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Research article

Univalence and convexity conditions for certain integral operators associated with the Lommel function of the first kind

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Abstract: A useful family of integral operators and special functions plays a crucial role on the study of mathematical and applied sciences. The purpose of the present paper is to give sufficient conditions for the families of integral operators, which involve the normalized forms of the generalized Lommel functions of the first kind to be univalent in the open unit disk. Furthermore, we determine the order of the convexity of the families of integral operators. In order to prove main results, we use differential inequalities for the Lommel functions of the first kind together with some known properties in connection with the integral operators which we have considered in this paper. We also indicate the connections of the results presented here with those in several earlier works on the subject of our investigation. Moreover, some graphical illustrations are provided in support of the results proved in this paper.

Keywords: analytic functions; univalent functions; integral operators; special functions; Lommel functions; univalence; convexity; quantum (or q-) calculus
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1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D}$ and satisfy the usual normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of the normalized analytic function class $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$. A function $f \in \mathcal{S}$ is said to be convex of order δ $(0 \leqq \delta < 1)$ if it satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta \qquad (z \in \mathbb{D}).$$

Let $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and

$$\mathcal{A}^n = \left\{(f_1, f_2, \cdots, f_n) : f_j \in \mathcal{A} \ \ (j = 1, 2, \cdots, n)\right\}.$$

For the functions $f_j \in \mathcal{A}$ $(j = 1,2,\cdots,n)$, the parameters $\alpha_j, \beta_j \in \mathbb{C}$ $(j = 1,2,\cdots,n)$ and $\gamma \in \mathbb{C}$, we define the following three integral operators:

$$\mathcal{J}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma},$$

$$\mathcal{K}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n}$$

and

$$\mathcal{L}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma} : \mathcal{A}^n \to \mathcal{A}$$

by

$$\mathcal{J}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma}[f_1, f_2, \cdots, f_n](z) := \left[\gamma \int_0^z t^{\gamma-1} \prod_{j=1}^{n} (f_j'(t))^{\alpha_j} \left(\frac{f_j(t)}{t}\right)^{\beta_j} dt\right]^{1/\gamma}, \tag{1.1}$$

$$\mathcal{K}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n}[f_1, f_2, \cdots, f_n](z) = \left[\left(1 + \sum_{j=1}^{n} \alpha_j\right) \int_0^z \prod_{j=1}^{n} \left(f_j(t) e^{f_j(t)}\right)^{\alpha_j} dt\right]^{1/\left(1+\sum_{j=1}^{n} \alpha_j\right)} \tag{1.2}$$

and

$$\mathcal{L}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma}[f_1, f_2, \cdots, f_n](z) := \left[\gamma \int_0^z t^{\gamma-1} \prod_{j=1}^{n} \left(f_j'(t)\right)^{\alpha_j} \left(e^{f_j(t)}\right)^{\beta_j} dt\right]^{1/\gamma}. \tag{1.3}$$

We note that, for some special parameters, the integral operators defined above have been extensively studied by many authors as follows (see also a recent investigation by Srivastava *et al.* [40] on the univalence of integral operators):

(1) $\mathcal{J}_{0,0,\cdots,0;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma}[f_1, f_2, \cdots, f_n] \equiv \mathcal{F}_{1/\beta_1,1/\beta_2,\cdots,1/\beta_n,\gamma}$
(Seenivasagan and Breaz [35]; see also [4, 37]);

(2) $\mathcal{J}_{\alpha_1,\alpha_2,\cdots,\alpha_n;0,0,\cdots,0;n;\gamma}[f_1, f_2, \cdots, f_n] \equiv \mathcal{L}_{\alpha_1,\alpha_2,\cdots,\alpha_n;0,0,\cdots,0;n;\gamma}[f_1, f_2, \cdots, f_n] \equiv \mathcal{H}_{\alpha_1,\alpha_2,\cdots,\alpha_n,\gamma}$ (Breaz and Breaz [10]);

(3) $\mathcal{J}_{\alpha_1,\alpha_2,\cdots,\alpha_n;0,0,\cdots,0;n;1}[f_1, f_2, \cdots, f_n] \equiv \mathcal{L}_{\alpha_1,\alpha_2,\cdots,\alpha_n;0,0,\cdots,0;n;1}[f_1, f_2, \cdots, f_n] \equiv \mathcal{H}_{\alpha_1,\alpha_2,\cdots,\alpha_n}$ (Breaz *et al.* [13]);

(4) $\mathcal{J}_{\alpha;0;1;1}[f] \equiv \mathcal{H}_\alpha$ (Pfaltzgraff [32]; see also Kim and Merkes [20]);

(5) $\mathcal{L}_{0;\beta;1;\gamma}[f] \equiv \mathcal{Q}_\beta$ (Pescar [31]);

(6) $\mathcal{K}_{\alpha,\alpha,\cdots,\alpha;n}[f_1, f_2, \cdots, f_n] \equiv \mathcal{G}_{n,\alpha}[f_1 e^{f_1}, f_2 e^{f_2}, \cdots, f_n e^{f_n}]$ (Breaz and Breaz [11]; see also [12]);

(7) $\mathcal{K}_{\alpha;1}[f] \equiv \mathcal{G}_{1,\alpha}[f e^f]$ (Moldoveanu and Pascu [24]).

Moreover, Deniz *et al.* [17] introduced certain integral operators by using an obvious parametric variation of the classical Bessel function $J_\nu(z)$ of the first kind and of order ν and studied the univalence criteria of the corresponding integral operators. On the other hand, Deniz [18] and Raza *et al.* [33] discussed the convexity, starlikeness and uniform convexity of integral operators involving these equivalent forms of the classical Bessel function $J_\nu(z)$. Recently, Al-Khrasani *et al.* [2] investigated some sufficient conditions for univalence of some linear fractional derivative operators involving the normalized forms of the same obvious parametric variation of the classical Bessel function $J_\nu(z)$ of the first kind and of order ν. Additionally, the theory of integrals and derivatives of an arbitrary real or complex order (see, for details, [42]; see also [36]) has been applied not only in geometric function theory of complex analysis, but has also emerged as a potentially useful direction in the mathematical modeling and analysis of real-world problems in applied sciences (see, for example, [39]).

Motivated by the works mentioned above, in this paper, we will investigate some mapping and geometric properties for the integral operators defined by (1.1), (1.2) and (1.3), associated with the Lommel function of the first kind which is defined as follows:

The Lommel function $s_{\mu,\nu}$ of the first kind, which is expressed here in terms of a hypergeometric series as follows:

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\ {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right),$$

where $\mu \pm \nu$ is not a negative odd integer, is a particular solution of the following inhomogeneous Bessel differential equation (see [23]):

$$z^2 w''(z) + z w'(z) + (z^2 - \nu^2) w(z) = z^{\mu+1}.$$

It is easily observed that the function $s_{\mu,\nu}$ does not belong to the class $\mathcal{A}$. Recently, Yağmur [45] and Baricz *et al.* [3] considered the function $h_{\mu,\nu}$ defined by

$$h_{\mu,\nu}(z) = (\mu-\nu+1)(\mu+\nu+1) z^{(1-\mu)/2} s_{\mu,\nu}(\sqrt{z})$$

and they obtained some geometric properties of the function $h_{\mu,\nu}$. For other interesting properties of the Lommel function, we refer the reader to [5, 6].

