamma}(t)) \neq 0$ , we have

$$\frac{1}{\sqrt{L}} ds_L = \sqrt{L^{-1} \left( \left(\frac{\dot{\gamma}_1}{f}\right)^2 + \dot{\gamma}_3^2 \right) + \omega(\dot{\gamma}(t))^2} dt.$$

Using the Taylor expansion, we can prove (4.3). From the definition of  $ds_L$  and  $\omega(\dot{\gamma}(t)) = 0$ , we get (4.4).  $\square$

Let  $\Sigma \subset (\mathbb{G}, g_L)$  be a Euclidean  $C^2$ -smooth surface and  $\Sigma = \{u = 0\}$ . Let  $d\sigma_{\Sigma, L}$  denote the surface measure on  $\Sigma$  with respect to the Riemannian metric  $g_L$ . Then similar to Proposition 4.2 in [7], we have

$$\lim_{L \rightarrow +\infty} \frac{1}{\sqrt{L}} \int_{\Sigma} d\sigma_{\Sigma, L} = d\sigma_{\Sigma} := (\bar{p}\omega_2 - \bar{q}\omega_1) \wedge \omega. \quad (4.5)$$

Similar to the proof of Theorem 1.1 in [1], we have

**Theorem 4.2.** *Let  $\Sigma \subset (\mathbb{G}, g_L)$  be a regular surface with finitely many boundary components  $(\partial\Sigma)_i$ ,  $i \in \{1, \dots, n\}$ , given by Euclidean  $C^2$ -smooth regular and closed curves  $\gamma_i : [0, 2\pi] \rightarrow (\partial\Sigma)_i$ . Let  $A_0$  be defined by (3.42) and  $d\sigma_{\Sigma}$ ,  $d\bar{\sigma}_{\Sigma}$  be defined by (4.5) and  $d\bar{s}$  be defined by (4.1) and  $k_{\gamma_i, \Sigma}^{\infty, s}$  be the sub-Riemannian signed geodesic curvature of  $\gamma_i$  relative to  $\Sigma$ . Suppose that the characteristic set  $C(\Sigma)$  satisfies  $\mathcal{H}^1(C(\Sigma)) = 0$  where  $\mathcal{H}^1(C(\Sigma))$  denotes the Euclidean 1-dimensional Hausdorff measure of*



$C(\Sigma)$  and that  $\|\nabla_H u\|_H^{-1}$  is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set  $C(\Sigma)$ , then

$$\int_{\Sigma} \mathcal{K}^{\Sigma, \infty} d\sigma_{\Sigma} + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma}^{\infty, s} ds = 0. \quad (4.6)$$

**Example 4.3.** Let  $f = x_1^2 + 1$ , then  $\mathbb{G} = R^3$ . Let  $u = x_1^2 + x_2^2 + x_3^2 - 1$  and  $\Sigma = S^2$ .  $\Sigma$  is a regular surface. By (2.1), we get

$$X_1(u) = 2(x_1^2 + 1)x_1; \quad X_2(u) = 2(x_1^2 + 1)x_2 + 2x_3. \quad (4.7)$$

Solve the equations  $X_1(u) = X_2(u) = 0$ , then we get

$$C(\Sigma) = \left\{ \left(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \right\}$$

and  $\mathcal{H}^1(C(\Sigma)) = 0$ .

A parametrization of  $\Sigma$  is

$$\begin{aligned} x_1 &= \cos(\phi)\cos(\theta), & x_2 &= \cos(\phi)\sin(\theta), \\ x_3 &= \sin(\phi), & \text{for } \phi &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \theta \in [0, 2\pi). \end{aligned} \quad (4.8)$$

Then

$$\begin{aligned} \|\nabla_H u\|_H^2 &= X_1(u)^2 + X_2(u)^2 \\ &= 4(x_1^2 + 1)^2 x_1^2 + 4(x_1^2 + 1)^2 x_2^2 + 4x_3^2 + 8(x_1^2 + 1)x_2 x_3 \\ &= 4(\cos(\phi)^2 \cos(\theta)^2 + 1)^2 \cos(\phi)^2 + 4\sin(\phi)^2 \\ &\quad + 8(\cos(\phi)^2 \cos(\theta)^2 + 1)\sin(\phi)\cos(\phi)\sin(\theta). \end{aligned} \quad (4.9)$$

By the definitions of  $w_j$  for  $1 \leq j \leq 3$  and (4.5), we have

$$\begin{aligned} d\sigma_{\Sigma} &= \frac{1}{\|\nabla_H u\|_H} [(X_1(u)dx_3 - (x_1^2 + 1)^{-1}(X_2(u))dx_1] \wedge ((x_1^2 + 1)^{-1}dx_2 - dx_3) \\ &= -\frac{1}{\|\nabla_H u\|_H} 2\cos(\phi)\lambda_0 d\theta \wedge d\phi. \end{aligned} \quad (4.10)$$

where

$$\lambda_0 = \cos(\phi)^2 + 2(\cos(\phi)^2 \cos(\theta)^2 + 1)^{-1} \cos(\phi)\sin(\phi)\sin(\theta) + (\cos(\phi)^2 \cos(\theta)^2 + 1)^{-2} \sin(\phi)^2$$

is a bounded smooth function on  $\Sigma$ . By (4.9) and (4.10), we have  $\|\nabla_H u\|_H^{-1}$  is locally summable around the isolated characteristic points with respect to the measure  $d\sigma_{\Sigma}$ .

## 5. The sub-Riemannian limit of curvature of curves in the generalized BCV spaces

We consider some notation on the generalized BCV spaces. Let  $f(x_2), \bar{f}(x_1), F(x_1, x_2, x_3)$  be smooth functions. The generalized BCV spaces  $M$  is the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid F(x_1, x_2, x_3) > 0\}$$

Let

$$X_1 = F\partial_{x_1} + f\partial_{x_3}, \quad X_2 = F\partial_{x_2} + \bar{f}\partial_{x_3}, \quad X_3 = \partial_{x_3}. \quad (5.1)$$

Then

$$\partial_{x_1} = \frac{1}{F}(X_1 - fX_3), \quad \partial_{x_2} = \frac{1}{F}(X_2 - \bar{f}X_3), \quad \partial_{x_3} = X_3, \quad (5.2)$$

and  $\text{span}\{X_1, X_2, X_3\} = TM$ . Let  $H = \text{span}\{X_1, X_2\}$  be the horizontal distribution on  $M$ . Let  $\omega_1 = \frac{1}{F}dx_1$ ,  $\omega_2 = \frac{1}{F}dx_2$ ,  $\omega = dx_3 - \frac{(fdx_1 + \bar{f}dx_2)}{F}$ . Then  $H = \text{Ker}\omega$ . The generalized BCV spaces have some well-known special case. When  $F = 1 + \frac{1}{4}(x_1^2 + x_2^2)$ ,  $f = -\tau x_2$ ,  $\bar{f} = \tau x_1$ , we get the BCV spaces. When  $F = 1$ ,  $f = f(x_2)$ ,  $\bar{f} = \bar{f}(x_1)$ , we can the Heisenberg manifolds. When  $F = 1$ ,  $f = \frac{1}{2}x_2^2$ ,  $\bar{f} = 0$ , we get the Martinet distribution. When  $F = \frac{1}{x_1}$ ,  $f = 0$ ,  $\bar{f} = -2$ , we get the Welyczko's example (see [5]). For the constant  $L > 0$ , let  $g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L\omega \otimes \omega$ ,  $g = g_1$  be the Riemannian metric on  $M$ . Then  $X_1, X_2, \bar{X}_3 := L^{-\frac{1}{2}}X_3$  are orthonormal basis on  $TM$  with respect to  $g_L$ . We have

$$\begin{aligned} [X_1, X_2] &= -\left(F_2 + \frac{\bar{f}F_3}{F}\right)X_1 + \left(F_1 + \frac{fF_3}{F}\right)X_2 + (F_2f - F_1\bar{f} + F\bar{f}' - Ff')X_3, \\ [X_2, X_3] &= -\frac{F_3}{F}X_2 + \frac{\bar{f}F_3}{F}X_3, \quad [X_1, X_3] = -\frac{F_3}{F}X_1 + \frac{fF_3}{F}X_3. \end{aligned} \quad (5.3)$$

where  $F_i = \frac{\partial F}{\partial x_i}$ , for  $1 \leq i \leq 3$ ,  $f' = \frac{\partial f}{\partial x_2}$ ,  $\bar{f}' = \frac{\partial \bar{f}}{\partial x_1}$ . Let  $\nabla^L$  be the Levi-Civita connection on  $M$  with respect to  $g_L$ . Then we have the following lemma

