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Research article

New Chebyshev type inequalities via a general family of fractional integral operators with a modified Mittag-Leffler kernel

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Abstract: The main goal of this article is first to introduce a new generalization of the fractional integral operators with a certain modified Mittag-Leffler kernel and then investigate the Chebyshev inequality via this general family of fractional integral operators. We improve our results and we investigate the Chebyshev inequality for more than two functions. We also derive some inequalities of this type for functions whose derivatives are bounded above and bounded below. In addition, we establish an estimate for the Chebyshev functional by using the new fractional integral operators. Finally, we find similar inequalities for some specialized fractional integrals keeping some of the earlier results in view.

Keywords: Chebyshev inequality; generalized fractional integrals; modified Mittag-Leffler kernel; Fox-Wright function; integral inequalities
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1. Introduction

For the last few decades, the study of integral inequalities has been a significant field of fractional calculus and its applications, connecting with such other areas as differential equations, mathematical analysis, mathematical physics, convexity theory, and discrete fractional calculus [1–13]. One important type of integral inequalities consists of the familiar Chebyshev inequality which is related to the synchronous functions. This has been intensively studied, with many book chapters and important research articles dedicated to the Chebyshev type inequalities [14–18]. The Chebyshev inequality is given as follows (see [16]):

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\zeta_1(\mathbf{z})\zeta_2(\mathbf{z})\,\mathrm{d}\mathbf{z} \geqq \left(\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\zeta_1(\mathbf{z})\,\mathrm{d}\mathbf{z}\right)\left(\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\zeta_2(\mathbf{z})\,\mathrm{d}\mathbf{z}\right), \tag{1.1}$$

where ζ_1 and ζ_2 are assumed to be integrable and synchronous functions on $[b_1, b_2]$. By definition, two functions are called synchronous on $[b_1, b_2]$ if the following inequality holds true:

$$(\zeta_1(\mathbf{z})-\zeta_1(y))(\zeta_2(\mathbf{z})-\zeta_2(y)) \geqq 0, \quad \forall\, \mathbf{z}, y \in [b_1, b_2].$$

In particular, the Chebyshev inequality (1.1) is useful due to its connections with fractional calculus and it arises naturally in existence of solutions to various integer-order or fractional-order differential equations including some which are useful in practical applications such as those in numerical quadrature, transform theory, statistics and probability [19–24].

In the context of fractional calculus, the study of the derivative and integral operators of calculus is extended to non-integer orders [25–27], but most (if not all) of the potentially useful studies come about only along the real line. The standard left-side and right-side Riemann-Liouville (RL) fractional integrals of order $\mu > 0$ are defined, respectively, by

$$\left(\mathcal{I}^{\mu}_{b_1+}\varphi\right)(\mathbf{z}) = \frac{1}{\Gamma(\mu)}\int_{b_1}^{\mathbf{z}}(\mathbf{z}-\xi)^{\mu-1}\,\varphi(\xi)\,\mathrm{d}\xi \qquad (\mathbf{z} > b_1) \tag{1.2}$$

and

$$\left(\mathcal{I}^{\mu}_{b_2-}\varphi\right)(\mathbf{z}) = \frac{1}{\Gamma(\mu)}\int_{\mathbf{z}}^{b_2}(\xi-\mathbf{z})^{\mu-1}\,\varphi(\xi)\,\mathrm{d}\xi \qquad (\mathbf{z} < b_2), \tag{1.3}$$

where $\varphi(\mathbf{z})$ is a function defined on $\mathbf{z} \in [b_1, b_2]$. Furthermore, the left-side and right-side Riemann-Liouville (RL) fractional derivatives are defined, respectively, by means of the following expressions for $\Re(\mu) \geqq 0$:

$$D^{\mu}_{b_1+}\varphi(\mathbf{z}) := \frac{\mathrm{d}^n}{\mathrm{d}\mathbf{z}^n}\mathcal{I}^{n-\mu}_{b_1+}\varphi(\mathbf{z})$$

and

$$D^{\mu}_{b_2-}\varphi(\mathbf{z}) := \frac{\mathrm{d}^n}{\mathrm{d}\mathbf{z}^n}\mathcal{I}^{n-\mu}_{b_2-}\varphi(\mathbf{z}),$$

in each of which $n := \lfloor \Re(\mu) \rfloor + 1$.

There are many ways to define fractional derivatives and fractional integrals, often related to or inspired by the RL definitions (see, for example, [28–30]), with reference to some general classes into which such fractional derivative and fractional integral operators can be classified. In pure mathematics, we always consider the most general possible setting in which a specific behaviour or result can be obtained. However, in applied mathematics, it is important to consider particular types of fractional calculus, which are suited to the model of a given real-world problem.

Some of these definitions of fractional calculus have properties which are from those of the standard RL definitions, and some of them can be used to the model of real-life data more effectively than the RL model [31–37]. As described in many recent articles which are cited herein, the fractional calculus definitions, which are discussed in this article, have been found to be useful, particularly in the modelling of real-world problems.

Special functions have many relations with fractional calculus [1, 25, 38]. In particular, the Mittag-Leffler (ML) type functions are remarkably significant in this area (see [39–42]).

The familiar Mittag-Leffler function $E_{\alpha}(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$ are defined, respectively, by

$$E_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \qquad \text{and} \qquad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{1.4}$$

$$(z, \alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0),$$

which were first considered by Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) in 1903 and Anders Wiman (1865–1959) in 1905.

