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Research article

Cubic nonlinear differential system, their periodic solutions and bifurcation analysis

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Abstract: In this article, periodic solutions from a fine focus $U = 0$, are accomplished for several classes. Some classes have polynomial coefficients, while the remaining classes $C_{14,7}$, $C_{16,8}$ and $C_{5,5}$, $C_{6,6}$ have non-homogeneous and homogenous trigonometric coefficients accordingly. By adopting a systematic procedure of bifurcation that occurs under perturbation of the coefficients, we have succeeded to find the highest known multiplicity 10 as an upper bound for the class $C_{9,4}$, $C_{11,3}$ with algebraic and $C_{5,5}$, $C_{6,6}$ with trigonometric coefficients. Polynomials of different degrees with various coefficients have been discussed using symbolic computation in Maple 18. All of the results are executed and validated by using past and present theory, and they were found to be novel and authentic in their respective domains.

Keywords: periodic solutions; limit cycle; bifurcation method; multiplicity; algebraic and trigonometric coefficients; focal values
Mathematics Subject Classification: 34C05, 34C07, 34C25

1. Introduction

Periodic behavior is essential to our life. The mathematical modeling of real-life problems mainly arises in functional equations, such as partial differential equations, ordinary differential equations, and integro-differential equations. It is a matter of the fact that periodic solutions can be created or destroyed only at infinity, or their stability can change. Such qualitative changes in the dynamics are called bifurcation. It is also possible for pairs of limit cycles (one stable and one unstable) to coalesce and vanish in a codimension two generalized Hopf bifurcation (see Guckenheimer and Holmes [1]). Many of these functions have the ability to adopt bifurcation, with which we can check the stability. If

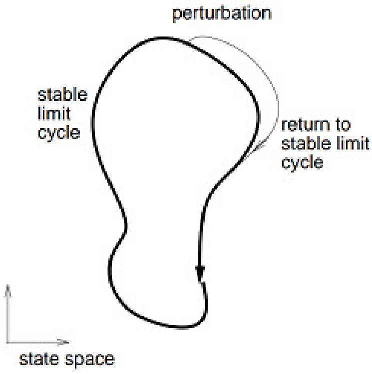

Figure 1. Passively stable limit cycle.

all the periodic solutions come out of the origin, then these are unstable, otherwise stable (isolated closed trajectory in periodic behavior). Stable limit cycles are an example of attractors. By the Jordan curve theorem, every closed trajectory divides the plane into two regions, an interior region bounded by the curve and an exterior containing all of the nearby and far away exterior points, so that every continuous path connecting a point of one region to a point of the other intersects with that loop somewhere. In the present article, we are mainly concerned with the nonlinear system because trajectories do not simply need to approach or leave a single point. They might even approach a large set, like a circle or another closed curve. It is a fact that limit cycles are a nonlinear phenomenon; because the linear system $x^{'} = Ax$ may have closed orbits, they would not be isolated, such that if they have a periodic solution $x(t)$, then all constant multiples should be, see also [2].

In engineering, power engineering, we can exploit oscillators for timing and sequencing. We can use them in locomotion control in robots. Van der Pol oscillator comes up in electrical circuits; it is an example of relaxation oscillation. Its equation is $x^{''} - \mu\left(1 - x^2\right)x^{'} + x = 0$, where μ is a positive constant. If $\mu = 1$, then trajectories fastly settle on a closed curve, while for $\mu = 0$, all trajectories become a circle. In analyzing nonlinear systems in the xy-plane, we have so far concentrated on analyzing how the system's trajectories look in the neighborhood of each critical point.

We also observed periodic behavior in our body like; heartbeats, breathing, chewing, locomotion, diverse rhythms inside the brain, etc. More generally in walking, which is also a periodic, a passive stable limit cycle exists. as shown in Figure 1, see [3] for more details.

Consider the system of the form:

$$\begin{aligned} \dot{\mathfrak{t}} &= \lambda\mathfrak{t} + \mathfrak{s} + a_w(\mathfrak{t}, \mathfrak{s}) \\ \dot{\mathfrak{s}} &= -\mathfrak{t} + \lambda\mathfrak{s} + b_w(\mathfrak{t}, \mathfrak{s}), \end{aligned} \tag{1.1}$$

where a_w and b_w are homogenous polynomials having degree w. The polar form of (1.1) is written as following:

$$\begin{aligned} \dot{r} &= \lambda r + x(\theta) r^w \\ \dot{\theta} &= -1 + y(\theta) r^{w-1}. \end{aligned} \tag{1.2}$$

Where x and y are polynomials in $\cos\theta$ and $\sin\theta$ with degree $(w+1)$. In Lloyd [4], it is presented that if:

$$\zeta = r^{w-1}\left(1 - r^{w-1}y(\theta)\right)^{-1}. \tag{1.3}$$

Then the Eq (1.3) can be transformed to non-autonomous first order differential equation as:

$$\frac{d\zeta}{d\theta} = \kappa(\theta)\,\zeta^3 + \rho(\theta)\,\zeta^2 - \lambda(w-1)\,\zeta, \tag{1.4}$$

where

$$\kappa(\theta) = -(w-1)\,y(\theta)\,(x(\theta) + \lambda y(\theta)),$$

and

$$\rho(\theta) = (w-1)\,x(\theta) + 2\lambda(w-1)\,y(\theta) - y'(\theta).$$

Here κ & ρ are homogeneous polynomials in $\sin\theta$ and $\cos\theta$.

We structured the paper as follows. In Section 2, we have discussed the transformation and some essential formulae to calculate periodic multiplicity. In Section 3, we recall some lemma and theorems for the origin. The significant results and conclusions are discussed in Sections 4 and 5 accordingly.

2. First order cubic system

In the present article, we are considering the differential equation given as:

$$\dot{U} = \kappa(\tau)U^3 + \rho(\tau)U^2 + \upsilon(\tau)U. \tag{2.1}$$

Where the coefficients κ, ρ, υ and involved variable are real-valued functions and $U \in \mathbb{C}$. This paper's central result is the determination of possible maximum periodic solutions with perturbation of the coefficients on the plane. This equation is the part of the following equation, described in [5]:

$$\dot{U} = \rho_0(\tau)U^n + \rho_1(\tau)U^{n-1} + \rho_2(\tau)U^{n-2} + ... + \rho_n(\tau). \tag{2.2}$$

