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Research article

On partial fuzzy k-(pseudo-)metric spaces

Yaoqiang Wu*

School of Mathematics and Arts, Suqian University, Suqian, 223800, Jiangsu, China

* **Correspondence:** Email: 23036@sqc.edu.cn.

Abstract: In this paper, we introduce the concept of partial fuzzy k-(pseudo-)metric spaces, which is a generalization of fuzzy metric type spaces which introduced by Saadati. Also, we study some properties in partial fuzzy k-metric spaces and give some examples to support our results. Furthermore, we investigate the topological structures of partial fuzzy k-pseudo-metric spaces. Finally, we prove the existence of fixed points in these spaces.

Keywords: partial metric; partial k-(pseudo-)metric; partial fuzzy k-pseudo-metric; partial fuzzy k-metric; fixed point theorem
Mathematics Subject Classification: 47H10, 54E70, 54H25

1. Introduction and preliminaries

Since Zadeh [24] introduced the notion of fuzzy sets in 1965, several different versions of fuzzy metric spaces were studied by topological researchers. In particular, Kramosi and Michalek [11] introduced the fuzzy metric spaces in 1975, basing on statistical metric spaces which introduced by Menger [14] in 1942, Schweizer and Sklar [19] in 1960, respectively. Kaleva and Seikkala [10] introduced the concept of fuzzy metric spaces and studied some fixed point theorems in these spaces in 1984. Later, by modifying the fuzzy metric spaces concept of Kramosi and Michálek (usually called KM-fuzzy metric spaces), George and Veeramani [4] introduced the notion of fuzzy metric spaces in 1994 (usually called GV-fuzzy metric spaces), and defined Hausdorff topologies on these spaces. In the last years, many authors devoted to study various types of generalizing fuzzy metric spaces by different approaches. For instance, fuzzy pseudo-metric spaces [1], fuzzy quasi-metric spaces [7, 8], fuzzy partial (pseudo-)metric spaces [6, 23], fuzzy cone metric spaces [17], fuzzy b-metric spaces [16], fuzzy k-pseudometric spaces [22] (also called fuzzy metric type spaces in [18]), etc.

Furthermore, since Grabiec extended fixed point theorems of Banach and Edelstein to KM-fuzzy metric spaces [5], many researchers investigated the contractive mappings and obtained some interesting fixed point theorems concerning fuzzy metric spaces [2,3,9,15,20,25–30].

In this paper, firstly, we introduce the concept of partial k-(pseudo-)metric spaces. By generalizing the notion of fuzzy k-pseudo-metric spaces, we introduce the concept of partial fuzzy k-(pseudo-) metric spaces and give some examples to support our results in these spaces. Moreover, we investigate the relationships between (partial) k-pseudo-metric spaces and partial fuzzy k-pseudo-metric spaces. Finally, we provide several fixed pointed theorems in partial fuzzy k-metric spaces.

We recall some basic notions and results that will be used in the following sections (see more details in [12,13,21]). Throughout this paper, the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}^+$ always denote the set of real numbers, of positive real numbers and of positive integers, respectively.

Definition 1.1. [13,21] Let X be a nonempty set and the mapping $p_k : X \times X \to [0, +\infty)$ satisfying the following conditions for some number $k \geq 1$: $\forall x, y, z \in X$,
(PK1) $p_k(x, x) \leq p_k(x, y)$;
(PK2) $p_k(x, y) = p_k(y, x)$;
(PK3) $p_k(x, z) \leq k[p_k(x, y) + p_k(y, z)] - p_k(y, y)$;
(KP3) $p_k(x, z) \leq k[p_k(x, y) + p_k(y, z)]$.

If p_k satisfies the conditions (PK1)–(PK3), then p_k is called a *partial k-pseudo-metric*. If p_k satisfies the conditions (PK1)–(KP3), then p_k is called a *k-pseudo-metric*.

A (partial) k-pseudo-metric space with coefficient $k \geq 1$ is a pair (X, p_k) such that p_k is a (partial) k-pseudo-metric on X.

Furthermore, if p_k satisfies (PK1)–(PK3) and the following condition:
(PK4) $x = y \Leftrightarrow p_k(x, x) = p_k(x, y) = p_k(y, y)$ for all $x, y \in X$;

Then it is called *partial k-metric* [21] and the pair (X, p_k) is called a partial k-metric space with a coefficient $k \geq 1$.

Particularly, a partial 1-metric (i.e. k=1) is called *partial metric* [13].

Example 1.2. Let $X = \{a, b, c\}$. Define p_k: $X \times X \to [0, +\infty)$ as follows:
$p_k(a, a) = p_k(a, b) = p_k(b, a) = p_k(b, c) = p_k(c, b) = 2, p_k(b, b) = p_k(c, c) = 0, p_k(c, a) = p_k(a, c) = 6$.
It is trivial to verify (X, p_k) is a partial k-pseudo-metric space with a coefficient $k = 2$.

Definition 1.3. [19] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *continuous triangular norm* (briefly t-norm) if it satisfies the following conditions:
(1) $*$ is associative and commutative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0, 1]$;
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

The following are the three basic t-norms: minimum, usual product and Lukasiewicz t-norm, which are given by, respectively: $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ and $a *_L b = \max\{0, a + b - 1\}$, $\forall a, b \in [0, 1]$.

Definition 1.4. [11] A triple $(X, M, *)$ is called a *KM-fuzzy metric space* if X is an arbitrary nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, +\infty)$, satisfying the following conditions: $\forall x, y, z \in X$ and $t, s > 0$,
(1) $M(x, y, 0) = 0$;
(2) $M(x, y, t) = 1$ if and only if $x = y$;
(3) $M(x, y, t) = M(y, x, t)$;

(4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(5) The function $M(x,y,\cdot) : [0,+\infty) \to [0,1]$ is left-continuous.

We note that A. George and P. Veeramani [4] modified the concept of KM-fuzzy metric spaces and defined a Hausdorff topology on this fuzzy space.

