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Research article

Minimum functional equation and some Pexider-type functional equation on any group

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Abstract: We discuss the solution to the minimum functional equation

$$\min\{\,\eta(xy^{-1}),\eta(xy)\,\} = \eta(x)\eta(y), \qquad x, y \in G,$$

for a real-valued function $\eta : G \to \mathbb{R}$ defined on arbitrary group G. In addition, we examine the Pexider-type functional equation

$$\max\{\,\eta(xy^{-1}),\eta(xy)\,\} = \chi(x)\eta(y) + \psi(x), \qquad x, y \in G,$$

where η, χ and ψ are real mappings acting on arbitrary group G. We also investigate this Pexiderized functional equation that generalizes two functional equations

$$\max\{\,\eta(xy^{-1}),\eta(xy)\,\} = \eta(x)\eta(y), \qquad x, y \in G,$$

and

$$\min\{\,\eta(xy^{-1}),\eta(xy)\,\} = \eta(x)\eta(y), \qquad x, y \in G,$$

with the restriction that the function η satisfies the Kannappan condition.

Keywords: minimum functional equation; Pexider functional equation; Kannappan condition; strictly positive solution
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1. Introduction

Simon and Volkmann considered in [1] the following two equations which are connected with the absolute values of some additive function $\gamma\colon G \to \mathbb{R}$, that is,

$$\eta(x) + \eta(y) = \max\{\, \eta(x - y), \eta(x + y) \,\}, \qquad x, y \in G, \tag{1.1}$$

$$|\eta(x) - \eta(y)| = \min\{\, \eta(x - y), \eta(x + y) \,\}, \qquad x, y \in G, \tag{1.2}$$

for a real function $\eta : G \to \mathbb{R}$ defined on an abelian group $(G, +)$ and both functional equations are satisfied by $\eta(x) = |\gamma(x)|$ where $\gamma(x + y) = \gamma(x) + \gamma(y)$. Moreover, solution of the equation

$$\eta(x)\eta(y) = \max\{\, \eta(x + y), \eta(x - y) \,\}, \tag{1.3}$$

with supposition about G to be an abelian group was presented in the following theorem as:

Theorem 1. *[1, Theorem 2] Let $\eta : G \to \mathbb{R}$, where every element of an abelian group G is divisible by 2 and 3. Then, η fulfills Eq* (1.3) *if and only if $\eta(x) = 0$ or $\eta(x) = e^{|\gamma(x)|}$, $x \in G$, where $\gamma : G \to \mathbb{R}$ is an additive function.*

The solutions of Eqs (1.1) and (1.2) presented by Jarczyk et al. [2] and are demonstrated as:

Theorem 2. *Let $\eta : G \to \mathbb{R}$, where η is defined on an abelian group G. Then η fulfills Eq* (1.1) *if and only if functional Eq* (1.2) *holds and also satisfies $\eta(2x) = 2\eta(x)$ for $x \in G$.*

Furthermore, the most comprehensive study of the equation

$$\eta(x) + \eta(y) = \max\{\, \eta(xy^{-1}), \eta(xy) \,\} \qquad x, y \in G, \tag{1.4}$$

on groups has been presented in [3, 4]. Volkmann has given the solution of Eq (1.4) with supposition that η fulfills the renowned condition called Kannappan condition [5], that is defined as, $\eta(xgy) = \eta(xyg)$ for $x, y, g \in G$. Following that, Toborg [3] gave the characterization of such mappings exhibited in Eq (1.4) without taking into account the Kannappan condition and abelian group G. Their key findings are as follows:

Theorem 3 (For the special case, see [3, 4] for the general case)**.** *Let $\eta : G \to \mathbb{R}$, where η is acting on any group G. Then, η fulfills Eq* (1.4) *if and only if $\eta(x) = |\gamma(x)|$ for every $x \in G$, where $\gamma\colon G \to \mathbb{R}$ is an additive function.*

We suggest the readers consult the articles [2, 6] and related cited references to get some inclusive results and solutions about the functional Eq (1.4). In addition, some stability results of Eqs (1.2) and (1.4) can be found in [7] and [6] respectively.

Recently, in [8], Eq (1.3) presented in a generalized form as

$$\max\{\, \eta(xy^{-1}), \eta(xy) \,\} = \eta(x)\eta(y), \qquad x, y \in G, \tag{1.5}$$

with the exception of additional suppositions that every element of the abelian group is divisible by 2 and 3. Their main result is demonstrated as:

Theorem 4 (see [8]). *Let $\eta : G \to \mathbb{R}$, where G is any group. Then a mapping $\eta\colon G \to \mathbb{R}$ fulfills the Eq* (1.5) *if and only if $\eta \equiv 0$ or there exists a normal subgroup N_η such that*

$$N_\eta = \{\, x \in G \mid \eta(x) = 1 \,\}$$

and

$$xy \in N_\eta \qquad or \qquad xy^{-1} \in N_\eta, \qquad x, y \in G \text{ and } x, y \notin N_\eta;$$

or $\eta(x) = e^{|\gamma(x)|}$, $x \in G$, where $\gamma\colon G \to \mathbb{R}$ is an additive function.

