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Research article

On Φ-powerful submodules and Φ-strongly prime submodules

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Abstract: Let R be a commutative ring with identity and N be a submodule of an R-module M. We say a nonnil submodule N of an R-module M is a Φ-powerful (resp., Φ-strongly prime) submodule, if $\Phi(N)$ is a powerful (resp., strongly prime) submodule of a module $\Phi(M)$. We show that a nonnil prime submodule N of an R-module M is a Φ-powerful submodule if and only if it is a Φ-strongly prime submodule. Similarly, if every prime submodule of an R-module M is a Φ-strongly prime, then we call it a Φ-pseudo-valuation module (Φ-PVM). We also prove that a faithful multiplication R-module M is Φ-PVM if and only if some maximal nonnil submodules of M are Φ-powerful. In this perspective, we analyze that M is Φ-PVM if and only if R is a PVD. In due course, we provide some characterizations of these submodules along with their relationships under certain conditions.

Keywords: powerful submodule; strongly prime submodule; Φ-powerful submodule; Φ-strongly prime submodule; Φ-pseudo-valuation module
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1. Introduction

A prime ideal P of a commutative ring R with identity is said to be divided if P is comparable to every principal ideal of R. Any two ideals I and J of a commutative ring R are said to be comparable if $I \subseteq J$ or $J \subseteq I$. If every prime ideal of R is divided then R is said to be a divided ring. Similarly, $Nil(R)$ is said to be divided if it is comparable to every principal ideal of R. If $Nil(R)$ is both divided and prime ideal then R is said to be a ϕ-ring. The name ϕ-ring is derived from the natural map ϕ from a total quotient ring $T(R)$ to a ring R localized at $Nil(R)$. In this perspective, ϕ-Prüfer, ϕ-Bezout, ϕ-Dedekind, ϕ-Krull, ϕ-pseudo-valuation and ϕ-Mori rings have been introduced [2, 3, 5, 6]. We will denote the class of all ϕ-rings by $\mathcal{H}$. Recently, many terms have been shifted from ring theory to module theory such as Dedekind rings towards Dedekind modules [17], Prüfer rings towards Prüfer modules [4] etc. Further to this, Motmaen and Darani [16] generalized the notions of Dedekind modules, Prüfer modules, Bezout modules and valuation modules towards Φ-Dedekind modules, Φ-Prüfer modules, Φ-Bezout modules and Φ-valuation modules, respectively. These all motivate us to generalize powerful submodules and strongly prime submodules towards Φ-powerful submodules and Φ-strongly prime submodules. If $Nil(M)$ is both divided and prime submodule then we call M as Φ-module. Throughout we will denote the class of all Φ-modules by $\mathbb{H}$.

In our work, we introduce and discuss the terms Φ-powerful submodules, Φ-strongly prime submodules and Φ-pseudo-valuation modules. During our study, we also shift some results which have been discussed in the literature about powerful submodules and strongly prime submodules towards Φ-powerful submodules and Φ-strongly prime submodules. In addition, we provide numerous characterizations and relationships of our newly established submodules. A nonnil submodule N of a module $M \in \mathbb{H}$ is said to be a Φ-powerful submodule if $\Phi(N)$ is powerful submodule of module $\Phi(M)$. In other words, a submodule N of $M \in \mathbb{H}$ is a Φ-powerful submodule if $ab\Phi(M) \subseteq \Phi(N)$ implies $a \in R$ or $b \in R$, where $a, b \in K$. We also introduce the concepts of Φ-strongly prime submodules and Φ-pseudo-valuation modules along with few of their characterizations. A nonnil submodule N of a module M is a Φ-strongly prime submodule if $\Phi(N)$ is strongly prime submodule of $\Phi(M)$. Similarly, a nonnil module M is said to be a Φ-pseudo-valuation module if each one of its submodules is a Φ-strongly prime submodule. We also provide some generalizations of strongly prime submodules and pseudo-valuation modules in the setting of Φ-strongly prime submodules and Φ-pseudo-valuation modules.

Throughout the paper, by R we mean a commutative ring with identity unless otherwise specified.

2. Preliminaries

In this section, we provide some useful terminologies, discussions and results.

