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*Research article*

# Oscillatory and asymptotic properties of higher-order quasilinear neutral differential equations

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**Abstract:** The objective of this paper is to study the oscillation criteria for odd-order neutral differential equations with several delays. We establish new oscillation criteria by using Riccati transformation. Our new criteria are interested in complementing and extending some results in the literature. An example is considered to illustrate our results.

**Keywords:** neutral differential equations; odd-order; oscillation criteria
**Mathematics Subject Classification:** 34C10, 34K11

## 1. Introduction

In this work, we consider the odd-order quasi-linear neutral differential equations of the form

$$\left(r(t)\left[(x(t)+p(t)\,x(\tau(t)))^{(n-1)}\right]^{\alpha}\right)' + \sum_{\kappa=1}^{m} q_{\kappa}(t)\,x^{\alpha}(\sigma_{\kappa}(t)) = 0, \text{ for } t \geq t_0, \tag{1.1}$$

where $n \geq 3$ is an odd integer, $\alpha$ is a ratio of positive odd integers and $m$ is a positive integer. Throughout this work, we assume the following:

**(H1)** $r \in C^1[t_0,\infty)$ and $r'(t) \geq 0$, where

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\,\mathrm{d}t = \infty;$$

**(H2)** $p, q_{\kappa} \in C[t_0,\infty)$, $p(t) \in [0, p_0]$ such that $p_0$ is a constant and $q_{\kappa}(t) > 0$;

**(H3)** $\tau, \sigma_\kappa \in C[t_0,\infty)$, $\tau(t) \le t$, $\sigma_\kappa(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$ and $\lim_{t\to\infty} \sigma_\kappa(t) = \infty$ for all $\kappa = 1, 2, ..., m$.

By a solution of (1.1), we mean $x \in C([T_x,\infty),\mathbb{R})$ with $T_x \ge t_0$, which satisfies the properties

$$(x + p \cdot x \circ \tau) \in C^{(n-1)}([T_x,\infty),\mathbb{R})$$

and

$$r \cdot \left((x + p \cdot x \circ \tau)^{(n-1)}\right)^{\alpha} \in C^1([T_x,\infty),\mathbb{R})$$

and moreover satisfies (1.1) on $[T_x,\infty)$. We consider the nontrivial solutions of (1.1) existing on some half-line $[T_x,\infty)$ and satisfying the condition $\sup\{|x(t)| : t \ge t_*\} > 0$ for any $t_* \ge T_x$. If there exists a $t_1 \ge t_0$ such that either $x(t) > 0$ or $x(t) < 0$ for all $t \ge t_1$, then $x$ is said to be a nonoscillatory solution; otherwise, it is said to be an oscillatory solution.

Delay differential equations as a subclass of functional differential equations take into account the dependence on the systems past history where the theory of delay differential equations has enhanced our understanding of the qualitative behavior of their solutions and it has benefited significantly and wide from it, where many applications showed in various fields as mathematical biology and epidemiology (for instance, transport phenomena, distributed networks, interaction of species) and other related fields, etc., see [1–3].

Neutral delay differential equations are differential equations with delays, where the delays can appear in both the state variables and their time derivatives. There is considerable interest in studying of this type of equation because they are deemed to be adequate prescribing tool in modelling of the countless processes in all areas including problems concerning electric networks containing lossless transmission lines (as in high speed computers where such lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar or in the solution of variational problems with time delays, or in the theory of automatic control and in neuro-mechanical systems in which inertia plays a major role, and in many areas of science as physical, biological and chemical, etc., see [4, 5]. In addition, systems of delay differential equations were used to study stability properties of electrical power systems also, properties of delay differential equations were used in the study of singular fractional order differential equations, see [6, 7] and the references cited therein.

As a matter of fact, quasilinear (i.e., half-linear) (neutral) differential equations with deviating arguments (delayed or advanced arguments or mixed arguments) have numerous applications in physics and engineering (e.g., quasilinear (i.e., half-linear) differential equations arise in a variety of real world problems such as in the study of $p$-Laplace equations, porous medium problems, chemotaxis models, and so forth), see [8–12].

For several years, an increasing interest in obtaining sufficient conditions for oscillatory and nonoscillatory behavior of different classes of differential equations has been observed, see [13–18] for second-order equations. While the development of the study of the second-order equations was in turn reflected on the even-order equations in the works [19–28]. The development of the study of the odd-order equations can also be traced through works [29–34], some of which are special cases of the studied equation.

Many authors as Ladde and Zhang in [22, 28] established a criterion for oscillatory behavior of the higher-order differential equation

$$\left(\left(x(t)^{(n-1)}\right)^{\alpha}\right)' + q(t)\, x^{\beta}(\tau(t)) = 0. \tag{1.2}$$

Grace [20] extended some new results to the equation

$$\left(r(t)\left(x(t)^{(n-1)}\right)^{\alpha}\right)' + q(t)\, x^{\beta}(\tau(t)) = 0, \tag{1.3}$$

under the assumptions that $\alpha$ is even,

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\, \mathrm{d}t = \infty \ \text{ and } r'(t) \geq 0.$$

Agarwal et al. [19] studied Eq (1.3) under conditions

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\, \mathrm{d}t < \infty \ \text{ and } \int_{t_0}^{\infty} q(t)\, \mathrm{d}t = \infty.$$

Karpuz et al. [21] investigated the oscillatory behavior of linear neutral differential equations

$$(x(t) + p(t)\, x(\tau(t)))^{(n)} + q(t)\, x(\tau(t)) = 0,$$

where $n$ is an odd integer and $0 \leq p(t) < 1$.