2. Partial fuzzy k-(pseudo)metric spaces

Definition 2.1. A triple $(X, M_{p_k}, *)$ is called a *partial fuzzy k-metric space* if X is an arbitrary nonempty set, $*$ is a continuous t-norm and M_{p_k} is a fuzzy set on $X \times X \times [0,+\infty)$, satisfying the following conditions for some number $k \geq 1$: $\forall x,y,z \in X$ and $t,s > 0$,
(PFK1) $M_{p_k}(x,y,0) = 0$;
(PFK2) $M_{p_k}(x,x,t) \geq M_{p_k}(x,y,t)$;
(PFK3) $M_{p_k}(x,y,t) = M_{p_k}(y,x,t)$;
(PFK4) $M_{p_k}(x,x,t) = M_{p_k}(x,y,t) = M_{p_k}(y,y,t)$ if and only if $x = y$;
(PFK5) $M_{p_k}(x,y,t) * M_{p_k}(y,z,s) \leq M_{p_k}(x,z,k(t+s))$;
(PFK6) The function $M_{p_k}(x,y,\cdot) : (0,+\infty) \to [0,1]$ is left-continuous.

If it only satisfies (PFK1)–(PFK3) and (PFK5)–(PFK6), then it is called a *partial fuzzy k-pseudometric space*.

Example 2.2. Let $X = X_a \cup X_b \cup X_c$, where $X_a = \{a\} \times [0,1], X_b = \{b\} \times [0,1], X_c = \{c\} \times [0,1]$. We denote $x \in \{a,b,c\} \times \bar{x}$, where $\bar{x} \in [0,1]$. Define a fuzzy set on $X \times X \times [0,+\infty)$ as follows:

$$M_{p_k}(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)}, & t > 0; \\ 0, & t = 0. \end{cases}$$

where

$$d(x,y) = \begin{cases} |\bar{x} - \bar{y}|, & x,y \in X_a \text{ or } x,y \in X_b \text{ or } x,y \in X_c \ ; \\ 1, & x \in X_a, y \in X_b \text{ or } x \in X_b, y \in X_a \ ; \\ 2, & x \in X_a, y \in X_c \text{ or } x \in X_c, y \in X_a \ ; \\ 5, & x \in X_b, y \in X_c \text{ or } x \in X_c, y \in X_b \ . \end{cases}$$

Then $(X, M_{p_k}, *)$ is a partial fuzzy k-metric space with a coefficient $k = 3$, where $x * y = x \wedge y$.

To verify this result, we have to check (PFK5).

(PFK5): Since the authors showed that $d(x,z) \leq 3[d(x,y) + d(y,z)]$ (see Example 7 in [26]). For any $x,y,z \in X$ and $t,s > 0$, without loss of generality, we assume that $M_{p_k}(x,y,t) \leq M_{p_k}(y,z,s)$. Namely, $\frac{t}{t+d(x,y)} \leq \frac{s}{s+d(y,z)}$, which implies that $sd(x,y) \geq td(y,z)$. Furthermore, we can deduce that$(t+s)d(x,y) \geq t[d(x,y) + d(y,z)]$.

Thus, we have$(t+s)[t+d(x,y)] \geq t[(t+s)+(d(x,y)+d(y,z))]$. Then

$$\begin{aligned}
M_{p_k}(x,z,3(t+s)) &= \frac{3(t+s)}{3(t+s)+d(x,z)} \\
&\geq \frac{3(t+s)}{3(t+s)+3[d(x,y)+d(y,z)]} \\
&= \frac{t+s}{t+s+d(x,y)+d(y,z)} \\
&\geq \frac{t}{t+d(x,y)} \\
&= M_{p_k}(x,y,t) = M_{p_k}(x,y,t) \wedge M_{p_k}(y,z,s),
\end{aligned}$$

for all $x,y \in X$, $t,s > 0$.

Remark 2.3. Let X be a nonempty set and p_k be a partial k-metric with a coefficient $k \geq 1$. Define a fuzzy set on $X \times X \times (0,+\infty)$ as follows:

$$M_{p_k}(x,y,t) = \begin{cases} \frac{lt^n}{lt^n + mp_k(x,y)}, & t > 0; \\ 0, & t = 0. \end{cases}$$

for all $x,y \in X$, $l,m > 0$ and $n \in \mathbb{N}^+$. Then $(X, M_{p_k}, *)$ is a partial fuzzy k-metric space with a coefficient k, where $x * y = x \wedge y$.

Here we only check (PFK5).

In fact, $\frac{a}{a+c} \geq \frac{b}{b+c}$ if $a \geq b$ for all $a,b,c > 0$, and $(t+s)^n \geq t^n + s^n$ for all $s,t > 0$ where $n \in \mathbb{N}^+$. Then, for some number $k \geq 1$, we have

$$\frac{lk^n(t+s)^n}{lk^n(t+s)^n + mp_k(x,z)} \geq \frac{lt^n + ls^n}{lt^n + ls^n + \frac{m[p_k(x,y)+p_k(y,z)]}{k^{n-1}} - \frac{mp_k(y,y)}{k^n}},$$

for all $x,y \in X, t,s,l,m > 0$, $n \in \mathbb{N}^+$. It is similar to the proof of Example 2.2.

Example 2.4. Let X be a nonempty set and p_k be a partial k-metric with a coefficient $k \geq 1$. Define a fuzzy set on $X \times X \times [0,+\infty)$ as follows:

$$M_{p_k}(x,y,t) = \begin{cases} e^{\frac{-(p_k(x,y))^q}{t}}, & t > 0; \\ 0, & t = 0. \end{cases}$$

for all $x,y \in X$, $t > 0$ and $q \geq 1$, where $x * y = x \cdot y$. Then $(X, M_{p_k}, *)$ is a partial fuzzy k-metric space with a coefficient $k(2k)^{(q-1)}$.

To verify this result, we have to check (PFK5). First, we claim that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$ for all $a,b \geq 0$ and $q \geq 1$.