Definition 2.1. *[6] A commutative ring R is said to be a ϕ-ring if its nilradical $Nil(R)$ is both prime and comparable with each principal ideal.*

Definition 2.2. *[15] A proper ideal I of R is said to be ϕ-prime if for $a, b \in R$ with $ab \in I \backslash \phi(I)$, $a \in I$ or $b \in I$.*

Definition 2.3. *[11] Let R be a domain with quotient field K. A prime ideal P of R is called strongly prime ideal, if for $a, b \in K$, $ab \in P$ implies either $a \in P$ or $b \in P$.*

Definition 2.4. *[11] A domain R is called a pseudo-valuation domain if every ideal of R is strongly prime.*

Definition 2.5. *[9] An ideal I of a ring R is said to be a powerful ideal if, whenever $x, y \in T(R)$ (quotient ring of R), $xy \in I$, then $x \in R$ or $y \in R$.*

Definition 2.6. *[14] A submodule N of M is called prime if $N \neq M$ and for an arbitrary $r \in R$ and $m \in M$, $rm \in N$ implies $m \in N$ or $r \in (N :_R M)$, where $(N :_R M) = \{r \in R \mid rM \subseteq N\}$.*

Definition 2.7. *[16] A prime submodule P of M is said to be a divided prime if $P \subseteq (x)$ for every $x \in M \backslash P$. Thus a divided prime submodule is comparable to every submodule of M.*

Definition 2.8. *[10] A nonzero element m of an R-module M is said to be a nilpotent element of degree k if there exist "a"$\in R$ and "k"$\in N$ such that $a^k m = 0$ and $am \neq 0$. We take every zero element of a module to be nilpotent.*

Theorem 2.9. *[1] Let R be a ring and M a faithful R-module. Then*
(1) $Nil(M)$ is a submodule of M.
(2) Assuming M is multplication then $Nil(M) = Nil(R)M = \bigcap_{Q prime} Q$.

Definition 2.10. *[16] A submodule N of a module M is said to be nonnil submodule if $N \nsubseteq Nil(M)$.*

Definition 2.11. *[16] An element $r \in R$ is said to be a zero-divisor on M if $rm = 0$ for some nonzero element $m \in M$.*

The set of all the zero-divisors of a module M denoted by $Z(M)$ need not be an ideal of ring R. However, $Z(M)$ has a unique characteristic i.e., for $x, y \in R$ and $xy \in Z(M)$ implies $x \in Z(M)$ or $y \in Z(M)$ [16].

Definition 2.12. *[12] Let R be an integral domain with quotient field K and M be an R-module. We define a non-zero submodule N of M to be powerful if, $xyM \subseteq N$, for elements $x, y \in K$, we have $x \in R$ or $y \in R$.*

Definition 2.13. *[12] A submodule N of a module M is referred as a strongly prime submodule of M, if $xyM \subseteq N$ for $x, y \in K$ implies $xM \subseteq N$ or $y^n M \subseteq N$ for $n \geq 1$.*

Recall from [14], for any integral domain R with its quotient field K and M being a torsion-free R-module. A prime submodule N of a module M is said to be a strongly prime submodule, if for any $y \in K$ and $x \in M_S$, $yx \in N$ gives $x \in N$ or $y \in (N : M)$, where $M_S = \{\frac{a}{t} \mid a \in M, t \in S\}$ and $S = R \setminus \{0\}$.

Lemma 2.14. *[14] Let P be a strongly prime submodule of M, then $(P : M)$ is a strongly prime ideal of R.*

Definition 2.15. *[14] An R-module M is termed pseudo-valuation module if each of its prime submodule is a strongly prime submodule.*

Let M be an R-module and $T = (R \setminus Z(M)) \cap S$ then $T \subseteq S$, in general. However, if M is a torsion-free module then $T = S$. Specifically, if M is a faithful multiplication R-module then T and S coincide [16]. Let $\xi(M) = T^{-1}M$, where M is an R-module with $M \in \mathbb{H}$ and $P = (Nil(M) :_R M)$ then the mapping $\Phi : \xi(M) \to M_P$ defined by $\Phi(x/s) = x/s$ is a module homomorphism.

Proposition 2.16. *[16] Let R be a ring and let M be a finitely generated faithful multiplication R-module. Let $M \in \mathbb{H}$. Then the following assertions hold.*
(i) $Nil(M_P) = \Phi(Nil(M)) = Nil(\Phi(M))$.
(ii) $Nil(\xi(M)) = Nil(M)$.
(iii) $\Phi(M) \in \mathbb{H}$.

3. Φ-powerful submodules

In this section, we introduce and discuss the concept of Φ-powerful submodules which is closely related to the class of powerful submodules.

Definition 3.1. *A nonnil submodule N of an R-module $M \in \mathbb{H}$ is said to be a Φ-powerful submodule of M if $\Phi(N)$ is a powerful submodule of $\Phi(M)$.*

We may say that a nonnil submodule N of an R-module $M \in \mathbb{H}$ is a Φ-powerful submodule of M, if $ab\Phi(M) \subseteq \Phi(N)$ then $a \in R$ or $b \in R$, where $a, b \in K$.

Lemma 3.2. *Let N be a submodule of an R-module M. Then N is a powerful submodule of M if and only if $N/Nil(M)$ is a powerful submodule of $M/Nil(M)$.*

Proof. Let N be a powerful submodule of an R-module M. Let $x, y \in T(R)$ and $xyM/Nil(M) \subseteq N/Nil(M)$. It implies that $xyM \subseteq N$. Since N is a powerful submodule, we have $x \in R$ or $y \in R$ and hence $N/Nil(M)$ is a powerful submodule of $M/Nil(M)$.