With $\rho_0(\tau) = 1$. For $n = 3$, the Eq (2.2) is known as Abel's differential equation, we focused on it because of its connection with the second part of Hilbert 16^{th} problem (maximal number of limit cycles and their relative locations of planar polynomial real vector fields of given degree); it is related to ODEs and dynamical systems. The fascination of the problem comes from the fact that it sits at the confluence of analysis, algebra, geometry and even logic. It is known, for instance, that when $\rho_3(t)$ does not change sign, the upper bound for the number of limit cycles is 3, see [6, 7]. When $\rho_3(t) \equiv 1$ this upper bound also holds taking into account complex limit cycles, see [5, 8]. In [6], we see that when $\rho_0(t) \equiv 0$ and $\rho_2(t)$ does not change sign, the maximum number of limit cycles of Abel's equation is also 3. We likewise refer reader to the articles [9–11], for additional data with respect to this issue. For the Eq (2.1), we are considering a complex dependent variable, so that the number of zeros of a function in a bounded region of the complex plane can't be changed by any small perturbations. We substitute $\upsilon(\tau) \cong 0$ in (2.1), as was in [12]. Consequently, the Eq (2.1) becomes as:

$$\dot{U} = \kappa(\tau)U^3 + \rho(\tau)U^2. \tag{2.3}$$

Here κ and ρ may be polynomials in (i) τ (ii) $cos\tau$ and $sin\tau$, for more detail see [12, 13]. Also suppose that $\exists\,\beta \in \mathbb{R}$ such that:

$$U(\beta) = U(0), \quad \text{are periodic.}$$

For $U = 0$, the method for computing multiplicity "μ" is explained in [9, 10, 12, 14, 15]. For t in $[0,\beta]$ and c small, let

$$U(t,0,c) = \sum_{i=1}^{\infty} \xi_i(t)\, c^i. \tag{2.4}$$

Since $U(0,0,c) = c$, we have $\xi_1(0) = 1$ and $\xi_i(0) = 0$ for $i > 1$. The solution $U = 0$ is a centre for the Eq (2.3) if all the solutions are periodic for c in neighborhood of 0. The solution $U = 0$ is a centre if $\xi_1(\beta) = 1$ and $\xi_i(\beta) = 0$ for $i > 1$. Substituting the Eq (2.4) in the Eq (2.3) and equating coefficients of c yields $\dot{\xi}_1(\tau) = 0$. Hence, $\xi_1(\tau) = 1$. Moreover, the functions $\dot{\xi}_i(\tau)$, for $i > 1$ are obtained with the help of following equation:

$$\dot{\xi}_i = \kappa \sum_{\substack{j+k+l=i \\ j,k,l \geq 1}} \xi_j \xi_k \xi_l + \rho \sum_{\substack{j+k=i \\ j,k \geq 1}} \xi_j \xi_k. \tag{2.5}$$

With $\xi_1(\tau) = 1$. Expect to be that $\varkappa_i = \xi_i(\beta)$, at that point $\varkappa = i$ if $\varkappa_1 = 1$ and $\varkappa_k = 0$ for $2 \leq k \leq i-2$ but $\varkappa_i \neq 0$, shown in Theorem 2.1. Alwash in [12], presented $\xi_i(\tau)$ and $\varkappa_i$ for $i \leq 8$, for $i = 9$ are in [16], for $i = 10$ we calculated $\xi_{10}(\tau)$ and $\varkappa_{10}$ in [9], also presented in Theorem 2.1.

The following Theorem is the modification of Theorem 2 in [12], with the help of this Theorem periodic multiplicity is calculated. Here, in integral; $\int \kappa(\tau)\overline{\rho(\tau)}d\tau$, bar " – " function is like $\overline{\rho(\tau)} = \int \rho(\tau)\, d\tau$.

Theorem 2.1. *The solution $U = 0$ of the Eq (2.3) has a multiplicity k, wherever $2 \leq k \leq 10$ if $\varkappa_n = 0$ for $2 \leq n \leq k-1$ and $\varkappa_n \neq 0$ where*

$$\varkappa_2 = \int_0^\beta \rho,$$
$$\varkappa_3 = \int_0^\beta \kappa,$$
$$\varkappa_4 = \int_0^\beta \kappa\overline{\rho},$$
$$\varkappa_5 = \int_0^\beta \kappa\overline{\rho}^2,$$
$$\varkappa_6 = \int_0^\beta \left(\kappa\overline{\rho}^3 - \tfrac{1}{2}\overline{\kappa}^2\rho\right),$$
$$\varkappa_7 = \int_0^\beta \left(\kappa\overline{\rho}^4 + 2\kappa\overline{\rho}^2\overline{\kappa}\right),$$
$$\varkappa_8 = \int_0^\beta \left(\kappa\overline{\rho}^5 + 3\kappa\overline{\rho}^3\overline{\kappa} + \kappa\overline{\rho}^2\overline{\overline{\rho}\kappa} - \tfrac{1}{2}\overline{\kappa}^3\rho\right),$$
$$\varkappa_9 = \int_0^\beta \left(\kappa\overline{\rho}^6 - 5\kappa\overline{\rho}^4\overline{\kappa} - 2\overline{\rho}^3\overline{\overline{\rho\kappa}} + 20\overline{\rho\overline{\kappa}^2} + 2\overline{\rho\overline{\kappa}}\rho\overline{\kappa}^2\right),$$

and

$$\varkappa_{10} = \int_0^\beta \left(\begin{array}{c} \kappa\overline{\rho}^7 - \frac{1235}{6}\overline{\kappa}\kappa\overline{\rho}^5 - \frac{970}{3}\kappa\overline{\kappa}^2\overline{\rho}^3 - 237\rho\overline{\rho}^2\overline{\kappa}^3 - 24\kappa\overline{\kappa}^2\rho\overline{\rho}^2 - 70\overline{\rho}^3\kappa^2\overline{\kappa} - 21\overline{\kappa}^4\rho - 74\kappa\overline{\kappa}^3\overline{\rho} + \\ \frac{5}{2}\overline{\kappa}^2\rho\overline{\rho}^4 + 32\overline{\rho}^4\kappa\overline{\rho\overline{\kappa}} - 16\overline{\rho}^4\rho\overline{\kappa} - 15\overline{\rho}^5\kappa^2 - 36\overline{\rho}\rho\overline{\kappa}^2\overline{\rho\overline{\kappa}} - 8\rho\overline{\rho}^4\kappa\overline{\kappa} \end{array} \right).$$

3. Conditions for centre

For $U = 0$ as a centre, conditions that are useful for calculating maximum multiplicity $\varkappa_k$, $2 \leq k \leq 10$ are from [12] and are defined below.

Theorem 3.1. *Consider that f, g are continuous functions defined on interval $I = \delta([0,\beta])$ and a differentiable function δ with $\delta(\beta) = \delta(0)$ such that:*

$$\kappa(\tau) = f(\delta(\tau))\,\dot{\delta}$$

$$\rho(\tau) = g(\delta(\tau))\dot{\delta},$$

then origin is the centre for the Eq (2.3).

Corollary 3.1. *If κ is a constant multiple of ρ and $\int_0^\beta \rho(\tau)\, d\tau = 0$, then the origin is a centre for the Eq (2.3).*

Corollary 3.2. *If any ρ or κ is identically zero and other has mean value zero then the origin is a centre.*

Remark 1. *In [12], bifurcation method is described that when the coefficients of $\rho(\tau)$ and $\kappa(\tau)$ are slightly perturbed, two periodic solutions bifurcate out of the origin, when bifurcation method is applied. For the number of real periodic solutions, we conclude that if multiplicity μ is even, the origin is stable $\varkappa_\mu < 0$ and unstable if $\varkappa_\mu > 0$. If μ is odd, then the origin is stable on the right and unstable on the left if $\varkappa_\mu < 0$; however, if $\varkappa_\mu > 0$, the origin is stable on the left and unstable on the right.*

4. Results

4.1. Polynomial coefficients

This section describes the method for computing the maximum number of limit cycles of a polynomial differential equation in a plane for various classes of different degrees. Suppose $C_{r,q}$ indicates the class for the Eq (2.3), with degree r, q for $\kappa(\tau)$ and $\rho(\tau)$ accordingly, for more examples see [9, 10, 13, 14, 17]. The confirmation of the accompanying theorems, stems from papers in [12, 18]. We use Theorem 2.1 with $\beta = 1$, as is done in Lloyd et al. If we use all coefficients for polynomials $\kappa(\tau)$ and $\rho(\tau)$ then we can easily see that the periodic solutions greater than 4 can't obtained. So, some possible suitable coefficients are restricted in the following classes to find as many periodic solutions as possible. All calculations regarding the different classes are carried out using Maple 18.