The main objective of this research article is to determine the solution to the generalized minimum functional equation

$$\chi(x)\chi(y) = \min\{\, \chi(xy^{-1}), \chi(xy) \,\}, \qquad x, y \in G. \tag{1.6}$$

With the exception of additional suppositions, we derive some results concerning Eq (1.6) that are appropriate for arbitrary group G rather than abelian group $(G, +)$.

Redheffer and Volkmann [9] determined the solution of the Pexider functional equation

$$\max\{\, h(x + y), h(x - y) \,\} = f(x) + g(x), \qquad x, y \in G, \tag{1.7}$$

for three unknown functions h, f and g acting on abelian group $(G, +)$, which is a generalization of Eq (1.1).

We will also examine the general solutions of the generalized Pexider-type functional equation

$$\max\{\, \eta(xy), \eta(xy^{-1}) \,\} = \chi(x)\eta(y) + \psi(x), \qquad x, y \in G, \tag{1.8}$$

where real functions η, χ, and ψ are defined on any group G. This Pexider functional Eq (1.8) is a common generalization of two previous Eqs (1.4) and (1.5). Readers can see renowned papers [10, 11] and associated references cited therein to obtain comprehensive results and discussions concerning the Pexider version of some functional equations.

2. Analysis of Eq (1.6)

In this research paper, our group G will in general $(G, \cdot)$ not be abelian $(G, +)$, therefore, the group operation will be described multiplicatively as xy for $x, y \in G$. Symbol e will be acknowledged as the neutral element.

Definition 1. *Assume that G is any group. A mapping $\eta : G \to \mathbb{R}$ fulfills the Kannappan condition [5] if*

$$\eta(xzg) = \eta(xgz) \qquad \text{for every } g, x, z \in G.$$

Remark 1. *For every abelian group G, a mapping $\eta : G \to \mathbb{R}$ fulfills the Kannappan condition but converse may not be true.*

Lemma 1. *Suppose that $\eta\colon G \to \mathbb{R}$, where G is an arbitrary group. Let η is a strictly positive solution of the functional Eq* (1.6)*, then $\eta(x) = e^{-|\beta(x)|}$, $x \in G$, where $\beta\colon G \to \mathbb{R}$ is an additive function.*

Proof. By given assumption, $\eta(x) > 0$ for every $x \in G$. Since η satisfies Eq (1.6), as a result, $\frac{1}{\eta}$ also satisfies the functional Eq (1.3), then by well-known theorem from [6], we can get that $\eta(x) = e^{-|\beta(x)|}$, $x \in G$, where $\beta\colon G \to \mathbb{R}$ is an additive function. □

First, we are going to prove the following important lemma which will be utilized several times during computations especially to prove Theorem 5.

Lemma 2. *Let $\eta\colon G \to \mathbb{R}$, where G is an arbitrary group and η is a non-zero solution of Eq* (1.6)*, then the following results hold:*
(1) $\eta(e) = 1$;
(2) $\eta(x^{-1}) = \eta(x)$;
(3) $\eta(x^{-1}yx) = \eta(y)$;
(4) η is central.

Proof. (1). Putting $y = e$ in (1.6), we can obtain that $\eta(x)\eta(e) = \eta(x)$. By given condition, η is non-zero, therefore, we obtain $\eta(e) = 1$.
(2). Using $x = e$ in functional Eq (1.6), we can deduce

$$\begin{aligned} \eta(e)\eta(y^{-1}) &= \min\{\,\eta(e.y^{-1}), \eta(e.y)\,\} \\ \eta(y^{-1}) &= \min\{\,\eta(y^{-1}), \eta(y)\,\}, \end{aligned} \tag{2.1}$$

replacing y^{-1} with y in Eq (2.1) provides that

$$\eta(y) = \min\{\,\eta(y), \eta(y^{-1})\,\}. \tag{2.2}$$

Eqs (2.1) and (2.2) give that $\eta(y^{-1}) = \eta(y)$. Since y is arbitrary, therefore, we have $\eta(x^{-1}) = \eta(x)$ for any $x \in G$.
(3). From functional Eq (1.6), the proof of property (3) can be obtained from the following simple calculation:

$$\begin{aligned} \eta(x)\eta(x^{-1}yx) &= \min\{\,\eta(x(x^{-1}yx)), \eta(x(x^{-1}yx)^{-1})\,\} \\ &= \min\{\,\eta(yx), \eta(xx^{-1}y^{-1}x)\,\} \\ &= \min\{\,\eta((yx)^{-1}), \eta(y^{-1}x)\,\} \\ &= \min\{\,\eta(x^{-1}y^{-1}), \eta((y^{-1}x)^{-1})\,\} \qquad \text{(by Lemma 2(2))} \\ &= \min\{\,\eta(x^{-1}y^{-1}), \eta(x^{-1}y)\,\} \\ \eta(x)\eta(x^{-1}yx) &= \eta(x^{-1})\eta(y) \\ \eta(x)\eta(x^{-1}yx) &= \eta(x)\eta(y) \\ \eta(x^{-1}yx) &= \eta(y). \end{aligned}$$