The above function $h_{\mu,\nu}$ belongs to the normalized analytic function class $\mathcal{A}$ and is expressed by

$$h_{\mu,\nu}(z) = \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{4}\right)^{n-1}}{\left(\frac{\mu-\nu+3}{2}\right)_{n-1}\left(\frac{\mu+\nu+3}{2}\right)_{n-1}} z^n = z + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{4}\right)^n}{(K)_n (F)_n} z^{n+1}, \tag{1.4}$$

where $K = (\mu - \nu + 3)/2 \notin \mathbb{N}$, $F = (\mu + \nu + 3)/2 \notin \mathbb{N}$, $(\lambda)_n$ being the Pochhammer symbol which is defined in terms of Euler's gamma function such that

$$(\lambda)_0 = 0 \quad \text{and} \quad (\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \lambda(\lambda + 1) \cdots (\lambda + n - 1) \quad (n \in \mathbb{N}).$$

We remark that, by choosing special values for μ and ν, we obtain the following functions:

$$h_{1/2,1/2}(z) = 2(1 - \cos\sqrt{z}), \quad h_{3/2,1/2}(z) = 6\left(1 - \frac{\sin\sqrt{z}}{\sqrt{z}}\right) \tag{1.5}$$

and

$$h_{5/2,1/2}(z) = 12\left(\frac{z + 2\cos\sqrt{z} - 2}{z}\right). \tag{1.6}$$

Let $j = 1, 2, \cdots, n$ and let μ_j and ν_j be real numbers such that $\mu_j \pm \nu_j$ are not negative odd integers. Consider the functions h_{μ_j,ν_j} $(j = 1, 2, \cdots, n)$ defined by

$$h_{\mu_j,\nu_j}(z) = (\mu_j - \nu_j + 1)(\mu_j + \nu_j + 1)z^{(1-\mu_j)/2} s_{\mu_j,\nu_j}(\sqrt{z}).$$

Using these functions h_{μ_j,ν_j} and the integral operators defined by (1.1), (1.2) and (1.3), we define the functions $\mathcal{J}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma}$, $\mathcal{K}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n}$ and $\mathcal{L}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma} : \mathbb{D} \to \mathbb{C}$ as follows:

$$\begin{aligned}\mathcal{J}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma}(z) &:= \mathcal{J}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma}[h_{\mu_1,\nu_1}, h_{\mu_2,\nu_2}, \cdots, h_{\mu_n,\nu_n}](z) \\ &= \left[\gamma \int_0^z t^{\gamma-1} \prod_{j=1}^{n} (h'_{\mu_j,\nu_j}(t))^{\alpha_j} \left(\frac{h_{\mu_j,\nu_j}(t)}{t}\right)^{\beta_j} dt\right]^{1/\gamma},\end{aligned} \tag{1.7}$$

$$\begin{aligned}\mathcal{K}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n}(z) &:= \mathcal{K}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n}[h_{\mu_1,\nu_1}, h_{\mu_2,\nu_2}, \cdots, h_{\mu_n,\nu_n}](z) \\ &= \left[\left(1 + \sum_{j=1}^{n} \alpha_j\right) \int_0^z \prod_{j=1}^{n} \left(h_{\mu_j,\nu_j}(t) e^{h_{\mu_j,\nu_j}(t)}\right)^{\alpha_j} dt\right]^{1/\left(1+\sum_{j=1}^{n} \alpha_j\right)}\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}\mathcal{L}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma}(z) &:= \mathcal{L}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma}[h_{\mu_1,\nu_1}, h_{\mu_2,\nu_2}, \cdots, h_{\mu_n,\nu_n}](z) \\ &= \left[\gamma \int_0^z t^{\gamma-1} \prod_{j=1}^{n} \left(h'_{\mu_j,\nu_j}(t)\right)^{\alpha_j} \left(e^{h_{\mu_j,\nu_j}(t)}\right)^{\beta_j} dt\right]^{1/\gamma}.\end{aligned} \tag{1.9}$$

In this paper, we derive some sufficient conditions for the following operators:

$$\mathcal{J}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma},$$

$$\mathcal{K}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n}$$

and

$$\mathcal{L}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma},$$

defined by (1.7), (1.8) and (1.9), respectively, to be univalent in $\mathbb{D}$. We also determine the order of convexity of the functions defined by using the above-mentioned integral operators (1.7), (1.8) and (1.9).

Some potentially useful developments on the class of analytic and univalent functions can be found in the earlier works (see, for example, [1,7,8]; see also [9,21]. For studies using a number of derivative, integral and convolution operators, based upon various special functions including the Kummer, the Gauss and the generalized hypergeometric functions, we refer the interested reader to such more recent works as (for example) [14–16,22,25–28,34,38,41–44], as well as to many references therein to the related earlier investigations on the usages of many families of the derivative, integral and convolution operators in Geometric Function Theory of Complex Analysis.

2. A set of lemmas

The following lemmas will be required in our present investigations.

Lemma 1. (see Pescar [30]) *Let $\eta \in \mathbb{C}$ and $c \in \mathbb{C}$ be such that*

$$\Re(\eta) > 0 \quad \text{and} \quad |c| \leqq 1 \ (c \neq -1).$$

If $f \in \mathcal{A}$ satisfies the following inequality:

$$\left| c|z|^{2\eta} + \left(1 - |z|^{2\eta}\right) \frac{zf''(z)}{\eta f'(z)} \right| \leqq 1 \quad (z \in \mathbb{D}),$$

then the function F_η defined by

$$F_\eta(z) = \left(\eta \int_0^z t^{\eta-1} f'(t)dt\right)^{1/\eta}$$

is in the normalized univalent function class $\mathcal{S}$.

Lemma 2. (see Pascu [29]) *Let $\alpha \in \mathbb{C}$ be such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies the following inequality*:

$$\left(\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)}\right) \left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \qquad (z \in \mathbb{D}),$$

then, for all $\beta \in \mathbb{C}$ such that

$$\Re(\beta) \geqq \Re(\alpha),$$

the function F_β defined by

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)dt\right]^{1/\beta}$$

is in the normalized univalent function class $\mathcal{S}$.