Theorem 4.1. *Suppose the class $C_{9,4}$ for the Eq (2.3), if*

$$\kappa(\tau) = a + b\tau + e\tau^4 + f\tau^5 + i\tau^8 + j\tau^9.$$

$$\rho(\tau) = m + q\tau^4.$$

Then we come to conclusions $\mu_{\max}(C_{9,4}) \geq 10$.

Proof. From Theorem 2.1, we extract that:

$$\varkappa_2 = m + \frac{1}{5}q,$$

$$\varkappa_3 = a + \frac{1}{2}b + \frac{1}{5}e + \frac{1}{6}f + \frac{1}{9}i + \frac{1}{10}j.$$

Thus multiplicity of $U = 0$ is $\mu = 2$, if $\varkappa_2 \neq 0$. And is $\mu = 3$, if $\varkappa_2 = 0$ but $\varkappa_3 \neq 0$. If $\varkappa_2 = \varkappa_3 = 0$, then $\kappa(\tau)$ and $\rho(\tau)$ are as below:

$$\rho(\tau) = q\left(\tau^4 - \frac{1}{5}\right). \tag{4.1}$$

$$\kappa(\tau) = b(\tau - \frac{1}{2}) + e\left(\tau^4 - \frac{1}{5}\right) + f\left(\tau^5 - \frac{1}{6}\right) + i\left(\tau^8 - \frac{1}{9}\right) + j\left(\tau^9 - \frac{1}{10}\right), \tag{4.2}$$

And also we compute $\varkappa_4$ as given below:

$$\varkappa_4 = -\frac{q(-189j - 176i - 75f + 495b)}{103950}.$$

If $\varkappa_4 = 0$, then either $q = 0$ or:

$$j = -\frac{176}{189}i - \frac{75}{189}f + \frac{495}{189}b. \tag{4.3}$$

If $q = 0$, then $\rho(\tau) = 0$ and for $\varkappa_3 = 0$, origin is the centre derived from Corollary 3.2. So, assuming (4.3) holds and $q \neq 0$, $\varkappa_5$ is given as:

$$\varkappa_5 = -\frac{q^2(992i + 1425f + 8550b)}{553014000}.$$

If $\varkappa_5 = 0$, as $q \neq 0$ implies:

$$i = -\frac{1425}{992}f - \frac{8550}{992}b. \tag{4.4}$$

And by using (4.4) we take $\varkappa_6$ as:

$$\varkappa_6 = -\frac{q(6b + f)(41753450b + 11105688q^2 - 753375f)}{2599788102912000}.$$

If $\varkappa_6 = 0$, then, as $q \neq 0$ either $f = -6b$ or

$$f = \frac{41753450}{753375} + \frac{11105688}{753375}q^2. \tag{4.5}$$

If $f = -6b$, then the Eqs (4.2) and (4.1) become:

$$\kappa(\tau) = \left[e + b\left(5\tau^5 - 5\tau\right)\right]\dot{\delta}(\tau),$$

$$\rho(\tau) = q\dot{\delta}(\tau).$$

Where, $\dot{\delta}(\tau) = \tau^4 - \frac{1}{5}$, also $\delta(0) = \delta(1)$. As a result of Theorem 3.1, the origin is the centre, as follows:

$$f(\delta) = \left[e + b\left(5\tau^5 - 5\tau\right)\right], \ and \ g(\delta) = q.$$

So, $f \neq -6b$. If (4.5) holds then $\varkappa_7$ is computed as:

$$\varkappa_7 = -\frac{1357q^2(6q^2 + 25b)(2064342502325b + 519733807086q^2 + 68060500200e)}{66000270843643500000000}.$$

If $\varkappa_7 = 0$, since $q \neq 0$, either $b = -\frac{6}{25}q^2$ or,

$$b = -\frac{519733807086}{2064342502325}q^2 - \frac{68060500200}{2064342502325}e. \tag{4.6}$$

If $b = -\frac{6}{25}q^2$ then,

$$\kappa(\tau) = e\left(\tau^4 - \frac{1}{5}\right) + q^2\left(-\frac{6}{5}\tau^9 + \frac{36}{25}\tau^5 - \frac{6}{25}\tau\right),$$

$$\rho(\tau) = q\left(\tau^4 - \frac{1}{5}\right).$$

The origin is the centre with $f(\delta) = \left[e + q^2\left(-\frac{6}{5}\tau^5 + 6\tau\right)\right]$ and $g(\delta) = q$ according to Theorem 3.1. So, consider $b \neq -\frac{6}{25}q^2$. Using (4.6) we calculate $\varkappa_8$ as:

$$\varkappa_8 = \frac{59q(167936q^2 + 470525e)\phi}{150634300488681571068724220482868399006689875000000}.$$

Where,
$\phi = -3325005947550744260462549014528q^4 - 345198526986114785414664480eq^2 + 21339337823946954208255625e^2$.
Now, if $\varkappa_8 = 0$ then either $\phi = 0$ or

$$e = -\frac{167936}{470525}q^2. \tag{4.7}$$

As $q \neq 0$. If (4.7) holds but $\phi \neq 0, q \neq 0$, $\varkappa_9$ is calculated as follows:

$$\varkappa_9 = -\frac{512q^5(286469068411235 + 2477977842432q)}{17399233601520864843 75}.$$

If $\varkappa_9 = 0$ then, as $q \neq 0$, it results $q^5 \neq 0$, we examine q as:

$$q = -\frac{286469068411235}{2477977842432}. \tag{4.8}$$

If Eq (4.7) $\neq 0$, $q \neq 0$, but $\phi = 0$ holds then $e = y_i m^2$ for $i = 1, 2$ with $y_1 = 133.1766309$, $y_2 = -116.9999242$. If (4.8) holds, we can calculate $\varkappa_{10}$ as:

$$\varkappa_{10} = -\frac{\begin{matrix}767351999099436603854005506200096530966398641767502826581981622906195233 6893\\ 5586809033728834650990720779908569009680 43\end{matrix}}{\begin{matrix}11853474951918400099122214493761615083774360743656615787259615581071495\\ 849560849865267140893236719872441495060480000\end{matrix}}.$$

Here $\varkappa_{10}$ is not zero. As a result, we can deduce that $\mu_{\max}(C_{9,4}) \geq 10$.

For non zero different but the same (either positive or negative) values of the constants in κ and ρ, we can see that only one root is real and rest all the zeros are complex and are also in conjugate pairs. It gives stable limit cycles from Remark 1. The stability analysis is shown in Figure 2. ∎

Theorem 4.2. *For given below equation:*

$$\dot{U} = \kappa(\tau)U^3 + \rho(\tau)U^2. \tag{4.9}$$

Consider:
$\kappa(\tau) = \frac{167936}{2352625}(-\frac{286469068411235}{2477977842432} + \epsilon_1)^2 + \frac{1989561074}{412868500465}\epsilon_2 - \frac{641861}{103320}\epsilon_3 - \frac{301}{2976}\epsilon_4 - \frac{17}{945}\epsilon_5 - \frac{1}{10}\epsilon_6 + \epsilon_7 +$

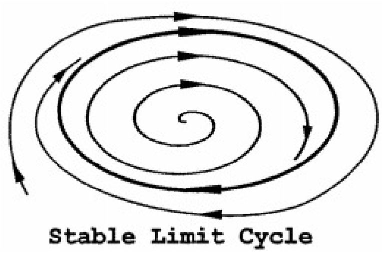

Figure 2. Stability of Class $C_{9,4}$.

