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*Research article*

# New Fujita type results for quasilinear parabolic differential inequalities with gradient dissipation terms

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**Abstract:** This paper deals with the new Fujita type results for Cauchy problem of a quasilinear parabolic differential inequality with both a source term and a gradient dissipation term. Specially, nonnegative weights may be singular or degenerate. Under the assumption of slow decay on initial data, we prove the existence of second critical exponents $\mu^*$, such that the nonexistence of solutions for the inequality occurs when $\mu < \mu^*$.

**Keywords:** parabolic differential inequalities; nonexistence of solutions; global solutions; second critical exponent
**Mathematics Subject Classification:** 35K59, 35R45, 35B33

## 1. Introduction

We consider the nonexistence theorems of nonnegative nontrivial solutions for Cauchy problem to a quasilinear parabolic differential inequality:

$$u_t - \Delta_p u \geq a(x)\, u^q - b(x)\, u^m |\nabla u|^s, \quad (x,t) \in S, \tag{1.1}$$

where $S = \mathbb{R}^N \times \mathbb{R}^+$, $\mathbb{R}^+ = (0, +\infty)$, $N \geq 2$, $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$. The parameters $p > 1$, $q > 0$, $0 \leq m < q$ and $0 < s \leq p(q-m)/(q+1)$, initial data $u_0 \in L^1\left(\mathbb{R}^N\right)$ is a nonnegative function and the nonnegative weights $a$, $b$ may be singular or degenerate.

As everyone knows, Fujita [1] studied the following Cauchy problem for the semilinear heat equation in 1966:

$$u_t - \Delta u = |u|^{q-1} u. \tag{1.2}$$

He obtained the critical exponent $q_F = 1 + 2/N$ on the existence versus nonexistence (i.e. blow-up) of nonnegative nontrivial global solutions, namely, every nontrivial solution blows up in finite time for any initial data when $1 < q < 1 + N/2$, while blow-up can occur when $q > 1 + 2/N$, which depends on the size of $u_0$. Noted that the critical case $q_F = 1 + 2/N$ belongs to blow-up case and was settled later by Hayakawa [2] and Weissler [3]. From then on, similar critical Fujita exponents have been appeared in various nonlinear parabolic equations, systems and inequalities, and have been studied by many mathematicians in many papers. There have been a number of literature and here we just refer to a survey paper [4] for models with gradient dissipative terms and [5–7] for recent progress. Particularly, Filippucci and Lombardi [7] derived the very interesting Fujita type theorems for the Cauchy problem to inequality (1.1). Their results can completely cover the two open problems in [4], and extend some partial results in [5, 6]. As already observed by authors in [7], the presence of weights is crucial in the validity of the Fujita type result under consideration. In addition, for the study of the nonexistence of solutions for stationary elliptic $p$-Laplacian differential inequalities involving gradient terms in the whole space, we refer to the papers by Mitidieri, Pohozaev and Filippucci et al. [8–10].

However, it is obvious that the Fujita critical exponent is not optimal for the Cauchy problem of classical semilinear parabolic Eq (1.2). Therefore, whether the global solution and nonglobal solution can be classified has become the key issue. The pioneering work in this subject is given by Lee and Ni [11]. They considered the Cauchy problem for Eq (1.2) and introduced *the second critical exponent* as $\mu^* = 2/(q-1)$ by virtue of the slow decay behavior of the initial data at spatial infinity. More precisely, with initial data $u_0(x) = \lambda\varphi(x)$ and $q > q_F = 1 + 2/N$, *there exist constants* $\eta$, $\Gamma$, $\Gamma_0$ *such that the solution blows up in finite time whenever* $\liminf_{|x|\to\infty} |x|^{\mu^*}\varphi(x) > \eta > 0$ *and* $\lambda > \Gamma$*, or exists globally if* $\limsup_{|x|\to\infty} |x|^{\mu}\varphi(x) < \infty$ *with* $\mu > \mu^*$ *and* $\lambda < \Gamma_0$. Afterward, the degenerate parabolic equation with $p$-Laplacian operator