In many recent investigations, the interest in the families of Mittag-Leffler type functions has grown considerably due mainly to their potential for applications in some reaction-diffusion and other applied problems and their various generalizations appear in the solutions of fractional-order differential and integral equations (see, for example, [43]; see also [44] and [45]). The following family of the multi-index Mittag-Leffler functions:

$$E_{\gamma,\kappa,\epsilon}\left[(\alpha_j, \beta_j)_{j=1}^m; z\right]$$

was considered and used as a kernel of some fractional-calculus operators by Srivastava *et al.* (see [46] and [47]; see also the references cited in each of these papers):

$$E^{\gamma,\kappa,\delta,\epsilon}_{(\alpha_j,\beta_j)_m}[z] = E_{\gamma,\kappa,\delta,\epsilon}\left[(\alpha_j, \beta_j)_{j=1}^m; z\right] := \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}\ (\delta)_{\epsilon n}}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \frac{z^n}{n!} \tag{1.5}$$

$$\left(\alpha_j, \beta_j, \gamma, \kappa, \delta, \epsilon \in \mathbb{C};\ \Re(\alpha_j) > 0\ \ (j = 1, \ldots, m);\ \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \Re(\kappa + \epsilon) - 1\right),$$

where $(\lambda)_\nu$ denotes the *general* Pochhammer symbol or the *shifted* factorial, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1,2,3,\ldots\}),$$

defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the familiar Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{1.6}$$

it being assumed *conventionally* that $(0)_0 := 1$ and understand *tacitly* that the Γ-quotient in (1.6) exists. Some of the special cases of the multi-index Mittag-Leffler function:

$$E_{\gamma,\kappa,\epsilon}\left[(\alpha_j,\beta_j)_{j=1}^m; z\right]$$

include (for example) the following generalizations of the Mittag-Leffler type functions:

(i) By using the relation between the Gamma function and the Pochhammer symbol in (1.6), the case when $m = 2$, $\delta = \epsilon = 1$, $\kappa = \mathfrak{q}$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, and $\alpha_2 = \mathfrak{p}$, and $\beta_2 = \delta$, the definition (1.5) would correspond to $[\Gamma(\delta)]^{-1}$ times the Mittag-Leffler type function $E_{\alpha,\beta,\mathfrak{p}}^{\gamma,\delta,\mathfrak{q}}(z)$, which was considered by Salim and Faraj [48].

(ii) A special case of the multi-index Mittag-Leffler function defined by (1.5) when $m = 2$ can be shown to correspond to the Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,\kappa}(z)$, which was introduced by Srivastava and Tomovski [49] (see also [50]).

(iii) For $m = 2$ and $\kappa = 1$, the multi-index Mittag-Leffler function defined by (1.5) would readily correspond to the Mittag-Leffler type function $E_{\alpha,\beta}^{\gamma}(z)$, which was studied by Prabhakar [51].

We now turn to the familiar Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ (with p numerator and q denominator parameters), which is given by the following series (see Fox [52] and Wright [53,54]; see also [1, p. 67, Eq (1.12 (68)] and [55, p. 21, Eq 1.2 (38)]):

$$\begin{aligned} {}_p\Psi_q\left[\begin{array}{c} (\alpha_1,A_1),\ldots,(\alpha_p,A_p); \\ \\ (\beta_1,B_1),\ldots,(\beta_q,B_q); \end{array} z\right] &:= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{k=1}^{q}\Gamma(\beta_k + B_k n)} \frac{z^n}{n!} \\ &= \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{k=1}^{q}\Gamma(\beta_k)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\left(\alpha_j\right)_{A_j n}}{\prod_{k=1}^{q}(\beta_k)_{B_k n}} \frac{z^n}{n!}, \end{aligned} \tag{1.7}$$

in which we have made use of the general Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ defined by (1.6), the parameters

$$\alpha_j, \beta_k \in \mathbb{C} \ \ (j = 1,\ldots,p;\ k = 1,\cdots,q)$$

and the coefficients

$$A_1,\ldots,A_p \in \mathbb{R}^+ \qquad \text{and} \qquad B_1,\ldots,B_q \in \mathbb{R}^+$$

are so constrained that

$$1+\sum_{k=1}^{q} B_k-\sum_{j=1}^{p} A_j \geqq 0, \tag{1.8}$$

with the equality for appropriately constrained values of the argument z. Thus, if we compare the definition (1.5) of the general multi-index Mittag-Leffler function:

$$E_{\gamma,\kappa,\delta,\epsilon}\left[(\alpha_j,\beta_j)_{j=1}^{m};z\right]$$

with the definition in (1.7), it immediately follows that

$$E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa,\delta,\epsilon}[z]=E_{\gamma,\kappa,\delta,\epsilon}\left[(\alpha_j,\beta_j)_{j=1}^{m};z\right]=\frac{1}{\Gamma(\gamma)\Gamma(\delta)}\,{}_2\Psi_m\left[\begin{array}{r}(\gamma,\kappa),(\delta,\epsilon);\\ \\ (\beta_1,\alpha_1),\ldots,(\beta_m,\alpha_m);\end{array}\ z\right]. \tag{1.9}$$

In particular, for the above-mentioned Mittag-Leffler type functions $E_{\alpha,\beta,\mathfrak{p}}^{\gamma,\delta,\mathfrak{q}}(z)$, $E_{\alpha,\beta}^{\gamma,\kappa}(z)$ and $E_{\alpha,\beta}^{\gamma}(z)$, we have the following relationships with the Fox-Wright hypergeometric function defined by (1.7):

$$E_{\alpha,\beta,\mathfrak{p}}^{\gamma,\delta,\mathfrak{q}}(z)=\frac{\Gamma(\delta)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{c}(1,1),(\gamma,\mathfrak{q});\\ \\ (\delta,\mathfrak{p}),(\beta,\alpha);\end{array}\ z\right], \tag{1.10}$$

$$E_{\alpha,\beta}^{\gamma,\kappa}(z)=\frac{1}{\Gamma(\gamma)}\,{}_1\Psi_1\left[\begin{array}{c}(\gamma,\kappa);\\ \\ (\beta,\alpha);\end{array}\ z\right] \tag{1.11}$$

and

$$E_{\alpha,\beta}^{\gamma}(z)=\frac{1}{\Gamma(\gamma)}\,{}_1\Psi_1\left[\begin{array}{c}(\gamma,1);\\ \\ (\beta,\alpha);\end{array}\ z\right]. \tag{1.12}$$

The relationships in (1.9), (1.10), (1.11) and (1.12) exhibit the fact that, not only this general multi-index Mittag-Leffler function defined by (1.5), but indeed also all of the above-mentioned Mittag-Leffler type functions *and many more*, are contained, as special cases, in the the extensively- and widely-investigated Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (1.7). The interested reader will find it to be worthwhile to refer also to the aforecited work of Srivastava and Tomovski [49, p. 199] for similar remarks about the much more general nature of the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ than any of these Mittag-Leffler type functions.