$(-\frac{6}{25}(-\frac{286469068411235}{2477977842432}+\epsilon_1)^2-\frac{2722420008}{82573700093}\epsilon_2+\epsilon_3)\tau+(-\frac{286469068411235}{2477977842432}+\epsilon_1)^2+\epsilon_2)\tau^4+(\frac{36}{25}(-\frac{286469068411235}{2477977842432}+\epsilon_1)^2-\frac{754408015152}{412868500465}\epsilon_2+\frac{1670138}{30135}\epsilon_3+\epsilon_4)\tau^5+(\frac{240204612285}{82573700093}\epsilon_2-\frac{1418065}{16072}\epsilon_3-\frac{1425}{992}\epsilon_4+\epsilon_5)\tau^8+(-\frac{170939097560}{82573700093}\epsilon_2+\frac{162185}{2583}\epsilon_3+\frac{175}{186}\epsilon_4-\frac{176}{189}\epsilon_5-\frac{6}{5}(-\frac{286469068411235}{2477977842432}+\epsilon_1)^2+\epsilon_6)\tau^9,$

$$\rho(\tau)=\frac{57293813682247}{2477977842432}-\frac{1}{5}\epsilon_1+\epsilon_8+(-\frac{286469068411235}{2477977842432}+\epsilon_1)\tau^4.$$

Choose $\epsilon_p \neq 0$ for $1 \leq p \leq 8$ and small as compared to ϵ_{p-1}. Then there are eight non-trivial real periodic solutions to Eq (4.9).

Proof. See [14]. ∎

Theorem 4.3. *For class $C_{9,5}$ with*

$$\kappa(\tau)=a+b\,(A_1)+e\,(A_1)^4+f\,(A_1)^5+j\,(A_1)^9\,. \tag{4.10}$$

$$\rho(\tau)=m+r\,(B_1)^5\,. \tag{4.11}$$

Then we conclude $\mu_{\max}\left(C_{9,5}\right)\geq 8$, where $A_1=B_1=(\tau-1)$.

Proof. By utilizing Theorem 2.1, we calculate:

$$\varkappa_2=m-\frac{1}{6}r,$$

$$\varkappa_3=a-\frac{1}{2}b+\frac{1}{5}e-\frac{1}{6}f-\frac{1}{10}j.$$

If $\varkappa_2=\varkappa_3=0$, we calculated $\varkappa_4$ as:

$$\varkappa_4=-\frac{r(-27j-16e+110b)}{22176}.$$

If $\varkappa_4=0$ then, either $r=0$ or:

$$j=-\frac{16}{27}e+\frac{110}{27}b. \tag{4.12}$$

If $r=0$ then, $\varkappa_2=0$ gives $m=0$, hence, $\rho(\tau)=0$. If $\varkappa_2=0$, using Corollary 3.2, origin is the centre. As a result, take $r\neq 0$. By using (4.12) we compute:

$$\varkappa_5=\frac{25r^2(10e+121b)}{132324192}.$$

If $\varkappa_5 = 0$ then:

$$e = -\frac{121}{10}b. \tag{4.13}$$

Because we have already suppose $r \neq 0$. If (4.13) holds then:

$$\varkappa_6 = -\frac{br(43163r^2 + 365769b)}{315412755840}.$$

If $\varkappa_6 = 0$, either $b = 0$ or:

$$b = -\frac{43163}{365769}r^2, \tag{4.14}$$

as $r \neq 0$. If $b = 0$, the Eqs (4.10) and (4.11) becomes:

$$\kappa(\tau) = f\left((\tau - 1)^5 + \frac{1}{6}\right),$$

$$\rho(\tau) = r\left((\tau - 1)^5 + \frac{1}{6}\right).$$

Let $\delta(\tau) = \frac{(\tau-1)^6}{6} + \frac{\tau}{6}$ then, $\dot{\delta}(\tau) = (\tau - 1)^5 + \frac{1}{6}$. Also $\delta(0) = \delta(1)$. Using these results we write as below:

$$\kappa(\tau) = f\dot{\delta}(\tau),$$

$$\rho(\tau) = r\dot{\delta}(\tau).$$

From Theorem 3.1, origin is the centre with $f(\delta) = f$ and $g(\delta) = r$. So, suppose $b \neq 0$. If (4.14) holds then $\varkappa_7$ is:

$$\varkappa_7 = \frac{2539r^4(-509767294467341r^2 + 197071523361900f)}{23238641778643142259408000}.$$

Now if $\varkappa_7 = 0$, as $r \neq 0$ then:

$$f = \frac{509767294467341}{197071523361900}r^2. \tag{4.15}$$

Using (4.15) we obtain:

$$\varkappa_8 = -\frac{80595412534966326909087343}{850991319676513383625189993294080000}r^7.$$

Thus, $\mu_{\max}(C_{9,5}) \geq 8$.

For non zero different but the same (either positive or negative) values of the constants in κ and ρ, we can see that only two roots are real and rest all the zeros are complex and are also in conjugate pairs. It gives stable limit cycles from Remark 1. The stability analysis is as shown in Figure 2. ∎

Theorem 4.4. *Consider class $C_{9,6}$ for the Eq (2.3), if*

$$\kappa(\tau) = a + bA_2 + e(A_2)^4 + g(A_2)^6 + j(A_2)^9.$$

$$\rho(\tau) = m + s(B_2)^6.$$

Then we conclude $\mu_{\max}(C_{9,6}) \geq 8$ with $A_2 = B_2 = (2\tau - 1)$.

Proof. By utilizing Theorem 2.1, we obtain:

$$\varkappa_2 = m + \frac{1}{7}s,$$

$$\varkappa_3 = a + \frac{1}{5}e + \frac{1}{7}g.$$

Thus, multiplicity of $U = 0$ is $\mu = 2$ if $\varkappa_2 \neq 0$, And is $\mu = 3$ if $\varkappa_2 = 0$ but $\varkappa_3 \neq 0$. If $\varkappa_2 = \varkappa_3 = 0$ then:

$$a = -\frac{1}{5}e - \frac{1}{7}g, \tag{4.16}$$

$$m = -\frac{1}{7}s. \tag{4.17}$$

By using Eqs (4.16) and (4.17) , we have:

$$\kappa(\tau) = b\,(A_2) + e\left((A_2)^4 - \frac{1}{5}\right) + g\left((A_2)^6 - \frac{1}{7}\right) + j\,(A_2)^9\,, \tag{4.18}$$

$$\rho(\tau) = s\left((2\tau - 1)^6 - \frac{1}{7}\right). \tag{4.19}$$

Also we calculate $\varkappa_4$ as:

$$\varkappa_4 = -\frac{s(187b + 27j)}{11781}.$$

If $\varkappa_4 = 0$ then, either $s = 0$ or:

$$j = -\frac{187}{27}b. \tag{4.20}$$

If $s = 0$ then, (4.17) gives $m = 0$ so $\rho(\tau) = 0$, and also for $\varkappa_2 = 0$, origin is the centre. So, $s \neq 0$ is taken. If (4.20) holds $\varkappa_5$ is given as:

$$\varkappa_5 = -\frac{592es^2}{19062225}.$$

If $\varkappa_5 = 0$, then, $e = 0$ because we had already seen that $s \neq 0$. Thus, by substituting $e = 0$, we have:

$$\varkappa_6 = -\frac{6848bs(3956283s^2 + 126422030b)}{5558447535465825}.$$

Now if $\varkappa_6 = 0$, as $s \neq 0$ either $b = 0$ or:

$$b = -\frac{3956283}{126422030}s^2. \tag{4.21}$$

If we take $b = 0$ gives $j = 0$ and $e = 0$ from $\varkappa_5$ then, Eqs (4.18) and (4.19) take given below form:

$$\kappa(\tau) = g\left((2\tau - 1)^6 - \frac{1}{7}\right),$$

and:

$$\rho(\tau) = s\left((2\tau - 1)^6 - \frac{1}{7}\right).$$

Let $\delta(\tau) = \frac{(2\tau-1)^7}{14} - \frac{\tau}{7}$, Also $\delta(0) = \delta(1)$. So:

$$\kappa(\tau) = g\dot{\delta}(\tau),$$

$$\rho(\tau) = s\dot{\delta}(\tau).$$

From Theorem 3.1, origin is the centre, with $f(\delta) = g$ and $g(\delta) = s$. By using (4.21), we have $\varkappa_7$ as:

$$\varkappa_7 = \frac{3630411881856}{17850688946859894125} g s^4.$$

If $\varkappa_7 = 0$, recalling that $s \neq 0$ (considered above) then by substituting $g = 0$, we found:

$$\varkappa_8 = -\frac{84179003432468973571503675648}{1174574376736233802469067895708738156 25} s^7.$$

As $s \neq 0$ considered above, we can't proceed further. So, $\mu_{\max}(C_{9,6}) \geq 8$.

For non zero different but the same (either positive or negative) values of the constants in κ and ρ, we can see that only one root is real and rest all the zeros are complex conjugate pairs. It gives stable limit cycles from Remark 1, as shown in Figure 2. ∎

Theorem 4.5. *If for the Eq (2.3),*

$$\kappa(\tau) = -\upsilon_1 + b(A_1) + e(A_1)^4 + f(A_1)^5 + j(A_1)^9,$$

$$\rho(\tau) = \frac{1}{6}r + \epsilon_6 + r(A_1)^5.$$

With $A_1 = \tau - 1$.

$$\begin{aligned}
\upsilon_1 &= -\frac{1255109217691397}{27195870223942200}r^2 + \frac{182}{45}\epsilon_2 - \frac{7}{27}\epsilon_3 + \frac{1}{10}\epsilon_4 + \frac{1}{6}\epsilon_1 + \epsilon_5,\\
b &= -\frac{43163}{365769}r^2 + \epsilon_2,\\
e &= \frac{5222723}{3657690}r^2 - \frac{121}{10}\epsilon_2 + \epsilon_3,\\
f &= \frac{509767294467341}{197071523361900}r^2 + \epsilon_1,
\end{aligned}$$

and

$$j = -\frac{949586}{715635}r^2 + \frac{506}{45}\epsilon_2 - \frac{16}{27}\epsilon_3 + \epsilon_4.$$

For $\epsilon_l \neq 0$, $(1 \leq l \leq 6)$ *if we take* $|\epsilon_6| \ll |\epsilon_5| \ll \ldots \ll |\epsilon_1|$. *Then there are six different real periodic solutions.*

Proof. See [9]. ∎

Theorem 4.6. *For the class* $C_{11,3}$, *consider that*

$$\kappa(\tau) = a + bt + dt^3 + et^4 + ht^7 + kt^{11},$$

$$\rho(\tau) = m + pt^3.$$

Then $\mu_{\max}(C_{11,3}) \geq 10$.

Proof. We write $w = a + \frac{1}{2}b + \frac{1}{4}d + \frac{1}{5}e + \frac{1}{8}h + \frac{1}{12}k$, $w_1 = m + \frac{1}{4}p$; using Theorem 2.1, we see $\varkappa_2 = w_1$, $\varkappa_3 = w$. If $w, w_1 = 0$, then by putting 'a' and 'm', $\varkappa_4$ is as shown below:

$$\varkappa_4 = -\frac{p(-495k - 455h - 208e + 780b)}{187200}.$$

Now, for $\varkappa_4 = 0$, we put

$$k = -\frac{455}{495}h - \frac{208}{495}e + \frac{780}{495}b,$$

and calculate

$$\varkappa_5 = \frac{p^2(-30065h - 21088e + 14820b)}{2405894400}.$$

From $\varkappa_5 = 0$, by substituting value of h, $\varkappa_6$ is

$$\varkappa_6 = -\frac{p\varpi(2574821080b - 154952704e + 1256094225p^2)}{253139462896320000}.$$

Here $\varpi = e + 5b$. If $\varkappa_6 = 0$; as $p \neq 0$, using value of e from $2574821080b - 154952704e + 1256094225p^2 = 0$, $\varkappa_7$ is

$$\varkappa_7 = -\frac{6499p^2\varpi_1(1739973262973695256b + 714825408941861737p^2 + 160220944663422720d)}{5824557256962807565368950784 0},$$

with $\varpi_1 = 3p^2+8b$. For $\varkappa_6, \varkappa_7 = 0$, suppose that $\varpi = \varpi_1 = 0$, (as possible) then for $e = -5b,\ b = -\frac{3}{8}p^2$; $\kappa(t)$ and $\rho(t)$ can be written as below:

$$\kappa(\tau) = d\left(\tau^3 - \frac{1}{4}\right) + p^2\left(\frac{3}{2}\tau^7 - \frac{15}{8}\tau^4 + \frac{3}{8}\tau\right),$$

$$\rho(\tau) = p\left(\tau^3 - \frac{1}{4}\right).$$

Let $\delta(\tau) = \frac{\tau^4}{4} - \frac{\tau}{4}$ then, $\dot{\delta}(\tau) = \tau^3 - \frac{1}{4}$, also $\delta(0) = \delta(1)$. So, above equations may written as

$$\kappa(\tau) = \left[d + p^2\left(\frac{3}{2}\tau^4 - \frac{3}{2}\tau\right)\right]\dot{\delta}(\tau),\ \rho(\tau) = p\dot{\delta}(\tau).$$

As a result of Theorem 3.1, the origin is the centre with $f(\delta) = \left[d + b\left(4\tau^4 - 4\tau\right)\right]$, $g(\delta) = p$; and for $b = -\frac{3}{8}p^2$ having $f(\delta) = \left[d + p^2\left(\frac{3}{2}\tau^4 - \frac{3}{2}\tau\right)\right]$, $g(\delta) = p$. Thus, suppose $e \neq -5b,\ b \neq -\frac{3}{8}p^2$. Now, we put

$$b = -\frac{714825408941861737}{1739973262973695256}p^2 - \frac{160220944663422720}{1739973262973695256}d,$$

and obtain

$$\varkappa_8 = \frac{p(15853669p^2 + 40748730d)\varpi_2}{415543773503741033001093945561146415899359361252226285384516927488}$$

where $\varpi_2 = -13428577611822818269376896810070582196773931p^4 - 17734094726830067851160472587240248614dp^2 +$