Li and Thandapani [31] established some oscillation criteria for certain higher-order neutral differential equation

$$(x(t) + p(t)\, x(a + bt))^{(n)} + q(t)\, x(c + dt) = 0,$$

with $0 \leq p(t) \leq p_0 < \infty$.

Yildiz et al. [27] examined the oscillation of odd-order neutral differential equation

$$(x(t) + p(t)\, x(\tau(t)))^{(n)} + q(t)\, x^{\alpha}(\tau(t)) = 0,$$

where $0 \leq p(t) \leq p_1 < 1$.

In the present paper, we aim to improve the results in previous studies and present some new sufficient conditions which ensure that every solution of (1.1) oscillates or tends to zero.

## 2. Auxiliary lemmas

Here are some lemmas that we need during the next results.

**Lemma 2.1.** *[18, Lemma (2.3)] Let $g(v) = Cv - Dv^{\alpha+1/\alpha}$ where $C, D > 0$. Then $g$ attains its maximum value on $\mathbb{R}$ at $v^* = (\alpha C/(\alpha+1) D)^{\alpha}$ and*

$$\max_{v \in \mathbb{R}} g(v) = g(v^*) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{C^{\alpha+1}}{D^{\alpha}}. \tag{2.1}$$

**Lemma 2.2.** *[34] Assume that $c_1, c_2 \in [0, \infty)$ and $\gamma > 0$. Then*

$$(c_1 + c_2)^{\gamma} \leq \mu\left(c_1^{\gamma} + c_2^{\gamma}\right), \tag{2.2}$$

*where*

$$\mu := \begin{cases} 1 & \text{if } \gamma \leq 1 \\ 2^{\gamma-1} & \text{if } \gamma > 1. \end{cases}$$

**Lemma 2.3.** *[35] Let $f \in C^n([t_0, \infty), (0, \infty))$. Assume that $f^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0, \infty)$ and that there exists a $t_1 \geq t_0$ such that $f^{(n-1)}(t) f^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} f(t) \neq 0$, then for every $\mu \in (0, 1)$ there exists $t_\mu \geq t_1$ such that*

$$f(t) \geq \frac{\mu}{(n-1)!} t^{n-1} \left|f^{(n-1)}(t)\right| \text{ for } t \geq t_\mu.$$

## 3. Main results

Through the rest of this paper, we will use the following definitions:

$$z := x + p \cdot x \circ \tau,$$

$$\eta(t) := \int_{t_0}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s$$

and

$$\sigma(t) = \min\{\sigma_\kappa(t) : \kappa = 1, 2, ..., m\}. \tag{3.1}$$

**Lemma 3.1.** *Let $x$ be a positive solution of (1.1). Then $z(t) > 0$, $\left(r\left((z)^{(n-1)}\right)^\alpha\right)' \leq 0$ and there are two possible cases for derivatives of $z$ :*

$$\begin{array}{ll} \textbf{(I)} & z'(t) > 0,\ z''(t) > 0,\ z^{(n-1)}(t) > 0,\ z^n(t) \leq 0; \\ \textbf{(II)} & z'(t) < 0,\ z''(t) > 0,\ z^{(n-1)}(t) > 0,\ z^n(t) \leq 0. \end{array}$$

*Proof.* Assume that $x$ is a positive solution of (1.1) on $[t_0, \infty)$. Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\sigma_\kappa(t)) > 0$ and $x(\tau(t)) > 0$, for $t \geq t_1$. By the definition of $z$, it is easy to see that $z(t) \geq x(t) > 0$. Furthermore, from (1.1), we have $\left(r\left((z)^{(n-1)}\right)^\alpha\right)' \leq 0$. The rest of the proof is similar to proof of Lemma in [29]. Thus, the proof is complete. □

**Lemma 3.2.** *Let $x(t)$ be a positive solution of (1.1) and $z(t)$ satisfy* **(II)**. *If*

$$\int_{t_0}^{\infty} \widetilde{\eta}(s)\, s^{n-2}\mathrm{d}s = \infty, \tag{3.2}$$

*then* $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$, *where*

$$\widetilde{\eta}(t) = \left(\frac{1}{r(t)} \int_t^\infty \sum_{\kappa=1}^{m} q_\kappa(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}}.$$

*Proof.* Let $x$ be a positive solution of (1.1) on $[t_0, \infty)$. Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\sigma_\kappa(t)) > 0$ and $x(\tau(t)) > 0$, for $t \geq t_1$. Since the corresponding function $z(t) > 0$ and $z'(t) < 0$, then there exists a finite limit $\lim_{t\to\infty} z(t) = c \geq 0$. Let $c > 0$. Then for any $\epsilon > 0$, we have $\epsilon + c > z(t) > c$, eventually. It is easy to see that

$$x(t) = z(t) - p(t)\,x(\tau(t)) \geq z(t) - p(t)\,z(\tau(t)),$$

thus,

$$x(t) \geq c - p_0(\epsilon + c) = \frac{c - p_0(\epsilon + c)}{\epsilon + c}(\epsilon + c).$$