(4). By Lemma 2(3) and replacing y with xy, we can see that $\eta(x^{-1}(xy)x) = \eta(xy)$, which gives $\eta(xy) = \eta(yx)$, therefore, η is central. □

In addition, we concentrate on the main theorem of Section 2 to describe the solutions η of Eq (1.6).

Theorem 5. *Let $\eta\colon G \to \mathbb{R}$, where G is an arbitrary group. A mapping η is a solution of Eq* (1.6) *if and only if $\eta \equiv 0$ or there exists a normal subgroup H_η of G defined as*

$$H_\eta = \{\, x \in G \mid \eta(x) = 1 \,\}$$

and fulfills the condition that

$$xy^{-1} \in H_\eta \vee xy \in H_\eta \qquad \text{for every} \quad x, y \in G \setminus H_\eta; \tag{2.3}$$

or there exists a normal subgroup H_η of G fulfills the condition that

$$xy^{-1} \in H_\eta \wedge xy \in H_\eta \qquad \text{for every} \quad x, y \in G \setminus H_\eta; \tag{2.4}$$

or $\eta(x) = e^{-|\beta(x)|}$, $x \in G$, where $\beta\colon G \to \mathbb{R}$ is some additive function.

Proof. The 'if' part obviously demonstrates that every mapping η determined in the statement of the theorem is a solution of Eq (1.6). Conversely, suppose that a function $\eta\colon G \to \mathbb{R}$ is a solution of (1.6), then putting $x = y = e$ in Eq (1.6), we get $\eta(e) = \eta(e)\eta(e)$, which gives that either $\eta(e) = 1$ or $\eta(e) = 0$. First, let $\eta(e) = 0$, and then put $y = e$ in (1.6) to get $\eta(x) = 0$ for every $x \in G$. Suppose that $\eta(e) = 1$, then there are the following different cases.

Suppose that there exists $z_\circ \in G$ such that $\eta(z_\circ) \leq 0$. Putting $x = y$ in (1.6), we have $\min\{\,\eta(x^2), \eta(e)\,\} = \eta(x)^2$, which gives that $\eta(x)^2 \leq 1$, so $-1 \leq \eta(x) \leq 1$ but $\eta(z_\circ) \leq 0$, therefore, $-1 \leq \eta(z_\circ) \leq 0$. Let $-1 < \eta(z_\circ) < 0$, then we can compute

$$\begin{aligned}\min\{\,\eta(z_\circ^2), \eta(e)\,\} &= \eta(z_\circ)^2\\ \min\{\,\eta(z_\circ^2), 1\,\} &= \eta(z_\circ)^2 < 1,\end{aligned}$$

which implies that $\eta(z_\circ^2) = \eta(z_\circ)^2$. Moreover,

$$\begin{aligned}\eta(z_\circ) &\geq \min\{\,\eta(z_\circ^3), \eta(z_\circ)\,\}\\ &= \eta(z_\circ^2)\eta(z_\circ)\\ &= \eta(z_\circ)^2\eta(z_\circ)\\ &= \eta(z_\circ)^3\\ \eta(z_\circ) &> \eta(z_\circ),\end{aligned}$$

which gives a contradiction, consequently, either $\eta(z_\circ) = 0$ or $\eta(z_\circ) = -1$. Additionally, it is not possible that $\eta(x) = 0$ and $\eta(y) = -1$ for some $x, y \in G$. Since $\eta(e) = 1$, therefore, either $\eta(x) \in \{\,0, 1\,\}$ or $\eta(x) \in \{\,-1, 1\,\}$. Moreover, define $H_\eta = \{\, x \in G \mid \eta(x) = 1 \,\}$.

It is obvious that $e \in H_\eta$ for the reason that $\eta(e) = 1$. Suppose that $h \in H_\eta$; then from Lemma 2(2) we obtain $\eta(h^{-1}) = \eta(h) = 1$; therefore, $h^{-1} \in H_\eta$. Let $h_1, h_2 \in H_\eta$; then, $\eta(h_1) = \eta(h_2) = 1$, and we can deduce from Eq (1.6) that

$$\begin{aligned}\eta(h_1h_2) &= \eta(h_1h_2)\eta(h_2)\\ &= \min\{\,\eta(h_1h_2^2), \eta(h_1)\,\}\\ &\leq \eta(h_1)\end{aligned}$$

$$\eta(h_1h_2) \leq \eta(h_1)\eta(h_2). \tag{2.5}$$

$$\begin{aligned}\eta(h_1)\eta(h_2) &= \min\{\,\eta(h_1h_2), \eta(h_1h_2^{-1})\,\}\\ &\leq \eta(h_1h_2)\\ \eta(h_1)\eta(h_2) &\leq \eta(h_1h_2).\end{aligned} \tag{2.6}$$