(PFK5): For all $x,y,z \in X$, $t,s > 0$, and some real number $k \geq 1$, we can deduce

$$\begin{aligned}
\frac{(p_k(x,z))^q}{t+s} &\leq \frac{(k[p_k(x,y)+p_k(y,z)] - p_k(y,y))^q}{t+s} \\
&\leq \frac{(k[p_k(x,y)+p_k(y,z)])^q}{t+s} \\
&\leq k^q \cdot 2^{q-1}[\frac{(p_k(x,y))^q + (p_k(y,z))^q}{t+s}] \\
&\leq k(2k)^{(q-1)}[\frac{(p_k(x,y))^q}{t} + \frac{(p_k(y,z))^q}{s}],
\end{aligned}$$

for all $q \geq 1$. Therefore, we have

$$\begin{aligned} M_{p_k}(x, z, k(2k)^{(q-1)}(t+s)) &= e^{\frac{-(P_k(x,z))^q}{k(2k)^{(q-1)}(t+s)}} \\ &\geq e^{\frac{-(P_k(x,y))^q}{t}} \cdot e^{\frac{-(P_k(y,z))^q}{t}} \\ &= M_{p_k}(x, y, t) \cdot M_{p_k}(y, z, s), \end{aligned}$$

for all $x, y, z \in X$, $t, s > 0$, $q \geq 1$.

Example 2.5. Let X be a nonempty set and p be a partial metric. Define a fuzzy set on $X \times X \times (0, +\infty)$ as follows:

$$M_{p_k}(x, y, t) = \begin{cases} \frac{t}{t+p(x,y)}, & t > 0; \\ 0, & t = 0. \end{cases}$$

for all $x, y \in X$. Then $(X, M_{p_k}, *)$ is a partial fuzzy k-metric space with a coefficient $k \geq 1$, where $x * y = x \wedge y$.

Indeed, $\frac{k(t+s)}{k(t+s)+p(x,z)} \geq \frac{t+s}{t+s+[p(x,y)+p(y,z)]}$ for all $x, y \in X$, $t, s > 0, k \geq 1$. It is similar to the proof of Example 2.2.

Apparently, if $(X, M_{p_k}, *)$ is a partial fuzzy k-metric space, then $(X, M_{p_k}, *)$ is not a KM-fuzzy metric space. In fact, it can be illustrated by Example 2.4 for $q = 2$.

Proposition 2.6. *Let X be a nonempty set. If $(X, M_{p_k}, *)$ is a partial fuzzy k-metric space with a coefficient $k \geq 1$ and $(X, P, *)$ is a KM-fuzzy metric space, respectively. Define a fuzzy set on $X \times X \times [0, +\infty)$ as follows:*

$$M(x, y, t) = P(x, y, t) * M_{p_k}(x, y, t)$$

for all $x, y \in X$ and $t \geq 0$. Then $(X, M, *)$ is a partial fuzzy k-metric space with a coefficient k.

Proof. It is trivial to prove that $(X, M, *)$ satisfied (PFK1)–(PFK3) and (PFK6). We verify condition (FPK4) and (PFK5) in the following.

(PFK4): $(\Rightarrow)$ Suppose that $M(x, x, t) = M(x, y, t) = M(y, y, t)$ for all $x, y \in X$ and $t > 0$. Then we have

$$P(x, x, t) * M_{p_k}(x, x, t) = P(x, y, t) * M_{p_k}(x, y, t) = P(y, y, t) * M_{p_k}(y, y, t).$$

By Definition 1.4 (2), we have

$$M_{p_k}(x, x, t) = P(x, y, t) * M_{p_k}(x, y, t) = M_{p_k}(y, y, t).$$

Since $0 \leq P(x, y, t) \leq 1$, it follows that $P(x, y, t) * M_{p_k}(x, y, t) \leq M_{p_k}(x, y, t)$. Thus $M_{p_k}(x, x, t) \leq M_{p_k}(x, y, t)$ and $M_{p_k}(y, y, t) \leq M_{p_k}(x, y, t)$. By (PFK2), we have $M_{p_k}(x, x, t) \geq M_{p_k}(x, y, t)$ and $M_{p_k}(y, y, t) \geq M_{p_k}(x, y, t)$. Therefore, $M_{p_k}(x, x, t) = M_{p_k}(x, y, t) = M_{p_k}(x, y, t)$. So $x = y$ by Definition 2.1.

$(\Leftarrow)$ First, we claim that $P(x, x, t) = P(y, y, t) = 1$ if $P(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$, where $(X, P, *)$ is a KM-fuzzy metric space. Suppose $x = y$. It is obvious that $M_{p_k}(x, x, t) = M_{p_k}(x, y, t) = M_{p_k}(y, y, t)$ by the Definition 2.1, and $P(x, y, t) = 1$ by Definition 1.4 (2). Hence, $P(x, x, t) * M_{p_k}(x, x, t) = P(x, y, t) * M_{p_k}(x, y, t) = P(y, y, t) * M_{p_k}(y, y, t)$. Namely, $M(x, x, t) = M(x, y, t) = M(y, y, t)$ for all $x, y \in X$ and $t > 0$.

(PFK5): First, we claim that $P(x, y, \cdot)$ is non-decreasing for all $x, y \in X$ (see Lemma [15]). By (PFK5), Definition 1.3 (4) and Definition 1.4 (4), we have

$$\begin{aligned} & P(x,z,k(t+s)) * M_{p_k}(x,z,k(t+s)) \\ \geq\ & P(x,z,t+s) * M_{p_k}(x,z,k(t+s)) \\ \geq\ & P(x,y,t) * P(y,z,s) * M_{p_k}(x,y,t) * M_{p_k}(y,z,s) \\ =\ & M(x,y,t) * M(y,z,s), \end{aligned}$$

for all $x, y, z \in X$, $t, s > 0$. □

Proposition 2.7. *Let X be a nonempty set. If $(X, M_{p_k}, *)$ is a partial fuzzy k-metric space with a coefficient $k \geq 1$, where $x * y = xy$. Define a function $\hat{p}_k : X \times X \to [0, +\infty)$ as follows:*

$$\hat{p}_k(x,y) = \begin{cases} -\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \ln M_{p_k}(x,y,t)dt, & t > 0; \\ 0, & t = 0. \end{cases}$$

for all $x, y \in X$.

Then $\hat{p}_k$ is a partial k-pseudo metric with a coefficient $3k$.

Proof. We verify the conditions (PK1)–(PK3) step by step.

(PK1) We prove it in the following two cases.