This implies that

$$x(t) \geq \varrho z(t), \tag{3.3}$$

where $\varrho = c - p_0(\epsilon + c)/\epsilon + c > 0$, that is,

$$x^\alpha(\sigma_\kappa(t)) \geq \varrho^\alpha z^\alpha(\sigma_\kappa(t)).$$

Using (3.3) in (1.1), we obtain

$$\left(r(t)\left((z(t))^{(n-1)}\right)^{\alpha}\right)' + \sum_{\kappa=1}^{m} q_{\kappa}(t)\varrho^{\alpha} z^{\alpha}(\sigma_{\kappa}(t)) \le 0.$$

By (3.1) and $\sigma(t) < t$, we see that

$$\left(r(t)\left((z(t))^{(n-1)}\right)^{\alpha}\right)' + \varrho^{\alpha} z^{\alpha}(\sigma(t)) \sum_{\kappa=1}^{m} q_{\kappa}(t) \le 0.$$

Integrating last inequality from $t$ to $\infty$, we get

$$r(t)\left((z(t))^{(n-1)}\right)^{\alpha} \ge \varrho^{\alpha} \int_{t}^{\infty} z^{\alpha}(\sigma(s)) \sum_{\kappa=1}^{m} q_{\kappa}(s)\,\mathrm{d}s.$$

By $\lim_{t\to\infty} z(\sigma(t)) > c$, it follows that

$$z^{(n-1)}(t) \ge \varrho c \widetilde{\eta}(t). \tag{3.4}$$

Integrating (3.4) twice from $t$ to $\infty$, we have

$$z^{(n-3)}(t) \ge \varrho c \int_{t}^{\infty}\int_{u}^{\infty} \widetilde{\eta}(s)\,\mathrm{d}s\mathrm{d}u = \varrho c \int_{t}^{\infty} \widetilde{\eta}(s)(s-t)\,\mathrm{d}s.$$

Repeating this procedure, we arrive at

$$-z'(t) \ge \frac{\varrho c}{(n-3)!} \int_{t}^{\infty} \widetilde{\eta}(s)(s-t)^{n-3}\,\mathrm{d}s.$$

Now, integrating from $t_1$ to $\infty$, we see that

$$z(t_1) \ge \frac{\varrho c}{(n-2)!} \int_{t_1}^{\infty} \widetilde{\eta}(s)(s-t_1)^{n-2}\,\mathrm{d}s \ge \frac{\varrho c}{2^{n-2}(n-2)!} \int_{2t_1}^{\infty} \widetilde{\eta}(s) s^{n-2}\mathrm{d}s.$$

This contradicts (3.2). Then we have $\lim_{t\to\infty} z(t) = 0$. □

In the following lemma, we will use the notions

$$\widetilde{q}_{\kappa 1}(t) := \min\{q_{\kappa}(t), q_{\kappa}(\tau(t))\}, \widetilde{q}_{\kappa 2}(t) := \min\left\{q_{\kappa}\left(\sigma^{-1}(t)\right), q_{\kappa}\left(\sigma^{-1}(\tau(t))\right)\right\}$$

and

$$\tau' \ge \tau_0 > 0; \tag{3.5}$$

$$\left(\sigma^{-1}(t)\right)' \ge \sigma_0 > 0. \tag{3.6}$$

**Lemma 3.3.** *If $x(t)$ is a positive solution of (1.1) and $z(t)$ satisfy* (**I**)*, (3.5) and $\sigma_{\kappa} \circ \tau = \tau \circ \sigma_{\kappa}$ hold, then*

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' + \frac{z^{\alpha}(\sigma(t))}{\mu} \sum_{\kappa=1}^{m} \widetilde{q}_{\kappa 1}(t) \le 0. \tag{3.7}$$

*Moreover, if (3.6) and* $\left(\sigma_\kappa \circ \sigma^{-1}\right) \circ \tau = \tau \circ \left(\sigma_\kappa \circ \sigma^{-1}\right)$ *hold, then*

$$
\begin{aligned}
0 \geq\ & \frac{\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^{\alpha}\right)'}{\sigma_0} \\
&+\frac{p_0^{\alpha}\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}\right)'}{\sigma_0\tau_0} \\
&+\frac{z^{\alpha}(t)}{\mu}\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 2}(t).
\end{aligned}
\tag{3.8}
$$

*Proof.* Let $x$ be a positive solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\sigma_\kappa(t)) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. By Lemma 2.2 , we see that

$$
z^{\alpha}(\sigma(t)) \leq \mu\left(x^{\alpha}(\sigma(t)) + p_0^{\alpha} x^{\alpha}(\tau(\sigma(t)))\right). \tag{3.9}
$$