Conversely, suppose $xyM \subseteq N$ and let $N/Nil(M)$ be a powerful submodule of $M/Nil(M)$. Then by [12, Corollary 2.19] $\sqrt{N/Nil(M) : M/Nil(M)}$ is a powerful ideal. Since $(N/Nil(M) : M/Nil(M)) \cong (N : M)$, we have that $\sqrt{N : M}$ is a powerful ideal. Since $xyM \subseteq N$ implies $xy \in (N : M)$. Since $I \subseteq rad(I)$ for any ideal I implies $xy \in (N : M) \subseteq \sqrt{N : M}$. Finally, as $\sqrt{N : M}$ is a powerful ideal so $x \in R$ or $y \in R$ and hence N is a powerful submodule of M. □

Theorem 3.3. *Let $M \in \mathbb{H}$ be an R-module M and N be its nonnil submodule. Then N is a Φ-powerful submodule if and only if $x^{-1}(\Phi(N) : \Phi(M)) \subseteq R$ for every $x \in K \setminus R$.*

Proof. Let N be a Φ-powerful submodule of an R-module $M \in \mathbb{H}$ and $x \in K \setminus R$. Then, for $a \in (\Phi(N) : \Phi(M))$, we have $xx^{-1}a = a \in (\Phi(N) : \Phi(M))$ it implies $x^{-1}a \in R$ so $x^{-1}(\Phi(N) : \Phi(M)) \subseteq R$.

For the converse, let $yz\Phi(M) \subseteq \Phi(N)$, for $x, y \in K$, where $y \notin R$. Then, $z = y^{-1}yz \in y^{-1}(\Phi(N) : \Phi(M)) \subseteq R \Rightarrow z \in R$. Hence N is a Φ-powerful submodule of M. □

Lemma 3.4. *Let $M \in \mathbb{H}$ be a faithful, finitely generated and multiplication R-module. Let N be a nonnil submodule of M. Then N is a Φ-powerful submodule if and only if $N/Nil(M)$ is a powerful submodule of $M/Nil(M)$.*

Proof. Let N be a Φ-powerful submodule of M. Then, by definition $\Phi(N)$ is a powerful submodule of a module $\Phi(M)$ and by Lemma 3.2, $\Phi(N)/Nil(\Phi(M))$ is a powerful submodule of $\Phi(M)/Nil(\Phi(M))$. Since M is a finitely generated, faithful and multiplication R-module, by [16, Lemma 2.6] $N/Nil(M)$ is a powerful submodule of $M/Nil(M)$. By the similar argument, we can easily prove the converse. □

Theorem 3.5. *Let $M \in \mathbb{H}$ be an R-module and N be its Φ-powerful submodule. Then, if Q is a submodule of N then N/Q is a Φ-powerful submodule of M/Q.*

Proof. Since we know that $(N/Q : M/Q) = Ann((M/Q)/(N/Q)) \cong Ann(M/N) = (N : M)$, and N is a Φ-powerful submodule of M implies N/Q is a Φ-powerful submodule of M/Q. □

Corollary 3.6. *Let $P = (\Phi(N) : \Phi(M))$ be a prime ideal of ring R. Then N/Q is a Φ-powerful R/P submodule of M/Q.*

Lemma 3.7. *Let $M \in \mathbb{H}$ be an R-module and N be its nonnil submodule containing a submodule Q. Let $\Phi(N)$ be a strongly prime submodule of $\Phi(M)$. Then, Q is a Φ-powerful submodule of M.*

Proof. Since $(\Phi(Q) : \Phi(M)) \subseteq (\Phi(N) : \Phi(M)) \subseteq R$. Consequently, $xy \in (\Phi(Q) : \Phi(M))$, for $x, y \in K$ implies $xy \in (\Phi(N) : \Phi(M))$. Since, $\Phi(N)$ is a strongly prime submodule of $\Phi(M)$ implies $x \in (\Phi(N) : \Phi(M))$ or $y \in (\Phi(N) : \Phi(M))$. Thus, $x \in R$ or $y \in R$ and hence $\Phi(Q)$ is a powerful submodule of $\Phi(M)$. Particularly, Q is a Φ-powerful submodule of M. □

Corollary 3.8. *Let N be a Φ-powerful submodule of an R-module $M \in \mathbb{H}$ and $Q \subseteq N$. Then Q is a Φ-powerful submodule of M.*

Proof. Since $(\Phi(Q) : \Phi(M)) \subseteq (\Phi(N) : \Phi(M))$. Thus, $xy \in (\Phi(Q) : \Phi(M))$ implies $xy \in (\Phi(N) : \Phi(M))$, for $x, y \in K$. As N is a Φ-powerful submodule of M implies $x \in R$ or $y \in R$. Thus, Q is a Φ-powerful submodule of M. □