$$u_t = \Delta_p u + u^q, \tag{1.3}$$

where $p > 2$, was considered by Mu et al. [12]. By constructing the proper auxiliary function and the radially symmetric forward self-similar supersolution, they got the second critical exponent $\mu^* = p/(q-p+1)$. When we move to the case of fast diffusion, the same result occurs with the same exponent $\mu^* = p/(q-p+1)$ by Zheng and Fang [13]. The authors proposed a modification of the method in [12] and discussed the singular Eq (1.3) with a more generalized nonlocal source term. On the Cauchy problem for quasilinear parabolic equation with singular coefficients

$$u_t = \Delta_p u + a(x)u^q, \tag{1.4}$$

where $a(x) \sim |x|^{-\alpha}$, Yang et al. [14] and Zheng and Mu [15] considered the slow and fast diffusion cases of Cauchy problem (1.4) in a more general form, respectively. While, they all derived a new second critical exponent $\mu^*(\alpha) = (p-\alpha)/(q-p+1)$.

For the study of quasilinear parabolic inequalities with gradient dissipative terms, we are particularly interested in the second critical exponent of Theorem 2.2 given by Mitidieri and Pokhozhaev in [5]. Using test function method in the whole space $\mathbb{R}^N$, they discussed the inequality

$$u_t - \Delta_p u \geq u^p - b_0|\nabla u|^s, \quad b_0 > 0, \tag{1.5}$$

where $p > 1$ and $s = pq/(q+1)$ and obtained: *Assume* $p > 1$*,* $q > \max\{1, p-1\}$*,* $s \in (0, pq/(q+1))$ *and* $b_0 > 0$ *sufficiently small. If there exist* $\mu > 0$ *such that*

$$\liminf_{|x|\to\infty} u_0(x)\,|x|^{\mu} > 0 \quad \textit{and} \quad \mu < \mu_1^*(0) = \frac{p}{q-p+1},$$

*then problem (1.5) does not admit nonnegative nontrivial global weak solutions belonging to appropriate set.*

The main goal of this paper is to show some non-optimality of Fujita type results of quasilinear parabolic inequality (1.1) in [7] without assuming the radial symmetry or monotonicity of solutions, and the proofs make no use of any comparison principles. In special cases, our Theorem 1.1 guarantees Theorem 2.2 in [5], and our main results are the second critical exponents corresponding to all the Fujita type results in [7]. Specifically, the following results are valid.

**Theorem 1.1.** *Let $1 < p < N$, $q > \max\{1, p-1\}$, $m \in [0, q)$. Assume that $b_0 > 0$ is sufficiently small, if there exist $\mu > 0$ such that*

$$\liminf_{|x|\to\infty} u_0(x)\,|x|^{\mu} > 0, \tag{1.6}$$

*and*

$$\mu < \mu_1^*(0) := \frac{p}{q-p+1}, \tag{1.7}$$

*then*

$$\begin{cases} u_t - \Delta_p u \geq u^q - b_0 u^m |\nabla u|^{p(q-m)/(q+1)}, & (x,t) \in S, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N \end{cases} \tag{1.8}$$

*does not admit nonnegative nontrivial solutions in*

$$S_d^0 := \{u \in W_{loc}^{1,p}(S) : u^{q-d}, u^{-d-1}|\nabla u|^p, u^{m-d}|\nabla u|^{p(q-m)/(q+1)} \in L_{loc}^1(S)\},$$

*where $d$ sufficiently small.*

**Remark 1.1.** *Theorem 1.1 shows that for model (1.9), if $b_0 > 0$ is sufficiently small, the second critical exponent $\mu_1^*(0) = \frac{p}{q-p+1}$ is identical with the second critical exponent of our model (1.3) in [12] and [13] and our model (1.5) in [5]. Meanwhile, when*

$$\max\{1, p-1\} < q < p - 1 + \frac{Np}{N(p-1)+p},$$

*it can be compared with the first critical exponent $q_{F,1}(0) = p - 1 + \frac{p}{N}$ of Theorem 1.1 in [7], and has the following relationship*

$$q_{F,1}(0) < \mu_1^*(0).$$

**Theorem 1.2.** *Assume*

$$1 < p < N, \quad q > \max\left\{1, p-1, \frac{mp+s}{p-s}\right\} \tag{1.9}$$

*and*

$$s \in \left(0, \frac{p(q-m)}{q+1}\right), \quad m \in \left[0, q - \frac{s(q+1)}{p}\right). \tag{1.10}$$