From (1.1), (3.5) and property $\sigma_\kappa \circ \tau = \tau \circ \sigma_\kappa$, we get

$$
\begin{aligned}
0 &= \frac{p_0^{\alpha}}{\tau'(t)}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' + p_0^{\alpha}\sum_{\kappa=1}^{m} q_\kappa(\tau(t))\, x^{\alpha}(\sigma_\kappa(\tau(t))) \\
&\geq \frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' + p_0^{\alpha}\sum_{\kappa=1}^{m} q_\kappa(\tau(t))\, x^{\alpha}(\tau(\sigma_\kappa(t))).
\end{aligned}
$$

Using (1.1) with above inequality and taking (3.9) into account, we have

$$
\begin{aligned}
0 &\geq \left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' + \frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' \\
&\quad + \sum_{\kappa=1}^{m} q_\kappa(t)\, x^{\alpha}(\sigma_\kappa(t)) + p_0^{\alpha}\sum_{\kappa=1}^{m} q_\kappa(\tau(t))\, x^{\alpha}(\tau(\sigma_\kappa(t))) \\
&\geq \left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' + \frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)' + \frac{1}{\mu}\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 1}(t)\, z^{\alpha}(\sigma_\kappa(t)) \\
&= \left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)\right)' + \frac{1}{\mu}\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 1}(t)\, z^{\alpha}(\sigma_\kappa(t)).
\end{aligned}
$$

By (3.1), we see that

$$
0 \geq \left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)\right)' + \frac{1}{\mu} z^{\alpha}(\sigma(t))\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 1}(t).
$$

Using (3.1) and (3.6) in (1.1), we are led to

$$
\begin{aligned}
0 &= \frac{1}{\left(\sigma^{-1}(t)\right)'}\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^{\alpha}\right)' + \sum_{\kappa=1}^{m} q_\kappa\left(\sigma^{-1}(t)\right) x^{\alpha}\left(\sigma_\kappa\left(\sigma^{-1}(t)\right)\right) \\
&\geq \frac{1}{\sigma_0}\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^{\alpha}\right)' + \sum_{\kappa=1}^{m} q_\kappa\left(\sigma^{-1}(t)\right) x^{\alpha}\left(\sigma_\kappa\left(\sigma^{-1}(t)\right)\right).
\end{aligned}
\tag{3.10}
$$

Also, using (3.1) and (3.5) in (1.1), we obtain

$$\begin{aligned} 0 &= \frac{p_0^\alpha}{\left(\sigma^{-1}(\tau(t))\right)'}\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^\alpha\right)' + p_0^\alpha \sum_{\kappa=1}^{m} q_\kappa\left(\sigma^{-1}(\tau(t))\right) x^\alpha\left(\sigma_\kappa\left(\sigma^{-1}(\tau(t))\right)\right) \\ &\geq \frac{p_0^\alpha}{\sigma_0\tau_0}\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^\alpha\right)' + p_0^\alpha \sum_{\kappa=1}^{m} q_\kappa\left(\sigma^{-1}(\tau(t))\right) x^\alpha\left(\tau\left(\sigma_\kappa\left(\sigma^{-1}(t)\right)\right)\right). \quad (3.11) \end{aligned}$$

Combining (3.10) with (3.11) and taking into account (3.9), one can see that

$$\begin{aligned} 0 \geq\ & \frac{1}{\sigma_0}\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha\right)' + \frac{p_0^\alpha}{\sigma_0\tau_0}\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^\alpha\right)' \\ & + \frac{1}{\mu}\sum_{\kappa=1}^{m} \widetilde{q}_{\kappa 2}(t)\left(x\left(\sigma_\kappa\left(\sigma^{-1}(t)\right)\right) + x\left(\tau\left(\sigma_\kappa\left(\sigma^{-1}(t)\right)\right)\right)\right)^\alpha. \end{aligned}$$

That is,

$$\begin{aligned} 0 \geq\ & \frac{1}{\sigma_0}\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha\right)' + \frac{p_0^\alpha}{\sigma_0\tau_0}\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^\alpha\right)' \\ & + \sum_{\kappa=1}^{m} \frac{1}{\mu}\widetilde{q}_{\kappa 2}(t)\, z^\alpha\left(\sigma_\kappa\left(\sigma^{-1}(t)\right)\right). \end{aligned}$$

By the fact $z' > 0$, we note that $z\left(\sigma_\kappa\left(\sigma^{-1}(t)\right)\right) > z(t)$ which implies that

$$\begin{aligned} 0 \geq\ & \frac{1}{\sigma_0}\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha\right)' + \frac{p_0^\alpha}{\sigma_0\tau_0}\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^\alpha\right)' \\ & + z^\alpha(t)\sum_{\kappa=1}^{m} \frac{1}{\mu}\widetilde{q}_{\kappa 2}(t). \end{aligned}$$