It should be mentioned in passing that, not only Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (1.7), but also much more general functions such as (for example) the Meijer G-function and the Fox H-function, have already been used as kernels of various families of fractional-calculus operators (see, for details, [56–58]; see also the references cited therein). In fact, Srivastava *et al.* [57] not only used the Riemann-Liouville type fractional integrals with the Fox H-function and the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ as kernels, but also applied their results to the substantially more general $\overline{H}$-function (see, for example, [59, 60]).

Our present investigation is based essentially upon the operators of the fractional integrals of the Riemann-Liouville type (1.2), which are defined below.

Definition 1.1 (see [61]). For a given L_1-function φ on an interval $[b_1, b_2]$, the general left-side and right-side fractional integral operators, applied to $\varphi(\mathbf{z})$, are defined for $\lambda, \rho > 0$ and $w \in R$ by

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b_1+;w}\varphi\right)(\mathbf{z}) = \int_{b_1}^{\mathbf{z}} (\mathbf{z}-\xi)^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(\mathbf{z}-\xi)^{\rho}\right]\varphi(\xi)\,\mathrm{d}\xi \qquad (\mathbf{z} > b_1) \tag{1.13}$$

and

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b_2-;w}\varphi\right)(\mathbf{z}) = \int_{\mathbf{z}}^{b_2} (\xi-\mathbf{z})^{\lambda-1}\,\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(\xi-\mathbf{z})^{\rho}\right]\varphi(\xi)\,\mathrm{d}\xi \qquad (\mathbf{z} < b_2), \tag{1.14}$$

where the function φ is so constrained that the integrals on the right-hand sides exit and $\mathcal{F}^{\sigma}_{\rho,\lambda}$ is the modified Mittag-Leffler function given by (see [62])

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(\mathbf{z}) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(\mathbf{z}) = \sum_{n=0}^{\infty} \frac{\sigma(n)}{\Gamma(\rho n+\lambda)}\,\mathbf{z}^n, \tag{1.15}$$

where $\rho, \lambda > 0,\ \ |\mathbf{z}| < R$, and $\{\sigma(n)\}_{n\in\mathbb{N}_0}$ is a bounded sequence in the real-number set $\mathbb{R}$.

The definition (1.15) should be credited, in fact, to Wright [63, p. 424] who studied this function rather systematically as long ago as 1940.

Remark 1.1. Obviously, if we set

$$\sigma(n) = \frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\Gamma(n+1)\ \prod\limits_{k=1}^{q}\Gamma(\beta_k + B_k n)} \qquad (n \in \mathbb{N}_0) \tag{1.16}$$

in the definition (1.15), we are led to the following special case:

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(\mathbf{z}) = {}_p\Psi_{q+1}\left[\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\lambda,\rho), (\beta_1, B_1), \ldots, (\beta_q, B_q); \end{array} z\right] \tag{1.17}$$

in terms of the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (1.7).

A slightly modified version of the fractional integrals in Definition 1.1, which we find to be convenient to use in this paper, is given by Definition 1.2 below.

Definition 1.2 (The ν-modified fractional integral operators). For a given L_1-function φ on an interval $[b_1, b_2]$, the general left-side and right-side fractional integral operators, applied to $\varphi(\mathbf{z})$, are defined for $\lambda, \rho, \nu > 0$ and $w \in \mathbb{R}$ by

$$\left(\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_1+;w}\varphi\right)(\mathbf{z}) = \int_{b_1}^{\mathbf{z}} (\mathbf{z}-\xi)^{\frac{\lambda}{\nu}-1}\,\mathcal{F}^{\sigma}_{\rho\nu,\lambda}\left[w(\mathbf{z}-\xi)^{\rho}\right]\varphi(\xi)\,\mathrm{d}\xi \qquad (\mathbf{z} > b_1) \tag{1.18}$$

and

$$\left(\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_2-;w}\varphi\right)(\mathbf{z}) = \int_{\mathbf{z}}^{b_2} (\xi-\mathbf{z})^{\frac{\lambda}{\nu}-1}\,\mathcal{F}^{\sigma}_{\rho\nu,\lambda}\left[w(\xi-\mathbf{z})^{\rho}\right]\varphi(\xi)\,\mathrm{d}\xi \qquad (\mathbf{z} < b_2), \tag{1.19}$$

provided that each of the integrals in (1.18) and (1.19) exists.

Remark 1.2. If we set $\nu = 1$ in Definition 1.2, then we can immediately obtain Definition 1.1.

Remark 1.3. It is easy to verify that

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_1+;w}\varphi \qquad \text{and} \qquad \mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_2-;w}\varphi$$

are bounded integral operators on $L_1(b_1, b_2)$ if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(b_2 - b_1)^{\rho}\right] < \infty.$$

In fact, for $\varphi \in L_1(b_1, b_2)$, we have

$$\left\|\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_1+;w}\varphi\right\|_1 \leqq \mathfrak{M}\,\|\varphi\|_1 \qquad \text{and} \qquad \left\|\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_2-;w}\varphi\right\|_1 \leqq \mathfrak{M}\,\|\varphi\|_1,$$

where

$$\|\varphi\|_p = \left(\int_{b_1}^{b_2} |\varphi(\mathbf{z})|^p \;\mathrm{d}\mathbf{z}\right)^{1/p}.$$

Remark 1.4. In view of the generality of the sequence $\{\sigma(n)\}_{n\in\mathbb{N}_0}$, the fractional integral operators given by Definition 1.1 and Definition 1.2 can be appropriately specialized to yield all those Riemann-Liouville type fractional integrals involving not only the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ kernel given by (1.17), but also involving all those multi-index Mittag-Leffler type kernels which are *further* special cases of the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (1.7).

There exist many classes integral inequalities related to the fractional integral operators given by Definition 1.1 (see, for example, [64–68]). Our objective in this work is to present a study of Chebyshev's inequality in terms of the fractional integrals given by Definition 1.2. We also apply our results to deduce several results by following the lines used in some of the earlier works.

2. Main results and their consequences

Throughout our study, we suppose that $\{\sigma(n)\}_{n\in\mathbb{N}_0}$ is a sequence of non-negative real numbers.