*Let $a, b : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}_0^+$ be continuous functions that satisfy*

$$a(x) \geq a_0|x|^{-\alpha}, \quad b(x) \leq b_0|x|^{-\beta}, \quad x \in \mathbb{R}^N \backslash \{0\} \tag{1.11}$$

*with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha < p \quad \text{and} \quad \alpha(p-s) < \beta p < N(p-s). \tag{1.12}$$

*When*

$$\beta \geq p - s - N\Lambda \tag{1.13}$$

*and*

$$\begin{cases} p - N(q-p+1) \leq \alpha < q, & \text{if} \quad \Lambda \leq 0, \\ \alpha \leq p - N(q-p+1), & \text{if} \quad \Lambda \geq 0, \end{cases} \tag{1.14}$$

*if there exist $\mu > 0$ such that (1.7) holds, namely*

$$\liminf_{|x|\to\infty} u_0(x)|x|^{\mu} > 0,$$

*and*

$$\mu < \mu_1^*(\alpha) := \frac{p-\alpha}{q-p+1}, \tag{1.15}$$

*then (1.1) does not admit nonnegative nontrivial solutions belonging to the class*

$$S_d := \{u \in W_{loc}^{1,p}(S) : a(x)u^{q-d}, u^{-d-1}|\nabla u|^p, b(x)u^{m-d}|\nabla u|^s \in L_{loc}^1(S)\},$$

*where d sufficiently small.*

**Remark 1.2.** *For the model inequality (1.1) with $\beta \geq p - s - N\Lambda$, the second critical exponent $\mu_1^*(\alpha) = \frac{p-\alpha}{q-p+1}$ is the same as the second critical exponent of our model (1.4) in references [14, 15]. At the same time, when*

$$\max\left\{1, p-1, \frac{mp+s}{p-s}\right\} < q < p - 1 + \frac{N(p-\alpha)}{N(p-1)+(p-\alpha)},$$

*it can be compared with the first critical exponent $q_{F,1}(\alpha) = p - 1 + \frac{p-\alpha}{N}$ given in Theorem 1.2 in [7], and has the following relationship*

$$q_{F,1}(\alpha) < \mu_1^*(\alpha).$$

**Theorem 1.3.** *Assume (1.9)-(1.12). When*

$$\beta \leq p - s - N\Lambda, \tag{1.16}$$

*and*

$$\begin{cases} p - N(q-p+1) \leq \alpha < q, & \text{if} \quad \Lambda \geq 0, \\ \alpha \leq p - N(q-p+1), & \text{if} \quad \Lambda \leq 0, \end{cases} \tag{1.17}$$

*if there exist* $\mu \geq 0$ *such that (1.6) holds and*

$$\mu < \mu_2^*(\alpha,\beta) := \frac{\beta p - \alpha(p-s)}{q(p-s) - mp - s}, \tag{1.18}$$

*then (1.1) does not admit nonnegative nontrivial solutions belonging to the class* $S_d$ *for d sufficiently small.*

**Remark 1.3.** *For the model inequality (1.1) with* $\beta \leq p - s - N\Lambda$*, when*

$$\max\left\{1, p-1, \frac{mp+s}{p-s}\right\} < q < \frac{mp+s}{p-s} + \frac{[\beta p - \alpha(p-s)]\,N}{\beta p - \alpha(p-s) - N(mp+s)},$$

*the second critical exponent* $\mu_2^*(\alpha,\beta) = \frac{\beta p - \alpha(p-s)}{q(p-s)-mp-s}$ *can be compared with the first critical exponent* $q_{F,2}(\alpha,\beta) = p - 1 - \frac{\alpha}{N} + \frac{p(N\Lambda+\beta)}{N(p-s)}$ *given in Theorem 1.3 in [7]. Actually, we have*

$$q_{F,2}(\alpha,\beta) < \mu_2^*(\alpha,\beta)\,.$$

**Remark 1.4.** *In particular,*

$$\mu_1^*(\alpha) = \mu_2^*(\alpha,\beta) = \frac{p-\alpha}{q-p+1}$$

*if* $\Lambda = m + 1 - p + s = 0$ *and* $\beta \geq p - s - N\Lambda$.