The proof of lemma is complete. □

**Theorem 3.1.** *Assume that (3.2), (3.5), $\sigma(t) \leq \tau(t)$, $\sigma'(t) > 0$ and $\sigma_\kappa \circ \tau = \tau \circ \sigma_\kappa$ hold. If there exists a function $\delta \in C^1([t_0,\infty),(0,\infty))$, such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\frac{\delta(s)}{\mu}\sum_{\kappa=1}^{m} \widetilde{q}_{\kappa 1}(s) - \frac{((n-2)!)^\alpha}{\mu^\alpha(\alpha+1)^{\alpha+1}}\left(1 + \frac{p_0^\alpha}{\tau_0}\right)\frac{r(s)\left(\delta'(s)\right)^{\alpha+1}}{\left(\delta(s)\sigma^{n-2}(s)\sigma'(s)\right)^\alpha}\right] \mathrm{d}s = \infty, \quad (3.12)$$

*then every solution of (1.1) is oscillatory or tends to zero.*

*Proof.* Let $x$ be a positive solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\sigma_\kappa(t)) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. Let $z$ satisfying case (**I**). Define the positive function $\omega(t)$ by

$$\omega(t) = \delta(t)\,\frac{r(t)\left(z^{(n-1)}(t)\right)^\alpha}{z^\alpha(\sigma(t))}. \quad (3.13)$$

Hence, by differentiating (3.13), we get

$$\omega'(t) = \delta'(t)\frac{r\left(\left(z\right)^{(n-1)}\right)^{\alpha}}{z^{\alpha}(\sigma(t))} + \delta(t)\frac{\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))} - \frac{\alpha\delta(t)r(t)\left(z^{(n-1)}(t)\right)^{\alpha}z^{\alpha-1}(\sigma(t))z'(\sigma(t))\sigma'(t)}{z^{2\alpha}(\sigma(t))}. \tag{3.14}$$

Since $z^{'} > 0, z^{''} > 0$, we see that $\lim_{t\to\infty} z^{'} \neq 0$, using Lemma 2.3 with $f = z^{'}$, we see that

$$z^{'}(t) \geq \frac{\mu}{(n-2)!}t^{n-2}z^{(n-1)}(t),$$

for every $\mu \in (0,1)$. By $z^{n}(t) \leq 0$, we get

$$z^{'}(\sigma(t)) \geq \frac{\mu}{(n-2)!}(\sigma(t))^{n-2}z^{(n-1)}(\sigma(t)) \geq \frac{\mu}{(n-2)!}(\sigma(t))^{n-2}z^{(n-1)}(t). \tag{3.15}$$

Substituting (3.13) and (3.15) into (3.14) implies

$$\begin{aligned}\omega'(t) &\leq \delta'(t)\frac{r(t)\left(z^{(n-1)}(t)\right)^{\alpha}}{z^{\alpha}(\sigma(t))} + \delta(t)\frac{\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))} \\ &\quad - \left(\frac{z^{(n-1)}(t)}{z(\sigma(t))}\right)^{\alpha+1}\frac{\alpha\delta(t)r(t)\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!} \\ &\leq \delta(t)\frac{\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))} + \frac{\delta'(t)}{\delta(t)}\omega(t) - \frac{\alpha\delta(t)r(t)\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!}\left(\frac{\omega(t)}{\delta(t)r(t)}\right)^{\frac{\alpha+1}{\alpha}},\end{aligned}$$

that is,

$$\omega'(t) \leq \delta(t)\frac{\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))} + \frac{\delta'(t)}{\delta(t)}\omega(t) - \frac{\alpha\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t). \tag{3.16}$$

Now, define another positive function $v(t)$ by

$$v(t) = \delta(t)\frac{r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}}{z^{\alpha}(\sigma(t))}. \tag{3.17}$$

By differentiating (3.17), we get

$$v'(t) = \delta'(t)\frac{r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}}{z^{\alpha}(\sigma(t))} + \frac{\delta(t)\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))} - \frac{\alpha\delta(t)r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}z^{\alpha-1}(\sigma(t))z'(\sigma(t))\sigma'(t)}{z^{2\alpha}(\sigma(t))}. \tag{3.18}$$

From (3.15), $\sigma(t) \leq \tau(t)$ and $z^{n}(t) \leq 0$, we have

$$z^{'}(\sigma(t)) \geq \frac{\mu}{(n-2)!}(\sigma(t))^{n-2}z^{(n-1)}(\sigma(t)) \geq \frac{\mu}{(n-2)!}(\sigma(t))^{n-2}z^{(n-1)}(\tau(t)). \tag{3.19}$$

Substituting (3.19) and (3.17) into (3.18), implies

$$\begin{aligned} v'(t) &\leq \delta'(t)\frac{r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}}{z^{\alpha}(\sigma(t))}+\delta(t)\frac{\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))} \\ &\quad -\left(\frac{z^{(n-1)}(\tau(t))}{z(\sigma(t))}\right)^{\alpha+1}\frac{\alpha\delta(t)r(\tau(t))\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!} \\ &\leq \delta(t)\frac{\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))}+\frac{\delta'(t)}{\delta(t)}v(t)-\frac{\alpha\delta(t)r(\tau(t))\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!}\left(\frac{v(t)}{\delta(t)r(\tau(t))}\right)^{\frac{\alpha+1}{\alpha}}. \end{aligned}$$

By $r'(t)>0$, we get

$$v'(t)\leq\delta(t)\frac{\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))}+\frac{\delta'(t)}{\delta(t)}v(t)-\frac{\alpha\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)}v^{(\alpha+1)/\alpha}(t). \tag{3.20}$$