**Lemma 5.1.** *Let  $M$  be the generalized BCV spaces, then*

$$\begin{aligned} \nabla_{X_1}^L X_1 &= \left(F_2 + \frac{F_3\bar{f}}{F}\right)X_2 + \frac{F_3}{LF}X_3, \\ \nabla_{X_1}^L X_2 &= -\left(F_2 + \frac{F_3\bar{f}}{F}\right)X_1 + \frac{1}{2}(fF_2 - F_1\bar{f} + F\bar{f}' - Ff')X_3, \\ \nabla_{X_1}^L X_3 &= -\frac{F_3}{F}X_1 - \frac{L}{2}(fF_2 - F_1\bar{f} + F\bar{f}' - Ff')X_2, \\ \nabla_{X_2}^L X_1 &= -\left(F_1 + \frac{F_3f}{F}\right)X_2 - \frac{1}{2}(fF_2 - F_1\bar{f} + F\bar{f}' - Ff')X_3, \\ \nabla_{X_2}^L X_2 &= \left(F_1 + \frac{F_3f}{F}\right)X_1 + \frac{F_3}{FL}X_3, \\ \nabla_{X_2}^L X_3 &= \frac{L}{2}(fF_2 - F_1\bar{f} + F\bar{f}' - Ff')X_1 - \frac{F_3}{F}X_2, \\ \nabla_{X_3}^L X_1 &= -\frac{L}{2}(fF_2 - F_1\bar{f} + F\bar{f}' - Ff')X_2 - \frac{fF_3}{F}X_3, \end{aligned} \quad (5.4)$$

$$\begin{aligned}\nabla_{X_3}^L X_2 &= \frac{L}{2}(fF_2 - F_1\bar{f} + F\bar{f}' - Ff')X_1 - \frac{\bar{f}F_3}{F}X_3, \\ \nabla_{X_3}^L X_3 &= \frac{LfF_3}{F}X_1 + \frac{L\bar{f}F_3}{F}X_2.\end{aligned}$$

*Proof.* By the Koszul formula, we have

$$2\langle \nabla_{X_i}^L X_j, X_k \rangle_L = \langle [X_i, X_j], X_k \rangle_L - \langle [X_j, X_k], X_i \rangle_L + \langle [X_k, X_i], X_j \rangle_L, \quad (5.5)$$

where  $i, j, k = 1, 2, 3$ . So lemma 5.1 holds.  $\square$

**Definition 5.2.** Let  $\gamma : [a, b] \rightarrow (M, g_L)$  be a Euclidean  $C^1$ -smooth curve. We say that  $\gamma(t)$  is a horizontal point of  $\gamma$  if

$$\omega(\dot{\gamma}(t)) = -\frac{f}{F}\dot{\gamma}_1(t) - \frac{\bar{f}}{F}\dot{\gamma}_2(t) + \dot{\gamma}_3(t) = 0.$$

where  $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$  and  $\dot{\gamma}_i(t) = \frac{\partial \gamma_i(t)}{\partial t}$ .

Similar to the definition 2.3 and definition 2.5, we can define  $k_\gamma^L$  and  $k_\gamma^\infty$  for the generalized BCV spaces, we have

**Lemma 5.3.** Let  $\gamma : [a, b] \rightarrow (M, g_L)$  be a Euclidean  $C^2$ -smooth regular curve in the Riemannian manifold  $(M, g_L)$ . Then

$$\begin{aligned}k_\gamma^\infty &= \left\{ \left[ -\frac{\dot{\gamma}_1}{F}(F_2f - F_1\bar{f} + F\bar{f}' - Ff') + \frac{F_3\bar{f}}{F}\omega(\dot{\gamma}(t)) \right]^2 \right. \\ &\quad \left. + \left[ \frac{\dot{\gamma}_2}{F}(F_2f - F_1\bar{f} + F\bar{f}' - Ff') + \frac{F_3f}{F}\omega(\dot{\gamma}(t)) \right]^2 \right\}^{\frac{1}{2}} |\omega(\dot{\gamma}(t))|^{-1}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0.\end{aligned} \quad (5.6)$$

$$\begin{aligned}k_\gamma^\infty &= \left\{ \left[ \left[ \frac{F'\dot{\gamma}_1 - F\ddot{\gamma}_1}{F^2} - \frac{\dot{\gamma}_1\dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3\bar{f}}{F} \right) + \frac{\dot{\gamma}_2^2}{F^2} \left( F_1 + \frac{F_3f}{F} \right) \right]^2 \right. \right. \\ &\quad \left. \left. + \left[ \frac{F'\dot{\gamma}_2 - F\ddot{\gamma}_2}{F^2} + \frac{\dot{\gamma}_1^2}{F^2} \left( F_2 + \frac{F_3\bar{f}}{F} \right) - \frac{\dot{\gamma}_1\dot{\gamma}_2}{F^2} \left( F_1 + \frac{F_3f}{F} \right) \right]^2 \right\} \left[ \frac{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)}{F^2} \right]^{-2} \\ &\quad + \left\{ \left[ \frac{F'\dot{\gamma}_1^2 - F\dot{\gamma}_1\ddot{\gamma}_1}{F^3} + \frac{F'\dot{\gamma}_2^2 - F\dot{\gamma}_2\ddot{\gamma}_2}{F^3} \right]^2 \right\} \left[ \frac{(\dot{\gamma}_1^2 + \dot{\gamma}_2^2)}{F^2} \right]^{-3} \right\}^{-\frac{1}{2}} \\ &\text{if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,\end{aligned} \quad (5.7)$$

where  $F' = \dot{\gamma}(F) = \frac{d}{dt}F(\gamma(t))$ .

$$\lim_{L \rightarrow +\infty} \frac{k_\gamma^L}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot{\gamma}(t)))|}{\frac{\dot{\gamma}_1^2}{F^2} + \frac{\dot{\gamma}_2^2}{F^2}}, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \quad (5.8)$$

*Proof.* By (5.2), we have

$$\dot{\gamma}(t) = \frac{\dot{\gamma}_1}{F}X_1 + \frac{\dot{\gamma}_2}{F}X_2 + \omega(\dot{\gamma}(t))X_3. \quad (5.9)$$