**Corollary 1.1.** *Assume (1.6), (1.9)–(1.11) with* $\beta = 0$ *and* $\alpha < 0$*. Then*

$$\begin{cases} u_t - \Delta_p u \geq a(x)u^q - b_0 u^m |\nabla u|^s & (x,t) \in S, \quad b_0 > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N \end{cases} \tag{1.19}$$

*does not admit nonnegative nontrivial solutions in* $S_d$*, with d sufficiently small, when* $0 < \mu < \mu_1^*(\alpha)$ *and the following conditions hold*

$$p - \frac{N(1+m)}{N+1} \leq s < \frac{p(q-m)}{q+1} \tag{1.20}$$

*and*

$$\alpha \leq p - N(q-p+1)\,. \tag{1.21}$$

Note that since $p - \frac{N(1+m)}{N+1} > p - 1 - m$, then it follows from condition (1.20) that $s > p - 1 - m$, that is $\Lambda > 0$, indeed the subcase $\Lambda \leq 0$ of Theorem 1.2 cannot occur when $\beta = 0$ since $p - s > 0$.

**Corollary 1.2.** *Assume (1.6), (1.9)–(1.11) with* $\beta = 0$ *and* $\alpha < 0$*. Then (1.19) does not admit nonnegative nontrivial solutions in* $S_d$*, with d sufficiently small, when* $0 < \mu < \mu_2^*(\alpha,\beta)$ *and the condition*

$$0 < s \leq p - \frac{N(1+m)}{N+1} \tag{1.22}$$

*and*

$$\begin{cases} p - N(q-p+1) \leq \alpha < q, & \text{if} \quad \Lambda \geq 0, \\ \alpha \leq p - N(q-p+1), & \text{if} \quad \Lambda \leq 0 \end{cases}$$

*hold.*

This paper is organized as follows. In Section 2, we introduce some basic definitions, assumptions, and key lemmas that will be used in the proofs of main results. In Section 3, we give the detailed proofs of main results, which include the proofs of Corollaries 1.1 and 1.2.

## 2. Preliminaries and a priori estimates

Throughout this paper, we will use $C$ to denote various constants independent of $u$, which may be different from line to line, and we denote $\mathbb{R}_0^+ := [0, \infty)$.

Meanwhile, we follow the definition of weak solution and the construction of test function introduced in [7]. Here we recall the preliminaries required for reader's convenience.

**Definition 2.1.** *[7] For a weak solution of (1.1), we mean a nonnegative function $u(x,t)$, belonging to the set $S$ given by those functions $u \in W_{loc}^{1,p}(S)$ with*

**(i)** $A(x, u, \nabla u) \in \left[L_{loc}^{p'}(S)\right]^N$,

**(ii)** $a(x)u^q$, $b(x)u^m|\nabla u|^s \in L_{loc}^1(S)$

*such that for any nonnegative test function $\varphi \in C_0^1\left(\mathbb{R}^N \times \mathbb{R}_0^+\right)$, we have*

$$\begin{aligned}\int_s a(x)u^q\varphi \mathrm{d}x\mathrm{d}t \le &- \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\mathrm{d}x - \int_s u\varphi_t \mathrm{d}x\mathrm{d}t \\ &+ \int_s A(x,u,\nabla u)\nabla u \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t \\ &+ \int_s b(x)u^m|\nabla u|^s\varphi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.1}$$

*Furthermore, when necessary, we make use of the following weak formulation of (1.1)*

$$\begin{aligned}\int_s a(x)u^q\varphi \mathrm{d}x\mathrm{d}t \le &\int_s u_t\varphi \mathrm{d}x\mathrm{d}t + \int_s A(x,u,\nabla u)\nabla u \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t \\ &+ \int_s b(x)u^m|\nabla u|^s\varphi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.2}$$

*for any nonnegative text function $\varphi \in C_0^1\left(\mathbb{R}^N \times \mathbb{R}_0^+\right)$.*

We introduce the test functions constructed in [7, Section 3], which have the form of a separation of the variables in detail.

Let be $B_{\mathbb{R}}(0)$ the ball of $\mathbb{R}^N$, centered at $x = 0$ and with radius $R > 0$. We introduce a cut-off function $\xi_0(s) \in C^1([0, +\infty))$ that satisfies

$$0 \le \xi_0(s) \le 1,\ \forall s \ge 0; \quad \xi_0(s) = 1,\ 0 \le s \le 1; \quad \xi_0(s) = 0,\ s \ge 2.$$

Moreover,

$$|\xi_0'(s)| \le C,\ s > 0,$$

where $C > 0$ is a constant.