Case 1: If $t = 0$, then $\hat{p}_k(x, y) = 0$. It is not difficult to show that $\hat{p}_k(x, x) = \hat{p}_k(x, y)$, for all $x, y \in X$.

Case 2: If $t > 0$ by the assumption, then we have $\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \ln M_{p_k}(x, y, t)dt$. Moreover, by (PFK2), we have $\ln M_{p_k}(x, x, t) \geq \ln M_{p_k}(x, y, t)$, which implies that

$$-\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \ln M_{p_k}(x,x,t)dt \leq -\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \ln M_{p_k}(x,y,t)dt,$$

for all $x, y \in X$, $t > 0$. Namely, $\hat{p}_k(x, x) \leq \hat{p}_k(x, y)$.

(PK2): By (PFK3), it is clear that $\hat{p}_k(x, y) = \hat{p}_k(y, x)$ for all $x, y \in X$.

(PK3): First, by (PK5), we claim that

$$M_{p_k}(x,z,t) = M_{p_k}(x,z,k(\frac{t}{2k}+\frac{t}{2k})) \geq M_{p_k}(x,y,\frac{t}{2k})M_{p_k}(y,z,\frac{t}{2k}),$$

for all $x, y, z \in X$, $t > 0$, which implies that

$$\int_\varepsilon^1 \ln M_{p_k}(x,z,t)dt \geq \int_\varepsilon^1 \ln M_{p_k}(x,y,\frac{t}{2k})dt + \int_\varepsilon^1 \ln M_{p_k}(y,z,\frac{t}{2k})dt.$$

Furthermore, set $u = \frac{t}{2k}$. We can deduce that

$$\int_\varepsilon^1 \ln M_{p_k}(x,y,\frac{t}{2k})dt = 2k\int_{\frac{\varepsilon}{2k}}^{\frac{1}{2k}} \ln M_{p_k}(x,y,u)du$$

and

$$\int_\varepsilon^1 \ln M_{p_k}(y,z,\frac{t}{2k})dt = 2k\int_{\frac{\varepsilon}{2k}}^{\frac{1}{2k}} \ln M_{p_k}(y,z,u)du.$$

On the other hand, since $\hat{p}_k(y,z) \geq \hat{p}_k(y,y)$ by (PK1), we have

$$\begin{aligned}
\hat{p}_k(x,z) &= -\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1} \ln M_{p_k}(x,z,u)du \\
&\leq -2k\lim_{\varepsilon\to 0^+}\int_{\frac{\varepsilon}{2k}}^{\frac{1}{2k}} \ln M_{p_k}(x,y,u)du - 2k\lim_{\varepsilon\to 0^+}\int_{\frac{\varepsilon}{2k}}^{\frac{1}{2k}} \ln M_{p_k}(y,z,u)du \\
&\leq -2k\lim_{\varepsilon\to 0^+}\int_{\frac{\varepsilon}{2k}}^{1} \ln M_{p_k}(x,y,u)du - 2k\lim_{\varepsilon\to 0^+}\int_{\frac{\varepsilon}{2k}}^{1} \ln M_{p_k}(y,z,u)du \\
&= 2k[\hat{p}_k(x,y) + \hat{p}_k(y,z)] \leq 3k[\hat{p}_k(x,y) + \hat{p}_k(y,z)] - \hat{p}_k(y,y),
\end{aligned}$$

for all $x, y, z \in X$, $u > 0$. Hence, $\hat{p}_k$ is a partial k-metric with a coefficient $3k$. □

Lemma 2.8. *Let X be a nonempty set and $(X, M_{p_k}, *)$ be a partial fuzzy k-metric space with a coefficient $k \geq 1$. If $M_{p_k}(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$, then $x = y$. But the converse may not be true.*

Proof. By (PFK2), we have $M_{p_k}(x,x,t) \geq M_{p_k}(x,y,t)$ and $M_{p_k}(y,y,t) \geq M_{p_k}(x,y,t)$. Suppose that $M_{p_k}(x,y,t) = 1$. It follows that $M_{p_k}(x,x,t) \geq 1$ and $M_{p_k}(y,y,t) \geq 1$, we can deduce that $M_{p_k}(x,x,t) = M_{p_k}(y,y,t) = 1$. Namely, $M_{p_k}(x,x,t) = M_{p_k}(y,y,t) = M_{p_k}(x,y,t)$. Thus, we have $x = y$ by (PFK5).

In addition, from Example 2.5, if $x = y$, then we have $M_{p_k}(x,x,t) = \frac{t}{t+p(x,x)}$. Since the distance of a point to itself may not be zero in partial metric spaces, so $M_{p_k}(x,x,t)$ may not be 1. □

3. Topological structures of partial fuzzy k-pseudo-metric spaces

In this section we begin by giving some basic notions that will be used in the following. First, we know that each fuzzy metric M on X generates a topology $\mathcal{T}_M$ on X with the basis $\mathcal{B} = \{B(x,r,t) : x \in X, 0 < r < 1, t > 0\}$, where the open ball $B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$ for all $0 < r < 1$ and $t > 0$. Also, we call that $\mathcal{T}_M$ is induced by the fuzzy metric M (see more details in [4]).

Theorem 3.1. *Let X be a nonempty set and $(X, M_{p_k}, *)$ be a partial fuzzy k-metric space with a coefficient $k \geq 1$. For any $x \in X$, $0 < r < 1$ and $t > 0$, we define the open ball as follows:*

$$B(x,r,t) = \{y \in X : M_{p_k}(x,y,t) > 1 - r\}.$$

Then $\mathcal{T}_{M_{p_k}} = \{V \subset X : \text{for each } x \in V, \text{there exist } 0 < r < 1, t > 0 \text{ such that } B(x,r,t) \subset V\}$ is a topology on X.

Proof. It is similar to the proof of Theorem 2.1 [16]. □

Furthermore, we can define another type topological structure on X as follows: $\mathcal{S}_{M_{p_k}} = \{V \subset X : \text{for each } i \in I, \text{there exist } B_{p_k}(x_i, r_i, t_i) \text{ such that } V = \bigcup_{i\in I} B_{p_k}(x_i, r_i, t_i)\}$. Then we can deduce that $\mathcal{S}_{M_{p_k}}$ is a suprartopology (see more details in [12]).