Theorem 3.9. *Let N be a Φ-powerful submodule of an R-module $M \in \mathbb{H}$. Then*
(1) *If K is a nonnil submodule of M, then $(K : M) \subseteq (N : M)$ or $(N : M)^2 \subseteq (K : M)$.*
(2) *If K is a nonnil prime submodule of M, then $(N : M)$ and $(K : M)$ are comparable.*
(3) *The nonnil prime submodules of M contained in $Rad(N)$ are linearly ordered.*

Proof. (1) Let K be a nonnil submodule of M and $(K : M) \nsubseteq (N : M)$. Then $(\Phi(K) : \Phi(M)) \nsubseteq (\Phi(N) : \Phi(M))$. Choose $\Phi(k) \in (\Phi(K) : \Phi(M)) \backslash (\Phi(N) : \Phi(M))$ and suppose that $\Phi(m), \Phi(q) \in (\Phi(N) : \Phi(M))$. Then $\frac{\Phi(m)\Phi(q)}{\Phi(k)} \frac{\Phi(k)}{\Phi(m)} \in (\Phi(N) : \Phi(M))$. Since N is Φ-powerful submodule with $\frac{\Phi(k)}{\Phi(m)} \notin R$, we have $\frac{\Phi(m)\Phi(q)}{\Phi(k)} \in R$. It means $\Phi(m)\Phi(q) \in \Phi(k)R \subseteq (\Phi(K) : \Phi(M))$. Hence $(\Phi(N) : \Phi(M))^2 \subseteq (\Phi(K) : \Phi(M))$ and therefore by [7, Lemma 3.3], $(N : M)^2 \subseteq (K : M)$.
(2) Let K be a prime submodule of M. Then $(K : M)$ is a prime ideal of ring R. By (1), we have $(K : M) \subseteq (N : M)$ or $(N : M)^2 \subseteq (K : M)$. Since $(K : M)$ is a prime ideal of R implies $(K : M) \subseteq (N : M)$ or $(N : M) \subseteq (K : M)$.
(3) Let P and K be two nonnil prime submodules of M properly contained in $Rad(N)$. Then, P and K are contained in N and hence by Corollary 3.8, both are Φ-powerful submodules and by (2), both are also comparable. □

Theorem 3.10. *Let N be a submodule of a finitely generated, faithful and multiplication R-module $M \in \mathbb{H}$. Then, a submodule N is a Φ-powerful submodule of M if and only if $\Phi(N)/Nil(\Phi(M))$ is a powerful submodule of $\Phi(M)/Nil(\Phi(M))$.*

Proof. Let N be a Φ-powerful submodule. Let $x, y \in K$ such that $xy\Phi(M)\ /Nil(\Phi(M)) \subseteq \Phi(N)/Nil(\Phi(M))$, it implies that $xy\Phi(M) \subseteq \Phi(N)$. Since N is a Φ-powerful submodule implies $x \in R$ or $y \in R$. It proves that $\Phi(N)/Nil(\Phi(M))$ is a powerful submodule of $\Phi(M)/Nil(\Phi(M))$.

Conversely, let $\Phi(N)/Nil(\Phi(M))$ be a powerful submodule of $\Phi(M)/Nil(\Phi(M))$. For $x, y \in K$, let $xy\Phi(M) \subseteq \Phi(N)$. Then, for $(x + Nil(\Phi(M))), (y + Nil(\Phi(M))) \in K/Nil(\Phi(M))$, $(x + Nil(\Phi(M)))\ (y + Nil(\Phi(M))) \in (\Phi(N) : \Phi(M))/Nil(\Phi(M))$ implies $(x + Nil(\Phi(M)))\ (y + Nil(\Phi(M)))\Phi(M)/Nil(\Phi(M)) \subseteq \Phi(N)/Nil(\Phi(M))$. Thus, $(x + Nil(\Phi(M))) \in R/Nil(\Phi(M))$ or $(y + Nil(\Phi(M))) \in R\ /\ Nil(\Phi(M))$, so $x \in R$ or $y \in R$ and hence N is a Φ-powerful submodule of M. □

Remark 3.11. *Let R be a ring and K be its ring of fractions. Let N be a powerful submodule of an R-module M. For $x, y \in K$, if $xy \in \sqrt{(N : M)}$ then there exists a positive number n such that $x^n \in (N : M)$ or $y^n \in (N : M)$.*

Theorem 3.12. *Let N be a nonnil Φ-powerful submodule of a module $M \in \mathbb{H}$. Then, $\sqrt{(\Phi(N) : \Phi(M))}$ is a ϕ-powerful ideal of ring R.*