For the space variable, we consider

$$\chi(x) := \xi_0(\frac{|x|}{R}),$$

thus $\chi(x) \in C_0^1(\mathbb{R}^N)$ and

$$\chi(x) = 1,\ x \in B_R(0), \quad \chi(x) = 0,\ x \in \mathbb{R}^N \backslash B_{2R}(0),$$

$$0 \le \chi(x) \le 1,\ \forall x \in \mathbb{R}^N, \quad |\nabla \chi(x)| \le \frac{C}{R},\ x \in \mathbb{R}^N,$$

where $C > 0$ is a constant.

For the time variable, take

$$\eta(t) := \xi_0(\frac{|x|}{R^\gamma}),$$

with $\gamma \ge 1$ to be chosen.

Now, we define, for all $R > 0$, a nonnegative cut-off function in $S$, given by

$$\psi(x,t) := \chi(x)\eta(t).$$

Clearly $\psi \in C_0^1(S)$.

We take the test function

$$\varphi(x,t) := \tilde{u}_\epsilon^{-d}\psi^k(x,t),$$

where

$$\tilde{u}_\epsilon := \tau + \int_{\mathbb{R}^N} \xi_\epsilon(x-y,t)u(y,t)\mathrm{d}y,$$

here $\tau > 0$, $k > 0$ large enough, $\epsilon > 0$ sufficiently small and $(\xi_\epsilon)_{\epsilon>0}$ a standard family of mollifiers.

From this, we can get the following two key lemmas. Owing that the proofs are similar to Lemmas 5.1 and 5.2 in [7], we omit here.

**Lemma 2.1.** *Let $p > 1$. Assume $(1.10)_2$ and*

$$s \in \left(0, \frac{p(q-m)}{q+1}\right], \quad q > \max\{1, p-1\}$$

*and*

$$0 < d < \min\left\{1, p-1, \frac{mp+s}{p-s}\right\}. \tag{2.3}$$

*If $u \in W_{loc}^{1,p}(S)$ is a nonnegative solution of (1.1) belonging to the class $S_d$, then the following inequality holds*

$$\begin{aligned}&\frac{1}{2}\int_S a(x)u^{q-d}\psi^k\mathrm{d}x\mathrm{d}t + \frac{c_1 d}{3}\int_S u^{-d-1}|\nabla u|^p\psi^k\mathrm{d}x\mathrm{d}t\\&+\frac{1}{1-d}\int_{\mathbb{R}^N} u_0^{1-d}\psi^k(x,0)\mathrm{d}x\mathrm{d}t\\ \le& C(R^{\sigma_1}+R^{\sigma_2}) + C\int_S b(x)^{\frac{p}{p-s}}u^{\frac{mp+s-d(p-s)}{p-s}}\psi^k\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.4}$$

*where*

$$\begin{aligned}\sigma_1 &:= N+\gamma+\frac{\alpha-\gamma q}{q-1}+d\frac{\gamma-\alpha}{q-1},\\ \sigma_2 &:= N+\gamma+\frac{\alpha(p-1)-pq}{q-p+1}+d\frac{p-\alpha}{q-p+1}\end{aligned} \tag{2.5}$$

*for some constant $C > 0$.*

**Lemma 2.2.** *Let $p > 1$. Assume (1.10), (2.3) and*

$$q > \max\left\{1, p-1, \frac{mp+s}{p-s}\right\}.$$

*If $u \in W^{1,p}_{loc}(S)$ is a nonnegative solution of (1.1) belonging to the class $S_d$, then the following inequality holds*

$$\begin{aligned}
&\frac{1}{4}\int_S a(x)u^{q-d}\psi^k \mathrm{d}x\mathrm{d}t + \frac{c_1 d}{3}\int_S u^{-d-1}|\nabla u|^p\psi^k \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{1-d}\int_{\mathbb{R}^N} u_0^{1-d}\psi^k(x,0)\mathrm{d}x\mathrm{d}t \\
&\leq C(R^{\sigma_1} + R^{\sigma_2} + R^{\sigma_3}),
\end{aligned} \tag{2.6}$$

*where $\sigma_1$ and $\sigma_2$ are given in (2.5),*

$$\sigma_3 := N + \gamma + \frac{\alpha(mp+s) - \beta pq}{q(p-s) - mp - s} + d\frac{\beta p - \alpha(p-s)}{q(p-s) - mp - s} \tag{2.7}$$

*for some constant $C > 0$.*

## 3. Proofs of main results

In this section, we consider the two cases that $a(x), b(x)$ are both positive constants, and $a(x), b(x)$ may be singular or degenerate. Based on the two lemmas established in Section 2, we give the detailed proofs of main results.