By (2.5) and (2.6) we can get that $\eta(h_1h_2) = \eta(h_1)\eta(h_2) = 1$, therefore, we have $h_1h_2 \in H_\eta$. Consequently, H_η is a subgroup of G. Assume that $h \in H_\eta$; then Lemma 2(3) yields that $\eta(x^{-1}hx) = \eta(h)$ for any $x \in G$ and $h \in H_\eta$; accordingly, H_η is a normal subgroup of G.

First, suppose that $\eta(x) \in \{0, 1\}$ and $x, y \in G \setminus H_\eta$; therefore, $\eta(x) = \eta(y) = 0$, then, by functional Eq (1.6), we have $\min\{\,\eta(xy), \eta(xy^{-1})\,\} = \eta(x)\eta(y) = 0$. In a consequence, we can determine that $xy^{-1} \in H_\eta \vee xy \in H_\eta$ for any $x, y \in G \setminus H_\eta$.

In addition, considering the second case, let $\eta(x) \in \{-1, 1\}$ and let $x, y \in G \setminus H_\eta$; thus, $\eta(x) \neq 1$ and $\eta(y) \neq 1$; then $\eta(x) = \eta(y) = -1$. Consequently, Eq (1.6) gives that $\min\{\,\eta(xy), \eta(xy^{-1})\,\} = \eta(x)\eta(y) = 1$. In either case, we can conclude that $\eta(xy) = 1$ and $\eta(xy^{-1}) = 1$, which infers that $xy^{-1} \in H_\eta \wedge xy \in H_\eta$ for all $x, y \in G \setminus H_\eta$.

Furthermore, let $\eta(x) > 0$ for all $x \in G$, then from Lemma 1, we can conclude that $\eta(x) = e^{-|\beta(x)|}$, $x \in G$. □

Corollary 1. *Let $\eta\colon G \to \mathbb{R}$, where G is an arbitrary group. Assume that η is a non-zero solution of the functional Eq* (1.6)*; then the commutator subgroup G' is a normal subgroup of H_η.*

Proof. Since η is a non-zero, then by the main theorem, we can derive the following cases:

Case 1. According to the main theorem, there exists a normal subgroup H_η defined as $\eta(x) = 1$ for every $x \in H_\eta$ and also satisfies the condition (2.4); consequently, by Lemma 2, we can compute that

$$\begin{aligned}(xy)^{-1} \in H_\eta &\wedge (xy^{-1})^{-1} \in H_\eta\\ y^{-1}x^{-1} \in H_\eta &\wedge yx^{-1} \in H_\eta\\ x^{-1}y^{-1} \in H_\eta &\wedge x^{-1}y \in H_\eta\\ xyx^{-1}y^{-1} \in H_\eta &\wedge xy^{-1}x^{-1}y \in H_\eta\\ [x, y] \in H_\eta &\wedge (xy^{-1}x^{-1}y)^{-1} \in H_\eta\\ [x, y] \in H_\eta &\wedge y^{-1}xyx^{-1} \in H_\eta\\ [x, y] \in H_\eta &\wedge xyx^{-1}y^{-1} \in H_\eta,\end{aligned}$$

which indicates that $\eta([x, y]) = 1$.

Case 2. There exists a normal subgroup H_η which satisfies the condition (2.3), that is $xy^{-1} \in H_\eta \vee xy \in H_\eta$ for all $x, y \in G \setminus H_\eta$; accordingly, applying Lemma 2 and Case 1, we can deduce that $\eta([x, y]) = 1$.

Case 3. Assume that $\eta(x) > 0$ for any $x \in G$; consequently by Theorem 5, we have $\eta(x) = e^{-|\beta(x)|}$ for any $x \in G$, where $\beta\colon G \to \mathbb{R}$ is an additive function, thus, $\eta([x, y]) = 1$ for the reason that $\beta([x, y]) = 0$ for any $x, y \in G$.

Hence, in either case, the required proof is completed. □

Corollary 2. *Any solution $\eta\colon G \to \mathbb{R}$ of Eq* (1.6) *on any group G fulfills the Kannappan condition.*

Proof. The proof relies on the following cases:

Case 1. Assume that $\eta \equiv 0$ on group G, then it is obvious that η fulfills the Kannappan condition.

Case 2. Let $\eta(x) \leq 0$ for all $x \in G$. Then from Theorem 5 and Corollary 1, there exists normal subgroup H_η such that $G' \subseteq H_\eta$, consequently, $\eta(xyg) = 1$ if and only if $xyg \in H_\eta$ if and only if $[y^{-1}, x^{-1}]xyg = xgy \in H_\eta$ if and only if $\eta(xgy) = 1$. It is sufficient to prove the Kannappan condition because η only takes the values 1, 0, and −1.