Lemma 3. (see Yağmur [45]) *Let $\mu, \nu \in \mathbb{R}$, where $\mu \pm \nu$ is not a negative odd integer*,

$$M = 4(K + 1)(F + 1) = (\mu + 5)^2 - \nu^2$$

and

$$N = 4KF = (\mu + 3)^2 - \nu^2.$$

Then, for all $z \in \mathbb{D}$, *the function* $h_{\mu,\nu}$ *defined by* (1.4) *satisfies the following inequalities*:

$$\left| h'_{\mu,\nu}(z) - \frac{h_{\mu,\nu}(z)}{z} \right| \leqq \frac{M}{N(M-2)} \qquad (M > 2), \tag{2.1}$$

$$\left| \frac{z h'_{\mu,\nu}(z)}{h_{\mu,\nu}(z)} - 1 \right| \leqq \frac{M(M-1)}{(M-2)(MN-M-N)} \qquad (M > 2) \tag{2.2}$$

and

$$\left| \frac{z h''_{\mu,\nu}(z)}{h'_{\mu,\nu}(z)} \right| \leqq \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} \qquad (M > 3). \tag{2.3}$$

Lemma 4. *Under the conditions in Lemma* 3, *the following inequality is satisfied for the function* $h_{\mu,\nu}$ *defined by* (1.4):

$$\left| z h'_{\mu,\nu}(z) \right| \leqq \frac{2MN-3N+4M}{N(2M-3)} \qquad \left(M > \frac{3}{2}\right). \tag{2.4}$$

Proof. By using the well-known triangle inequality and the following equalities:

$$(3/2)^{n-1} \geqq \frac{n+1}{2} \quad (n \in \mathbb{N};\ K+1 > 0;\ F+1 > 0)$$

and

$$(K+1)_{n-1}(F+1)_{n-1} \geqq (K+1)^{n-1}(F+1)^{n-1} \quad (n \in \mathbb{N};\ K+1 > 0;\ F+1 > 0),$$

we have

$$\begin{aligned}
\left| z h'_{\mu,\nu}(z) \right| &= \left| z + \sum_{n=1}^{\infty} \frac{(n+1)\,(-1/4)^n}{(K)_n (F)_n} z^{n+1} \right| \\
&\leqq 1 + \sum_{n=1}^{\infty} \frac{n+1}{4^n (K)_n (F)_n} \\
&= 1 + \frac{1}{2KF} \sum_{n=1}^{\infty} \frac{(n+1)/2}{4^{n-1}(K+1)_{n-1}(F+1)_{n-1}} \\
&= 1 + \frac{1}{2KF} \sum_{n=1}^{\infty} \frac{(n+1)/2}{(3/2)^{n-1}(8/3)^{n-1}(K+1)_{n-1}(F+1)_{n-1}} \\
&\leqq 1 + \frac{1}{2KF} \sum_{n=1}^{\infty} \left(\frac{3}{8(K+1)(F+1)} \right)^{n-1} \\
&= 1 + \frac{2}{N} \sum_{n=1}^{\infty} \left(\frac{3}{2M} \right)^{n-1} \\
&= \frac{2MN-3N+4M}{N(2M-3)} \quad (M > 3/2).
\end{aligned}$$

□

3. Univalence and convexity conditions for the integral operator in (1.7)

Firstly, we consider the integral operator defined by (1.7).

Theorem 1. *Let* $j = 1, 2, \cdots, n$ *and let* $\mu_j,$, ν_j, M_j, N_j, F_j *and* $K_j \in \mathbb{R}$ *be such that* $\mu_j \pm \nu_j$ *is not a negative odd integer*,

$$M_j = 4(F_j + 1)(K_j + 1) = (\mu_j + 5)^2 - \nu_j^2 > 3$$

and

$$N_j = 4K_jF_j = (\mu_j + 3)^2 - \nu_j^2 > 0.$$

Also let γ, c, α_j *and* β_j *be in* $\mathbb{C}$ *such that*

$$\Re(\gamma) > 0, \quad |c| \leq 1 \; (c \neq -1), \quad \alpha_j, \beta_j \neq 0.$$

Suppose that these numbers satisfy the following inequality:

$$|c| + \frac{1}{|\gamma|}\left(\frac{2M(2M-3)}{(M-3)(2MN-4M-3N)}\sum_{j=1}^{n}|\alpha_j| + \frac{M(M-1)}{(M-2)(MN-M-N)}\sum_{j=1}^{n}|\beta_j|\right) \leqq 1, \tag{3.1}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \qquad \textit{and} \qquad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{J}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma},$$

defined by (1.7)*, is in the normalized univalent function class* $\mathcal{S}$.

Proof. Let us define the function φ by

$$\varphi(z) := \mathcal{J}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;1}(z) = \int_0^z \prod_{j=1}^{n}(h'_{\mu_j,\nu_j}(t))^{\alpha_j}\left(\frac{h_{\mu_j,\nu_j}(t)}{t}\right)^{\beta_j} dt. \tag{3.2}$$

First of all, we observe that $h_{\mu_j,\nu_j}(0) = h'_{\mu_j,\nu_j}(0) - 1 = 0$, since $h_{\mu_j,\nu_j} \in \mathcal{A}$ for all $j = 1, 2, \cdots, n$. Therefore, clearly, $\varphi(z) \in \mathcal{A}$, that is, $\varphi(0) = \varphi'(0) - 1 = 0$. On the other hand, we have

$$\varphi'(z) = \prod_{j=1}^{n}(h'_{\mu_j,\nu_j}(z))^{\alpha_j}\left(\frac{h_{\mu_j,\nu_j}(z)}{z}\right)^{\beta_j}. \tag{3.3}$$

Differentiating both sides of (3.3), we obtain

$$\frac{z\varphi''(z)}{\varphi'(z)} = \sum_{j=1}^{n}\alpha_j\frac{zh''_{\mu_j,\nu_j}(z)}{h'_{\mu_j,\nu_j}(z)} + \sum_{j=1}^{n}\beta_j\left(\frac{zh'_{\mu_j,\nu_j}(z)}{h_{\mu_j,\nu_j}(z)} - 1\right) \tag{3.4}$$

and, from (2.2) and (2.3), we have

$$\begin{aligned}\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| &\leqq \sum_{j=1}^{n}\left(|\alpha_j|\left|\frac{zh''_{\mu_j,\nu_j}(z)}{h'_{\mu_j,\nu_j}(z)}\right| + |\beta_j|\left|\frac{zh'_{\mu_j,\nu_j}(z)}{h_{\mu_j,\nu_j}(z)} - 1\right|\right)\\ &\leqq \sum_{j=1}^{n}\left(|\alpha_j|\frac{2M_j(2M_j-3)}{(M_j-3)(2M_jN_j-4M_j-3N_j)} + |\beta_j|\frac{M_j(M_j-1)}{(M_j-2)(M_jN_j-M_j-N_j)}\right)\\ &\leqq \sum_{j=1}^{n}\left(|\alpha_j|\frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + |\beta_j|\frac{M(M-1)}{(M-2)(MN-M-N)}\right).\end{aligned} \tag{3.5}$$

Therefore, we get

$$\begin{aligned}
&\left|c|z|^{2\gamma} + (1-|z|^{2\gamma})\frac{z\varphi''(z)}{\gamma\varphi'(z)}\right| \\
&\leq |c| + \frac{1}{|\gamma|}\left(\frac{2M(2M-3)}{(M-3)(2MN-4M-3N)}\sum_{j=1}^{n}|\alpha_j| + \frac{M(M-1)}{(M-2)(MN-M-N)}\sum_{j=1}^{n}|\beta_j|\right) \\
&\leq 1.
\end{aligned} \tag{3.6}$$