Theorem 2.1. *Let $\lambda, \rho, \nu > 0$ and $w \in \mathbb{R}$. Also let ζ_1 and ζ_2 be two synchronous functions on $[0, \infty)$. Then*

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) \geqq \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) \qquad (\forall\ \xi > 0).$$

Proof. Since the functions ζ_1 and ζ_2 are synchronous on $[0, \infty)$, we find for $r, s \geqq 0$ that

$$(\zeta_1(r) - \zeta_1(s))(\zeta_2(r) - \zeta_2(s)) \geqq 0.$$

It follows that

$$\zeta_1(r)\zeta_2(r) + \zeta_1(s)\zeta_2(s) \geqq \zeta_1(r)\zeta_2(s) + \zeta_1(s)\zeta_2(r). \tag{2.1}$$

By multiplying both sides of (2.1) by

$$(\xi - r)^{\frac{\lambda}{\nu}-1}\,\mathcal{F}^{\sigma}_{\rho\nu,\lambda}\left[w(\xi - r)^{\rho}\right]$$

with $r \in (0, \xi)$, we can deduce that

$$
\begin{aligned}
&(\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \zeta_1(r)\zeta_2(r) + (\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \zeta_1(s)\zeta_2(s) \\
&\qquad \geqq (\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \zeta_1(r)\zeta_2(s) + (\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \zeta_1(s)\zeta_2(r),
\end{aligned}
$$

which, upon integration over $r \in (0, \xi)$, yields

$$
\begin{aligned}
&\int_0^{\xi} (\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \zeta_1(r)\zeta_2(r) \, \mathrm{d}r + \int_0^{\xi} (\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \zeta_1(s)\zeta_2(s) \, \mathrm{d}r \\
&\quad \geqq \int_0^{\xi} (\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \zeta_1(r)\zeta_2(s) \, \mathrm{d}r \\
&\qquad + \int_0^{\xi} (\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \zeta_1(s)\zeta_2(r) \, \mathrm{d}r
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) + \zeta_1(s)\zeta_2(s) \int_0^{\xi} (\xi - r)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - r)^{\rho} \right] \, \mathrm{d}r \\
&\qquad \geqq \zeta_2(s)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi) + \zeta_1(s)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) + \nu\zeta_1(s)\zeta_2(s)\xi^{\frac{\lambda}{\nu}} \mathcal{F}^{\sigma}_{\rho\nu,\lambda+1} \left[w(\xi)^{\rho} \right] \\
&\qquad \geqq \zeta_2(s)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi) + \zeta_1(s)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi).
\end{aligned}
$$

We now multiply this last inequality by

$$
(\xi - s)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - s)^{\rho} \right]
$$

with $s \in (0, \xi)$, so that

$$
\begin{aligned}
&(\xi - s)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - s)^{\rho} \right] \mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) \\
&\qquad + \nu\,(\xi - s)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - s)^{\rho} \right] \zeta_1(s)\zeta_2(s)\xi^{\frac{\lambda}{\nu}} \mathcal{F}^{\sigma}_{\rho\nu,\lambda+1} \left[w(\xi)^{\rho} \right] \\
&\quad \geqq (\xi - s)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - s)^{\rho} \right] \zeta_2(s)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi) \\
&\qquad + (\xi - s)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - s)^{\rho} \right] \zeta_1(s)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi),
\end{aligned}
$$

which, by integrating over $s \in (0, \xi)$, yields

$$
\begin{aligned}
&\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) \int_0^{\xi} (\xi - s)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - s)^{\rho} \right] \, \mathrm{d}s \\
&\qquad + \nu\xi^{\frac{\lambda}{\nu}} \mathcal{F}^{\sigma}_{\rho\nu,\lambda+1} \left[w(\xi)^{\rho} \right] \int_0^{\xi} (\xi - s)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - s)^{\rho} \right] \zeta_1(s)\zeta_2(s) \, \mathrm{d}s \\
&\quad \geqq \mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi) \int_0^{\xi} (\xi - s)^{\frac{\lambda}{\nu}-1} \mathcal{F}^{\sigma}_{\rho\nu,\lambda} \left[w(\xi - s)^{\rho} \right] \zeta_2(s) \, \mathrm{d}s
\end{aligned}
$$

$$+\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi)\int_0^{\xi}(\xi-s)^{\frac{\lambda}{\nu}-1}\mathcal{F}^{\sigma}_{\rho\nu,\lambda}\left[w(\xi-s)^{\rho}\right]\zeta_1(s)\,\mathrm{d}s.$$

If we simplify this last inequality, we get

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi)\geqq\frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi),$$

which completes our proof of Theorem 2.1. □

Remark 2.1. If we take $\nu = 1$ in Theorem 2.1, we obtain [21, Theorem 2] or [22, Corollary 3.11].

Remark 2.2. If we consider the interval $[b_1,\infty)$ $(b_1 > 0$ instead of the interval $[0,\infty)$ in Theorem 2.1 with slight modifications in the proof, then we can find that

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_1+;w}(\zeta_1\zeta_2)(\xi)\geqq\frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_1+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,b_1+;w}(\zeta_2)(\xi)\qquad(\xi>b_1).$$

Remark 2.3. By appropriately specializing the parameters involved in Theorem 2.1 or Remark 2.2, we can derive a number of known or new results including (for example) the known result [69, Theorem 3.1] Moreover, if we set $\lambda = \mu$ $(\lambda,\mu > 0)$, $\sigma(0) = \nu = 1$ and $w = 0$ in Remark 2.2, we can obtain

$$\mathcal{I}^{\mu}_{b_1+}(\zeta_1\zeta_2)(\xi)\geqq\frac{\Gamma(\mu+1)}{\xi^{\mu}}\mathcal{I}^{\mu}_{b_1+}(\zeta_1)(\xi)\mathcal{I}^{\mu}_{b_1+}(\zeta_2)(\xi).$$

Additionally, if $\mu = \nu = 1$ and $\xi = b_2$ with $b_2 > b_1$, then we can obtain (2.1). Furthermore, as we pointed out in Remark 1.4, with appropriate choices of, *and* under sufficient conditions on, the arguments and the parameters involved, we can express the result of Theorem 2.1 in terms of fractional integrals with the Fox-Wright hypergeometric function ${}_p\Psi_q(z)$, given by (1.7), (1.16) and (1.17), but also in terms of the aforementioned Mittag-leffler type kernels such as

$$E_{\gamma,\kappa,\epsilon}\left[(\alpha_j,\beta_j)_{j=1}^m;z\right],$$

given by (1.5) and (1.9), as well as its *further* special cases:

$$E^{\gamma,\delta,\mathfrak{q}}_{\alpha,\beta,\mathfrak{p}}(z),\quad E^{\gamma,\kappa}_{\alpha,\beta}(z)\qquad\text{and}\qquad E^{\gamma}_{\alpha,\beta}(z),$$

given by (1.10), (1.11) and (1.12), respectively. The details of these and other derivations from Theorem 2.1 or Remark 2.2 are fairly straightforward, so we choose to omit the details involved.