By Lemma 5.1 and (5.8), we have

$$\begin{aligned} \nabla_{\dot{\gamma}}^L X_1 &= \left[ \frac{\dot{\gamma}_1}{F} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \frac{\dot{\gamma}_2}{F} \left( F_1 + \frac{F_3 f}{F} \right) - \frac{L}{2} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') \omega(\dot{\gamma}(t)) \right] X_2 \\ &\quad + \left[ \frac{F_3 \dot{\gamma}_1}{LF^2} - \frac{\dot{\gamma}_2}{2F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') - \frac{F_3 f}{F} \omega(\dot{\gamma}(t)) \right] X_3, \\ \nabla_{\dot{\gamma}}^L X_2 &= \left[ -\frac{\dot{\gamma}_1}{F} + \left( F_2 + \frac{F_3 \bar{f}}{F} \right) + \frac{\dot{\gamma}_2}{F} \left( F_1 + \frac{F_3 f}{F} \right) + \frac{L}{2} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') \omega(\dot{\gamma}(t)) \right] X_1 \\ &\quad + \left[ \frac{F_3 \dot{\gamma}_2}{LF^2} + \frac{\dot{\gamma}_1}{2F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') - \frac{F_3 \bar{f}}{F} \omega(\dot{\gamma}(t)) \right] X_3, \\ \nabla_{\dot{\gamma}}^L X_3 &= \left[ -\frac{\dot{\gamma}_1 F_3}{F^2} + \frac{\dot{\gamma}_2 L}{2F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{LF_3 f}{F} \omega(\dot{\gamma}(t)) \right] X_1 \\ &\quad + \left[ -\frac{\dot{\gamma}_1 L}{2F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') - \frac{\dot{\gamma}_2 F_3}{F^2} + \frac{LF_3 \bar{f}}{F} \omega(\dot{\gamma}(t)) \right] X_2. \end{aligned} \quad (5.10)$$

By (5.8) and (5.9), we have

$$\begin{aligned} \nabla_{\dot{\gamma}}^L \dot{\gamma} &= \left\{ \frac{F' \dot{\gamma}_1 - F \ddot{\gamma}_1}{F^2} - \frac{\dot{\gamma}_1 \dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) + \frac{\dot{\gamma}_2^2}{F^2} \left( F_1 + \frac{F_3 f}{F} \right) \right. \\ &\quad \left. + \left[ -\frac{F_3 \dot{\gamma}_1}{F^2} + \frac{\dot{\gamma}_2 L}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{f L F_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \right\} X_1 \\ &\quad + \left\{ \frac{F' \dot{\gamma}_2 - F \ddot{\gamma}_2}{F^2} + \frac{\dot{\gamma}_1^2}{F^2} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \frac{\dot{\gamma}_1 \dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3 f}{F} \right) \right. \\ &\quad \left. + \left[ -\frac{F_3 \dot{\gamma}_2}{F^2} - \frac{\dot{\gamma}_1 L}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{\bar{f} L F_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \right\} X_2 \\ &\quad + \left\{ \frac{(\dot{\gamma}_2^2 + \dot{\gamma}_1^2) F_3}{F^3 L} - \frac{(F_3 f \dot{\gamma}_1 + F_3 \bar{f} \dot{\gamma}_2)}{F^2} \omega(\dot{\gamma}(t)) + \frac{d}{dt} \omega(\dot{\gamma}(t)) \right\} X_3. \end{aligned} \quad (5.11)$$

By (5.8) and (5.10), when  $\omega(\dot{\gamma}(t)) \neq 0$ , we have

$$\begin{aligned} \|\nabla_{\dot{\gamma}}^L \dot{\gamma}\|_L^2 &\sim \left\{ \left[ \frac{\dot{\gamma}_2}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{F_3 f}{F} \omega(\dot{\gamma}(t)) \right]^2 \right. \\ &\quad \left. + \left[ -\frac{\dot{\gamma}_1}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{F_3 \bar{f}}{F} \omega(\dot{\gamma}(t)) \right]^2 \right\} \omega(\dot{\gamma}(t))^2 L^2, \text{ as } L \rightarrow +\infty, \\ \|\dot{\gamma}\|_L^2 &\sim L \omega(\dot{\gamma}(t))^2, \text{ as } L \rightarrow +\infty, \\ \langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma} \rangle_L^2 &\sim O(L^2) \text{ as } L \rightarrow +\infty. \end{aligned} \quad (5.12)$$

Therefore

$$\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\|_L^2 \sim \frac{\left[ \frac{\dot{\gamma}_2}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{F_3 f}{F} \omega(\dot{\gamma}(t)) \right]^2}{\omega(\dot{\gamma}(t))^2} \quad (5.13)$$

$$+ \frac{\left[ -\frac{\dot{\gamma}_1}{F}(F_2f - F_1\bar{f} + F\bar{f}' - Ff') + \frac{F_3\bar{f}}{F}\omega(\dot{\gamma}(t)) \right]^2}{\omega(\dot{\gamma}(t))^2}, \text{ as } L \rightarrow +\infty,$$

$$\frac{\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma} \rangle_L^2}{\|\dot{\gamma}\|_L^6} \rightarrow 0, \text{ as } L \rightarrow +\infty.$$

So by (2.6), we have (5.5). (5.6) comes from (5.8), (5.10), (2.6) and  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$ . When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$ , we have

$$\|\nabla_{\dot{\gamma}}^L \dot{\gamma}\|_L^2 \sim L \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right]^2, \text{ as } L \rightarrow +\infty,$$

$$\|\dot{\gamma}\|_L^2 = \left[ \frac{\dot{\gamma}_1^2}{F^2} + \frac{\dot{\gamma}_2^2}{F^2} \right]^2,$$

$$\langle \nabla_{\dot{\gamma}}^L \dot{\gamma}, \dot{\gamma} \rangle_L^2 = O(1) \text{ as } L \rightarrow +\infty.$$

By (2.6), we get (5.7). □

## 6. The sub-Riemannian limit of geodesic curvature of curves on surfaces in the generalized BCV spaces

We will consider a regular surface  $\Sigma_1 \subset (M, g_L)$  and regular curve  $\gamma \subset \Sigma_1$ . We will assume that there exists a Euclidean  $C^2$ -smooth function  $u : M \rightarrow \mathbb{R}$  such that

$$\Sigma_1 = \{(x_1, x_2, x_3) \in M : u(x_1, x_2, x_3) = 0\}.$$

Similar to Section 3, we define  $p, q, r, l, l_L, \bar{p}, \bar{q}, \bar{p}_L, \bar{q}_L, \bar{r}_L, v_L, e_1, e_2, J_L, k_{\gamma, \Sigma_1}^L, k_{\gamma, \Sigma_1}^\infty, k_{\gamma, \Sigma_1}^{L,s}, k_{\gamma, \Sigma_1}^{\infty,s}$ . By (3.4) and (5.10), we have