Now, using inequalities (3.16) and (3.20), we get

$$\begin{aligned} \omega'(t)+\frac{p_0^{\alpha}}{\tau_0}v'(t) &\leq \delta(t)\frac{\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)'+\frac{p_0^{\alpha}}{\tau_0}\left(r(\tau(t))\left(z^{(n-1)}(\tau(t))\right)^{\alpha}\right)'}{z^{\alpha}(\sigma(t))} \\ &\quad +\frac{\delta'(t)}{\delta(t)}\omega(t)-\frac{\alpha\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t) \\ &\quad +\frac{p_0^{\alpha}}{\tau_0}\left(\frac{\delta'(t)}{\delta(t)}v(t)-\frac{\alpha\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)}v^{(\alpha+1)/\alpha}(t)\right). \end{aligned} \tag{3.21}$$

By (3.7), we obtain

$$\begin{aligned} \omega'(t)+\frac{p_0^{\alpha}}{\tau_0}v'(t) &\leq -\delta(t)\frac{\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 1}(t)}{\mu}+\frac{\delta'(t)}{\delta(t)}\omega(t)-\frac{\alpha\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t) \\ &\quad +\frac{p_0^{\alpha}}{\tau_0}\left(\frac{\delta'(t)}{\delta(t)}v(t)-\frac{\alpha\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)}v^{(\alpha+1)/\alpha}(t)\right). \end{aligned}$$

Applying the following inequality inequality (2.1) with

$$A=\frac{\alpha\mu\sigma^{n-2}(t)\sigma'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)} \text{ and } B=\frac{\delta'(t)}{\delta(t)},$$

we get

$$\begin{aligned} \omega'(t)+\frac{p_0^{\alpha}}{\tau_0}v'(t) &\leq -\frac{\delta(t)}{\mu}\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 1}(t)+\frac{((n-2)!)^{\alpha}}{\mu^{\alpha}(\alpha+1)^{\alpha+1}}\frac{r(t)(\delta'(t))^{\alpha+1}}{(\delta(t)\sigma^{n-2}(t)\sigma'(t))^{\alpha}} \\ &\quad +\frac{p_0^{\alpha}((n-2)!)^{\alpha}}{\tau_0\mu^{\alpha}(\alpha+1)^{\alpha+1}}\frac{r(t)(\delta'(t))^{\alpha+1}}{(\delta(t)\sigma^{n-2}(t)\sigma'(t))^{\alpha}}. \end{aligned}$$

Integrating the last inequality from $t_2$ to $t$, we obtain

$$\int_{t_2}^{t}\left(\frac{\delta(s)}{\mu}\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 1}(s)-\frac{((n-2)!)^{\alpha}}{\mu^{\alpha}(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r(s)(\delta'(s))^{\alpha+1}}{(\delta(s)\sigma^{n-2}(s)\sigma'(s))^{\alpha}}\right)\mathrm{d}s\leq\omega(t_2)+\frac{p_0^{\alpha}}{\tau_0}v(t_2).$$

The proof is complete. □

**Theorem 3.2.** *Assume that (3.2), (3.5), (3.6), $\sigma(t) \leq \tau(t)$ and $\sigma_\kappa \circ \sigma^{-1} \circ \tau = \tau \circ \sigma_\kappa \circ \sigma^{-1}$ hold. If there exists a function $\delta \in C^1([t_0, \infty), (0, \infty))$, such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[ \frac{\delta(s)}{\mu} \sum_{\kappa=1}^{m} \widetilde{q}_{\kappa 2}(s) - \frac{((n-2)!)^\alpha}{\mu^\alpha \sigma_0 (\alpha+1)^{\alpha+1}} \left(1 + \frac{p_0^\alpha}{\tau_0}\right) \frac{r\left(\sigma^{-1}(s)\right)(\delta'(s))^{\alpha+1}}{\left(\delta(s) s^{n-2}\right)^\alpha} \right] \mathrm{d}s = \infty, \tag{3.22}$$

*then every solution of (1.1) is oscillatory or tends to zero.*

*Proof.* Let $x$ be a positive solution of (1.1). Then, there exist $t_1 \geq t_0$ such that $x(t) > 0$, $x(\sigma_\kappa(t)) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. Let $z$ satisfying case (**I**). Define the positive function by

$$\omega(t) = \delta(t) \frac{r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha}{z^\alpha(t)}. \tag{3.23}$$

Hence, by differentiating (3.23), we get

$$\begin{aligned} \omega'(t) = {} & \delta'(t) \frac{r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha}{z^\alpha(t)} + \delta(t) \frac{\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha\right)'}{z^\alpha(t)} \\ & - \frac{\alpha\delta(t) r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha z^{\alpha-1}(t) z'(t)}{z^{2\alpha}(t)}. \end{aligned} \tag{3.24}$$