By Lemma 1, the inequalities in (3.6) imply that the function $\varphi \in \mathcal{S}$. □

Theorem 2. *Let $j = 1, 2, \cdots, n$ and let μ_j, ν_j, M_j, N_j, F_j and $K_j \in \mathbb{R}$ be such that $\mu_j \pm \nu_j$ is not a negative odd integer,*

$$M_j = 4(F_j+1)(K_j+1) = (\mu_j+5)^2 - \nu_j^2 > 3$$

and

$$N_j = 4K_jF_j = (\mu_j+3)^2 - \nu_j^2 > 0.$$

Also let γ, α_j and β_j be in $\mathbb{C}$ such that

$$\Re(\gamma) > 0 \qquad \text{and} \qquad \alpha_j, \beta_j \neq 0.$$

Suppose that these numbers satisfy the following inequality:

$$\Re\{\gamma\} \geqq \sum_{j=1}^{n}|\alpha_j|\frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n}|\beta_j|\frac{M(M-1)}{(M-2)(MN-M-N)}, \tag{3.7}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \qquad \text{and} \qquad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{J}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma},$$

defined by (1.7), *is in the normalized univalent function class $\mathcal{S}$.*

Proof. Let us define the function φ as in (3.2). Then we have (3.3) and (3.4). By using similar method as in our detivation of (3.5), we obtain

$$\begin{aligned}
&\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\left|\frac{z\varphi''(z)}{\varphi'(z)}\right| \\
&\leqq \frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\left\{\sum_{j=1}^{n}|\alpha_j|\frac{2M_j(2M_j-3)}{(M_j-3)(2M_jN_j-4M_j-3N_j)} + \sum_{j=1}^{n}|\beta_j|\frac{M_j(M_j-1)}{(M_j-2)(M_jN_j-M_j-N_j)}\right\} \\
&\leqq \frac{1}{\Re(\gamma)}\left\{\sum_{j=1}^{n}|\alpha_j|\frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n}|\beta_j|\frac{M(M-1)}{(M-2)(MN-M-N)}\right\} \\
&\leqq 1.
\end{aligned} \tag{3.8}$$

By Lemma 2, the inequalities in (3.8) imply that the function $\varphi \in \mathcal{S}$. □

Theorem 3. *Let $j = 1, 2, \cdots, n$ and let μ_j, ν_j, M_j, N_j, F_j and $K_j \in \mathbb{R}$ be such that $\mu_j \pm \nu_j$ is not a negative odd integer,*

$$M_j = 4(F_j + 1)(K_j + 1) = (\mu_j + 5)^2 - \nu_j^2 > 3$$

and

$$N_j = 4K_jF_j = (\mu_j + 3)^2 - \nu_j^2 > 0.$$

Also let γ, α_j and β_j be in $\mathbb{C}$ such that

$$\Re(\gamma) > 0 \qquad \text{and} \qquad \alpha_j, \beta_j \neq 0.$$

Suppose that these numbers satisfy the following inequality:

$$0 < \sum_{j=1}^{n} \left\{ |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + |\beta_j| \frac{M(M-1)}{(M-2)(MN-M-N)} \right\} \leqq 1, \tag{3.9}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \qquad \text{and} \qquad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{J}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;1},$$

defined by (1.7) *with $\gamma = 1$, is convex of order δ given by*

$$\delta = 1 - \sum_{j=1}^{n} \left\{ |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + |\beta_j| \frac{M(M-1)}{(M-2)(MN-M-N)} \right\}. \tag{3.10}$$

Proof. Let us define the function φ by (3.2). Then we readily have (3.3) and (3.4). By using similar method as in the derivation of (3.5), we obtain

$$\begin{aligned} \left| \frac{z\varphi''(z)}{\varphi'(z)} \right| &\leqq \sum_{j=1}^{n} |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n} |\beta_j| \frac{M(M-1)}{(M-2)(MN-M-N)} \\ &= 1 - \delta. \end{aligned} \tag{3.11}$$

Therefore, the function φ is convex of order

$$\delta = 1 - \sum_{j=1}^{n} \left\{ |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + |\beta_j| \frac{M(M-1)}{(M-2)(MN-M-N)} \right\}. \tag{3.12}$$

□

From Theorem 1 with $n = 1$, $\mu_1 = 1/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 1. *Let γ, c, α and β be in $\mathbb{C}$ such that $\Re(\gamma) > 0$, $|c| \leq 1 (c \neq -1)$, $\alpha \neq 0$ and $\beta \neq 0$. If these numbers satisfy the following inequality:*

$$|c| + \frac{1}{|\gamma|} \left(\frac{95}{423} |\alpha| + \frac{145}{1484} |\beta| \right) \leqq 1,$$

then the function:

$$\left[\gamma \int_0^z t^{-\frac{1}{2}\alpha - \beta + \gamma - 1} (\sin \sqrt{t})^{\alpha} (2 - 2\cos \sqrt{t})^{\beta} dt \right]^{1/\gamma}$$

is in the normalized univalent function class $\mathcal{S}$.

Example 1. *From Corollary* 1, *we can easily get the following consequence* (*see Figure* 1 *below*) :

$$f_1(z) := \int_0^z t^{-\frac{3}{2}} \sin \sqrt{t}(2 - 2\cos \sqrt{t})dt \in \mathcal{S}.$$

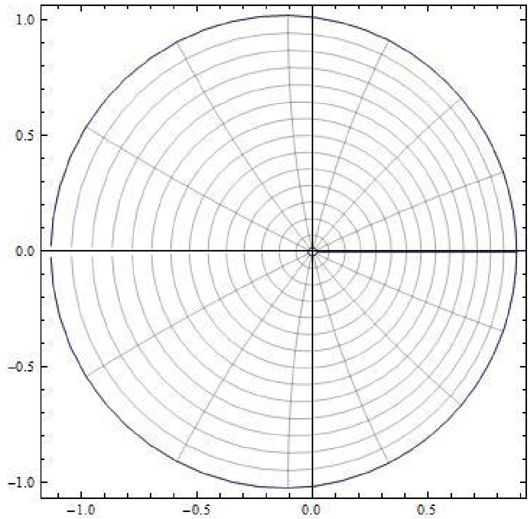

Figure 1. The image of f_1 on $\mathbb{D}$.

From Theorem 1 with $n = 1$, $\mu_1 = 3/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 2. *Let* γ, c, α *and* β *be in* $\mathbb{C}$ *such that* $\Re(\gamma) > 0$, $|c| \leq 1 (c \neq -1)$, $\alpha \neq 0$ *and* $\beta \neq 0$. *If these numbers satisfy the following inequality*:

$$|c| + \frac{1}{|\gamma|}\left(\frac{189}{1573}|\alpha| + \frac{861}{15560}|\beta|\right) \leqq 1,$$

then the integral operator:

$$\left[3^{\alpha+\beta}2^{\beta}\gamma \int_0^z t^{-\frac{3}{2}\alpha-\beta+\gamma-1}(\sin \sqrt{t} - \sqrt{t}\cos \sqrt{t})^{\alpha}(\sqrt{t} - \sin \sqrt{t})^{\beta} dt\right]^{1/\gamma}$$

is in the normalized univalent function class $\mathcal{S}$.