$$\begin{aligned} \nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma} = & \left\{ \bar{q} \left[ \frac{F'\dot{\gamma}_1 - F\ddot{\gamma}_1}{F^2} - \frac{\dot{\gamma}_1\dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3\bar{f}}{F} \right) + \frac{\dot{\gamma}_2^2}{F^2} \left( F_1 + \frac{F_3f}{F} \right) \right] \right. \\ & + \bar{q} \left[ -\frac{F_3\dot{\gamma}_1}{F^2} + \frac{\dot{\gamma}_2L}{F} (F_2f - F_1\bar{f} + F\bar{f}' - Ff') + \frac{fLF_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \\ & - \bar{p} \left[ \frac{F'\dot{\gamma}_2 - F\ddot{\gamma}_2}{F^2} + \frac{\dot{\gamma}_1^2}{F^2} \left( F_2 + \frac{F_3\bar{f}}{F} \right) - \frac{\dot{\gamma}_1\dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3f}{F} \right) \right] \\ & + \bar{p} \left[ -\frac{F_3\dot{\gamma}_2}{F^2} - \frac{\dot{\gamma}_1L}{F} (F_2f - F_1\bar{f} + F\bar{f}' - Ff') + \frac{\bar{f}LF_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \left. \right\} e_1 \\ & + \left\{ \bar{r}_L \bar{p} \left[ \frac{F'\dot{\gamma}_1 - F\ddot{\gamma}_1}{F^2} - \frac{\dot{\gamma}_1\dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3\bar{f}}{F} \right) + \frac{\dot{\gamma}_2^2}{F^2} \left( F_1 + \frac{F_3f}{F} \right) \right] \right. \\ & + \bar{r}_L \bar{p} \left[ -\frac{F_3\dot{\gamma}_1}{F^2} + \frac{\dot{\gamma}_2L}{F} (F_2f - F_1\bar{f} + F\bar{f}' - Ff') + \frac{fLF_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \\ & + \bar{r}_L \bar{q} \left[ \frac{F'\dot{\gamma}_2 - F\ddot{\gamma}_2}{F^2} + \frac{\dot{\gamma}_1^2}{F^2} \left( F_2 + \frac{F_3\bar{f}}{F} \right) - \frac{\dot{\gamma}_1\dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3f}{F} \right) \right] \end{aligned} \quad (6.1)$$

$$\begin{aligned}
& + \bar{r}_L \bar{q} \left[ -\frac{F_3 \dot{\gamma}_2}{F^2} - \frac{\dot{\gamma}_1 L}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{\bar{f} L F_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \\
& - \frac{l}{l_L} L^{\frac{1}{2}} \left[ \frac{(\dot{\gamma}_2^2 + \dot{\gamma}_1^2) F_3}{F^3 L} - \frac{(f \dot{\gamma}_1 + \bar{f} \dot{\gamma}_2) F_3}{F^2} \omega(\dot{\gamma}(t)) + \frac{d}{dt} (\omega(\dot{\gamma}(t))) \right] \Big\} e_2 \\
& = B_1 e_1 + B_2 e_2.
\end{aligned}$$

By (5.8) and  $\dot{\gamma}(t) \in T\Sigma_1$ , we have

$$\dot{\gamma}(t) = \left[ \frac{\bar{q} \dot{\gamma}_1}{F} - \frac{\bar{p} \dot{\gamma}_2}{F} \right] e_1 - \frac{l_L}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) e_2. \quad (6.2)$$

We have

**Lemma 6.1.** *Let  $\Sigma_1 \subset (M, g_L)$  be a regular surface. Let  $\gamma : [a, b] \rightarrow \Sigma_1$  be a Euclidean  $C^2$ -smooth regular curve. Then*

$$\begin{aligned}
k_{\gamma, \Sigma_1}^\infty &= \left\{ \left[ q \dot{\gamma}_2 (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + F_3 f \bar{q} \omega(\dot{\gamma}(t)) \right] \right. \\
&\quad \left. + \left[ \bar{p} \dot{\gamma}_1 (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') - F_3 f \bar{p} \omega(\dot{\gamma}(t)) \right] \right\} |F \omega(\dot{\gamma}(t))|^{-1}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0,
\end{aligned} \quad (6.3)$$

$$k_{\gamma, \Sigma_1}^\infty = 0, \text{ if } \omega(\dot{\gamma}(t)) = 0, \text{ and } \frac{d}{dt} (\omega(\dot{\gamma}(t))) = 0,$$

$$\lim_{L \rightarrow +\infty} \frac{k_{\gamma, \Sigma_1}^L}{\sqrt{L}} = \frac{|\frac{d}{dt} (\omega(\dot{\gamma}(t)))|}{\left[ \frac{\bar{q} \dot{\gamma}_1}{F} - \frac{\bar{p} \dot{\gamma}_2}{F} \right]^2}, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt} (\omega(\dot{\gamma}(t))) \neq 0. \quad (6.4)$$

*Proof.* By (6.1), we have

$$\begin{aligned}
\|\nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}\|_{L, \Sigma_1}^2 &= B_1^2 + B_2^2 \\
&\sim L^2 \omega(\dot{\gamma}(t))^2 \left\{ \left[ \frac{\bar{q} \dot{\gamma}_2}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{F_3 f \bar{q}}{F} \omega(\dot{\gamma}(t)) \right] \right. \\
&\quad \left. + \left[ \frac{\bar{p} \dot{\gamma}_1}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') - \frac{F_3 f \bar{p}}{F} \omega(\dot{\gamma}(t)) \right] \right\}^2, \text{ as } L \rightarrow +\infty.
\end{aligned} \quad (6.5)$$

By (6.2), we have that when  $\omega(\dot{\gamma}(t)) \neq 0$ ,

$$\|\dot{\gamma}\|_{\Sigma_1, L} \sim L^{\frac{1}{2}} |\omega(\dot{\gamma}(t))|, \text{ as } L \rightarrow +\infty. \quad (6.6)$$

By (6.1) and (6.2), we have

$$\langle \nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma_1, L} \sim M_0 L, \quad (6.7)$$

where  $M_0$  does not depend on  $L$ .

By (3.7), (6.5)–(6.7), we get (6.3). When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$ , we have

$$\|\nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}\|_{L, \Sigma_1}^2 \sim C_0 := \left\{ \bar{q} \left[ \frac{F' \dot{\gamma}_1 - F \ddot{\gamma}_1}{F^2} - \frac{\dot{\gamma}_1 \dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) + \frac{\dot{\gamma}_2^2}{F^2} \left( F_1 + \frac{F_3 f}{F} \right) \right] \right. \\ \left. - \bar{p} \left[ \frac{F' \dot{\gamma}_2 - F \ddot{\gamma}_2}{F^2} + \frac{\dot{\gamma}_1^2}{F^2} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \frac{\dot{\gamma}_1 \dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3 f}{F} \right) \right] \right\}^2, \quad (6.8)$$

as  $L \rightarrow +\infty$ .

and

$$\|\dot{\gamma}\|_{\Sigma_1, L}^2 = \left[ \frac{\bar{q} \dot{\gamma}_1}{F} - \frac{\bar{p} \dot{\gamma}_2}{F} \right]^2, \quad (6.9)$$

$$\langle \nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma_1, L} = \left[ \frac{\bar{q} \dot{\gamma}_1}{F} - \frac{\bar{p} \dot{\gamma}_2}{F} \right] C_0. \quad (6.10)$$

By (6.8)–(6.10) and (3.7), we get  $k_{\gamma, \Sigma_1}^{\infty} = 0$ . When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$ , we have

$$\|\nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}\|_{L, \Sigma_1}^2 \sim L \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right]^2,$$

$$\langle \nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma_1, L} = O(1),$$

so we get (6.4).  $\square$

**Lemma 6.2.** *Let  $\Sigma_1 \subset (M, g_L)$  be a regular surface. Let  $\gamma : [a, b] \rightarrow \Sigma_1$  be a Euclidean  $C^2$ -smooth regular curve. Then*

$$k_{\gamma, \Sigma_1}^{\infty, s} = \left\{ \left[ q \dot{\gamma}_2 (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + F_3 f \bar{q} \omega(\dot{\gamma}(t)) \right] \right. \\ \left. + \left[ \bar{p} \dot{\gamma}_1 (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') - F_3 f \bar{p} \omega(\dot{\gamma}(t)) \right] \right\} |F \omega(\dot{\gamma}(t))|^{-1}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0, \quad (6.11)$$

$$k_{\gamma, \Sigma_1}^{\infty, s} = 0, \text{ if } \omega(\dot{\gamma}(t)) = 0, \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,$$

$$\lim_{L \rightarrow +\infty} \frac{k_{\gamma, \Sigma_1}^{L, s}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot{\gamma}(t)))|}{\left[ \frac{\bar{q} \dot{\gamma}_1}{F} - \frac{\bar{p} \dot{\gamma}_2}{F} \right]^2}, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \quad (6.12)$$