We next state and prove Theorem 2.2 below.

Theorem 2.2. *Let $\lambda,\rho,\nu > 0$ and $w \in \mathbb{R}$. Also let $\{\zeta_i\}_{i=1}^n$ be n positive and increasing functions defined on* $[0,\infty)$. *Then*

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\prod_{i=1}^n\zeta_i\right)(\xi)\geqq\left(\frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\right)^{n-1}\prod_{i=1}^n\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_i)(\xi)\qquad(\forall\ \xi>0).\tag{2.2}$$

Proof. The proof will make use of the principle of mathematical induction. Firstly, for $n = 1$, we have

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\zeta_1\right)(\xi) \geqq \mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\zeta_1\right)(\xi) \qquad (\forall\ \xi > 0).$$

In the case when $n = 2$, by making use of Theorem 2.1, we have

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) \geqq \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) \qquad (\forall\ \xi > 0).$$

We now assume that the inequality (2.2) holds true for some $n \in \mathbb{N}$. Then, since the n functions $\{\zeta_i\}_{i=1}^{n}$ are positive and increasing on $[0,\infty)$, $\prod_{i=1}^{n}\zeta_i$ is also an increasing function. Hence, we can apply Theorem 2.1 with

$$\zeta_1 = \prod_{i=1}^{n-1}\zeta_i \qquad \text{and} \qquad \zeta_2 = \zeta_n$$

in order to obtain

$$\begin{aligned}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\prod_{i=1}^{n}\zeta_i\right)(\xi) &= \mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\zeta_1\zeta_2\right)(\xi)\\ &\geqq \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\zeta_1\right)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\zeta_2\right)(\xi)\\ &= \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\prod_{i=1}^{n-1}\zeta_i\right)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\zeta_n\right)(\xi).\end{aligned}$$

Thus, if we make use of our assumed inequality (2.2) in the last inequality, we have

$$\begin{aligned}&\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\prod_{i=1}^{n}\zeta_i\right)(\xi)\\ &\geqq \left(\frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\right)\left(\frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\right)^{(n-1)-1}\\ &\qquad \cdot\prod_{i=1}^{n-1}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\zeta_i\right)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\zeta_n\right)(\xi)\\ &= \left(\frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\right)^{n-1}\ \mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\prod_{i=1}^{n}\zeta_i\right)(\xi).\end{aligned}$$

This completes our proof of Theorem 2.2. □

Remark 2.4. If we set $\nu = 1$ in Theorem 2.2, we obtain [21, Theorem 4].

Remark 2.5. Several particular cases can be obtained from Theorem 2.2 for the right-side Riemann-Liouville fractional integral operator in Definition 1.1. For example, if we put $\lambda = \mu \quad (\lambda, \mu > 0)$, $\sigma(0) = \nu = 1$ and $w = 0$ in Theorem 2.2, we can obtain the following result:

$$\mathcal{I}^{\mu}_{0+}\left(\prod_{i=1}^{n}\zeta_i\right)(\xi) \geqq \left(\frac{\Gamma(\mu+1)}{\xi^{\mu}}\right)^{n-1}\ \prod_{i=1}^{n}\mathcal{I}^{\mu}_{0+}\left(\zeta_i\right)(\xi),$$

which was considered in in [69, Theorem 3.3]. Moreover, just as we pointed out in Remark 2.3, with appropriate choices of, *and* under sufficient conditions on, the parameters and the arguments involved, we can express the result of Theorem 2.2 in terms of fractional integrals with the aforementioned Mittag-leffler type kernels:

$$E^{\gamma,\delta,\mathfrak{q}}_{\alpha,\beta,\mathfrak{p}}(z), \quad E^{\gamma,\kappa}_{\alpha,\beta}(z) \qquad \text{and} \qquad E^{\gamma}_{\alpha,\beta}(z),$$

given by (1.10), (1.11) and (1.12), respectively. The details involved are being skipped here.

We next state and prove Theorem 2.3 below.

Theorem 2.3. *Let $\lambda, \rho, \nu > 0$ and $w \in \mathbb{R}$. Also let ζ_1, ζ_2 be two functions such that ζ_1 is increasing and ζ_2 is differentiable. If there exists a real number $\mathfrak{m}$ with $\mathfrak{m} = \inf_{\xi \geqq 0} \zeta_2{}'(\xi)$, then*

$$\begin{aligned}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) \geqq{}& \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) \\ &- \frac{\mathfrak{m}\xi\,\mathcal{F}^{\sigma}_{\rho\nu,\lambda+2}[w\xi^{\rho}]}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi) + \mathfrak{m}\,\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id}\,\zeta_1)(\xi) \qquad (\forall\ \xi > 0),\end{aligned}$$

where

$$\mathrm{Id}(\xi) = \xi \qquad \textit{and} \qquad (\mathrm{Id}\,\zeta_1)(\xi) = \mathrm{Id}(\xi)\cdot\zeta_1(\xi) = \xi\cdot\zeta_1(\xi).$$

Proof. Let us define the following function:

$$h(\xi) := \zeta_2(\xi) - \mathfrak{m}\,\mathrm{Id}(\xi),$$

where $\mathrm{Id}(\xi) = \xi$. One can easily verify that h is an increasing and differentiable function on $[0, \infty)$. Then, by using Theorem 2.1, we have

$$\begin{aligned}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1 h)(\xi) &\geqq \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(h)(\xi) \\ &= \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\left(\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) - \mathfrak{m}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id})(\xi)\right) \\ &= \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) \\ &\quad - \frac{\mathfrak{m}\xi\mathcal{F}^{\sigma}_{\rho\nu,\lambda+2}[w\xi^{\rho}]}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi).\end{aligned}$$

Moreover, since

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1 h)(\xi) = \mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) - \mathfrak{m}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id}\zeta_1)(\xi),$$

it follows that

$$\begin{aligned}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) \geqq{}& \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) \\ &- \frac{\mathfrak{m}\xi\mathcal{F}^{\sigma}_{\rho\nu,\lambda+2}[w\xi^{\rho}]}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}\left[w(\xi)^{\rho}\right]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi) + \mathfrak{m}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id}\zeta_1)(\xi).\end{aligned}$$

This evidently completes our proof of Theorem 2.3. □

Remark 2.6. Upon setting $\nu = 1$ in Theorem 2.3, we obtain [21, Theorem 5].