Proof. Let N be a Φ-powerful submodule of M. Let $xy \in \sqrt{(\Phi(N) : \Phi(M))}$, it implies that $(xy)^m \in (\Phi(N) : \Phi(M))$, for some $m > 0$. Consider $(\frac{x^{3m}}{x^m y^m}) \cdot (\frac{y^{3m}}{x^m y^m}) = x^m y^m \in (\Phi(N) : \Phi(M))$. Since N is a Φ-powerful submodule of a module M it implies $(\frac{x^{3m}}{x^m y^m}) \in R$ or $(\frac{y^{3m}}{x^m y^m}) \in R$. Thus, we have either $x^{3m} \in (x^m y^m)R$ or $y^{3m} \in (x^m y^m)R$ implies $x^{3m} \in (\Phi(N) : \Phi(M))$ or $y^{3m} \in (\Phi(N) : \Phi(M))$. Hence $x \in \sqrt{(\Phi(N) : \Phi(M))}$ or $y \in \sqrt{(\Phi(N) : \Phi(M))}$ which completes the proof. □

Theorem 3.13. *Let N be a nonnil powerful submodule of an R-module $M \in \mathbb{H}$ containing a submodule Q. Then N/Q is a Φ-powerful R/P-submodule of M/Q, where $P = (\Phi(N) : \Phi(M))$.*

Proof. Let N be a powerful submodule of an R-module M containing Q. Let $xy\Phi(M/Q) \subseteq \Phi(N/Q)$, for some $x, y \in K$. Then, $xy \in (\Phi(\frac{N}{Q}) : \Phi(\frac{M}{Q})) = Ann(\Phi(\frac{M}{Q})/(\Phi(\frac{N}{Q})))$. Since $\Phi(A)/\Phi(B) \cong A/B$, if $B \subseteq A$. It implies $\Phi(\frac{M}{Q})/\Phi(\frac{N}{Q}) \cong ((\frac{M}{Q})/(\frac{N}{Q}))$, so $Ann(\Phi(\frac{M}{Q})/(\Phi(\frac{N}{Q}))) \cong Ann((\frac{M}{Q})\ /(\frac{N}{Q})) \cong Ann(\frac{M}{N}) = (N : M)$. Hence $xyM \subseteq N$. Since N is a powerful submodule of M implies $x \in R$ or $y \in R$. Thus, N/Q is a Φ-powerful submodule of M/Q. On the other hand, let $P = (\Phi(N) : \Phi(M))$. Then, P is a prime ideal of ring R and hence by Corollary 3.6, N/Q is a Φ-powerful R/P-submodule of M/Q. □

4. Φ-strongly prime submodules and Φ-pseudo-valuation modules

Recall that a submodule N of an R-module M is a strongly prime submodule, if $xyM \subseteq N$, for $x, y \in K$ implies $xM \subseteq N$ or $y^n M \subseteq N$, for $n \geq 1$. In this section, we generalize this concept toward Φ-strongly prime submodules. In our study, we show that a nonnil submodule N is a Φ-powerful submodule if and only if N is a Φ-strongly prime submodule. Consequently, we introduce and discuss the notion of Φ-pseudo-valuation modules.

Definition 4.1. *A submodule N of an R-module $M \in \mathbb{H}$ is said to be a Φ-strongly prime submodule if $\Phi(N)$ is a strongly prime submodule of $\Phi(M)$.*

In other words, a submodule N is a Φ-strongly prime submodule if $xy\Phi(M) \subseteq \Phi(N)$, for $x, y \in K = R_{Nil(R)}$ implies $x\Phi(M) \subseteq \Phi(N)$ or $y^n\Phi(M) \subseteq \Phi(N)$ for $n \geq 1$.

Lemma 4.2. *Let $M \in \mathbb{H}$ be a module over an integral domain R and N be its submodule. Then N is a strongly prime submodule of M if and only if $N/Nil(M)$ is a strongly prime submodule of $M/Nil(M)$.*

Proof. Obvious from [12, Theorem 2.6] and Lemma 3.2. □

Theorem 4.3. *Let N be a nonnil prime submodule of an R-module $M \in \mathbb{H}$. Then N is a Φ-strongly prime submodule of M if and only if it is a Φ-powerful submodule of M.*

Proof. Let N be a Φ-strongly prime submodule of M. Let $x \in K \setminus R$ and $n \in (\Phi(N) : \Phi(M))$, then $n = nx^{-1}\ x \in (\Phi(N) : \Phi(M))$, hence $nx^{-1} \in (\Phi(N) : \Phi(M))$ or $x \in (\Phi(N) : \Phi(M))$. Since $x \notin R$, we must have $nx^{-1} \in (\Phi(N) : \Phi(M))$. Thus, $x^{-1}(\Phi(N) : \Phi(M)) \subseteq (\Phi(N) : \Phi(M)) \subseteq R$. Hence by Theorem 3.3, N is a Φ-powerful submodule of M.