*Proof.* By (3.3) and (6.2), we have

$$J_L(\dot{\gamma}) = \frac{l_L}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) e_1 + \left[ \frac{\bar{q} \dot{\gamma}_1}{F} - \frac{\bar{p} \dot{\gamma}_2}{F} \right] e_2. \quad (6.13)$$

By (6.1) and (6.13), we have

$$\begin{aligned}
\langle \nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L, \Sigma_1} &= \frac{l_L}{L} L^{\frac{1}{2}} \omega(\dot{\gamma}(t)) \left\{ \bar{q} \left[ \frac{F' \dot{\gamma}_1 - F \ddot{\gamma}_1}{F^2} - \frac{\dot{\gamma}_1 \dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) + \frac{\dot{\gamma}_2^2}{F^2} \left( F_1 + \frac{F_3 f}{F} \right) \right] \right. \\
&+ \bar{q} \left[ -\frac{F_3 \dot{\gamma}_1}{F^2} + \frac{\dot{\gamma}_2 L}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{f L F_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \\
&- \bar{p} \left[ \frac{F' \dot{\gamma}_2 - F \ddot{\gamma}_2}{F^2} + \frac{\dot{\gamma}_1^2}{F^2} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \frac{\dot{\gamma}_1 \dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3 f}{F} \right) \right] \\
&+ \bar{p} \left[ -\frac{F_3 \dot{\gamma}_2}{F^2} - \frac{\dot{\gamma}_1 L}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{\bar{f} L F_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \left. \right\} \\
&+ \left[ \frac{\bar{q} \dot{\gamma}_1}{F} - \frac{\bar{p} \dot{\gamma}_2}{F} \right] \left\{ \bar{r}_L \bar{p} \left[ \frac{F' \dot{\gamma}_1 - F \ddot{\gamma}_1}{F^2} - \frac{\dot{\gamma}_1 \dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) + \frac{\dot{\gamma}_2^2}{F^2} \left( F_1 + \frac{F_3 f}{F} \right) \right] \right. \\
&+ \bar{r}_L \bar{p} \left[ -\frac{F_3 \dot{\gamma}_1}{F^2} + \frac{\dot{\gamma}_2 L}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{f L F_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \\
&+ \bar{r}_L \bar{q} \left[ \frac{F' \dot{\gamma}_2 - F \ddot{\gamma}_2}{F^2} + \frac{\dot{\gamma}_1^2}{F^2} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \frac{\dot{\gamma}_1 \dot{\gamma}_2}{F^2} \left( F_2 + \frac{F_3 f}{F} \right) \right] \\
&+ \bar{r}_L \bar{q} \left[ -\frac{F_3 \dot{\gamma}_2}{F^2} - \frac{\dot{\gamma}_1 L}{F} (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') + \frac{\bar{f} L F_3}{F} \omega(\dot{\gamma}(t)) \right] \omega(\dot{\gamma}(t)) \\
&\left. - \frac{l}{l_L} L^{\frac{1}{2}} \left[ \frac{(\dot{\gamma}_2^2 + \dot{\gamma}_1^2) F_3}{F^3 L} - \frac{(f \dot{\gamma}_1 + \bar{f} \dot{\gamma}_2) F_3}{F^2} \omega(\dot{\gamma}(t)) + \frac{d}{dt} (\omega(\dot{\gamma}(t))) \right] \right\}
\end{aligned} \tag{6.14}$$

So by (3.17), (6.6) and (6.14), we get (6.11). When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$ , we get

$$\langle \nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L, \Sigma_1} \sim M_0 L^{-\frac{1}{2}} \text{ as } L \rightarrow +\infty. \tag{6.15}$$

So  $k_{\gamma, \Sigma_1}^{\infty, s} = 0$ . When  $\omega(\dot{\gamma}(t)) = 0$  and  $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$ , we have

$$\langle \nabla_{\dot{\gamma}}^{\Sigma_1, L} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L, \Sigma_1} \sim L^{\frac{1}{2}} \left[ \frac{\bar{p} \dot{\gamma}_2}{F} - \frac{\bar{q} \dot{\gamma}_1}{F} \right] \frac{d}{dt} (\omega(\dot{\gamma}(t))), \text{ as } L \rightarrow +\infty. \tag{6.16}$$

So we get (6.12).  $\square$

In the following, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in the generalized BCV spaces. Similarly to Theorem 4.3 in [3], we have

**Theorem 6.3.** *The second fundamental form  $II_1^L$  of the embedding of  $\Sigma_1$  into  $(M, g_L)$  is given by*

$$II_1^L = \begin{pmatrix} h_{11}, & h_{12} \\ h_{21}, & h_{22} \end{pmatrix}, \tag{6.17}$$

where

$$h_{11} = \frac{l}{l_L} [X_1(\bar{p}) + X_2(\bar{q})] - \bar{q}_L \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \bar{p}_L \left( F_1 + \frac{F_3 f}{F} \right) + \bar{r}_L L^{-\frac{1}{2}} \frac{F_3}{F},$$



$$h_{12} = h_{21} = -\frac{l_L}{l} \langle e_1, \nabla_H(\overline{r_L}) \rangle_L - \frac{1}{2} L^{\frac{1}{2}} (fF_2 - F_1\overline{f} + \overline{f}'F - Ff'),$$

$$h_{22} = -\frac{l^2}{l_L^2} \langle e_2, \nabla_H(\frac{r}{l}) \rangle_L + \widetilde{X}_3(\overline{r_L}) - \overline{p_L} \frac{fF_3}{F} - \overline{q_L} \frac{\overline{f}F_3}{F} - \overline{r_L} \frac{F_3}{F} L^{-\frac{1}{2}}.$$

*Proof.* By lemma 5.1 and (3.2),

$$\begin{aligned} \nabla_{e_1}^L e_1 &= \nabla_{(\overline{q}X_1 - \overline{p}X_2)}^L (\overline{q}X_1 - \overline{p}X_2) \\ &= [\overline{q}X_1(\overline{q}) - \overline{p}X_2(\overline{p}) + \overline{p}\overline{q}(F_2 + \frac{F_3\overline{f}}{F}) + \overline{p}^2(F_1 + \frac{F_3f}{F})]X_1 \\ &\quad - [\overline{q}X_1(\overline{p}) - \overline{p}X_2(\overline{p}) - \overline{q}^2(F_2 + \frac{F_3\overline{f}}{F}) - \overline{p}\overline{q}(F_2 + \frac{F_3f}{F})]X_2 + \frac{F_3}{F} L^{-\frac{1}{2}} \widetilde{X}_3. \end{aligned} \tag{6.18}$$