$2205020838128046275374234026162794502d^2$.
If $\varkappa_8 = 0$, then we substitute $d = -\frac{15853669}{40748730}p^2$ and obtain $\varkappa_9$ as follows:

$$\varkappa_9 = -\frac{p^5(2120781350311700544 + 21005802551120299p)}{1578227753631840030720 0}.$$

If $\varkappa_9 = 0$, recalling that $p \neq 0$, we put $p = -\frac{2120781350311700544}{21005802551120299}$, and calculate

$$\varkappa_{10} = -\frac{\begin{array}{c}148312985671995988003729017715939409655677312353027592856538937285314571147\\ 6888499651640748819426810675724829205416748438845463673100954752931696476 16\end{array}}{\begin{array}{c}337550822945425779544304546672699987291432678249406618814296950972604047945\\ 2572614597625736233466957885006352405207528733624321549580247373225685625\end{array}}.$$

It is a non-zero constant number. Therefore, it is concluded that $\mu_{\max}(C_{11,3}) \geq 10$.

For non zero different but the same (either positive or negative) values of the constants in κ, and ρ, we can see that one root is real and others are complex. It gives stable limit cycles from Remark 1, as shown in Figure 2. ∎

Corollary 4.1. *For the equation:*

$$\dot{U} = \kappa(\tau)U^3 + \rho(\tau)U^2 + \upsilon + \upsilon_1. \tag{4.22}$$

If the polynomials $\kappa(\tau)$, $\rho(\tau)$ are as used in Theorem 4.2 and Theorem 4.6. If υ, υ_1 are enough small, then (4.22) has ten real periodic solutions.

Theorem 4.7. *For the Eq (2.3) consider that*

$$\kappa(\tau) = -\omega_1 + bt + dt^3 + et^4 + ht^7 + kt^{11},$$

$$\rho(\tau) = \frac{530195337577925136}{21005802551120299} - \frac{1}{4}\epsilon_1 + \epsilon_8 + (-\frac{2120781350311700544}{21005802551120299} + \epsilon_1)t^3.$$

With
$\upsilon_1 = (-\frac{2120781350311700544}{21005802551120299} + \epsilon_1)^2 + \frac{4709898985434859}{434993315743423814}\epsilon_2 - \frac{4224512475}{1491419776}\epsilon_3 - \frac{8667}{66143}\epsilon_4 - \frac{115}{2376}\epsilon_5 - \frac{1}{12}\epsilon_6 + \epsilon_7$
$b = -(-\frac{2120781350311700544}{21005802551120299} + \epsilon_1)^2 - \frac{20027618082927840}{217496657871711907}\epsilon_2 + \epsilon_3,$
$d = -(-\frac{2120781350311700544}{21005802551120299} + \epsilon_1)^2 + \epsilon_2,$
$h = (-\frac{2120781350311700544}{21005802551120299} + \epsilon_1)^2 + \frac{10115606540780 40}{984147773175167}\epsilon_2 - \frac{47294663}{4236988}\epsilon_3 - \frac{21088}{30065}\epsilon_4 + \epsilon_5,$
$e = -(-\frac{2120781350311700544}{21005802551120299} + \epsilon_1)^2 - \frac{102398557040420025}{66922048575911356}\epsilon_2 + \frac{321852635}{19369088}\epsilon_3 + \epsilon_4,$
and
$k = -\frac{439851904713900}{984147773175167}\epsilon_2 + \frac{64632555}{13316248}\epsilon_3 + \frac{10608}{47245}\epsilon_4 - \frac{91}{99}\epsilon_5 + \epsilon_6.$
If ϵ_l for $(1 \leq l \leq 8)$, have the property that

$$|\epsilon_8| \ll |\epsilon_5| \ll \ldots \ll |\epsilon_1|.$$

Then there exists eight distinct non-trivial periodic solutions.

4.2. Trigonometric coefficients

Now, we consider the Eq (2.3), with polynomials $\kappa(\tau)$, $\rho(\tau)$ in $\sin\tau$ and $\cos\tau$; here upper limit β is 2π (period of these trigonometric functions) for Theorem 2.1.

4.2.1. Non-homogeneous coefficients

Theorem 4.8. *Consider the class $C_{14,7}$, If the coefficients are:*

$$\kappa(\tau) = \Big((c)\cos\tau\sin^5\tau + (e)\cos^5\tau\sin\tau\Big)\Big(\cos^2\tau + \sin^2\tau\Big)^4,$$

$$\rho(\tau) = (a)\cos^6\tau\sin\tau + (b)\cos\tau\sin^6\tau.$$

Then $\varkappa_{\max}(C_{14,7}) \geq 9$.

Proof. Using Theorem 2.1, we calculate

$$\varkappa_2 = \varkappa_3 = \varkappa_4 = 0, \text{ and } \varkappa_5 = -\frac{11ab\pi(c+e)}{458752}.$$

If $\varkappa_5 = 0$, either $a, b = 0$ or:

$$c + e = 0. \tag{4.23}$$

If $a, b = 0$ then $\rho(\tau) = 0$, and $\varkappa_3 = 0$, gives origin is the centre. Using (4.23) we calculate $\varkappa_6 = 0$ and $\varkappa_7$ as:

$$\varkappa_7 = \frac{323abc\pi\left(a^2 - b^2\right)}{6576668672}.$$

If $\varkappa_7 = 0$ as $abc \neq 0$, we put $a = -b$, and calculate $\varkappa_8 = 0$, and $\varkappa_9$ as shown below:

$$\varkappa_9 = \frac{393321cb^4\pi}{1012806974588}.$$

If $\varkappa_9 = 0$, for $c, b = 0$ the origin is the centre. Hence, concluded that $\mu_{\max}(C_{14,7}) \geq 9$. ∎

Theorem 4.9. *For the Eq* (2.3) , *consider coefficients as:*

$$\kappa(\tau) = \Big[(c)\cos^2\tau\sin^6\tau + (e)\cos^6\tau\sin^2\tau\Big]\Big(\cos^2\tau + \sin^2\tau\Big)^4,$$

$$\rho(\tau) = (a)\cos^7\tau\sin\tau + (b)\sin^7\tau\cos\tau.$$

Then $\mu_{\max}(C_{16,8}) \geq 9$ is presented.

Proof. With Theorem 2.1, we calculate $\varkappa_2 = 0$ and:

$$\varkappa_3 = \frac{5\pi(c+e)}{64}.$$

If $\varkappa_2 = \varkappa_3 = 0$, as $\pi \neq 0$, we calculate $\varkappa_4$ as:

$$\varkappa_4 = -\frac{3e\pi(a+b)}{1024}.$$

For $\varkappa_4 = 0$, as $\pi \neq 0$, either $e = 0$ or $a = -b$. If we substitute $e = 0$ then $\kappa(\tau) = 0$, and $\varkappa_2 = 0$, origin is centre. Hence suppose that $e \neq 0$. We substitute for $a = -b$ and take $\varkappa_5 = \varkappa_6 = \varkappa_7 = \varkappa_8 = 0$ with $\varkappa_9$ as:

$$\varkappa_9 = \frac{287bc^2\pi}{589824}.$$

If $\varkappa_9 = 0$, then $c, b = 0$ it can be easily seen that origin is the centre. Hence, concluded that $\mu_{\max}(C_{16,8}) \geq 9$. ∎

4.2.2. Homogeneous coefficients

Theorem 4.10. *Let the class $C_{5,5}$ with:*

$$\kappa(\tau) = (a)\cos^4\tau\sin\tau + (c)\cos^3\tau\sin^2\tau + (d)\cos\tau\sin^4\tau + (f)\cos^5\tau,$$

$$\rho(\tau) = (m)\sin^5\tau.$$

Then $\mu_{\max}(C_{5,5}) \geq 10$ is calculated as follows.