Conversely, let N be a Φ-powerful submodule of an R-module M. Let $xy \in (\Phi(N) : \Phi(M))$, for $x, y \in K$. Since $(\Phi(N) : \Phi(M))$ is an ideal of ring R implies $x^2y^2 \in (\Phi(N) : \Phi(M))$. Let $x \notin R$ and $y \in R$.

Now if $x^2 \in R$ and since $x \notin R$, we assume $x^2 \notin (\Phi(N) : \Phi(M))$. By definition, $x^2y^2 \in (\Phi(N) : \Phi(M))$, and $y^2 \in (\Phi(N) : \Phi(M))$ implies $y \in (\Phi(N) : \Phi(M))$. However, if $x^2 \notin R$ then $\frac{y^2}{xy}x^2 \in (\Phi(N) : \Phi(M))$ implies $\frac{y^2}{xy} \in R$. We have $y^2 = \frac{y^2}{xy}xy \in (\Phi(N) : \Phi(M))$ implies $y \in (\Phi(N) : \Phi(M))$. □

Theorem 4.4. *Let N be a nonnil Φ-powerful submodule of an R-module $M \in \mathbb{H}$. Then $\phi(P) = \bigcap_k (\Phi(N) : \Phi(M))^k$ is a strongly prime ideal of ring $\phi(R) \in \mathcal{H}$.*

Proof. Following Theorem 4.3 and Corollary 3.8, we only have to prove that P is ϕ-prime ideal. For $x, y \in T(R)$, let $xy \in \phi(P)$ with $x \notin \phi(P)$. Hence $x \notin (\Phi(N) : \Phi(M))^n$, for some $n > 0$, following Theorem 3.9 (1), $(\Phi(N) : \Phi(M))^{2n} \subseteq xR$. Thus, for each $k > 0$, we have $xy \in \phi(P) \subseteq (\Phi(N) : \Phi(M))^{2n+k} \subseteq x(\Phi(N) : \Phi(M))^k$. Hence, $y \in (\Phi(N) : \Phi(M))^k$, for each $k > 0$. It implies that $y \in \phi(P)$, it proves that $\phi(P)$ is a prime ideal, and hence by definition P is a ϕ-prime ideal of ring $R \in \mathcal{H}$. □

Theorem 4.5. *Let N be a nonnil submodule of an R-module $M \in \mathbb{H}$. Then N is a Φ-strongly prime submodule if and only if for some fractional ideal I of a ring R and for some fractional submodule U of a module M, $IU \subseteq \Phi(N)$ implies $U \subseteq \Phi(N)$ or $I \subseteq (\Phi(N) : \Phi(M))$.*

Proof. Let N be a Φ-strongly prime submodule of an R-module M. For $x \in U \setminus \Phi(N)$ and $y \in I$, we have $xy \in IU$. Since $IU \subseteq \Phi(N)$ for $x \in M_S \setminus N$ and $y \in K$. Thus we have $y \in (\Phi(N) : \Phi(M))$, it implies that $I \subseteq (\Phi(N) : \Phi(M))$.

Conversely, as submodule N is a prime submodule of a module M. Let $yx \in N$ for $y \in K$ and $x \in M_S$. Take $I = Ry$, a fractional ideal of ring R and $U = Rx$, a fractional submodule of a module M. Since $IU \subseteq \Phi(N)$ implies either $I = Ry \subseteq \Phi(N)$ or $U = Rx \subseteq \Phi(N)$. Thus, $x \in \Phi(N)$ or $y \in (\Phi(N) : \Phi(M))$. It implies $\Phi(N)$ is a strongly prime submodule of $\Phi(M)$. Hence N is a Φ-strongly prime submodule of M. □

Corollary 4.6. *Let N be a nonnil submodule of $M \in \mathbb{H}$. Then N is a Φ-strongly prime submodule if and only if for any element $y \in K$ and any fractional submodule U of module M, $yU \subseteq \Phi(N)$ implies $U \subseteq \Phi(N)$ or $y \in (\Phi(N) : \Phi(M))$.*

Definition 4.7. *An R-module $M \in \mathbb{H}$ is said to be a Φ-pseudo-valuation module (Φ-PVM) if each one of its prime submodules is a Φ-strongly prime submodule.*

Theorem 4.8. *Let $M \in \mathbb{H}$ be a faithful and multiplication R-module. Then M is Φ-PVM if and only if $M/Nil(M)$ is a PVM.*

Proof. Let M be a Φ-PVM and $P/Nil(M)$ be a prime submodule of $M/Nil(M)$. Let $(x + Nil(M))(y + Nil(M)) \in (P : M)/Nil(M)$, for $x + Nil(M)$, $y + Nil(M) \in K/Nil(M)$. Since P is a nonnil prime submodule of M, $(x + Nil(M))\, M \subseteq P + Nil(M)$ or $(y^n + Nil(M))\, M \subseteq P + Nil(M)$, for $n \geq 1$. Hence $P/Nil(M)$ is a Φ-strongly prime and $M/Nil(M)$ is a pseudo-valuation module. The converse is easy to prove. □

Proposition 4.9. *Let $M \in \mathbb{H}$ be a faithful and multiplication R-module. Then M is Φ-PVM if and only if $\Phi(M)$ is a PVM.*

Proof. Let M be Φ-PVM and L be a nonnil prime submodule of $\Phi(M)$. Then, $L = \Phi(N)$, for any nonnil prime submodule of module M. Hence $\Phi(N)$ is a strongly prime submodule of module $\Phi(M)$. It means $\Phi(M)$ is a PVM.