Theorem 3.2. *Let X be a nonempty set and $(X, M_{p_k}, \wedge)$ be a partial fuzzy k-pseudo-metric space with a coefficient $k \geq 1$. Define a function $d_\alpha : X \times X \to [0, +\infty)$ as follows:*

$$d_\alpha(x,y) = \bigwedge\{t > 0 : M_{p_k}(x,y,t) \geq \alpha\}, \forall x, y, \in X.$$

Then the following statements hold:

(1) $\{d_\alpha : \alpha \in (0,1)\}$ *is non-increasing with respect to* α.

(2) If $d_\alpha(x,y) > t$*, then* $M_{P_k}(x,y,t) < \alpha$.

(3) $\{d_\alpha : \alpha \in (0,1)\}$ *is a k-pseudo-metric family on X.*

Proof. By the Definition of d_α, it is not difficult to prove (1) and (2).

(3) We verify the conditions (PK1), (PK2) and (KP3) step by step.

(PK1): By (PFK2), we have $M_{p_k}(x,x,t) \geq M_{p_k}(x,y,t)$ for all $x,y \in X$. Then $\{t > 0 : M_{p_k}(x,x,t) \geq \alpha\} \subset \{t > 0 : M_{p_k}(x,y,t) \geq \alpha\}$, which implies that $\bigwedge\{t > 0 : M_{p_k}(x,x,t) \geq \alpha\} \geq \bigwedge\{t > 0 : M_{p_k}(x,y,t) \geq \alpha\}$. Namely, $d_\alpha(x,x) \geq d_\alpha(x,y)$.

(PK2): It is trivial by (PFK3).

(KP3): Since $M_{p_k}(x,y,t) \wedge M_{p_k}(y,z,s) \leq M_{p_k}(x,z,k(t+s))$ by (PFK5), we have

$$\begin{aligned}
&k[d_\alpha(x,y) + d_\alpha(y,z)] \\
&= k[\bigwedge\{t > 0 : M_{p_k}(x,y,t) \geq \alpha\} + \bigwedge\{s > 0 : M_{p_k}(y,z,s) \geq \alpha\}] \\
&\geq k\{\bigwedge(t+s) > 0 : M_{p_k}(x,y,t) \geq \alpha, M_{p_k}(y,z,s) \geq \alpha\} \\
&\geq \bigwedge\{k(t+s) > 0 : M_{p_k}(x,y,t) \wedge M_{p_k}(y,z,s) \geq \alpha\} \\
&\geq \bigwedge\{k(t+s) > 0 : M_{p_k}(x,z,k(t+s)) \geq \alpha\} = d_\alpha(x,z),
\end{aligned}$$

for all $x,y,z \in X$, $t,s > 0$. □

Theorem 3.3. *Let X be a nonempty set and* (X, d_α) *be a generating space of k-pseudo-metric family for all* $\alpha \in (0,1)$ *and some number* $k \geq 1$*, where* $\{d_\alpha : \alpha \in (0,1)\}$ *is a family of mapping from* $X \times X \to [0,+\infty)$ *and* (X, d_α) *satisfies the following conditions:* $\forall x,y,z \in X$ *and for any* $\alpha, \beta \in (0,1)$*,*

(GPKP1) $d_\alpha(x,x) \leq d_\alpha(x,y)$;

(GPKP2) $d_\alpha(x,y) = d_\alpha(y,x)$;

(GPKP3) $d_{\alpha\wedge\beta}(x,z) \leq k[d_\alpha(x,y) + d_\beta(y,z)]$;

(GPKP4) $d_\alpha(x,y)$ *non-increasing with respect to* α.

Define a function $M_D : X \times X \times [0,\infty) \to [0,1]$ *as follows:*

$$M_D(x,y,t) = \begin{cases} 0, & t = 0; \\ \bigvee\{\alpha \in (0,1) : d_\alpha(x,y) < t\}, & t > 0. \end{cases}$$

Then $(X, M_D, \wedge)$ *is a partial fuzzy k-pseudo-metric space with a coefficient* $k \geq 1$.

Proof. We verify the conditions (PFK1)–(PFK3), (PFK5) and (PFK6) step by step.

(PFK1): It is clear that $M_D(x,y,0) = 0$.

(PFK2): By (GPKP1), we have $d_\alpha(x,x) \leq d_\alpha(x,y)$. Then

$$\{\alpha \in (0,1) : d_\alpha(x,x) < t\} \supset \{\alpha \in (0,1) : d_\alpha(x,y) < t\},$$

which implies that $\bigvee\{\alpha \in (0,1) : d_\alpha(x,x) < t\} \geq \bigvee\{\alpha \in (0,1) : d_\alpha(x,y) < t\}$. Namely, $M_D(x,x,t) \geq M_D(x,y,t)$.

(PFK3): By (GPKP2), it is easy to show $M_D(x,y,t) = M_D(y,x,t)$.

(PFK5): We prove $M_D(x,z,k(t+s)) \geq M_D(x,y,t) \wedge M_D(y,z,s)$ as follows:

Case 1: If $M_D(x,y,t) = 0$ or $M_D(y,z,s) = 0$ for all $t, s > 0$, then $M_D(x,y,t) \wedge M_D(y,z,s) = 0$. It is easy to see that the above relation holds.

Case 2: Suppose that $M_D(x,y,t) \wedge M_D(y,z,s) > 0$ for all $s,t > 0$, i.e., $M_D(x,y,t) > 0$ and $M_D(y,z,s) > 0$. Set $M_D(x,y,t) = \beta$ and $M_D(y,z,s) = \gamma$. For any $\varepsilon > 0$, where $\varepsilon < \beta \wedge \gamma$. Then there exist $\alpha_1, \alpha_2 \in (0,1)$, such that $\alpha_1 > \beta - \varepsilon$, $\alpha_2 > \gamma - \varepsilon$, and $d_{\alpha_1}(x,y) < t$, $d_{\alpha_2}(y,z) < s$. By (GPKP4), we have $d_{\beta-\varepsilon}(x,y) < t$, $d_{\gamma-\varepsilon}(y,z) < s$. Furthermore, by (GPKP3), we can deduce

$$d_{(\beta-\varepsilon)\wedge(\gamma-\varepsilon)}(x,z) \leq k[d_{\beta-\varepsilon}(x,y) + d_{\gamma-\varepsilon}(y,z)] < k(t+s).$$

Therefore, it follows that $M_D(x,z,k(t+s)) \geq (\beta-\varepsilon) \wedge (\gamma-\varepsilon)$. From the arbitrariness of α,β and the continuity of $\wedge$, it follows that $M_D(x,z,k(t+s)) \geq \beta \wedge \gamma$, namely, $M_D(x,z,k(t+s)) \geq M_D(x,y,t) \wedge M_D(y,z,s)$.