Then

$$\begin{aligned} h_{11} &= -\langle \nabla_{e_1}^L e_1, V_L \rangle_L \\ &= -\overline{p_L} [\overline{q}X_1(\overline{q}) - \overline{p}X_2(\overline{p})] + \overline{q_L} [\overline{q}X_1(\overline{p}) - \overline{p}X_2(\overline{p})] \\ &\quad - \overline{q_L} \left( F_2 + \frac{F_3\overline{f}}{F} \right) - \overline{p_L} \left( F_1 + \frac{F_3f}{F} \right) + \overline{r_L} L^{-\frac{1}{2}} \frac{F_3}{F} \\ &= \frac{l}{l_L} [X_1(\overline{p}) + X_2(\overline{q})] - \overline{q_L} \left( F_2 + \frac{F_3\overline{f}}{F} \right) - \overline{p_L} \left( F_1 + \frac{F_3f}{F} \right) + \overline{r_L} L^{-\frac{1}{2}} \frac{F_3}{F}. \end{aligned} \tag{6.19}$$

Similarly,

$$\begin{aligned} \nabla_{e_1}^L e_2 &= \nabla_{(\overline{q}X_1 - \overline{p}X_2)}^L (\overline{r_L}\overline{p}X_1 + \overline{r_L}\overline{q}X_2 - \frac{l}{l_L} \widetilde{X}_3) \\ &= [\overline{q}X_1(\overline{r_L}\overline{p}) - \overline{p}X_2(\overline{r_L}\overline{p}) + \frac{(fF_2 - F_1\overline{f} + F\overline{f}' - Ff')\overline{p_L}\sqrt{L}}{2} \\ &\quad - \overline{r_L}\overline{p}\overline{q}(F_1 + \frac{F_3f}{F}) - \overline{r_L}\overline{q}^2(F_2 + \frac{F_3\overline{f}}{F}) + \overline{q_L}\frac{F_3}{F}L^{-\frac{1}{2}}]X_1 \\ &\quad + [\overline{q}X_1(\overline{q}) - \overline{p}X_2(\overline{q}) + \frac{(fF_2 - F_1\overline{f} + F\overline{f}' - Ff')\overline{q_L}\sqrt{L}}{2} \\ &\quad - \overline{r_L}\overline{p}\overline{q}(F_2 + \frac{F_3\overline{f}}{F}) + \overline{r_L}\overline{p}^2(F_1 + \frac{F_3f}{F}) + \overline{p_L}\frac{F_3}{F}L^{-\frac{1}{2}}]X_2 \\ &\quad + [\overline{p}X_2(\frac{l}{l_L}) - \overline{q}X_1(\frac{l}{l_L}) + \frac{(fF_2 - F_1\overline{f} + F\overline{f}' - Ff')\overline{r_L}\sqrt{L}}{2}] \widetilde{X}_3. \end{aligned} \tag{6.20}$$

Then

$$h_{12} = -\langle \nabla_{e_1}^L e_2, V_L \rangle_L$$

$$\begin{aligned}
&= -\frac{l}{l_L}[\bar{q}X_1(\bar{r}_L) - \bar{p}X_2(\bar{r}_L)] + \bar{r}_L[\bar{q}X_1(\frac{l}{l_L}) - \bar{p}X_2(\frac{l}{l_L})] - \frac{1}{2}L^{\frac{1}{2}}(fF_2 - F_1\bar{f} + \bar{f}'F - Ff') \\
&= -\frac{l_L}{l}\langle e_1, \nabla_H(\bar{r}_L)\rangle_L - \frac{1}{2}L^{\frac{1}{2}}(fF_2 - F_1\bar{f} + \bar{f}'F - Ff').
\end{aligned} \tag{6.21}$$

Since

$$\begin{aligned}
\langle \nabla_{e_2}^L V_L, e_1 \rangle_L &= -\langle \nabla_{e_2}^L e_1, V_L \rangle_L = -\langle \nabla_{e_1}^L e_2 + [e_2, e_1], V_L \rangle_L \\
&= -\langle \nabla_{e_1}^L e_2, V_L \rangle = \langle \nabla_{e_1}^L V_L, e_2 \rangle_L.
\end{aligned} \tag{6.22}$$

Then,

$$h_{21} = h_{12} = -\frac{l_L}{l}\langle e_1, \nabla_H(\bar{r}_L)\rangle_L - \frac{1}{2}L^{\frac{1}{2}}(fF_2 - F_1\bar{f} + \bar{f}'F - Ff'). \tag{6.23}$$

Since

$$\begin{aligned}
\nabla_{e_2}^L e_2 &= \nabla_{(\bar{r}_L \bar{p} X_1 + \bar{r}_L \bar{q} X_2 - \frac{l}{l_L} \widetilde{X}_3)}^L (\bar{r}_L \bar{p} X_1 + \bar{r}_L \bar{q} X_2 - \frac{l}{l_L} \widetilde{X}_3) \\
&= [\bar{q}X_1(\bar{r}_L \bar{p}) - \bar{p}X_2(\bar{r}_L \bar{p}) + (fF_2 - F_1\bar{f} + \bar{f}'F - Ff')\bar{q}_L \sqrt{L} - \bar{r}_L^2 \bar{p}\bar{q}(F_2 + \frac{F_3 \bar{f}}{F}) \\
&\quad + \frac{\bar{r}_L \bar{p} L}{F} L^{-\frac{1}{2}} + \bar{r}_L^2 \bar{q}^2 (F_1 + \frac{F_3 \bar{f}}{F}) + (\frac{l}{l_L})^2 \frac{F_3 \bar{f}}{F}] X_1 \\
&\quad + [\bar{q}X_1(\bar{q}) - \bar{p}X_2(\bar{q}) + (fF_2 - F_1\bar{f} + \bar{f}'F - Ff')\bar{p}_L \sqrt{L} + \bar{r}_L^2 \bar{p}\bar{q}(F_2 + \frac{F_3 \bar{f}}{F}) \\
&\quad - \frac{\bar{r}_L \bar{q} L}{F} L^{-\frac{1}{2}} - \bar{r}_L^2 \bar{p}\bar{q}(F_1 + \frac{F_3 \bar{f}}{F}) + (\frac{l}{l_L})^2 \frac{F_3 \bar{f}}{F}] X_2 \\
&\quad + [\bar{p}X_2(\frac{l}{l_L}) - \bar{q}X_1(\frac{l}{l_L}) + \bar{r}_L^2 \bar{p}\bar{q} \frac{F_3 \bar{f}}{F} L^{-\frac{1}{2}} + \bar{r}_L^2 \bar{q}^2 \frac{F_3 \bar{f}}{F} L^{-\frac{1}{2}} + \frac{\bar{r}_L \bar{p} L}{F} + \frac{\bar{r}_L \bar{q} L}{F} \frac{\bar{f} F_3}{F}] \widetilde{X}_3.
\end{aligned} \tag{6.24}$$

Then,

$$\begin{aligned}
h_{22} &= -\langle \nabla_{e_2}^L e_2, V_L \rangle_L \\
&= -\bar{p} \frac{r}{l} X_1(\bar{r}_L) - \bar{q} \frac{r}{l} X_2(\bar{r}_L) + \widetilde{X}_3(\bar{r}_L) - \bar{p}_L \frac{f F_3}{F} - \bar{q}_L \frac{\bar{f} F_3}{F} - \bar{r}_L \frac{F_3}{F} L^{-\frac{1}{2}} \\
&= -\frac{l^2}{l_L} \langle e_2, \nabla_H(\frac{r}{l}) \rangle_L + \widetilde{X}_3(\bar{r}_L) - \bar{p}_L \frac{f F_3}{F} - \bar{q}_L \frac{\bar{f} F_3}{F} - \bar{r}_L \frac{F_3}{F} L^{-\frac{1}{2}}.
\end{aligned} \tag{6.25}$$