Corollary 2.1. *Let $\lambda, \rho, \nu > 0$ and $w \in \mathbb{R}$. Also let ζ_1 and ζ_2 be two functions such that ζ_1 is increasing and ζ_2 is differentiable. If there is a real number M with $M = \sup_{\xi \geqq 0} \zeta_2{}'(\xi)$, then*

$$\begin{aligned}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) \geqq{}& \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}[w(\xi)^{\rho}]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi)\\ &- \frac{M\xi\mathcal{F}^{\sigma}_{\rho\nu,\lambda+2}[w\xi^{\rho}]}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}[w(\xi)^{\rho}]}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi) + M\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id}\zeta_1)(\xi) \qquad (\forall\ \xi > 0),\end{aligned}$$

where $\mathrm{Id}(\xi)$ *is as defined in Theorem* 2.3.

Proof. By the same technique as that used for proving Theorem 2.3, together with

$$h(\xi) \coloneqq \zeta_2(\xi) - M\mathrm{Id}(\xi),$$

we can obtain the desired result asserted by Corollary 2.1. □

Corollary 2.2. *Let $\lambda, \rho, \nu > 0$ and $w \in \mathbb{R}$. Also let ζ_1 and ζ_2 be two functions such that ζ_1 is increasing and both ζ_1 and ζ_2 are differentiable. If there exist real numbers m_1 and m_2 with*

$$m_1 = \inf_{\xi \geqq 0} \zeta_1{}'(\xi) \qquad \text{and} \qquad m_2 = \inf_{\xi \geqq 0} \zeta_2{}'(\xi),$$

then

$$\begin{aligned}&\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) - m_1\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id}\,\zeta_2)(\xi)\\ &\qquad - m_2\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id}\zeta_1)(\xi) + m_1m_2\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\mathrm{Id}^2\right)(\xi)\\ &\geqq \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}[w(\xi)^{\rho}]}\Bigg[\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) - m_1\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id})(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\\ &\qquad - m_2\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id})(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) + m_1m_2\left(\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id})(\xi)\right)^2\Bigg],\end{aligned}$$

where $(\mathrm{Id})(\xi)$ *is defined as in Theorem* 2.3.

Proof. By the same technique used for Theorem 2.3 with the setting

$$h_1(\xi) \coloneqq \zeta_2(\xi) - m_1\mathrm{Id}(\xi) \qquad \text{and} \qquad h_2(\xi) \coloneqq \zeta_2(\xi) - m_2\mathrm{Id}(\xi),$$

we can obtain the desired result asserted by Corollary 2.2. □

Corollary 2.3. *Let $\lambda, \rho, \nu > 0$ and $w \in \mathbb{R}$. Also let ζ_1 and ζ_2 be such functions that ζ_1 is increasing and both ζ_1 and ζ_2 are differentiable. If there exist real numbers*

$$M_1 = \sup_{\xi \geqq 0} \zeta_1{}'(\xi) \qquad \text{and} \qquad M_2 = \sup_{\xi \geqq 0} \zeta_2{}'(\xi),$$

then

$$\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\xi) - M_1\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id}\zeta_2)(\xi)$$

$$
\begin{aligned}
&- M_2\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id}\zeta_1)(\xi) + M_1M_2\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}\left(\mathrm{Id}^2\right)(\xi) \\
\geqq\ & \frac{\xi^{-\frac{\lambda}{\nu}}}{\nu\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}[w(\xi)^{\rho}]}\Bigg[\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) - M_1\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id})(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\xi) \\
&- M_2\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id})(\xi)\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\xi) + M_1M_2\left(\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\mathrm{Id})(\xi)\right)^2\Bigg],
\end{aligned}
$$

where $(\mathrm{Id})(\xi)$ *is defined as in Theorem* 2.3.

Proof. By the same technique used for proving Theorem 2.3 with the setting

$$h_1(\xi) := \zeta_2(\xi) - M_1\mathrm{Id}(\xi) \qquad \text{and} \qquad h_2(\xi) := \zeta_2(\xi) - M_2\mathrm{Id}(\xi),$$

we can derive the desired result asserted by Corollary 2.3. □

Theorem 2.4. *Let* $\lambda, \rho, \nu_1 > 0$ *and* $w \in \mathbb{R}$. *Also let* h *be a positive function on* $[0, \infty)$ *and suppose that* ζ_1 *and* ζ_2 *are two differentiable functions on* $[0, \infty)$. *If* $\zeta_1' \in L_r[0, \infty)$ *and* $\zeta_2' \in L_s[0, \infty)$ *with* $r > 1$ *and* $r^{-1} + s^{-1} = 1$, *then*

$$
\begin{aligned}
2&\left|\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_1\zeta_2)(\mathbf{z})\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h)(\mathbf{z}) - \mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_1)(\mathbf{z})\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_2)(\mathbf{z})\right| \\
&\leqq \|\zeta_1'\|_r \cdot \|\zeta_2'\|_s \cdot \mathbf{z}\int_0^{\mathbf{z}}\int_0^{\mathbf{z}}(\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^{\rho}] \\
&\qquad\qquad \times \mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]h(\nu)h(\tau)\,\mathrm{d}\tau\,\mathrm{d}\nu \\
&\leqq \|\zeta_1'\|_r \cdot \|\zeta_2'\|_s \cdot \mathbf{z}\left(\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h)(\mathbf{z})\right)^2.
\end{aligned} \tag{2.3}
$$