Case 3. Suppose that $\eta(x) > 0$, $x \in G$, then $\eta(x) = e^{-|\beta(x)|}$, therefore $\eta(xyg) = \eta(xgy)$ for any $x, g, y \in G$ because β is an additive function. □

Corollary 3. *If η is a strictly positive solution of* (1.6)*, then* $\max\{\, \eta(xy^{-1}), \eta(xy) \,\} \in (0, 1]$.

Theorem 6. *Let $\eta\colon G \to \mathbb{R}$ and η is a non-zero solution of* (1.6)*, then:*

(1) Assume that $g \in G$ and $\eta(gx^{-1}) = \eta(gx)$ for some elements $x \in G$ with the restriction that $\eta(x^2) \neq 1$. Then $\eta(g^2) = 1$.

(2) Suppose that $G_\eta = \{\, g \in G \mid \eta(g^2) = 1 \,\}$, then G_η is a normal subgroup of G.

(3) If η is strictly positive, then $G_\eta = H_\eta$.

Proof. Assume that $x, y \in G$, then by Eq (1.6) and Corollary 2, we have

$$\begin{aligned} \eta(gx)\eta(gx^{-1}) &= \min\{\, \eta(gxgx^{-1}), \eta(gx(gx^{-1})^{-1}) \,\} \\ &= \min\{\, \eta(gxgx^{-1}), \eta(gxxg^{-1}) \,\} \\ &= \min\{\, \eta(g^2), \eta(gx^2g^{-1}) \,\} \\ \eta(gx)\eta(gx^{-1}) &= \min\{\, \eta(g^2), \eta(x^2) \,\}. \end{aligned} \tag{2.7}$$

(1). By given condition $\eta(gx) = \eta(gx^{-1})$ for some $x \in G$ and by Eq (2.7), we can see that either $\eta(x^2) = 1$ or $\eta(g^2) = 1$, therefore, given condition $\eta(x^2) \neq 1$ implies that $\eta(g^2) = 1$.

(2). Since $\eta(e) = 1$, therefore $e \in G_\eta$. Let $g \in G_\eta$; then $\eta(g^2) = 1$. Moreover, $\eta(x^{-1}) = \eta(x)$ gives that $\eta(g^{-2}) = \eta(g^2) = 1$, therefore $g^{-1} \in G_\eta$. Let $g, y \in G_\eta$; then, $\eta(y^2) = 1$ and $\eta(g^2) = 1$, therefore, a simple calculation yields

$$\begin{aligned} \eta(y^2g^2) &= \eta(y^2g^2)\eta(g^2) \\ &= \min\{\, \eta(y^2g^4), \eta(y^2) \,\} \\ &\leq \eta(y^2) \\ \eta(y^2g^2) &\leq \eta(y^2)\eta(g^2). \end{aligned} \tag{2.8}$$

$$\begin{aligned} \eta(y^2)\eta(g^2) &= \min\{\, \eta(y^2g^2), \eta(y^2g^{-2}) \,\} \\ &\leq \eta(y^2g^2) \\ \eta(y^2)\eta(g^2) &\leq \eta(y^2g^2), \end{aligned} \tag{2.9}$$

So, inequalities (2.8) and (2.9) implies that $\eta(y^2g^2) = \eta(y^2)\eta(g^2) = 1$, which yields that $yg \in G_\eta$, consequently, G_η is a subgroup of G. Additionally, Lemma 2 provides that $\eta(xg) = \eta(gx)$ for every $x \in G$ and $g \in G_\eta$. As a result, G_η is a normal subgroup of a group G.

(3). As $\eta(x) > 0$ for every $x \in G$, then the proof can be seen easily from Lemma 1 and Theorem 6 (2). □

Definition 2. *Suppose that group G is abelian. Then a mapping $\eta : G \to \mathbb{R}$ is called a discrete norm if $\eta(x) > \gamma$, where $\gamma > 0$ and $x \in G \setminus \{e\}$. Then (G, η, e) is said to be a discretely normed abelian group [12].*

Theorem 7. *Assume that (G, η, e) is a discretely normed abelian group. A mapping $\eta : G \to \mathbb{R}$ is a solution of* (1.6) *if and only if $\eta(x) = e^{-|\beta(x)|}$, $x \in G \setminus \{e\}$, where $\beta : G \to \mathbb{R}$ is some additive function.*

Proof. Since (G, η, e) is a discretely normed, then there exists a mapping $\eta : G \to \mathbb{R}$ such that $\eta(x) > \gamma$, where $\gamma > 0$ and $x \in G \setminus \{e\}$. Setting $\eta(x) = \log \eta(x)$, and using Lemma 1, we get

$$\min\{\, \eta(xy^{-1}), \eta(xy)\,\} = \eta(x)\eta(y)$$

if and only if $\eta(x) = e^{-|\beta(x)|}$, $x \in G \setminus \{e\}$, where $\beta : G \to \mathbb{R}$ is some additive function. □