□

Similar to Proposition 3.8, we have

**Proposition 6.4.** *Away from characteristic points, the horizontal mean curvature  $\mathcal{H}_\infty^1$  of  $\Sigma_1 \subset M$  is given by*

$$\mathcal{H}_\infty^1 = -\left(\bar{p}\frac{fF_3}{F} + \bar{q}\frac{\bar{f}F_3}{F}\right) + X_1(\bar{p}) + X_2(\bar{q}) - \bar{q}\left(F_2 + \frac{F_3\bar{f}}{F}\right) - \bar{p}\left(F_1 + \frac{F_3f}{F}\right). \quad (6.26)$$

By Lemma 5.1, we have

**Lemma 6.5.** *Let  $M$  be the the generalized BCV spaces, then*

$$\begin{aligned} R^L(X_1, X_2)X_1 &= \left[-X_1(A) - X_2(B) + \frac{3L}{4}C^2 + \frac{F^3}{LF^2} + A^2 + B^2\right]X_2 \\ &+ \left[-\frac{1}{2}X_1(C) - X_2\left(\frac{F_3}{LF}\right) + \frac{F_3fC}{F}\right]X_3, \\ R^L(X_1, X_2)X_2 &= \left[X_1(A) + X_2(B) - \frac{3L}{4}C^2 - \frac{F_3^2}{LF^2} - A^2 - B^2\right]X_1 \\ &+ \left[-\frac{1}{2}X_2(C) + X_1\left(\frac{F_3}{LF}\right) + \frac{F_3\bar{f}C}{F}\right]X_3, \\ R^L(X_1, X_2)X_3 &= \left[\frac{1}{2}X_2(LC) + X_2\left(\frac{F_3}{F}\right) - \frac{F_3fCL}{F}\right]X_1 \\ &+ \left[\frac{1}{2}X_2(LC) - X_1\left(\frac{F_3}{F}\right) - \frac{F_3\bar{f}CL}{F}\right]X_2, \\ R^L(X_1, X_3)X_1 &= \left[-X_1\left(\frac{LC}{2}\right) - X_3(B) + \frac{BF_3}{F} - \frac{F_3^2f^2}{F^2}\right]X_2 \\ &+ \left[-\frac{LC^2}{4} - X_1\left(\frac{F_3f}{F}\right) - X_3\left(\frac{F_3}{LF}\right) + \frac{F_3\bar{f}B}{F} + \frac{F_3^2}{LF^2} + \frac{F^3f^2}{F^2}\right]X_3, \\ R^L(X_1, X_3)X_2 &= \left[X_1\left(\frac{LC}{2}\right) + X_3(B) - \frac{CF_3fL}{F} + \frac{F^3\bar{f}^2}{F^2} + \frac{BF_3}{F}\right]X_1 \\ &+ \left[-X_1\left(\frac{F_3\bar{f}}{F}\right) - \frac{1}{2}X_3(C) + \frac{F_3C}{F} - \frac{fF_3B}{F} + \frac{F^3f\bar{f}}{F^2}\right]X_3, \\ R^L(X_1, X_3)X_3 &= \left[X_1\left(\frac{LF_3f}{F}\right) + X_3\left(\frac{F_3}{F}\right) - \frac{F_3\bar{f}L}{F}B + \frac{L^2C^2}{4} - \frac{F_3^2}{F^2} - \frac{LF_3^2f^2}{F^2}\right]X_1 \\ &+ \left[X_1\left(\frac{LF_3\bar{f}}{F}\right) - \frac{LCF_3}{F} + X_3\left(\frac{LC}{2}\right) + \frac{F_3fL}{F}B - \frac{LF_3^2f\bar{f}}{F^2}\right]X_2, \\ R^L(X_2, X_3)X_1 &= \left[-X_2\left(\frac{LC}{2}\right) + X_3(A) + \frac{F_3\bar{f}LC}{F} + \frac{fF_3^2}{F^2} - \frac{AF_3}{F}\right]X_2 \\ &+ \left[-X_2\left(\frac{fF_3}{F}\right) - \frac{CF_3}{F} + X_3\left(\frac{C}{2}\right) - \frac{F_3\bar{f}A}{F} + \frac{F_3^2f\bar{f}}{F^2}\right]X_3, \end{aligned} \quad (6.27)$$

$$\begin{aligned}
 R^L(X_2, X_3)X_2 &= \left[ X_2 \left( \frac{LC}{2} \right) - X_3(A) - \frac{F_3 \bar{f} LC}{F} - \frac{f F_3^2}{F^2} + \frac{A F_3}{F} \right] X_1 \\
 &+ \left[ -X_2 \left( \frac{\bar{f} F_3}{F} \right) + \frac{A f F_3}{F} - X_3 \left( \frac{F_3}{FL} \right) - \frac{LC^2}{4} + \frac{F_3^2}{F^2 L} + \frac{F_3^2 \bar{f}^2}{F^2} \right] X_3, \\
 R^L(X_2, X_3)X_3 &= \left[ X_2 \left( \frac{L F_3 f}{2} \right) - X_3 \left( \frac{LC}{2} \right) + \frac{A F_3 \bar{f} L}{F} - \frac{L f \bar{f} F_3^2}{F^2} + \frac{L C F_3}{F} \right] X_2 \\
 &+ \left[ X_2 \left( \frac{L F_3 \bar{f}}{2} \right) + X_3 \left( \frac{F_3}{F} \right) - \frac{A F_3 f L}{F} + \frac{L^2 C^2}{4} + \frac{L F_3^2 \bar{f}^2}{F^2} - \frac{F_3^2}{F^2} \right] X_3.
 \end{aligned}$$

where

$$\left( F_1 + \frac{F_3 f}{F} \right) = A, \left( F_2 + \frac{F_3 \bar{f}}{F} \right) = B, (F_2 f - F_1 \bar{f} + F \bar{f}' - F f') = C.$$

**Proposition 6.6.** *Away from characteristic points, we have*

$$\mathcal{K}^{\Sigma_1, \infty}(e_1, e_2) = -C \langle e_1, \nabla_H \left( \frac{X_3 u}{|\nabla_H u|} \right) \rangle + \bar{N} + O(L^{-\frac{1}{2}}). \tag{6.28}$$

where  $\bar{N} = N_0 + N$ ,

$$\begin{aligned}
 N &= - \left( \bar{p} \frac{f F_3}{F} + \bar{q} \frac{\bar{f} F_3}{F} \right) \left[ X_1(\bar{p}) + X_2(\bar{q}) - \bar{q} \left( F_2 + \frac{F_3 \bar{f}}{F} \right) - \bar{p} \left( F_1 + \frac{F_3 f}{F} \right) \right], \\
 N_0 &= 2\bar{q} \left[ -\frac{1}{2} X_1(C) + \frac{F_3 C f}{F} \right] - 2\bar{p} \left[ X_1 \left( -\frac{1}{2} X_2(C) + \frac{F_3 C \bar{f}}{F} \right) + \bar{p}^2 \left[ X_2 \left( \frac{F_3 \bar{f}}{F} \right) - \frac{F_3 A f}{F} - \frac{F_3^2 \bar{f}^2}{F^2} \right] \right. \\
 &\quad \left. - 2\bar{p}\bar{q} \left[ X_1 \left( \frac{F_3 \bar{f}}{F} \right) - \frac{1}{2} X_3(C) + \frac{F_3 B f}{F} - \frac{F_3 C}{F} \right] + \bar{q}^2 \left[ X_1 \left( \frac{F_3 f}{F} \right) - \frac{F_3 B \bar{f}}{F} - \frac{F_3^2 \bar{f}^2}{F^2} \right] \right].
 \end{aligned}$$