Proof. Using Theorem 2.1, $\varkappa_2 = \varkappa_3 = 0$ and by proceeding further we calculate as:

$$\varkappa_4 = -\frac{m\pi(161c + 189d + 689f)}{1920}.$$

If $\varkappa_4 = 0$, either $m = 0$ or:

$$d = -\frac{167}{189}c - \frac{689}{189}f. \tag{4.24}$$

If $m = 0$, then $\rho(\tau) = 0$ and for $\varkappa_3 = 0$, $\kappa(\tau)$ has mean value as zero. So, origin is a centre, by Corollary 3.2. Also with holding the Eq (4.24), we calculate $\varkappa_5 = 0$ and $\varkappa_6$ as:

$$\varkappa_6 = -\frac{187m^3\pi(247c + 1510f)}{8847360}.$$

For $\varkappa_6 = 0$, we substitute value of c as:

$$c = -\frac{1510}{247}f,$$

and obtained:

$$\varkappa_7 = -\frac{1753m^2af\pi}{2655744}.$$

If $\varkappa_7 = 0$, then we substitute $a = 0$, because $fm^2\pi \neq 0$ and get:

$$\varkappa_8 = -\frac{29mf\pi(46125464931m^4 + 3146842112f^2)}{1667637037760512}.$$

For $\varkappa_8 = 0$, we put $f^2 = -\frac{46125464931}{3146842112}m^4$ and obtained $\varkappa_9 = 0$ whereas $\varkappa_{10}$ comes out as:

$$\varkappa_{10} = -\frac{1840750191m^9\pi(59976855709863203909055069m^6+21955499866730475384586 24m^2-10435252197114333352151040)}{10211153272928270623002967 7127270400}.$$

Hence we conclude that $\mu_{\max}(C_{5,5}) \geq 10$. ∎

Theorem 4.11. *For the class $C_{6,6}$, if:*

$$\kappa(\tau) = (a)\sin^6\tau + (b)\cos^5\tau\sin\tau + (d)\cos\tau\sin^5\tau + (e)\cos^6\tau,$$

$$\rho(\tau) = (g)\cos\tau\sin^5\tau + (i)\cos^5\tau\sin\tau.$$

Then $\mu_{\max}(C_{6,6}) \geq 10$ is given below.

Proof. By using Theorem 2.1, it is calculated that $\varkappa_2 = 0$, $\varkappa_3 = \frac{5\pi(e+a)}{8}$ and by substituting $a = -e$ from $\varkappa_3 = 0$, we calculate $\varkappa_4$ as:

$$\varkappa_4 = -\frac{113e\pi(i+g)}{1536}.$$

If $\varkappa_4 = 0$ then, as π is nonzero, either $e = 0$ or:

$$i = -g. \tag{4.25}$$

If $e = 0$, then $e, a = 0$, gives $\kappa(\tau)$ and $\rho(\tau)$ are same with only different coefficients; i.e. $\kappa(\tau)$ is constant multiple of $\rho(\tau)$. As a result of Corollary 3.1, the origin is a centre. So, suppose $e \neq 0$. Also with holding Eq (4.25) , we calculate $\varkappa_5 = 0$ and $\varkappa_6$ as:

$$\varkappa_6 = -\frac{215ge\pi(d+b)}{49152}.$$

If $\varkappa_6 = 0$, consider that $g, e \neq 0$, because for $g = 0$, the Eq (4.25) gives $i = 0$; so $\rho(\tau) = 0$. For $\varkappa_3 = 0$, origin is the centre. By substituting $d = -b$, we calculate $\varkappa_7 = 0$, $\varkappa_8 = 0$ and by proceeding further, we get $\varkappa_9$ as:

$$\varkappa_9 = \frac{5g\pi(29896704b^2 + 34927bg^3 - 43473024e^2)}{1019215872}.$$

If $\varkappa_9 = 0$, then the only possible substitution is as follows:

$$e^2 = \frac{29896704}{43473024}b^2 + \frac{34927}{43473024}bg^3. \tag{4.26}$$

With holding the Eq (4.26) , $\varkappa_{10}$ comes out as:

$$\varkappa_{10} = -\frac{g^3b\pi(361424202582272bg^3 - 20937295726323g^5 + 341930657292484608b^2 - 18978246855069696bg^2)}{30247865046792142848}.$$

Now, due to lack of formula for $\varkappa_{11}$, we can't proceed further to calculate focal value greater than 10. Hence we conclude that $\mu_{\max}(C_{6,6}) \geq 10$. ∎

5. Further examples

In this section, we have discussed various examples. Here, we are considering the series expansion of some functions. For finding the maximum possible periodic solutions, we shall restrict these expansions up to some extent, for $n = 5$ (say). All the coefficients used below in series expansion are the equal to 1, in general.

Example 5.1. *For the Eq (2.3) , suppose the series expansion upto power five of the functions;* $\kappa(t) = \frac{1}{1+t} = c - dt + et^2 - ft^3 + gt^4 - ht^5$ *and* $\rho(t) = \tanh^{-1}(t) = it + \frac{1}{3}kt^3 + \frac{1}{5}mt^5$*. Then we calculate the periodic multiplicity.*

Solution 5.1. *For finding maximum periodic solutions, we make suitable restrictions of the coefficients and put* $e, k = 0$*. We write* $w = c - \frac{1}{2}d - \frac{1}{4}f + \frac{1}{5}g - \frac{1}{6}h$, $w_1 = \frac{1}{2}i + \frac{1}{30}m$*; using Theorem 2.1, we see* $\varkappa_2 = w_1$, $\varkappa_3 = w$*. If* $w, w_1 = 0$*, then by putting 'c' and 'i',* $\varkappa_4$ *is as shown below:*

$$\varkappa_4 = \frac{m(275h - 384g + 528f + 825d)}{831600}.$$

Now, for $\varkappa_4 = 0$, as $m = 0$ gives origin is the centre. We put

$$h = \frac{384}{275}g - \frac{528}{275}f - \frac{825}{275}d,$$

and calculate

$$\varkappa_5 = -\frac{m^2(-4096g + 7837f)}{25061400000}.$$

From $\varkappa_5 = 0$, by substituting $g = \frac{7837}{4096}f$, $\varkappa_6$ is

$$\varkappa_6 = \frac{fm(-45701267456m^2 + 244257686775f)}{653902675653427200000}.$$

As for $f, m = 0$, origin is centre, using value of f, $\varkappa_7$ is

$$\varkappa_7 = -\frac{3433088m^4(35352838974082506928m^2 + 2309277728312691976125d)}{30313492252257758311549607950072265625}.$$

For $\varkappa_7 = 0$, by substituting 'd' in terms of 'm^2' we obtain:

$$\varkappa_8 = \frac{549673210800336313478854285851894994 12}{3702946397309775159394260734781620687595844233 3984375}m^7.$$

Thus, the multiplicity is 8.