(PFK6): For any $\varepsilon > 0$, we have $M_D(x,y,t) - \varepsilon < M_D(x,y,t)$. Then there exists $\alpha_0 \in (0,1)$, such that $d_{\alpha_0}(x,y) < t$ and $M_D(x,y,t) - \varepsilon < \alpha_0$. Furthermore, we have $M_D(x,y,t_0) \geq \alpha_0$ whenever $d_{\alpha_0}(x,y) < t_0 < t$. Therefore, it follows that $M_D(x,y,t) - M_D(x,y,t_0) \leq M_D(x,y,t) - \alpha_0 < \varepsilon$. Hence, $M_D(x,y,\cdot)$ is left-continuous. □

4. Fixed point theorem on partial fuzzy k-metric spaces

In this section, we investigate fixed point theorems for a self-mappings in partial fuzzy k-metric spaces, following the method given by Shen Yonghong et al. [20].

Definition 4.1. Let X be a nonempty set and $(X, M_{p_k}, *)$ be a partial fuzzy k-metric space with a coefficient $k \geq 1$.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}^+}$ in $(X, M_{p_k}, *)$ *converges to a point* $x \in X$ if for any $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}^+$ such that $M_{p_k}(x_n, x, t) > 1 - \varepsilon$ for all $n > n_0$ (or equivalently for any open ball $B(x,r,t)$, there exists $n_0 \in \mathbb{N}^+$ such that $x_n \in B(x,r,t)$ for all $n \geq n_0$), we denote $\lim_{n\to+\infty} x_n = x$.

(2) A sequence $\{x_n\}_{n\in\mathbb{N}^+}$ is called a *Cauchy sequence* if for any $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}^+$ such that $M_{p_k}(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

(3) $(X, M_{p_k}, *)$ is said to be *complete* if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}^+}$ in X converges to a point $x \in X$.

Indeed, we can give another definition type of sequence convergence as follows: a sequence $\{x_n\}$ in $(X, M_{p_k}, *)$ converges to a point $x \in X$ if for any open set V containing x there exists $n_0 \in \mathbb{N}^+$ such that $x_n \in V$ for all $n \geq n_0$ in $\mathcal{S}_{M_{p_k}}$, we denote $\mathrm{Lim}_{n\to+\infty} x_n = x$.

Theorem 4.2. *Let X be a nonempty set, $(X, M_{p_k}, *)$ be a partial fuzzy k-metric space with a coefficient $k \geq 1$ and $\{x_n\}_{n\in\mathbb{N}^+}$ be a sequence in X. Then $\lim_{n\to+\infty} x_n = x$ if and only if $\lim_{n\to+\infty} M_{p_k}(x_n, x, t) = 1$ for all $t > 0$.*

Proof. ($\Rightarrow$) Suppose that $\lim_{n\to+\infty} x_n = x$. Then for any open ball $B(x,r,t)$, there exists $n_0 \in \mathbb{N}^+$ such that $x_n \in B(x,r,t)$ for all $n > n_0$. Thus $M_{p_k}(x_n, x, t) > 1 - r$ for all $n > n_0$ and $t, r > 0$, namely, $1 - M_{p_k}(x_n, x, t) < r$. Hence $\lim_{n\to+\infty} M_{p_k}(x_n, x, t) = 1$.

($\Leftarrow$) Suppose that $\lim_{n\to+\infty} M_{p_k}(x_n, x, t) = 1$. Then for each $t > 0$, there exists $n_0 \in \mathbb{N}^+$ such that $1 - M_{p_k}(x_n, x, t) < r$ for all $n \geq n_0$. Namely, $M_{p_k}(x_n, x, t) > 1 - r$ for all $n > n_0$. Therefore, $x_n \in B(x,r,t)$ for all $n > n_0$. Thus $\lim_{n\to+\infty} x_n = x$. □

Corollary 4.3. *Let X be a nonempty set, $(X, M_{p_k}, *)$ be a partial fuzzy k-metric space with a coefficient $k \geq 1$ and $\{x_n\}_{n\in\mathbb{N}^+}$ be a sequence in X. If $\lim_{n\to+\infty} M_{p_k}(x_n, x, t) = 1$ for all $t > 0$, then $\mathrm{Lim}_{n\to+\infty}x_n = \lim_{n\to+\infty} x_n$.*

Proof. Indeed, by Definition 4.1 and Theorem 3.1, it is not difficult to see that $\mathrm{Lim}_{n\to+\infty}x_n = x$ if $\lim_{n\to+\infty} x_n = x$. Then it is trivial by Theorem 4.2. □

Theorem 4.4. *Let X be a nonempty set, $(X, M_{p_k}, *)$ be a partial fuzzy k-metric space with a coefficient $k \geq 1$ and $\{x_n\}_{n\in\mathbb{N}^+}$ be a sequence in X. If $\{x_n\}$ is convergent, then it is a Cauchy sequence.*