From Theorem 3 with $n = 1$, $\mu_1 = 1/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 3. *Let* α *and* β *be nonzero complex numbers such that*

$$0 < \frac{95}{423}|\alpha| + \frac{145}{1484}|\beta| \leqq 1.$$

Then the function:

$$\int_0^z t^{-\frac{1}{2}\alpha-\beta}(\sin \sqrt{t})^{\alpha}(2 - 2\cos \sqrt{t})^{\beta} dt$$

is convex of order δ *given by*

$$\delta = 1 - \frac{95}{423}|\alpha| - \frac{145}{1484}|\beta|.$$

From Theorem 3 with $n = 1$, $\mu_1 = 3/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 4. *Let α and β be nonzero complex numbers such that*

$$0 < \frac{189}{1573}|\alpha| + \frac{861}{15560}|\beta| \leqq 1.$$

Then the function:

$$3^{\alpha+\beta}2^{\beta}\int_0^z t^{-\frac{3}{2}\alpha-\beta}(\sin\sqrt{t} - \sqrt{t}\cos\sqrt{t})^{\alpha}(\sqrt{t} - \sin\sqrt{t})^{\beta}dt$$

is convex of order δ given by

$$\delta = 1 - \frac{189}{1573}|\alpha| - \frac{861}{15560}|\beta|.$$

4. Univalence and convexity conditions for the integral operator in (1.8)

In this section, we investigate univalence and convexity conditions for the integral operator defined by (1.8).

Theorem 4. *Let $j = 1, 2, \cdots, n$ and let μ_j, ν_j, M_j, N_j, F_j, and $K_j \in \mathbb{R}$ be such that $\mu_j \pm \nu_j$ is not a negative odd integer*,

$$M_j = 4(F_j + 1)(K_j + 1) = (\mu_j + 5)^2 - \nu_j^2 > 2$$

and

$$N_j = 4K_jF_j = (\mu_j + 3)^2 - \nu_j^2 > 0.$$

Also let c and α_j be in $\mathbb{C}$ such that

$$|c| \leq 1 \ (c \neq -1) \qquad and \qquad \Re\left(1 + \sum_{j=1}^{n}\alpha_j\right) > 0.$$

Suppose that these numbers satisfy the following inequality:

$$|c| + \frac{\sum_{j=1}^{n}|\alpha_j|}{\left|1 + \sum_{j=1}^{n}\alpha_j\right|}\left(\frac{M(M-1)}{(M-2)(MN-M-N)} + \frac{2MN-3N+4M}{N(2M-3)}\right) \leqq 1, \tag{4.1}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \qquad and \qquad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{K}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n},$$

defined by (1.8), *is in the normalized univalent function class $\mathcal{S}$.*

Proof. Let us define the functions ψ by

$$\psi(z) = \int_0^z \prod_{j=1}^{n}\left(\frac{h_{\mu_j,\nu_j}(t)e^{h_{\mu_j,\nu_j}(t)}}{t}\right)^{\alpha_j} dt. \tag{4.2}$$

Then $\psi(0) = \psi'(0) - 1 = 0$. Differentiating both sides of (4.2) logarithmically, we obtain

$$\frac{z\psi''(z)}{\psi'(z)} = \sum_{j=1}^{n} \alpha_j \left(\frac{zh'_{\mu_j,\nu_j}(z)}{h_{\mu_j,\nu_j}(z)} + zh'_{\mu_j,\nu_j}(z) - 1 \right) \tag{4.3}$$

and, from (2.2) in Lemma 3 and Lemma 4, we have

$$\begin{aligned} \left| \frac{z\psi''(z)}{\psi'(z)} \right| &\leqq \sum_{j=1}^{n} |\alpha_j| \left\{ \left| \frac{zh'_{\mu_j,\nu_j}(z)}{h_{\mu_j,\nu_j}(z)} - 1 \right| + |zh'_{\mu_j,\nu_j}(z)| \right\} \\ &\leqq \sum_{j=1}^{n} |\alpha_j| \left\{ \frac{M_j(M_j-1)}{(M_j-2)(M_jN_j - M_j - N_j)} + \frac{2M_jN_j - 3N_j + 4M_j}{N_j(2M_j-3)} \right\} \\ &\leqq \sum_{j=1}^{n} |\alpha_j| \left\{ \frac{M(M-1)}{(M-2)(MN-M-N)} + \frac{2MN-3N+4M}{N(2M-3)} \right\}. \end{aligned} \tag{4.4}$$

Therefore, we get

$$\begin{aligned} &\left| c|z|^{2\left(1+\sum_{j=1}^{n}\alpha_j\right)} + \left(1 - |z|^{2\left(1+\sum_{j=1}^{n}\alpha_j\right)}\right) \frac{z\psi''(z)}{\left(1+\sum_{j=1}^{n}\alpha_j\right)\psi'(z)} \right| \\ &\leq |c| + \left| \frac{z\psi''(z)}{\left(1+\sum_{j=1}^{n}\alpha_j\right)\psi'(z)} \right| \\ &\leq |c| + \frac{\sum_{j=1}^{n}|\alpha_j|}{\left|1+\sum_{j=1}^{n}\alpha_j\right|} \left(\frac{M(M-1)}{(M-2)(MN-M-N)} + \frac{2MN-3N+4M}{N(2M-3)} \right) \\ &\leq 1. \end{aligned} \tag{4.5}$$

By Lemma 1 with

$$\eta = 1 + \sum_{j=1}^{n} \alpha_j,$$

the inequalities in (4.5) imply that the function $\psi \in \mathcal{S}$. □

Theorem 5. *Let $j = 1, 2, \cdots, n$ and let μ_j, ν_j, M_j, N_j, F_j and $K_j \in \mathbb{R}$ be such that $\mu_j \pm \nu_j$ is not a negative odd integer,*

$$M_j = 4(F_j+1)(K_j+1) = (\mu_j+5)^2 - \nu_j^2 > 2$$

and

$$N_j = 4K_jF_j = (\mu_j+3)^2 - \nu_j^2 > 0.$$

Also let α_j be in $\mathbb{C}$ such that $\Re\left(\alpha_j\right) \geqq 0$. Suppose that these numbers satisfy the following inequality:

$$\sum_{j=1}^{n} |\alpha_j| \left(\frac{M(M-1)}{(M-2)(MN-M-N)} + \frac{2MN-3N+4M}{N(2M-3)} \right) \leqq 1, \tag{4.6}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \quad \text{and} \quad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{K}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n},$$

defined by (1.8), *is in the normalized univalent function class $\mathcal{S}$.*

Proof. Let us define the function ψ as in (4.2). By using similar methods in the proof of Theorem 4, we obtain (4.4). Therefore, we have

$$\begin{aligned}&\left(1-|z|^2\right)\left|\frac{z\psi''(z)}{\psi'(z)}\right|\\ &\leqq \sum_{j=1}^{n}|\alpha_j|\left(\frac{M(M-1)}{(M-2)(MN-M-N)}+\frac{2MN-3N+4M}{N(2M-3)}\right)\\ &\leqq 1.\end{aligned} \tag{4.7}$$