Example 5.2. *For the Eq (2.3), suppose the series expansion upto power five of the functions; $\kappa(t) = \ln(1+t) = ct - \frac{e}{2}t^2 + \frac{f}{3}t^3 - \frac{k}{4}t^4 + \frac{h}{5}t^5$ and $\rho(t) = \sinh(t) = it + \frac{1}{3!}lt^3 + \frac{1}{5!}mt^5$. Then we calculate the periodic multiplicity.*

Solution 5.2. *For finding maximum periodic solutions, we make suitable restrictions of the coefficients and put $l = 0$. We write $w = \frac{1}{2}c - \frac{1}{3}e - \frac{1}{4}f + \frac{1}{5}k - \frac{1}{6}h$, $w_1 = \frac{1}{2}i + \frac{1}{30}m$; using Theorem 2.1, we see $\varkappa_2 = w_1$, $\varkappa_3 = w$. If $w = 0 = w_1$, then by putting values of 'c' and 'i', $\varkappa_4$ is as shown below:*

$$\varkappa_4 = \frac{m(27k - 77f + 154e)}{39916800}.$$

If $\varkappa_4 = 0$, we put $k = \frac{77}{27}f - \frac{154}{27}e$, and calculate

$$\varkappa_5 = \frac{m^2(-18293f + 96256e)}{857460764160000}.$$

Now, if $\varkappa_5 = 0$, we substitute $f = \frac{96256}{18293}e$ and obtained

$$\varkappa_6 = \frac{em(4028768833m^2 + 7178039836650e)}{189986012118785570688000 0}.$$

From $\varkappa_6 = 0$, if $e, m = 0$, origin is the centre. So, we put value of 'e' and calculate

$$\varkappa_7 = -\frac{2422591m^4(3613155317291143m^2 + 1238164754507743536h)}{78757396676696854269380552528627424000000}.$$

and

$$\varkappa_8 = \frac{269093522331877797101235997079544588 67}{117724822732769444947775178271894319926764517651230720000000000}m^7.$$

Thus, the multiplicity is 8.

6. Conclusions

We obtained periodic solutions for two-dimensional non-autonomous differential equations. We discussed two types of coefficients called algebraic and trigonometric (non-homogeneous and homogeneous) coefficients for various classes. Maximum possible upper bound is executed for classes $C_{14,7}$, $C_{16,8}$ with non-homogeneous and $C_{5,5}$ with homogeneous trigonometric coefficients, while $C_{9,4}$, $C_{9,5}$, $C_{9,6}$ and $C_{11,3}$ with the polynomial coefficients. We attained the highest multiplicity as 10 for classes $C_{9,4}$, $C_{11,3}$ $C_{5,5}$, and $C_{6,6}$ which is the highest one. All the result has been done using computer algebra package Maple 18. We have concluded from this extensive analysis that there is no connection between the degree of polynomial and the number of limit cycles. The higher degree classes like $C_{14,7}$, $C_{16,8}$ ends up with maximum multiplicity eight in contrast to lower degree polynomial classes $C_{5,5}$, $C_{6,6}$ and $C_{9,4}, C_{11,3}$ having ten limit cycles. This conclusion is in accordance with the 2nd part of Hilbert's 16th problem. We have calculated the formulae in Section 2; which extends previous works of Alwash [12] and Yasmin [16] and attempt to step forward in the comprehension of the case n > 9, in the Eq (2.5). In future, one can generalize the same concept of multiplicity and can calculate multiplicity greater than ten by extending Theorem 2.1. The same results can be validated by the use of simulation against time domain.

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Conflict of interest

The authors declare that they have no known financial interest or personal relationship that could have appeared to influence the work reported in this paper.

References

1. J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields, *Springer,* (1983), 353–420.
2. Z. Wang, D. Liu, M. Song, Existence of three periodic solutions for a quasilinear periodic boundary value problem, *AIMS Math.,* **5** (2020), 6061–6072.
3. J. Laszlo, M. V. Panne, E. Fiume, *Limit cycle control and its application to the animation of balancing and walking*, Proceedings of the 23rd annual conference on Computer graphics and interactive techniques, 1996, 155–162.
4. N. G. Lloyd, *Small amplitude limit cycles of polynomial differential equations*, In: Ordinary differential equations and operators, 1982, 346–357.
5. N. G. Lloyd, The number of periodic solutions of the equation $Z^{\cdot} = z^n + p_1(s)z^{n-1} + P_2(s)z^{n-2}+...+P_n(s)$, *Proc. London Math. Soc.,* **27** (1973), 667–700.

6. A. Gasull, J. Llibre, Limit cycles for a class of Abel equations, *J. Math. Anal.,* **21** (1990), 1235–1244.

7. L. Neto, On the number of solutions of the equations $\frac{dX}{ds} = \sum_{j=0}^{n} P_j(s)\, s' \; 0 \le s \le 1$ for which $X(0) = X(1)$, *Invent. Math.,* **59** (1980), 67–76.

8. N. G. Lloyd, *Limit cycles of certain polynomial systems,* In: Non-linear functional analysis and its applications, NATO ASI Series, **173** (1986), 317–326.

9. S. Akram, A. Nawaz, T. Abdeljawad, A. Ghaffar, K. S. Nisar, Calculation of focal values for first-order non-autonomous equation with algebraic and trigonometric coefficients, *Open Phys.,* **18** (2020), 738–750.

10. S. Akram, A. Nawaz, N. Yasmin, H. Kalsoom, Y. M. Chu, Periodic solutions for first order cubic non-autonomous differential equation with bifurcation analysis, *J. Taibah Univ. Sci.,* **14** (2020), 1208–1217.

11. D. Hilbert, Mathematical problems, *Bull. Amer. Math. Soc.*, **8** (1902), 437–479.

12. M. A. M. Alwash, N. G. Lloyd, Non-autonomous equation related to polynomial two-dimensional system, *Proceed. Royal Soc. Edin.,* **5** (1987), 129–152.

13. S. Akram, A. Nawaz, N. Yasmin, A. Ghaffar, D. Baleanu, Periodic solutions of some classes of one dimensional non-autonomous system, *Front. Phys.,* **8** (2020), 264.

14. S. Akram, A. Nawaz, H. Kalsoom, M. Idrees, Y. M. Chu, Existence of multiple periodic solutions for cubic nonautonomous differential equation, *Math. Prob. Eng.,* (2020), 7618097.

15. S. Akram, A. Nawaz, M. B. Riaz, M. Rehman, Periodic solutions for two dimensional quartic nonautonomous differential equation, *Intell. Autom. Soft Comput.,* (2021), 019767.

16. N. Yasmin, *Closed orbits of certain two dimensional cubic systems (Ph.D. thesis)*, University College of Wales Aberystwyth, United Kingdom, 1989, 1–169.

17. G. Nallappan, S. Sabarathinam, Z. Guisheng, Y. Qiang, Dynamical analysis and sampled-data stabilization of memristor-based chuas circuits, *IEEE Access,* (2021), 25648–25658.

18. M. A. M. Alwash, Periodic solutions of polynomial non-autonomous differential equations, *Electron. J. Differ. Eq.,* **2005** (2005), 1–8.

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