Proof. Let h, ζ_1 and ζ_2 be three functions that fulfill the hypotheses of Theorem 2.4. We define

$$H(\tau, \nu) := (\zeta_1(\tau) - \zeta_1(\nu))(\zeta_2(\tau) - \zeta_2(\nu)) \qquad (\tau, \nu \in (0, \mathbf{z});\ \mathbf{z} > 0). \tag{2.4}$$

If we first multiply (2.4) by

$$(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]h(\tau)$$

with $\tau \in (0, \mathbf{z})$, and then integrate over $\tau \in (0, \mathbf{z})$, we get

$$
\begin{aligned}
&\int_0^{\mathbf{z}}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]h(\tau)H(\tau,\nu)\,\mathrm{d}\tau \\
&\quad = \mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_1\zeta_2)(\mathbf{z}) - \zeta_1(\nu)\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_2)(\mathbf{z}) \\
&\qquad - \zeta_2(\nu)\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_1)(\mathbf{z}) + \zeta_1(\nu)\zeta_2(\nu)\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h)(\mathbf{z}).
\end{aligned} \tag{2.5}
$$

We now multiply both sides of (2.5) by

$$(\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^{\rho}]h(\nu)$$

with $\nu \in (0, \mathbf{z})$, and then integrate over $\nu \in (0, \mathbf{z})$. Upon some simplication, we thus find that

$$\int_0^{\mathbf{z}}\int_0^{\mathbf{z}}(\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^{\rho}]\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]h(\nu)h(\tau)H(\tau,\nu)\,\mathrm{d}\tau\,\mathrm{d}\nu$$

$$= 2\left(\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_1\zeta_2)(\mathbf{z})\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h)(\mathbf{z}) - \mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_1)(\mathbf{z})\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h\zeta_2)(\mathbf{z})\right). \tag{2.6}$$

In view of the following known result:

$$H(\tau,\nu) = \int_\tau^\nu \int_\tau^\nu \zeta_1'(u)\zeta_2'(v)\, \mathrm{d}u\, \mathrm{d}v,$$

if we use the Hölder's inequality for double integrals, we have

$$\begin{aligned}
|H(\tau,\nu)| &\leqq \left|\int_\tau^\nu \int_\tau^\nu |\zeta_1'(u)|^r\, dudv\right|^{1/r} \left|\int_\tau^\nu \int_\tau^\nu |\zeta_2'(u)|^s\, \mathrm{d}u\, \mathrm{d}v\right|^{1/s} \\
&= |\tau-\nu| \left|\int_\tau^\nu |\zeta_1'(u)|^r\, du\right|^{1/r} \left|\int_\tau^\nu |\zeta_2'(v)|^s\, \mathrm{d}v\right|^{1/s}.
\end{aligned} \tag{2.7}$$

By using (2.7) in (2.6), we can deduce that

$$\begin{aligned}
&\left|\int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^\rho]h(\nu)h(\tau)H(\tau,\nu)\, \mathrm{d}\tau\, \mathrm{d}\nu\right| \\
&\quad\leqq \int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^\rho]h(\nu)h(\tau)\,|H(\tau,\nu)|\, \mathrm{d}\tau\, \mathrm{d}\nu \\
&\quad\leqq \int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^\rho] \\
&\qquad\times |\tau-\nu|\, h(\nu)h(\tau) \left|\int_\tau^\nu |\zeta_1'(u)|^r\, \mathrm{d}u\right|^{1/r} \left|\int_\tau^\nu |\zeta_2'(v)|^s\, dv\right|^{1/s} \mathrm{d}\tau\, \mathrm{d}\nu.
\end{aligned} \tag{2.8}$$

By applying the Hölder's inequality to the right-hand side of (2.8), we get

$$\begin{aligned}
&\int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^\rho]h(\nu)h(\tau)\,|H(\tau,\nu)|\, \mathrm{d}\tau\, \mathrm{d}\nu \\
&\quad\leqq \left(\int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^\rho]\right. \\
&\qquad\left.\times |\tau-\nu|\, h(\nu)h(\tau) \left|\int_\tau^\nu |\zeta_1'(u)|^r\, \mathrm{d}u\right|\, d\tau\, d\nu\right)^{1/r} \left(\int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\right. \\
&\qquad\left.\times \mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^\rho]\,|\tau-\nu|\, h(\nu)h(\tau) \left|\int_\tau^\nu |\zeta_2'(v)|^s\, \mathrm{d}v\right|\, \mathrm{d}\tau\, \mathrm{d}\nu\right)^{1/s},
\end{aligned} \tag{2.9}$$

which, by using the fact that $\zeta_1' \in L_r[0,\infty)$ and $\zeta_2' \in L_s[0,\infty)$, yields

$$\begin{aligned}
&\int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^\rho]h(\nu)h(\tau)\,|H(\tau,\nu)|\, \mathrm{d}\tau\, \mathrm{d}\nu \\
&\leqq \left(\|\zeta_1'\|_r^r \int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^\rho]\,|\tau-\nu|\, h(\nu)h(\tau)\, \mathrm{d}\tau\, \mathrm{d}\nu\right)^{1/r} \\
&\qquad\times \left(\|\zeta_2'\|_s^s \int_0^{\mathbf{z}} \int_0^{\mathbf{z}} (\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^\sigma_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^\rho]\right.
\end{aligned}$$

$$\times \mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]\,|\tau-\nu|\,h(\nu)h(\tau)\,\mathrm{d}\tau\,\mathrm{d}\nu\Bigg)^{1/s}. \quad (2.10)$$

Since $r^{-1}+s^{-1}=1$, it follows that

$$\begin{aligned}
&\int_0^{\mathbf{z}}\int_0^{\mathbf{z}}(\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^{\rho}]\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]h(\nu)h(\tau)\,|H(\tau,\nu)|\,\mathrm{d}\tau\,\mathrm{d}\nu\\
&\quad\leqq \|\zeta_1{}'\|_r\cdot\|\zeta_2{}'\|_s\Bigg(\int_0^{\mathbf{z}}\int_0^{\mathbf{z}}(\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\\
&\qquad\times \mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^{\rho}]\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]\,|\tau-\nu|\,h(\nu)h(\tau)\,\mathrm{d}\tau\,\mathrm{d}\nu\Bigg). \qquad (2.11)
\end{aligned}$$

Therefore, by using (2.8) and (2.11), we can obtain the first inequality in (2.3).