Since $z' > 0$, $z'' > 0$, we see that $\lim_{t\to\infty} z' \neq 0$, using Lemma 2.3 with $f = z'$, we obtain

$$z'(t) \geq \frac{\mu}{(n-2)!} t^{n-2} z^{(n-1)}(t), \tag{3.25}$$

for every $\mu \in (0, 1)$. Thus, by $\sigma^{-1}(t) > t$ and $z^n(t) \leq 0$, we get

$$z'(t) \geq \frac{\mu}{(n-2)!} t^{n-2} z^{(n-1)}(t) \geq \frac{\mu}{(n-2)!} t^{n-2} z^{(n-1)}\left(\sigma^{-1}(t)\right). \tag{3.26}$$

Substituting (3.23) and (3.26) into (3.24) implies

$$\begin{aligned} \omega'(t) \leq {} & \delta'(t) \frac{r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha}{z^\alpha(t)} + \delta(t) \frac{\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha\right)'}{z^\alpha(t)} \\ & - \left(\frac{z^{(n-1)}\left(\sigma^{-1}(t)\right)}{z(t)}\right)^{\alpha+1} \frac{\alpha\delta(t) r\left(\sigma^{-1}(t)\right)\mu t^{n-2}}{(n-2)!} \\ \leq {} & \delta(t) \frac{\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha\right)'}{z^\alpha(t)} + \frac{\delta'(t)}{\delta(t)} \omega(t) \\ & - \frac{\alpha\delta(t) r\left(\sigma^{-1}(t)\right)\mu t^{n-2}}{(n-2)!} \left(\frac{\omega(t)}{\delta(t) r\left(\sigma^{-1}(t)\right)}\right)^{\frac{\alpha+1}{\alpha}}, \end{aligned}$$

that is,

$$\omega'(t) \leq \delta(t) \frac{\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^\alpha\right)'}{z^\alpha(t)} + \frac{\delta'(t)}{\delta(t)} \omega(t) - \frac{\alpha\mu t^{n-2}}{(n-2)!\delta^{1/\alpha}(t) r^{1/\alpha}\left(\sigma^{-1}(t)\right)} \omega^{(\alpha+1)/\alpha}(t). \tag{3.27}$$

Now, define another positive function $v(t)$ by

$$v(t)=\delta(t)\frac{r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}}{z^{\alpha}(t)}. \tag{3.28}$$

By differentiating (3.28), we get

$$\begin{aligned} v'(t) &= \delta'(t)\frac{r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}}{z^{\alpha}(t)}+\frac{\delta(t)\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}\right)'}{z^{\alpha}(t)} \\ &\quad -\frac{\alpha\delta(t)r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}z^{\alpha-1}(t)z'(t)}{z^{2\alpha}(t)}. \end{aligned} \tag{3.29}$$

From (3.25), $\sigma^{-1}(\tau(t))\geq t$ and $z^{n}(t)\leq 0$, we have

$$z'(t)\geq\frac{\mu}{(n-2)!}t^{n-2}z^{(n-1)}(t)\geq\frac{\mu}{(n-2)!}t^{n-2}z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right). \tag{3.30}$$

Substituting (3.30) and (3.28) into (3.29), implies

$$\begin{aligned} v'(t) &\leq \delta'(t)\frac{r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}}{z^{\alpha}(t)}+\frac{\delta(t)\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}\right)'}{z^{\alpha}(t)} \\ &\quad -\left(\frac{z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)}{z(t)}\right)^{\alpha+1}\frac{\alpha\delta(t)r\left(\sigma^{-1}(\tau(t))\right)\mu t^{n-2}}{(n-2)!} \\ &\leq \frac{\delta(t)\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}\right)'}{z^{\alpha}(t)}+\frac{\delta'(t)}{\delta(t)}v(t) \\ &\quad -\frac{\alpha\delta(t)r\left(\sigma^{-1}(\tau(t))\right)\mu t^{n-2}}{(n-2)!}\left(\frac{v(t)}{\delta(t)r\left(\sigma^{-1}(\tau(t))\right)}\right)^{\frac{\alpha+1}{\alpha}}, \end{aligned}$$

By $r'(t)>0$, we get

$$v'(t)\leq\frac{\delta(t)\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}\right)'}{z^{\alpha}(t)}+\frac{\delta'(t)}{\delta(t)}v(t)-\frac{\alpha\mu t^{n-2}}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}\left(\sigma^{-1}(t)\right)}v^{(\alpha+1)/\alpha}(t). \tag{3.31}$$

Now, using inequalities (3.27) and (3.31), we get

$$\begin{aligned} \frac{1}{\sigma_0}\omega'(t)+\frac{p_0^{\alpha}}{\sigma_0\tau_0}v'(t) &\leq \delta(t)\frac{\frac{1}{\sigma_0}\left(r\left(\sigma^{-1}(t)\right)\left(z^{(n-1)}\left(\sigma^{-1}(t)\right)\right)^{\alpha}\right)'}{z^{\alpha}(t)} \\ &\quad +\delta(t)\frac{p_0^{\alpha}}{\sigma_0\tau_0}\frac{\left(r\left(\sigma^{-1}(\tau(t))\right)\left(z^{(n-1)}\left(\sigma^{-1}(\tau(t))\right)\right)^{\alpha}\right)'}{z^{\alpha}(t)} \\ &\quad +\frac{\delta'(t)}{\sigma_0\delta(t)}\omega(t)-\frac{\alpha\mu t^{n-2}}{\sigma_0(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}\left(\sigma^{-1}(t)\right)}\omega^{(\alpha+1)/\alpha}(t) \\ &\quad +\frac{p_0^{\alpha}}{\sigma_0\tau_0}\left(\frac{\delta'(t)}{\delta(t)}v(t)-\frac{\alpha\mu t^{n-2}}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}\left(\sigma^{-1}(t)\right)}v^{(\alpha+1)/\alpha}(t)\right). \end{aligned}$$