By Lemma 2 with $\alpha = 1$ and $\beta = 1 + \sum_{j=1}^{n}\alpha_j$, the inequalities in (4.7) imply that the function $\psi \in \mathcal{S}$. □

Theorem 6. *Let $j = 1, 2, \cdots, n$ and let μ_j, ν_j, M_j, N_j, F_j and $K_j \in \mathbb{R}$ be such that $\mu_j \pm \nu_j$ is not a negative odd integer,*

$$M_j = 4(F_j+1)(K_j+1) = (\mu_j+5)^2 - \nu_j^2 > 2$$

and

$$N_j = 4K_jF_j = (\mu_j+3)^2 - \nu_j^2 > 0.$$

Also let α_j be in $\mathbb{C}$ such that

$$\Re\left(1+\sum_{j=1}^{n}\alpha_j\right) > 0.$$

Suppose that these numbers satisfy the following inequality:

$$0 < \sum_{j=1}^{n}|\alpha_j|\left(\frac{M(M-1)}{(M-2)(MN-M-N)}+\frac{2MN-3N+4M}{N(2M-3)}\right) \leqq 1. \tag{4.8}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \qquad \textit{and} \qquad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{K}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;n},$$

defined by (1.8), *is convex of order δ given by*

$$\delta = 1 - \sum_{j=1}^{n}|\alpha_j|\left(\frac{M(M-1)}{(M-2)(MN-M-N)}+\frac{2MN-3N+4M}{N(2M-3)}\right). \tag{4.9}$$

Proof. Let us define the function ψ as given by (4.2). Using similar methods in the proof of Theorem 4, we obtain

$$\begin{aligned}\left|\frac{z\psi''(z)}{\psi'(z)}\right| &\leqq \sum_{j=1}^{n}|\alpha_j|\left(\left|\frac{zh'_{\mu_j,\nu_j}(z)}{h_{\mu_j,\nu_j}(z)}-1\right|+\left|zh'_{\mu_j,\nu_j}(z)\right|\right)\\ &\leqq \sum_{j=1}^{n}|\alpha_j|\left(\frac{M(M-1)}{(M-2)(MN-M-N)}+\frac{2MN-3N+4M}{N(2M-3)}\right)\\ &= 1-\delta.\end{aligned} \tag{4.10}$$

Therefore, the function ψ is convex of order

$$\delta = 1 - \sum_{j=1}^{n} |\alpha_j| \left(\frac{M(M-1)}{(M-2)(MN-M-N)} + \frac{2MN-3N+4M}{N(2M-3)} \right).$$

□

From Theorem 4 with $n = 1$, $\mu_1 = 1/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 5. *Let α and c be in $\mathbb{C}$ such that $\Re(1+\alpha) > 0$ and $|c| \le 1 (c \neq -1)$. If these numbers satisfy the following inequality*:

$$|c| + \frac{107693}{84588} \frac{|\alpha|}{|1+\alpha|} \leqq 1,$$

then the function:

$$\left[2^{\alpha}(1+\alpha) \int_0^z (1 - \cos\sqrt{t})^{\alpha} e^{2\alpha(1-\cos\sqrt{t})} dt \right]^{1/(1+\alpha)} \tag{4.11}$$

is in the normalized univalent function class $\mathcal{S}$.

Example 2. *From Corollary 5 with $\alpha = 1$, we can easily get the following consequence*:

$$f_2(z) := 2 \int_0^z \left[(1 - \cos\sqrt{t}) e^{2(1-\cos\sqrt{t})} dt \right]^{1/2} \in \mathcal{S}.$$

In fact, by a simple computation, we obtain

$$\frac{zf_2'(z)}{f_2(z)} + 1 + \frac{zf_2''(z)}{f_2'(z)} = 1 + \frac{\sqrt{z}\sin\sqrt{z}}{2(1-\cos\sqrt{z})} + \sqrt{z}\sin\sqrt{z} =: g(z).$$

It also holds true that $\Re(g(z)) > 0$ for all $z \in \mathbb{D}$ (see Figure 2). Therefore, f_2 is a $1/2$-convex function [19, Vol. I, p. 142]. *Thus it follows from* [19, Vol. I, p. 142] *that f_2 belongs to the class $\mathcal{S}$.*

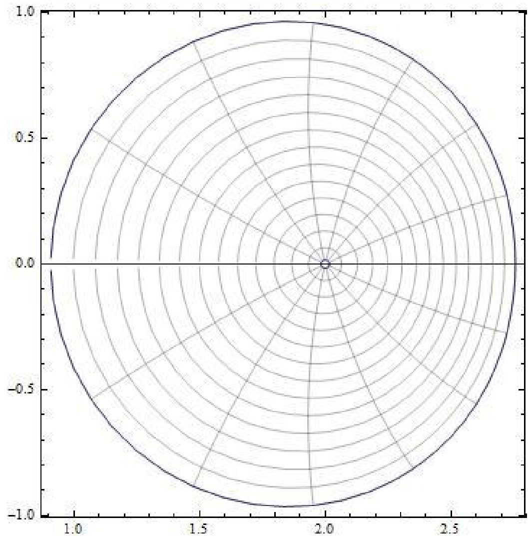

Figure 2. The image of g on $\mathbb{D}$.

From Theorem 4 with $n = 1$, $\mu_1 = 3/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 6. *Let* α *and* c *be in* $\mathbb{C}$ *such that* $\Re(1+\alpha) > 0$ *and* $|c| \leq 1 (c \neq -1)$. *If these numbers satisfy the following inequality*:

$$|c| + \frac{97387}{84024}\frac{|\alpha|}{|1+\alpha|} \leqq 1,$$

then the function:

$$\left[6^{\alpha}(1+\alpha)\int_0^z \left(1 - \frac{\sin\sqrt{t}}{\sqrt{t}}\right)^{\alpha} e^{6\alpha\left(1-\frac{\sin\sqrt{t}}{\sqrt{t}}\right)}dt\right]^{1/(1+\alpha)} \tag{4.12}$$

is in the normalized univalent function class $\mathcal{S}$.

From Theorem 6 with $n = 1$, $\mu_1 = 1/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 7. *Let* α *be a complex number such that*

$$\Re(1+\alpha) > 0 \qquad \textit{and} \qquad 0 < \frac{107693}{84588}|\alpha| \leqq 1.$$

Then the function defined by (4.11) *is convex of order* δ *given by*

$$\delta = 1 - \frac{107693}{84588}|\alpha|.$$

From Theorem 6 with $n = 1$, $\mu_1 = 3/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 8. *Let* α *be a complex number such that*

$$\Re(1+\alpha) > 0 \qquad \textit{and} \qquad 0 < \frac{97387}{84024}|\alpha| \leqq 1.$$

Then the function defined by (4.12) *is convex of order* δ *given by*

$$\delta = 1 - \frac{97387}{84024}|\alpha|.$$

5. Univalence and convexity conditions for the integral operator in (1.9)

Here, in this section, we derive univalence and convexity conditions for the integral operator defined by (1.9).