Proof. Suppose that $\{x_n\}$ is convergent. By Definition 4.1 and Theorem 4.2, there exists $n_0 \in \mathbb{N}^+$ such that $M_{p_k}(x_n, x, t) > 1 - r$ for all $n \geq n_0$, $t > 0$ and some number $k \geq 1$. Set $s = \frac{t}{2k}$. By (PFK5), we have

$$\begin{aligned} M_{p_k}(x_n, x_m, t) &= M_{p_k}(x_n, x_m, k(\frac{t}{2k} + \frac{t}{2k})) \\ &\geq M_{p_k}(x_n, x, \frac{t}{2k}) * M_{p_k}(x, x_m \frac{t}{2k}), \end{aligned}$$

for all $n, m \geq n_0$. Furthermore, since $0 < \frac{t}{2k} < t$, we have $M_{p_k}(x, x_m \frac{t}{2k}) > 1 - r$. Set $r_0 = M_{p_k}(x, x_m \frac{t}{2k})$. Then $r_0 > 1 - r$. By continuity of the t-norm, we can find some $s > 0$, such that $r_0 > 1 - s > 1 - r$. Thus, there exists $0 < r_1 < 1$, such that $r_1 * r_1 \geq 1 - s$, from which, we can deduce that $M_{p_k}(x_n, x_m, t) \geq r_1 * r_1 \geq 1 - s > 1 - r$, for all $n, m \geq n_0$, $t > 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. □

Theorem 4.5. *Let X be a nonempty set and $(X, M_{p_k}, *)$ be a complete partial fuzzy k-metric space with a coefficient $k \geq 1$, and let $T : X \to X$ be a function satisfying the following conditions:*

(1) $\varphi(M_{p_k}(Tx, Ty, t)) \leq \lambda\varphi(M_{p_k}(x, y, t))$ for all $x, y \in X$ and $x \neq y$, where $t > 0, \lambda \in (0, 1)$, and $\varphi : [0, 1] \to [0, 1]$ is a strictly decreasing and continuous mapping;

(2) $\varphi(M_{p_k}(x, y, t)) = 0$ if and only if $M_{p_k}(x, y, t) = 1$.

Then T has a unique fixed point.

Proof. We define a sequence in the following way : $x_0 = x$, and $x_{n+1} = Tx_n$, $f_n(t) = M_{p_k}(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}^+$, $t > 0, x \in X$, and some number $k \geq 1$.

Case 1: If $x_{n+1} = x_n$ for some $n \in \mathbb{N}^+$, then we have $Tx_n = x_n$, which shows that x_n is a fixed point.

Case 2: If $x_{n+1} \neq x_n$, then we have $\varphi(f_n(t)) = \varphi(M_{p_k}(x_n, x_{n+1}, t)) = \varphi(M_{p_k}(T_{x_{n-1}}, T_{x_n}, t))$. Since $\varphi(M_{p_k}(T_{x_{n-1}}, T_{x_n}, t)) \leq \lambda\varphi(M_{p_k}(x_{n-1}, x_n, t)) = \varphi(f_{n-1}(t))$ by the condition (1), this follows that $\varphi(f_n(t)) < \varphi(f_{n-1}(t))$ for all $n \in \mathbb{N}^+$ and $t > 0$. By assumption, φ is strictly decreasing, which implies that $\{f_n(t)\}$ is an increasing sequence with respect to n for all $t > 0$. Furthermore, since $0 \leq f_n(t) \leq 1$ for all $t > 0$, $\{f_n(t)\}$ is bounded. Therefore, $\{f_n(t)\}$ is convergent. We denote $\lim_{n\to+\infty} f_n(t) = f(t)$. Namely, there exists $n_0 \in \mathbb{N}^+$, such that $f_n(t) \leq f(t)$ for all $n \geq n_0$ and $t > 0$. On the other hand, for all $n \in \mathbb{N}^+$ and $t > 0$, we have $\varphi(f_{n+1}(t)) \leq \lambda\varphi(f_n(t))$. It follows that $\lim_{n\to+\infty} \varphi(f_{n+1}(t)) \leq \lim_{n\to+\infty} \lambda\varphi(f_n(t))$, which implies that $\varphi(f(t)) \leq \lambda\varphi(f(t))$. Thus $(1 - \lambda)\varphi(f(t)) \leq 0$. So $\varphi(f(t)) = 0$ for all $t > 0$. Namely, $\lim_{n\to+\infty} M_{p_k}(x_n, x_{n+1}, t) = 1$.

To prove the existence and uniqueness of the fixed point, we consider the following steps:

Step 1: We claim that $\{x_n\}$ is a Cauchy sequence in $(X, M_{P_k}, *)$. Otherwise, for some $0 < \varepsilon < 1$, we can find two sequences $\{i_n\}$ and $\{j_n\}$, such that $M_{p_k}(x_{i_n}, x_{j_n}, t) \leq 1 - \varepsilon$, $M_{p_k}(x_{i_n-1}, x_{j_n-1}, t) > 1 - \varepsilon$ and $M_{p_k}(x_{i_n-1}, x_{j_n}, t) > 1 - \varepsilon$ for all $n \in \mathbb{N}^+$ and $t > 0$, where $i_n > j_n \geq n$. Set $g_{(i_n,j_n)}(t) = M_{p_k}(x_{i_n}, x_{j_n}, t)$. It is not difficult to show that $g_{(i_n,j_n)}(t) \leq 1 - \varepsilon$ by (PFK5). Moreover, we have

$\varphi(g_{(i_n,j_n)}(t)) \leq \lambda\varphi(g_{(i_n-1,j_n-1)}(t)) < \varphi(g_{(i_n-1,j_n-1)}(t))$. Thus, $g_{(i_n-1,j_n-1)}(t) < g_{(i_n,j_n)}(t)$ by the monotonicity of φ, it follows that

$$1 - \varepsilon < g_{(i_n-1,j_n-1)}(t) < g_{(i_n,j_n)}(t) \leq 1 - \varepsilon,$$

which is a contradiction. In addition, suppose for some $n_0 \in \mathbb{N}^+$, any $p \in \mathbb{N}^+$ and $t > 0$. We can deduce that $\{f_{n_0+p}(t)\}$ is convergent by the monotonicity of f. We denote $\lim_{p\to+\infty} f_{n_0+p}(t) = f_{n_0}(t)$. By assumption, we have $\varphi(f_{n_0+p}(t)) \leq \lambda\varphi(f_{n_0+p-1}(t))$. By repeating the above process, it follows that $\varphi(f_{n_0+p}(t)) \leq (\lambda)^p f_{n_0}(t)$. Thus, we can deduce $\varphi(f_{n_0}(t)) = 0$. Namely, $\varphi(M_{p_k}(x_{n_0}, x_{n_0+1}, t) = 0$. By the condition (2), we have $M_{p_k}(x_{n_0}, x_{n_{0+1}}, t) = 1$. It is clear to see that $x_{n_0} = x_{n_0+1}$ by Lemma 2.8, which is a contradiction.