Corollary 4. *For free abelian group G, a mapping η is a solution of Eq* (1.6) *if and only if $\eta(x) = e^{-|\beta(x)|}$, $x \in G \setminus \{e\}$, where $\beta : G \to \mathbb{R}$ is some additive function.*

3. Generalized Pexider-type functional Eq (1.8)

Theorem 8. *Let $\eta\colon G \to \mathbb{R}$ fulfills the Kannappan condition, where G is an arbitrary group. Then η, χ, ψ are solutions of the functional Eq* (1.8) *if and only if*

$$\begin{cases} \eta(x) = \lambda_1, \qquad x \in G,\ \lambda_1 \in \mathbb{R}, \\ \psi \; is\; an\; arbitrary\; function, \\ \chi(x) = 1 - \lambda_1^{-1}\psi(x); \end{cases}$$

or

$$\begin{cases} \eta(x) = \xi(x) + \lambda_1, \qquad x \in G,\ \lambda_1 \in \mathbb{R}, \\ \chi(x) = 1, \\ \psi(x) = \xi(x), \end{cases}$$

where $\xi\colon G \to \mathbb{R}$ is a solution of Eq (1.4)*;*

or

$$\begin{cases} \eta(x) = \lambda_2\xi(x) + \lambda_1, \qquad x \in G,\ \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_2 > 0, \\ \chi(x) = \xi(x), \\ \psi(x) = \lambda_1(1 - \xi(x)), \end{cases}$$

where $\xi\colon G \to \mathbb{R}$ is a solution of Eq (1.5)*;*

or

$$\begin{cases} \eta(x) = \lambda_2\xi(x) + \lambda_1, \qquad x \in G,\ \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_2 < 0, \\ \chi(x) = \xi(x), \\ \psi(x) = \lambda_1(1 - \xi(x)), \end{cases}$$

where $\xi\colon G \to \mathbb{R}$ is a solution of Eq (1.6)*.*

Proof. The 'if' part of the theorem can easily be seen that every function η, χ, and ξ presented in the statement is a solution of Eq (1.8). Conversely, suppose that η, χ, ψ are solutions of Eq (1.8), then we have the following cases:

(1). η is constant.

Assuming that $\eta(x) = \lambda_1$ for $x \in G$ and $\lambda_1 \in \mathbb{R}$, we may deduce from Eq (1.8) that $\chi(x) = 1 - \lambda_1^{-1}\psi(x)$ when ψ is an arbitrary function, which is required result described in the statement.

(2). η is not constant.

Setting $y = e$ in Eq (1.8) gives that

$$\begin{aligned} \max\{\,\eta(x), \eta(x)\,\} &= \chi(x)\eta(e) + \psi(x) \\ \eta(x) &= \chi(x)\eta(e) + \psi(x) \\ \psi(x) &= \eta(x) - \chi(x)\eta(e). \end{aligned} \tag{3.1}$$

Using Eq (3.1) in (1.8), we conclude that

$$\max\{\,\eta(xy), \eta(xy^{-1})\,\} = \chi(x)\eta(y) + \eta(x) - \chi(x)\eta(e). \tag{3.2}$$

Setting $x = e$, we can obtain

$$\max\{\,\eta(y), \eta(y^{-1})\,\} = \chi(e)\eta(y) + \eta(e) - \chi(e)\eta(e). \tag{3.3}$$

We are going to show that $\chi(e) = 1$, but on the contrary, assume that $\chi(e) \neq 1$. Setting

$$H := \{y \in G :\ \eta(y^{-1}) \leq \eta(y)\}, \qquad H' := G \setminus H.$$

If $y \in H'$ then $y^{-1} \in H$. Also, if $y \in H$, then from Eq (3.3), we have

$$\begin{aligned} \eta(y) &= \chi(e)\eta(y) + \eta(e) - \chi(e)\eta(e) \\ (\chi(e) - 1)(\eta(e) - \eta(y)) &= 0, \end{aligned}$$

which implies that $\eta(y) = \eta(e)$ for all $y \in H$. Moreover, $H' \neq \emptyset$ because η is not constant. Assume that $y' \in H'$ then $\eta(y') < \eta(y'^{-1}) = \eta(e)$, which implies that

$$\eta(y') - \eta(e) < 0. \tag{3.4}$$

Writing y' instead of y in Eq (3.3) and using (3.4) we can get that

$$\eta(e) = \eta(y'^{-1}) = \chi(e)\eta(y') + \eta(e) - \chi(e)\eta(e),$$

which implies that $(\eta(y') - \eta(e))\chi(e) = 0$, so $\chi(e) = 0$. Setting $x = y'$ and $y = y'^{-1}$ in (3.2) we have