Theorem 7. *Let* $j = 1, 2, \cdots, n$ *and let* μ_j, ν_j, M_j, N_j, F_j *and* $K_j \in \mathbb{R}$ *be such that* $\mu_j \pm \nu_j$ *is not a negative odd integer*,

$$M_j = 4(F_j + 1)(K_j + 1) = (\mu_j + 5)^2 - \nu_j^2 > 3$$

and

$$N_j = 4K_jF_j = (\mu_j + 3)^2 - \nu_j^2 > 0.$$

Also let γ, c, α_j *and* β_j *be in* $\mathbb{C}$ *such that*

$$\Re(\gamma) > 0, \quad |c| \leq 1\ (c \neq -1) \quad \textit{and} \quad \alpha_j, \beta_j \neq 0.$$

Suppose that these numbers satisfy the following inequality:

$$|c| + \frac{1}{|\gamma|}\left\{\sum_{j=1}^{n}|\alpha_j|\frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n}|\beta_j|\frac{2MN-3N+4M}{N(2M-3)}\right\} \leqq 1, \tag{5.1}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \qquad \textit{and} \qquad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{L}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma},$$

defined by (1.9), *is in the normalized univalent function class* $\mathcal{S}$.

Proof. Let us define the function ϕ by

$$\phi(z) := \mathcal{L}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;1}(z) = \int_0^z \prod_{j=1}^{n}\left(h'_{\mu_j,\nu_j}(t)\right)^{\alpha_j}\left(e^{h_{\mu_j,\nu_j}(t)}\right)^{\beta_j} dt, \tag{5.2}$$

so that

$$\phi'(z) = \prod_{j=1}^{n}\left(h'_{\mu_j,\nu_j}(z)\right)^{\alpha_j}\left(e^{h_{\mu_j,\nu_j}(z)}\right)^{\beta_j}. \tag{5.3}$$

We observe that

$$h_{\mu_j,\nu_j}(0) = h'_{\mu_j,\nu_j}(0) - 1 = 0 \qquad (\forall\ j = 1, 2, \cdots, n),$$

since $h_{\mu_j,\nu_j} \in \mathcal{A}$ for all $j = 1, 2, \cdots, n$. Therefore, clearly, $\phi(z) \in \mathcal{A}$, that is, $\phi(0) = \phi'(0) - 1 = 0$.

Now, upon differentiating both sides of (5.3) logarithmically, we obtain

$$\frac{z\phi''(z)}{\phi'(z)} = \sum_{j=1}^{n}\alpha_j\frac{zh''_{\mu_j,\nu_j}(z)}{h'_{\mu_j,\nu_j}(z)} + \sum_{j=1}^{n}\beta_j z h'_{\mu_j,\nu_j}(z). \tag{5.4}$$

Furthermore, by (2.3) and (2.4), we have

$$\begin{aligned}\left|\frac{z\phi''(z)}{\phi'(z)}\right| &\leqq \sum_{j=1}^{n}|\alpha_j|\left|\frac{zh''_{\mu_j,\nu_j}(z)}{h'_{\mu_j,\nu_j}(z)}\right| + \sum_{j=1}^{n}|\beta_j||zh'_{\mu_j,\nu_j}(z)| \\ &\leqq \sum_{j=1}^{n}|\alpha_j|\left(\frac{2M_j(2M_j-3)}{(M_j-3)(2M_jN_j-4M_j-3N_j)}\right) + \sum_{j=1}^{n}|\beta_j|\left(\frac{2M_jN_j-3N_j+4M_j}{N_j(2M_j-3)}\right) \\ &\leqq \sum_{j=1}^{n}|\alpha_j|\left(\frac{2M(2M-3)}{(M-3)(2MN-4M-3N)}\right) + \sum_{j=1}^{n}|\beta_j|\left(\frac{2MN-3N+4M}{N(2M-3)}\right).\end{aligned} \tag{5.5}$$

Therefore, we have

$$\begin{aligned}&\left|c|z|^{2\gamma} + (1-|z|^{2\gamma})\frac{z\phi''(z)}{\gamma\phi'(z)}\right| \\ &\leqq |c| + \left|\frac{z\phi''(z)}{\phi'(z)}\right| \\ &\leqq |c| + \frac{1}{|\gamma|}\left\{\sum_{j=1}^{n}|\alpha_j|\frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n}|\beta_j|\frac{2MN-3N+4M}{N(2M-3)}\right\} \\ &\leqq 1.\end{aligned} \tag{5.6}$$

By Lemma 1, the inequalities in (5.6) imply that the function $\phi \in \mathcal{S}$. □

Theorem 8. *Let $j = 1, 2, \cdots, n$ and let μ_j, ν_j, M_j, N_j, F_j and $K_j \in \mathbb{R}$ be such that $\mu_j \pm \nu_j$ is not a negative odd integer,*

$$M_j = 4(F_j + 1)(K_j + 1) = (\mu_j + 5)^2 - \nu_j^2 > 3$$

and

$$N_j = 4K_jF_j = (\mu_j + 3)^2 - \nu_j^2 > 0.$$

Also let γ, α_j and β_j be in $\mathbb{C}$ such that $\Re(\gamma) > 0$ and $\alpha_j, \beta_j \neq 0$. Suppose that these numbers satisfy the following inequality:

$$\Re\{\gamma\} \geqq \sum_{j=1}^{n} |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n} |\beta_j| \frac{2MN-3N+4M}{N(2M-3)}, \tag{5.7}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \qquad \text{and} \qquad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{L}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;\gamma},$$

defined by (1.9), *is in the normalized univalent function class $\mathcal{S}$.*

Proof. Let us define the function ϕ as in (5.2). By using similar methods in (5.5), we obtain

$$\begin{aligned}
&\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)} \left| \frac{z\phi''(z)}{\phi'(z)} \right| \\
&\leqq \frac{1-|z|^{\Re(\gamma)}}{\Re(\gamma)} \left\{ \sum_{j=1}^{n} |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n} |\beta_j| \frac{2MN-3N+4M}{N(2M-3)} \right\} \\
&\leqq \frac{1}{\Re(\gamma)} \left\{ \sum_{j=1}^{n} |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n} |\beta_j| \frac{2MN-3N+4M}{N(2M-3)} \right\} \\
&\leqq 1.
\end{aligned} \tag{5.8}$$

By Lemma 2, the inequalities in (5.8) imply that the integral operator $\phi \in \mathcal{S}$. □