**Proof of Theorem 1.1:** Let $u$ be a nonnegative solution of (1.8), with $u \in S^0_d$. Replacing $\alpha = \beta = 0$ in (1.11) and taking $a_0 = 1$ for simplicity, that is $a(x) = a_0 = 1$ and $b(x) = b_0$, and applying Lemma 1 with $s = \frac{p(q-m)}{q+1}$, so that

$$\frac{p}{p-s} = \frac{q+1}{m+1} \quad \text{and} \quad q - d = \frac{mp + s - d(p-s)}{p-s},$$

then (2.4) gives

$$\begin{aligned}
&\left[\frac{1}{2} - Cb_0^{\frac{q+1}{m+1}}\right]\int_S u^{q-d}\psi^k \mathrm{d}x\mathrm{d}t + \frac{c_1 d}{3}\int_S u^{-d-1}|\nabla u|^p\psi^k \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{1-d}\int_{\mathbb{R}^N} u_0^{1-d}\psi^k(x,0)\mathrm{d}x\mathrm{d}t \leq C(R^{\sigma_1} + R^{\sigma_2}),
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1 &= N + \gamma - \frac{\gamma q}{q-1} + d\frac{\gamma}{q-1}, \\
\sigma_2 &= N + \gamma - \frac{pq}{q-p+1} + d\frac{p}{q-p+1}.
\end{aligned}$$

Hence if $b_0 > 0$ is sufficiently small, then $\frac{1}{2} - Cb_0^{\frac{q+1}{m+1}} > 0$, so that we arrive to

$$\int_{\mathbb{R}^N} u_0^{1-d}\psi^k(x,0)\mathrm{d}x\mathrm{d}t \leq C(R^{\sigma_1} + R^{\sigma_2}).$$

Thanks to (1.6), it becomes

$$R^{N-\mu(1-d)} \leq C(R^{\sigma_1} + R^{\sigma_2}).$$

Taking $\gamma = \frac{p(q-1)}{q-p+1}$ so that $\sigma_1 = \sigma_2$, therefore we get

$$R^{N-\mu} \leq CR^{N-\frac{p}{q-p+1}}$$

with $d$ sufficiently small. Finally by our assumption (1.7), let $R \to \infty$, the contradiction obtained completes the proof of the theorem.

**Proofs of Theorem 1.2 and Theorem 1.3:** Let $u$ be a nonnegative solution of (1.1), with $u \in S_d$. By the definition of the weak solution of (1.1) and Lemma 2.2, we derive

$$\begin{aligned} &\frac{1}{4}\int_S a(x)u^{q-d}\psi^k\mathrm{d}x\mathrm{d}t + \frac{c_1 d}{3}\int_S u^{-d-1}|\nabla u|^p\psi^k\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{1-d}\int_{\mathbb{R}^N} u_0^{1-d}\psi^k(x,0)\mathrm{d}x\mathrm{d}t \leq C(R^{\sigma_1} + R^{\sigma_2} + R^{\sigma_3}), \end{aligned}$$

thus

$$\int_{\mathbb{R}^N} u_0^{1-d}\psi^k(x,0)\mathrm{d}x\mathrm{d}t \leq C(R^{\sigma_1} + R^{\sigma_2} + R^{\sigma_3}),$$

where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are given in (2.5) and (2.7), specifically,

$$\begin{aligned} \sigma_1 &= N + \gamma + \frac{\alpha - \gamma q}{q-1} + d\frac{\gamma - \alpha}{q-1}, \\ \sigma_2 &= N + \gamma + \frac{\alpha(p-1) - pq}{q-p+1} + d\frac{p-\alpha}{q-p+1}, \\ \sigma_3 &= N + \gamma + \frac{\alpha(mp+s) - \beta pq}{q(p-s) - mp - s} + d\frac{\beta p - \alpha(p-s)}{q(p-s) - mp - s}. \end{aligned}$$