Conversely, let every nonnil prime submodule of $\Phi(M)$ be a strongly prime submodule. Then, for any nonnil submodule N of R-module M, $\Phi(N)$ is a nonnil prime submodule of R-module $\Phi(M)$. Thus, $\Phi(N)$ is a strongly prime submodule of $\Phi(M)$. By definition, the submodule N is Φ-strongly prime submodule, it implies M is a Φ-PVM. □

Proposition 4.10. *Let $M \in \mathbb{H}$ be a Φ-PVM. Then $\{(\Phi(P) : \Phi(M)) \mid P \in Spec(M)\}$ is totally ordered.*

Proof. Let M be Φ-PVM. By Proposition 4.9, $\Phi(M)$ is PVM. Following [14, Lemma 3.3] $\{(\Phi(P) : \Phi(M)) \mid P \in Spec(M)\}$ is totally ordered. □

Corollary 4.11. *Let $M \in \mathbb{H}$ be a faithful and the multiplication R-module. Then M is a Φ-PVM if and only if some maximal nonnil submodules of a module M are Φ-powerful submodules.*

Lemma 4.12. *Let $M \in \mathbb{H}$ be a faithful and multiplication R-module over an integral domain R. Then M is Φ-PVM if and only if R is a PVD.*

Proof. Let M be Φ-PVM. Following the definition, $\Phi(M)$ is PVM. Let $U \in Spec(R)$. Since M is multiplication R-module, $UM \in Spec(M)$. Since $\Phi(M)$ is PVM, $U(\Phi(M))$ is a strongly prime submodule of $\Phi(M)$. Following [14, Proposition 2.10], U is a strongly prime ideal of R and hence R is a PVD.

Conversely, let $N \in Spec(M)$. Since M is a multiplication R-module, it implies that $\Phi(N) = U(\Phi(M))$, for any prime ideal U of a ring R. Again, since R is a PVD, U is a strongly prime ideal of R. Following [14, Proposition 2.10], $\Phi(N)$ is a strongly prime submodule of M. Thus M is a Φ-PVM. □

Theorem 4.13. *Let M be a finitely generated, faithful and multiplication R-module. Then the following statements are satisfied.*
1) *If a ring $R \in \mathcal{H}$ is a ϕ-PVR, then the module M is a Φ-PVM.*
2) *If an R-module $M \in \mathbb{H}$ is a Φ-PVM, then the ring R is a ϕ-PVR.*

Proof. 1) Let $R \in \mathcal{H}$. Then by [8, Proposition 4], $M \in \mathbb{H}$. If a ring R is ϕ-PVR, then by [9, Theorem 21] $R/Nil(R)$ is a pseudo-valuation domain (PVD). Hence, by [14, Lemma 3.6], $M/Nil(M)$ is a pseudo-valuation module (PVM). Finally, by Theorem 4.8, module M is a Φ-PVM.

2) If $M \in \mathbb{H}$, then by [8, Proposition 4], $R \in \mathcal{H}$. And if a module M is a Φ-PVM then by Theorem 4.8, $M/Nil(M)$ is a pseudo-valuation module (PVM). Also, following [14, Lemma 3.6], $R/Nil(R)$ is a pseudo-valuation domain (PVD). Finally, by [9, Theorem 21] ring R is a ϕ-PVR. □

Theorem 4.14. *Let $M \in \mathbb{H}$ be a multiplication and faithful module over an integral domain R and I be a prime ideal of R. Then I is a strongly prime ideal of R if and only if IM is a Φ-strongly prime submodule of a module M.*

Proof. Let M be a multiplication and faithful R-module over an integral domain R and I be a strongly prime ideal of R. Then by [14, Proposition 2.10], IM is a strongly prime submodule of a module M. Since $M/Ker(\Phi) \cong \Phi(M)$ and by [16, Proposition 2.1 (1)], $Ker(\Phi)$ is a subset of $Nil(M)$. Since R is an integral domain we have $Nil(M) = Nil(R)M = \{0\}$ it implies that $M \cong \Phi(M)$. Since $I\Phi(M)$ is a strongly prime submodule of $\Phi(M)$, $\Phi(IM)$ is a strongly prime submodule of $\Phi(M)$. Hence, by definition IM is a Φ-strongly prime submodule of module M.