Theorem 9. *Let $j = 1, 2, \cdots, n$ and let μ_j, ν_j, M_j, N_j, F_j and $K_j \in \mathbb{R}$ be such that $\mu_j \pm \nu_j$ is not a negative odd integer,*

$$M_j = 4(F_j + 1)(K_j + 1) = (\mu_j + 5)^2 - \nu_j^2 > 3$$

and

$$N_j = 4K_jF_j = (\mu_j + 3)^2 - \nu_j^2 > 0.$$

And let γ, α_j and β_j be in $\mathbb{C}$ such that $\Re(\gamma) > 0$ and $\alpha_j, \beta_j \neq 0$. Suppose that these numbers satisfy the following inequality:

$$0 < \sum_{j=1}^{n} \left\{ |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + |\beta_j| \frac{2MN-3N+4M}{N(2M-3)} \right\} \leqq 1, \tag{5.9}$$

where

$$M = \min\{M_1, M_2, \cdots, M_n\} \qquad \textit{and} \qquad N = \min\{N_1, N_2, \cdots, N_n\}.$$

Then the function:

$$\mathcal{L}^{\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots,\nu_n}_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n;n;1},$$

defined by (1.9) *with* $\gamma = 1$, *is convex of order* δ *given by*

$$\delta = 1 - \sum_{j=1}^{n} \left\{ |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + |\beta_j| \frac{2MN-3N+4M}{N(2M-3)} \right\}. \tag{5.10}$$

Proof. Let us define the function ϕ by (5.2). Then, by using similar methods as above in deriving (5.5), we obtain

$$\begin{aligned}\left|\frac{z\phi''(z)}{\phi'(z)}\right| &\leqq \sum_{j=1}^{n} |\alpha_j| \frac{2M(2M-3)}{(M-3)(2MN-4M-3N)} + \sum_{j=1}^{n} |\beta_j| \frac{2MN-3N+4M}{N(2M-3)} \\ &= 1-\delta.\end{aligned}$$

Therefore, the function ϕ is convex of order δ. □

From Theorem 7 with $n = 1$, $\mu_1 = 1/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 9. *Let* γ, c, α *and* β *be in* $\mathbb{C}$ *such that* $\Re(\gamma) > 0$ *and* $|c| \leq 1 (c \neq -1)$. *If these numbers satisfy the following inequality*:

$$|c| + \frac{1}{|\gamma|}\left(\frac{95}{423}|\alpha| + \frac{67}{57}|\beta|\right) \leqq 1,$$

then the function:

$$\left[\gamma \int_0^z t^{-\frac{1}{2}\alpha+\gamma-1} (\sin\sqrt{t})^{\alpha} e^{2\beta(1-\cos\sqrt{t})} dt\right]^{1/\gamma}$$

is in the normalized univalent function class $\mathcal{S}$.

Example 3. *From Corollary* 9 *with* $c = 0$, $\alpha = \gamma = 1$ *and* $\beta = 1/2$, *we can easily get the following consequence* (*see Figure* 3 *below*) :

$$f_3(z) := \int_0^z t^{-\frac{1}{2}} \sin\sqrt{t} e^{1-\cos\sqrt{t}} dt \in \mathcal{S}.$$

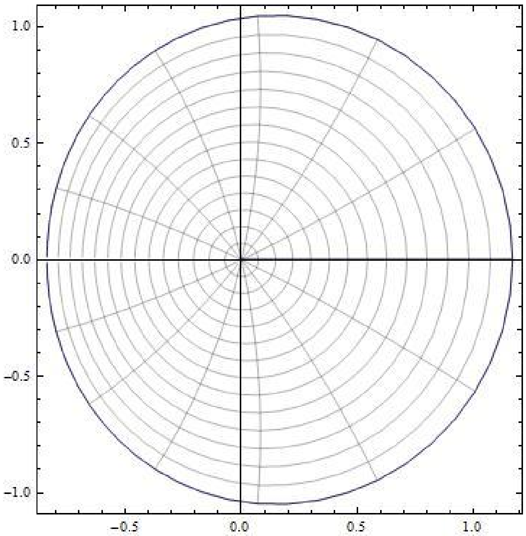

Figure 3. The image of f_3 on $\mathbb{D}$.

From Theorem 7 with $n = 1$, $\mu_1 = 3/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 10. *Let γ, c, α and β be in $\mathbb{C}$ such that $\Re(\gamma) > 0$ and $c \neq -1$. If these numbers satisfy the following inequality*:

$$|c| + \frac{1}{|\gamma|}\left(\frac{189}{1573}|\alpha| + \frac{149}{135}|\beta|\right) \leqq 1,$$

then the function:

$$\left[3^{\alpha}\gamma\int_0^z t^{-\frac{3}{2}\alpha+\gamma-1}(\sin\sqrt{t} - \sqrt{t}\cos\sqrt{t})^{\alpha}e^{6\beta\left(1-\frac{\sin\sqrt{t}}{\sqrt{t}}\right)}dt\right]^{1/\gamma}$$

is in the normalized univalent function class $\mathcal{S}$.

From Theorem 9 with $n = 1$, $\mu_1 = 1/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 11. *Let α and β be nonzero complex numbers such that*

$$0 < \frac{95}{423}|\alpha| + \frac{67}{57}|\beta| \leqq 1.$$

Then the function:

$$\int_0^z t^{-\frac{1}{2}\alpha}(\sin\sqrt{t})^{\alpha}e^{2\beta(1-\cos\sqrt{t})}dt$$

is convex of order δ given by

$$\delta = 1 - \frac{95}{423}|\alpha| - \frac{67}{57}|\beta|.$$

From Theorem 9 with $n = 1$, $\mu_1 = 3/2$ and $\nu_1 = 1/2$, we can obtain the following result.

Corollary 12. *Let α and β be nonzero complex numbers such that*

$$0 < \frac{189}{1573}|\alpha| + \frac{149}{135}|\beta| \leqq 1.$$

Then the function:

$$3^{\alpha}\int_{0}^{z} t^{-\frac{3}{2}\alpha}(\sin\sqrt{t}-\sqrt{t}\cos\sqrt{t})^{\alpha}e^{6\beta\left(1-\frac{\sin\sqrt{t}}{\sqrt{t}}\right)}dt$$

is convex of order δ *given by*

$$\delta = 1-\frac{189}{1573}|\alpha|-\frac{149}{135}|\beta|.$$

6. Conclusions

In the present investigation, we have first introduced a family of integral operators and the Lommel functions of the first kind which, in particular, plays a very important role in the study of pure and applied mathematical sciences. We have then successfully obtained various interesting mapping and geometric properties, such as univalence and convexity conditions, for the integral operators, introduced in the paper and associated with the Lommel function of the first kind, by using the known techniques which were used in the literature. Finally, we have highlighted a number of corollaries and examples together with the associated graphical illustrations that are potentially useful for motivating further researches in this subject and on other related topics.

Usages of the quantum (or q-) calculus happens to provide another popular direction for researches in geometric function theory of complex analysis. This is evidenced by the recently-published survey-cum-expository review article by Srivastava [36]. As a matter of fact, in this survey article, two function classes, using a fractional q-calculus operator, are introduced and one can find the associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points and growth and distortion theorems for each of these function classes which play a very crucial role in the study of the geometric function theory. Furthermore, as already demonstrated by Srivastava [36, p. 340], whereas the quantum (or q-) extensions of the results, which we have presented in this paper, are worthy of investigation, the so-called (p, q)-variations of these suggested q-results would be trivially inconsequential, because the additional parameter p is obviously redundant or superfluous.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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