By (3.9), we obtain

$$\frac{1}{\sigma_0}\omega'(t)+\frac{p_0^\alpha}{\sigma_0\tau_0}v'(t) \leq -\frac{\delta(t)}{\mu}\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 2}(t)+\frac{\delta'(t)}{\sigma_0\delta(t)}\omega(t)-\frac{\alpha\mu t^{n-2}}{\sigma_0(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(\sigma^{-1}(t))}\omega^{(\alpha+1)/\alpha}(t)$$
$$+\frac{p_0^\alpha}{\tau_0}\left(\frac{\delta'(t)}{\sigma_0\delta(t)}v(t)-\frac{\alpha\mu t^{n-2}}{\sigma_0(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(\sigma^{-1}(\tau(t)))}v^{(\alpha+1)/\alpha}(t)\right).$$

Applying the following inequality (2.1) with

$$A=\frac{\alpha\mu t^{n-2}}{\sigma_0(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(\sigma^{-1}(t))} \text{ and } B=\frac{\delta'(t)}{\sigma_0\delta(t)},$$

we get

$$\frac{1}{\sigma_0}\omega'(t)+\frac{p_0^\alpha}{\sigma_0\tau_0}v'(t) \leq -\frac{\delta(t)}{\mu}\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 2}(t)+\frac{((n-2)!)^\alpha}{\mu^\alpha\sigma_0(\alpha+1)^{\alpha+1}}\frac{r\left(\sigma^{-1}(t)\right)(\delta'(t))^{\alpha+1}}{(\delta(t)t^{n-2})^\alpha}$$
$$+\frac{p_0^\alpha((n-2)!)^\alpha}{\tau_0\sigma_0\mu^\alpha(\alpha+1)^{\alpha+1}}\frac{r\left(\sigma^{-1}(t)\right)(\delta'(t))^{\alpha+1}}{(\delta(t)t^{n-2})^\alpha}.$$

Integrating last the inequality from $t_2$ to $t$, we obtain

$$\int_{t_2}^{t}\left[\frac{\delta(s)}{\mu}\sum_{\kappa=1}^{m}\widetilde{q}_{\kappa 2}(s)-\frac{((n-2)!)^\alpha}{\mu^\alpha\sigma_0(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{r\left(\sigma^{-1}(s)\right)(\delta'(s))^{\alpha+1}}{(\delta(s)s^{n-2})^\alpha}\right]\mathrm{d}s\leq\frac{1}{\sigma_0}\omega(t_2)+\frac{p_0^\alpha}{\sigma_0\tau_0}v(t_2).$$

The proof is complete. □

**Example 3.1.** *Consider the odd order neutral delay differential equation*

$$\left(x(t)+\frac{17}{18}x\left(\frac{t}{b}\right)\right)^{(n)}+\sum_{k=1}^{m}\frac{q_0}{t^n}x\left(\frac{t}{b_\kappa^2}\right)=0,\ n\geq 3,\ t\geq 1, \tag{3.32}$$

*we note that*

$$\mu=\alpha=r(t)=1,\ b=b_1>1,\ \widetilde{q}_{\kappa 2}(s)=\frac{q_0}{b^{2n}t^n},\ \sigma(t)=\frac{t}{b^2},\ \tau(t)=\frac{t}{b} \text{ and set } \delta(t)=t^{n-1}.$$

*It is easy to see that the conditions (3.5), (3.6) and (3.2) hold.*

*Applying Theorem 3.2, we have that every solution of (3.32) is oscillatory or tends to zero as $t\to\infty$ when*

$$q_0>\frac{(n-2)!(n-1)^2b^{2n-2}}{4m}\left(1+\frac{17}{18}b\right).$$

**Remark 3.1.** *If we consider the special case $\left(x(t)+\frac{17}{18}x(t/2)\right)^{(3)}+\frac{q_0}{t^3}x\left(t/2^2\right)=0$, then every solution is oscillatory or tends to zero if $q_0>46.22$, while by using the result in [21], we have that every solution is oscillatory or tends to zero if $q_0>144$. Consequently, our results apply to the equation $\left(x(t)+\frac{17}{18}x(t/2)\right)^{(3)}+\frac{70}{t^3}x\left(t/2^2\right)=0$, while the other results fail to study this equation.*

## 4. Conclusions

In this study, oscillatory properties of a class of odd-order quasi-linear neutral differential equations are established. By introducing some Riccati substitution, we obtained new conditions that guarantee that all nonoscillatory solutions of (1.1) converge to zero. Our results extend and complement the previous results in the literature. An interesting issue is obtaining new criteria that ensure that all solutions of (1.1) oscillate.

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## Conflict of interest

There are no competing interests

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