Therefore, $\{x_n\}$ is a Cauchy sequence in $(X, M_{P_k}, *)$. By the completeness of $(X, M_{P_k}, *)$, there exists a point $x^* \in X$, such that $\lim_{n\to+\infty} x_n = x^*$.

Step 2: By Step 1, there exists a subsequence $\{x_{n_k}\}$, where $x_{n_k} \neq x_n$ for all $n \in \mathbb{N}^+$. Then we have

$$\varphi(M_{p_k}(x_{n_k+1}, Tx^*, t)) = \varphi(M_{p_k}(Tx_{n_k}, Tx^*, t)) < \lambda\varphi(M_{p_k}(x_{n_k}, x^*, t)),$$

for all $n_k \in \mathbb{N}^+$, $t > 0$. We can deduce that $\varphi(M_{p_k}(x^*, Tx^*, t)) = 0$. By the condition (2), we have $M_{p_k}(x^*, Tx^*, t) = 1$. It is clear to see $x^* = Tx^*$ by Lemma 2.8.

Step 3: Suppose that $x^* \neq y^*$, where $Ty^* = y^*$. We have $\varphi(M_{p_k}(x^*, y^*, t)) = \varphi(M_{p_k}(Tx^*, Ty^*, t)) \leq \lambda\varphi(M_{p_k}(x^*, y^*, t)) < \varphi(M_{p_k}(x^*, y^*, t))$, which is a contradiction. Hence, $x^* = y^*$. □

To conclude this section, we illustrate our result by the following examples.

Example 4.6. Let $X = \{1, 2, 3, \cdots\}$. Define a fuzzy set set on $X \times X \times [0, +\infty)$ by $M_{p_k}(x, y, t) = \frac{x\wedge y}{x\vee y}$ for all $x, y \in X$, $t > 0$, and $M_{p_k} = 0$ when $t = 0$. It is trivial to verify that $(X, M_{p_k}, *)$ is a partial fuzzy k-metric space with $a * b = ab$ for all $a, b \in [0, 1]$. Define mappings $T : [0, +\infty) \to [0, +\infty)$, $\varphi : [0, 1] \to [0, 1]$, respectively, where $T(v) = \sqrt{v}$ and $\varphi(u) = 1 - \sqrt{u}$. Set $\lambda(t) = \frac{1}{1+t^2}$ for all $t > 0$. Thus, all the conditions of Theorem 4.5 are satisfied and obviously $x = 1$ is a fixed point of T.

Now, similar to the basic form of the functional equations in dynamic programming, which investigated by Bellman and Lee [31], the existence and uniqueness of solution and common solution for a functional equation and system of functional equations are discussed by using Theorem 4.5 as follows.

Let X and Y be Banach spaces, $S \subset X$ be the state space and $D \subset Y$ be the decision space. $B(S)$ denotes the set of all real-valued bounded functions on S. Define $u : S\times D \to \mathbb{R}$, $T : S\times D \to S$, $H : S\times D\times\mathbb{R} \to \mathbb{R}$. Moreover, define a fuzzy set set on $B(S)\times B(S)\times[0, +\infty)$ as follows: $M_{p_k}(x, y, t) = e^{-\frac{d(h,k)}{t}}$, for all $h, k \in B(S), t > 0$ and $M_{p_k}(x, y, t) = 0$ when $t = 0$, where $d(h, k) = \bigvee_{x\in S} |h(x) - k(x)|, \forall a, b, \in S$, $a * b = a \cdot b$, $\forall a, b \in [0, 1]$. It is obvious that $(X, M_{p_k}, *)$ is a complete partial fuzzy k-metric space by Example 2.4. Suppose that the following conditions hold: $\varphi(e^{-\frac{|H(x,y,g(\xi))-H(x,y,h(\xi))|}{t}}) \leq \lambda\varphi(e^{-\frac{|g(\xi)-h(\xi)|}{t}})$, for $t > 0, \lambda \in (0, 1)$, where $\varphi : [0, 1] \to [0, 1]$ is a strictly decreasing and continuous mapping. In fact, set the system of functional equations $Ag(x) = opt_{y\in D}\{u(x, y) + H(x, y, g(T(x, y))\}$, for all $x \in S, g \in B(S)$, where opt represents $\wedge$ or $\vee$. For all $g, h \in B(S), x \in S$, there exist $y, z \in D$ such that $d(Ag, Ah) \leq \max\{|H(x, y, g(T(x, y))) - H(x, y, h(T(x, y)))|, |H(x, z, g(T(x, z))) - H(x, z, h(T(x, z)))|\}$. It implies that $\varphi(e^{-\frac{|Ag(\xi)-Ah(\xi)|}{t}}) \leq \varphi(e^{-\frac{|H(x,y,g(\xi))-H(x,y,h(\xi))|}{t}})$. By the above condition, we have that $\varphi(e^{-\frac{d(Ag,Ah)}{t}}) \leq \lambda\varphi(e^{-\frac{d(g,h)}{t}})$. Hence, $\varphi(M_{p_k}(Ag, Ah, t)) \leq \lambda\varphi(M_{p_k}(g, h, t))$. Thus, Theorem 4.5 ensures that A has a unique common fixed pointed $w \in B(S)$. That is, the system of functional equations $q(x) = opt_{y\in D}\{u(x, y) + H(x, y, g(T(x, y))\}$ possesses a unique common solution $w \in B(S)$.

5. Conclusions

In this paper, by introducing the notion of weak partial-quasi k-metric spaces, we generalized and unified weak partial metric spaces and partial k-metric spaces. Moreover, we provided some examples of weak partial-quasi k-metric spaces, and illustrated the relationships between weak partial-quasi k-metric spaces and weak partial metric spaces. Additionally, another purpose of this paper to obtain the constitution of k-metric in weak partial-quasi k-metric spaces. In Section 4, we discussed the existence of fixed point on partial fuzzy k-metric spaces, and presented application of the revealed fixed point theorems.

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Conflict of interest

The authors declare that they have no competing interest.

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