Conversely, let IM be a Φ-strongly prime submodule of M. Then, by definition of Φ-strongly prime submodule, $\Phi(IM)$ is a strongly prime submodule of $\Phi(M)$. Since $\Phi(IM) = I\Phi(M)$, $I\Phi(M)$ is a strongly prime submodule of $\Phi(M)$. Hence by [14, Proposition 2.10], I is a strongly prime ideal of ring R. □

Theorem 4.15. *Let $M \in \mathbb{H}$ be a finitely generated, faithful and multiplication R-module and M_S be a Φ-PVM, where $S = R \setminus Z(R)$. Then, if T is an extension of R with $Spec(R) = Spec(T)$ then M is a Φ-PVM.*

Proof. Let $M \in \mathbb{H}$. Then, by [8, Proposition 4], $R \in \mathcal{H}$. Suppose that $T = S^{-1}R = R_S$, where $S = R \setminus Z(R)$ is the ring extension of R. Let M_S be a Φ-PVM and every nonnil element of M lies in finite number of maximal submodules of M. Following Theorem 4.13, $T = R_S$ is a ϕ-PVR. Again, since $T = R_S$ and Spec(R)=Spec(T) then by [13, Proposition 2.4], R is a ϕ-PVR and hence M is a Φ-PVM by Theorem 4.13. □

5. Conclusions

In this manuscript, we have introduced and discussed Φ-powerful submodules and Φ-strongly prime submodules which are the generalizations of powerful and strongly prime submodules. Numerous associated concepts of powerful submodules and strongly prime submodules are shifted towards Φ-powerful submodules and Φ-strongly prime submodules. During our study, we explored that a nonnil prime submodule N of an R-module M is Φ-powerful submodule if and only if it is Φ-strongly prime submodule of M. In this pursuance, the notion of Φ-pseudo-valuation module is also presented. Based on the obtained results, one can further discuss the other types of submodules in the same setting such as strongly primary submodules, almost strongly prime submodules, almost strongly prime submodules and almost pseudo-valuation modules.

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Conflict of interest

The authors declare no conflict of interest.

References

1. M. M. Ali, Idempotent and nilpotent submodules of multiplication modules, *Commun. Algebra*, **36** (2008), 4620–4642.
2. D. F. Anderson, A. Badawi, On ϕ-Prüfer rings and ϕ-Bezout rings, *Houston J. Math.*, **30** (2004), 331–343.
3. D. F. Anderson, A. Badawi, On ϕ-Dedekind rings and ϕ-Krull rings, *Houston J. Math.*, **31** (2005), 1007–1022.
4. S. E. Atani, S. D. Pishhesari, M. Khoramdel, Some remarks on Prüfer modules, *Discuss. Math. Gen. Algebra Appl.*, **33** (2013), 121–128.

5. A. Badawi, On ϕ-Pseudo-valuation rings, In: *Advances in commutative ring theory*, New York/Basel: Dekker, 1999, 101–110.

6. A. Badawi, On ϕ-Mori rings, *Houston J. Math.*, **32** (2006), 1–32.

7. A. Y. Darani, M. Rahmatinia, On Φ-Mori modules, *New York J. Math.*, **21** (2015), 1269–1282.

8. A. Y. Darani, Nonnil-noetherian modules over commutative rings, *J. Algebr. Syst.*, **3** (2016), 201–210.

9. A. Y. Darani, The study of ϕ-Powerful ideals, Research project at Faculty of Science, Department of Mathematics and Applications, University of Mohaghegh Ardabili, 2019. Available from: http://repository.uma.ac.ir/id/eprint/8260.

10. N. J. Groenewald, D. Ssevviiri, Generalization of nilpotency of ring elements to module elements, *Commun. Algebra*, **42** (2014), 571–577.

11. J. R. Hedstrom, E. G. Houston, Pseudo-valuation domains, *Pac. J. Math.*, **75** (1978), 137–147.

12. A. Khaksari, S. Mehry, R. Safakish, On special submodule of modules, *B. Iran. Math. Soc.*, **40** (2014), 1441–1451.

13. R. Kumar, A. Gaur, A note on pairs of rings with same prime ideals, 2020, *arXiv:2005.05959v1*.

14. J. Moghaderi, R. Nekooei, Strongly prime submodules and pseudo-valuation modules, *Int. Electron. J. Algebra*, **10** (2011), 65–75.

15. H. Mostafanasab, A. Y. Darani, On ϕ-n-absorbing primary ideals of commutative rings, *J. Korean Math. Soc.*, **53** (2016), 549–582.

16. S. Motmaen, A. Y. Darani, On Φ-Dedekind, Φ-Prüfer and Φ-Bezout modules, *Georgian Math. J.*, **27** (2020), 103–110.

17. A. G. Naoum, F. H. Al-Alwan, Dedekind modules, *Commun. Algebra*, **24** (1996), 397–412.

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