*Proof.* By (3.43) and Lemma 6.5, we have

$$\begin{aligned}
 \mathcal{K}^{M,L}(e_1, e_2) &= \bar{r}_L^2 \left[ X_1(A) + X_2(B) - \frac{3LC^2}{4} - \frac{F_3^2}{F^2 L} - A^2 - B^2 \right] \\
 &+ 2 \frac{l}{l_L} \bar{q} r_L L^{\frac{1}{2}} \left[ -\frac{1}{2} X_1(C) - X_2 \left( \frac{F_3}{FL} \right) + \frac{F_3 C f}{F} \right] \\
 &- 2 \frac{l}{l_L} \bar{p} r_L L^{\frac{1}{2}} \left[ X_1 \left( \frac{F_3}{FL} \right) - \frac{1}{2} X_2(C) + \frac{F_3 C \bar{f}}{F} \right] \\
 &- 2 \frac{l^2}{l_L^2} \bar{p} \bar{q} \left[ X_1 \left( \frac{F_3 \bar{f}}{F} \right) - \frac{1}{2} X_3(C) + \frac{F_3 B f}{F} - \frac{F_3 C}{F} \right] \\
 &+ \frac{l^2}{l_L^2} \bar{q}^2 \left[ \frac{LC^2}{4} + X_1 \left( \frac{F_3 f}{F} \right) - \frac{F_3 B \bar{f}}{F} + X_3 \left( \frac{F_3}{FL} \right) - \frac{F_3^2}{F^2 L} - \frac{F_3^2 \bar{f}^2}{F^2} \right]
 \end{aligned} \tag{6.29}$$

$$\begin{aligned}
& + \frac{l^2}{l_L^2} \bar{P}^2 \left[ \frac{LC^2}{4} + X_2 \left( \frac{F_3 \bar{f}}{F} \right) - \frac{F_3 A f}{F} + X_3 \left( \frac{F_3}{FL} \right) - \frac{F_3^2}{F^2 L} - \frac{F_3^2 \bar{f}^2}{F^2} \right] \\
& \sim \frac{LC^2}{4} + N_0, \quad \text{as } L \rightarrow +\infty.
\end{aligned}$$

Similar to (3.45), we have

$$\begin{aligned}
\det(II_1^L) &= h_{11}h_{22} - h_{12}h_{21} \\
&= -\frac{LC^2}{4} - C \langle e_1, \nabla_H \left( \frac{X_3 u}{|\nabla_H u|} \right) \rangle + N + O(L^{-\frac{1}{2}}) \quad \text{as } L \rightarrow +\infty.
\end{aligned} \tag{6.30}$$

By (6.21) and (6.22), we have (6.20).  $\square$

Similar to (4.2) and (4.5), for the generalized BCV spaces, we have

$$\lim_{L \rightarrow +\infty} \frac{1}{\sqrt{L}} ds_L = ds, \quad \lim_{L \rightarrow +\infty} \frac{1}{\sqrt{L}} d\sigma_{\Sigma_1, L} = d\sigma_{\Sigma_1}. \tag{6.31}$$

By (6.20), (6.23) and Lemma 6.2, similar to the proof of Theorem 1 in [1], we have

**Theorem 6.7.** *Let  $\Sigma_1 \subset (M, g_L)$  be a regular surface with finitely many boundary components  $(\partial\Sigma_1)_i$ ,  $i \in \{1, \dots, n\}$ , given by Euclidean  $C^2$ -smooth regular and closed curves  $\gamma_i : [0, 2\pi] \rightarrow (\partial\Sigma_1)_i$ . Suppose that the characteristic set  $C(\Sigma_1)$  satisfies  $\mathcal{H}^1(C(\Sigma_1)) = 0$  and that  $\|\nabla_H u\|_H^{-1}$  is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set  $C(\Sigma_1)$ , then*

$$\int_{\Sigma_1} \mathcal{K}^{\Sigma_1, \infty} d\sigma_{\Sigma_1} + \sum_{i=1}^n \int_{\gamma_i} k_{\gamma_i, \Sigma_1}^{\infty, s} ds = 0. \tag{6.32}$$

**Example 6.8.** *Let  $F = 1$ ,  $f = -x_2^2$ ,  $\bar{f} = x_1^2$ . Consider  $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid F > 0\} = \mathbb{R}^3$ , let  $u = x_1^2 + x_2^2 + x_3^2 - 1$  and  $\Sigma_1 = S^2$ .  $\Sigma_1$  is a regular surface. By (4.1), we get*

$$X_1(u) = 2x_1 - 2x_2^2 x_3; \quad X_2(u) = 2x_2 + 2x_1^2 x_3. \tag{6.33}$$

*Solve the equations  $X_1(u) = X_2(u) = 0$ , then we get  $C(\Sigma) = \{(0, 0, 1), (0, 0, -1)\}$  and  $\mathcal{H}^1(C(\Sigma_1)) = 0$ . A parametrization of  $\Sigma$  is*

$$\begin{aligned}
x_1 &= \cos(\phi)\cos(\theta), \quad x_2 = \cos(\phi)\sin(\theta), \\
x_3 &= \sin(\phi), \quad \text{for } \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \theta \in [0, 2\pi).
\end{aligned} \tag{6.34}$$

Then

$$\begin{aligned}
\|\nabla_H u\|_H^2 &= X_1(u)^2 + X_2(u)^2 = 4(x_1^2 + x_2^2) + 4(x_1^4 + x_2^4)x_3^2 \\
&= 4\cos^2(\phi) + 4\sin^2(\phi)\cos^2(\phi)^4(\cos^2(\theta) + \sin^2(\theta)^4).
\end{aligned} \tag{6.35}$$

By the definitions of  $w_j$  for  $1 \leq j \leq 3$  and (6.23), we have

$$d\sigma_{\Sigma_1} = \frac{1}{\|\nabla_H u\|_H} [(X_1(u)dx_2 - (X_2(u))dx_1] \wedge (dx_3 + x_2^2 dx_1 - x_1^2 dx_2) \tag{6.36}$$

$$= \frac{1}{\|\nabla_H u\|_H} [2\cos(\phi)^3 + 2\sin(\phi)^2\cos(\phi)^5(\cos(\theta)^4 + \sin(\theta)^4) - 4\cos(\phi)^4\sin(\theta)^2\sin(\phi)\cos(\theta) + 4\cos(\phi)^4\sin(\theta)\sin(\phi)\cos(\theta)^2]d\theta \wedge d\phi.$$

By (6.27) and (6.28), we have  $\|\nabla_H u\|_H^{-1}$  is locally summable around the isolated characteristic points with respect to the measure  $d\sigma_{\Sigma_1}$ .

## 7. Conclusions

Firstly, We give some basic definitions of two kinds of spaces, such as 2.3, 2.4 and 2.5. By computation, we get sub-Riemannian limits of Gaussian curvature for a Euclidean  $C^2$ -smooth surface in the generalized affine group and the generalized BCV spaces away from characteristic points and signed geodesic curvature for Euclidean  $C^2$ -smooth curves on surfaces, respectively. Then, by the second fundamental form  $II^L$  and the Gauss equation  $\mathcal{K}^{\Sigma, L}(e_1, e_2) = \mathcal{K}^L(e_1, e_2) + \det(II^L)$ , we find the Gauss curvature on the surface is convergent in two cases. Therefore, a good result is obtained. Finally, we give the proof of Gauss-Bonnet theorems in the generalized affine group and the generalized BCV spaces.

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## Conflict of interest

The authors declare no conflict of interest.

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