$$\begin{aligned} \eta(e) \leq \max\{\,\eta(e), \eta(y'^2)\,\} &= \chi(y')\eta(y'^{-1}) + \eta(y') - \chi(y')\eta(e) \\ &= \chi(y')\eta(e) + \eta(y') - \chi(y')\eta(e) \\ \eta(e) \leq \eta(y') &< \eta(e), \end{aligned}$$

which is a contradiction, thus, we have $\chi(e) = 1$. Moreover, from Eq (3.3), we can see that

$$\max\{\,\eta(y), \eta(y^{-1})\,\} = \eta(y), \tag{3.5}$$

writting y^{-1} instead of y in (3.5) we have

$$\max\{\eta(y^{-1}), \eta(y)\} = \eta(y^{-1}), \tag{3.6}$$

from (3.5) and (3.6) we can get that $\eta(y^{-1}) = \eta(y)$.

Since $\eta(x^{-1}) = \eta(x)$ for every $x \in G$, then from Eq (3.2) and Kannappan condition we have

$$\begin{aligned}
\chi(x)\eta(y) + \eta(x) - \chi(x)\eta(e) &= \max\{\eta(xy), \eta(xy^{-1})\} \\
&= \max\{\eta(y^{-1}x^{-1}), \eta(yx^{-1})\} \\
&= \max\{\eta(ey^{-1}x^{-1}), \eta(eyx^{-1})\} \\
&= \max\{\eta(x^{-1}y^{-1}), \eta(x^{-1}y)\} \\
\chi(x)\eta(y) + \eta(x) - \chi(x)\eta(e) &= \chi(x^{-1})\eta(y) + \eta(x^{-1}) - \chi(x^{-1})\eta(e) \\
(\eta(y) - \eta(e))(\chi(x) - \chi(x^{-1})) &= 0,
\end{aligned}$$

which infers that $\chi(x^{-1}) - \chi(x) = 0$ because η is not constant. Moreover, when η is not constant then $\eta(x^{-1}) = \eta(x)$ and $\chi(x^{-1}) = \chi(x)$ for every $x \in G$, consequently, by Eq (3.1) we can get that $\psi(x^{-1}) = \psi(x)$. Also, by Eq (3.2) and Kannappan condition we can see that

$$\begin{aligned}
\chi(x)\eta(y) + \eta(x) - \chi(x)\eta(e) &= \max\{\eta(xy), \eta(xy^{-1})\} \\
&= \max\{\eta(exy), \eta(yx^{-1})\} \\
&= \max\{\eta(yx), \eta(yx^{-1})\} \\
\chi(x)\eta(y) + \eta(x) - \chi(x)\eta(e) &= \chi(y)\eta(x) + \eta(y) - \chi(y)\eta(e) \\
\chi(x)(\eta(y) - \eta(e)) + \eta(x) &= \chi(y)(\eta(x) - \eta(e)) + \eta(y) \\
\chi(x)(\eta(y) - \eta(e)) - (\eta(y) - \eta(e)) &= \chi(y)(\eta(x) - \eta(e)) - (\eta(y) - \eta(e)) \\
(\eta(y) - \eta(e))(\chi(x) - 1) &= (\eta(x) - \eta(e))(\chi(y) - 1).
\end{aligned}$$

Suppose that $\eta(y') \neq \eta(e)$ for $y' \in G$, then we can obtain that

$$\chi(x) - 1 = \frac{\chi(y') - 1}{\eta(y') - \eta(e)}(\eta(x) - \eta(e)).$$

Moreover, assume that $\beta := \frac{\chi(y')-1}{\eta(y')-\eta(e)}$, then $\chi(x) - 1 = \beta(\eta(x) - \eta(e))$, so we can write as $\chi_1(x) = \beta\eta_1(x)$, where

$$\chi_1(x) := \chi(x) - 1, \qquad x \in G, \tag{3.7}$$

$$\eta_1(x) := \eta(x) - \eta(e), \qquad x \in G. \tag{3.8}$$

Also $\eta_1(e) = 0$. By functional Eq (3.2) and definition of η_1, we have

$$\begin{aligned}
\max\{\eta_1(xy), \eta_1(xy^{-1})\} &= \max\{\eta(xy), \eta(xy^{-1})\} - \eta(e) \\
&= \chi(x)\eta(y) + \eta(x) - \chi(x)\eta(e) - \eta(e) \\
&= \chi(x)(\eta(y) - \eta(e)) + \eta(x) - \eta(e)
\end{aligned}$$

$$
\begin{aligned}
&= \chi(x)\eta_1(y) + \eta_1(x) \\
&= (\beta\eta_1(x) + 1)\eta_1(y) + \eta_1(x) \\
\max\{\,\eta_1(xy), \eta_1(xy^{-1})\,\} &= \beta\eta_1(x)\eta_1(y) + \eta_1(x) + \eta_1(y). \qquad (3.9)
\end{aligned}
$$

According to the different values of β, we can discuss the following three different cases.