By by our assumption (1.6), we have

$$R^{-\mu(1-d)} \leq C\left(R^{\gamma + \frac{\alpha-\gamma q}{q-1}} + R^{\gamma + l}\right), \tag{3.1}$$

with $d$ sufficiently small, here

$$l = \max\left\{\frac{\alpha(p-1) - pq}{q-p+1}, \frac{\alpha(mp+s) - \beta pq}{q(p-s) - mp - s}\right\}.$$

One can see that the expression on the right in (3.1) attains its minimum at

$$R^{\gamma} = \left(\frac{1}{q-1}\right)^{\frac{q-1}{q}} R^{\left(\frac{\alpha}{q-1}-l\right)\frac{q-1}{q}} = c(q)R^{\frac{\alpha-l(q-1)}{q}}. \tag{3.2}$$

Noting that $c(q) = \left(\frac{1}{q-1}\right)^{\frac{q-1}{q}}$ and substituting (3.2) into (3.1), we obtain

$$R^{-\mu} \leq CR^{\frac{\alpha+l}{q}}. \tag{3.3}$$

Now, we discuss the value of $l$. We consider the following difference:

$$\frac{\alpha(p-1)-pq}{q-p+1} - \frac{\alpha(mp+s)-\beta pq}{q(p-s)-mp-s}.$$

Due to the conditions we assumed, obviously, $q-p+1 > 0$, $q(p-s)-mp-s > 0$.

Considering now only the numerator, we write

$$\begin{aligned} M :=& \left[\alpha(p-1)-pq\right]\left[q(p-s)-mp-s\right] - \left[\alpha(mp+s)-\beta pq\right](q-p+1) \\ =&\alpha q\left[(p-s)(p-1)-mp-s\right] + \beta pq(q-p+1) \\ &- pq^2(p-s) + pq(mp+s). \end{aligned}$$

On one hand, substituting (1.13), that is, $\beta \geq p-s-N\Lambda$ into the formula above and replacing the value of $\Lambda$, after simple calculation one can get

$$\begin{aligned} \frac{M}{q} \geq& \alpha\left[(p-s)(p-1)-mp-s\right] + (p-s-N\Lambda)p(q-p+1) \\ &- pq(p-s) + p(mp+s) \\ =&(\alpha-p)\left[(p-s)(p-1)-mp-s\right] - pN\Lambda(q-p+1) \\ =&-p\Lambda\left[\alpha-p+N(q-p+1)\right]. \end{aligned} \tag{3.4}$$

When the condition (1.14) holds, summarizing (1.9) and (1.10), we have $\frac{M}{q} \geq 0$, thus, $l = \frac{\alpha(p-1)-pq}{q-p+1}$. Now (3.3) can be rewritten as

$$R^{-\mu} \leq CR^{\frac{\alpha-p}{q-p+1}}.$$

Meanwhile, by (1.15), $\mu < \mu_1^*(\alpha) = \frac{p-\alpha}{q-p+1}$, and let $R \to \infty$, the contradiction can be derived.

On the other hand, when $\beta \leq p-s-N\Lambda$, Similar to the comparison in (3.4), we can get

$$\frac{M}{q} \leq -p\Lambda\left[\alpha-p+N(q-p+1)\right].$$

When the condition (1.17) holds, we have $\frac{M}{q} \leq 0$ and $l = \frac{\alpha(mp+s)-\beta pq}{q(p-s)-mp-s}$. Consequently, (3.3) can be rewritten as

$$R^{-\mu} \leq CR^{\frac{\alpha(p-s)-\beta p}{q(p-s)-mp-s}}.$$

Meanwhile, by (1.18), $\mu < \mu_1^*(\alpha) = \frac{p-\alpha}{q-p+1}$, and let $R \to \infty$, the contradiction can be derived.

## 4. Conclusions

In this paper, by using the test function method, we derive the second Fujita type critical exponents for quasilinear parabolic differential inequality (1.1) with weighted coefficients under the assumption of slow decay on initial data at infinity. Our results correspond to all the results obtained in [7] for the first Fujita type critical exponents and can also cover the relevant critical exponents in existing literature. Meanwhile, its analytical method can be used in other models.

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## Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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