On the other hand, by using the fact that $0<|\tau-\nu|<\mathbf{z}$, we can write

$$\begin{aligned}
&\int_0^{\mathbf{z}}\int_0^{\mathbf{z}}(\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^{\rho}]\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]h(\nu)h(\tau)\,|H(\tau,\nu)|\,\mathrm{d}\tau\,\mathrm{d}\nu\\
&\quad\leqq \|\zeta_1{}'\|_r\cdot\|\zeta_2{}'\|_s\cdot\mathbf{z}\Bigg(\int_0^{\mathbf{z}}\int_0^{\mathbf{z}}(\mathbf{z}-\nu)^{\frac{\lambda}{\nu_1}-1}(\mathbf{z}-\tau)^{\frac{\lambda}{\nu_1}-1}\\
&\qquad\times \mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\nu)^{\rho}]\mathcal{F}^{\sigma}_{\rho\nu_1,\lambda}[w(\mathbf{z}-\tau)^{\rho}]h(\nu)h(\tau)\,\mathrm{d}\tau\,\mathrm{d}\nu\Bigg)\\
&\quad= \|\zeta_1{}'\|_r\cdot\|\zeta_2{}'\|_s\cdot\mathbf{z}\left(\mathcal{J}^{\sigma,\nu_1}_{\rho,\lambda,0+;w}(h)(\mathbf{z})\right)^2, \qquad (2.12)
\end{aligned}$$

which gives the second inequality in (2.3). The proof of Theorem 2.4 is thus completed. □

Corollary 2.4. *Let $\lambda,\rho,\nu>0$ and $w\in\mathbb{R}$. Also let the functions ζ_1 and ζ_2 be differentiable on $[0,\infty)$. If*

$$\zeta_1{}'\in L_r[0,\infty) \qquad \text{and} \qquad \zeta_2{}'\in L_s[0,\infty)$$

with $r>1$ and $r^{-1}+s^{-1}=1$, then

$$\begin{aligned}
&\left|\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1\zeta_2)(\mathbf{z})-\frac{1}{\left(\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}(\mathbf{z})\right)}\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_1)(\mathbf{z})\mathcal{J}^{\sigma,\nu}_{\rho,\lambda,0+;w}(\zeta_2)(\mathbf{z})\right|\\
&\qquad\leqq \frac{1}{2}\|\zeta_1{}'\|_r\cdot\|\zeta_2{}'\|_s\cdot\mathbf{z}\left(\mathcal{F}^{\sigma}_{\rho\nu,\lambda+1}(\mathbf{z})\right). \qquad (2.13)
\end{aligned}$$

Proof. The proof of Corollary 2.4 follows by applying Theorem 2.4 for $h=1$. □

Remark 2.7. Some particularly simple cases of Theorem 2.4 are given below.

- If $\lambda=\mu$, $\sigma(0)=\nu_1=1$, $\sigma(k)=0$ $(k\in\mathbb{N})$ and $w=0$ in Theorem 2.4, then we obtain the following inequality for the Riemann–Liouville fractional integral:

$$|\mathcal{I}^{\mu}(\zeta_1\zeta_2)(\mathbf{z})\mathcal{I}^{\mu}(h)(\mathbf{z})-\mathcal{I}^{\mu}\zeta_1(\mathbf{z})\mathcal{I}^{\mu}(\zeta_2)(\mathbf{z})|\leqq\frac{1}{2}\|\zeta_1{}'\|_r\cdot\|\zeta_2{}'\|_s\cdot\mathbf{z}\,(\mathcal{I}^{\mu}(h)(\mathbf{z}))^2,$$

  which was given in [70, Theorem 3.1].

- If we take $h = 1$ and $\nu_1 = 1$ in Theorem 2.4, we get

$$\left|\frac{\mathbf{z}^{\mu}}{\Gamma(\mu+1)}\mathcal{I}^{\mu}(\zeta_1\zeta_2)(\mathbf{z}) - \mathcal{I}^{\mu}\zeta_1(\mathbf{z})\mathcal{I}^{\mu}(\zeta_2)(\mathbf{z})\right| \leqq \frac{1}{2}\|\zeta_1{}'\|_r \cdot \|\zeta_2{}'\|_s \,\frac{\mathbf{z}^{2\mu+1}}{[\Gamma(\mu+1)]^2},$$

  which was derived in [70, Corollary 3.3].
- Just as we pointed out in Remark 1.4, with appropriate choices of, *and* under sufficient conditions on, the arguments and the parameters involved, we can express the result of Theorem 2.4 in terms of fractional integrals with kernels involving not only the Fox–Wright hypergeometric function ${}_p\Psi_q(z)$, given by (1.7), (1.16) and (1.17), but also in terms of the aforementioned Mittag–Leffler type kernels such as

$$E_{\gamma,\kappa,\epsilon}\left[(\alpha_j,\beta_j)_{j=1}^{m};z\right],$$

  given by (1.5) and (1.9), as well as its *further* special cases:

$$E_{\alpha,\beta,\mathfrak{p}}^{\gamma,\delta,\mathfrak{q}}(z), \quad E_{\alpha,\beta}^{\gamma,\kappa}(z) \qquad \text{and} \qquad E_{\alpha,\beta}^{\gamma}(z),$$

  given by (1.10), (1.11) and (1.12), respectively. The details of these and various other deductions and derivations from Theorem 2.4 are being left as an exercise for the interested reader.

3. Conclusions

In the development of the present work, the Chebyshev inequality was established via a certain family of modified fractional integral operators in Theorem 2.1. Moreover, Chebyshev's inequality was proved for more than two functions in Theorem 2.2. Several inequalities of this type were established in Theorem 2.3 as well as in and Corollaries 2.1, Corollary 2.2 and 2.3 for functions whose derivatives are bounded above or bounded below. Furthermore, an estimate for the Chebyshev functional was established in Theorem 2.4 by using the above-mentioned family of modified fractional integrals. Finally, from the main results, similar inequalities can be deduced for each of the aforementioned simpler Riemann-Liouville fractional integrals with other specialized Fox-Wright and Mittag-Leffler type kernels.

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Conflict of interest

The authors declare no conflicts of interest.

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