Case 1. $\beta = 0$.

By Eq (3.7), we see that $\chi(x) = 1$, $x \in G$. Furthermore, by functional Eq (3.9), we have

$$\max\{\,\eta_1(xy), \eta_1(xy^{-1})\,\} = \eta_1(x) + \eta_1(y),$$

for all $x, y \in G$ and also η_1 satisfies the functional Eq (1.4), then from well-known theorem of Toborg [3], there exists some additive function $g\colon G \to \mathbb{R}$ such that $\eta_1(x) = |g(x)|$ for all $x \in G$, then from Eqs (3.1), (3.7) and (3.8), we can deduce

$$
\begin{cases}
\eta(x) = \lambda_1 + \xi(x), \\
\chi(x) = 1, \\
\psi(x) = \xi(x),
\end{cases}
$$

where $\lambda_1 = \eta(e)$ and $\xi\colon G \to \mathbb{R}$ is a solution of Eq (1.4) such that $\xi(x) = |g(x)|$.

Case 2. $\beta > 0$.

Let $\eta_2 := \beta\eta_1(x)$ for all $x \in G$, then multiplying functional Eq (3.9) by β, we conclude that

$$
\begin{aligned}
\max\{\beta\eta_1(xy), \beta\eta_1(xy^{-1})\,\} &= (\beta\eta_1(x))(\beta\eta_1(y)) + \beta\eta_1(x) + \beta\eta_1(y) \\
\max\{\,\eta_2(xy), \eta_2(xy^{-1})\,\} &= \eta_2(x)\eta_2(y) + \eta_2(x) + \eta_2(y) \\
&= (\eta_2(x) + 1)(\eta_2(y) + 1) - 1 \\
\max\{\,\eta_2(xy), \eta_2(xy^{-1})\,\} + 1 &= (\eta_2(x) + 1)(\eta_2(y) + 1),
\end{aligned}
$$

then by setting $\xi(x) := \eta_2(x) + 1$ for $x \in G$, we get

$$\max\{\,\xi(xy), \xi(xy^{-1})\,\} = \xi(x)\xi(y), \qquad x, y \in G.$$

It is clear that $\xi\colon G \to \mathbb{R}$ satisfies Eq (1.5), then from Eq (3.8) we get

$$\xi(x) = \eta_2(x) + 1 = \beta\eta_1(x) + 1 = \beta(\eta(x) - \eta(e)) + 1,$$

which gives that $\eta(x) = \lambda_2\xi(x) + \lambda_1$ where $\lambda_2 = \beta^{-1}, \lambda_1 = \eta(e) - \beta^{-1}$.

Also, from Eqs (3.1), (3.7) and (3.8), we can see that

$$
\begin{cases}
\eta(x) = \lambda_2\xi(x) + \lambda_1, \quad x \in G,\ \lambda_2 > 0, \\
\chi(x) = \xi(x), \\
\psi(x) = \lambda_1(1 - \xi(x)),
\end{cases}
$$

where $\xi\colon G \to \mathbb{R}$ is a solution of (1.5).

Case 3. $\beta < 0$.

Assume that $\eta_2 := -\beta\eta_1(x)$ for every $x \in G$, then multiplying functional Eq (3.9) by $-\beta$, we have

$$\begin{aligned}
\max\{-\beta\eta_1(xy), -\beta\eta_1(xy^{-1})\} &= (-\beta\eta_1(x))(\beta\eta_1(y)) - \beta\eta_1(x) - \beta\eta_1(y) \\
\max\{\eta_2(xy), \eta_2(xy^{-1})\} &= -\eta_2(x)\eta_2(y) + \eta_2(x) + \eta_2(y) \\
&= -(\eta_2(x) - 1)(\eta_2(y) - 1) + 1 \\
\max\{\eta_2(xy), \eta_2(xy^{-1})\} - 1 &= -(\eta_2(x) - 1)(\eta_2(y) - 1),
\end{aligned}$$

for any $x, y \in G$, then by setting $\xi_1(x) := \eta_2(x) - 1$ for $x \in G$, we have

$$\max\{\xi_1(xy), \xi_1(xy^{-1})\} = -\xi_1(x)\xi_1(y), \qquad x, y \in G,$$

then by setting $\xi(x) := -\xi_1(x)$, $x \in G$, we can see that $\xi\colon G \to \mathbb{R}$ satisfies the Eq (1.6), then from Eqs (3.1), (3.7) and (3.8), we have

$$\begin{cases}
\eta(x) = \lambda_2\xi(x) + \lambda_1, \quad x \in G,\ \lambda_2 < 0, \\
\chi(x) = \xi(x), \\
\psi(x) = \lambda_1(1 - \xi(x)),
\end{cases}$$

which completes the proof. □

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Conflict of interest

All authors declare no conflict of interest in this paper.

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