Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(10): 11109–11123.
DOI: 10.3934/math.2021645
Received: 07 February 2021
Accepted: 21 July 2021
Published: 03 August 2021

Research article

A study on a line congruence as surface in the space of lines

Rashad A. Abdel-Baky[1,*] **and Monia F. Naghi**[2]

[1] Department of Mathematics, Faculty of Science, University of Assiut, Assiut 71516, Egypt
[2] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

* **Correspondence:** Email: baky1960@aun.edu.eg.

Abstract: In this work, we introduce a line congruence as surface in the space of lines in terms of the E. Study map. This provides the ability to derive some formulae of surfaces theory into line spaces. In addition, the well known equation of the Plucker's conoid has been obtained and its kinematic-geometry are examined in details. At last, an example of application is investigated and explained in detail.

Keywords: ruled and developable surfaces; the dual angle of pitch
Mathematics Subject Classification: 53A04, 53A05, 53A17

1. Introduction

Differential line geometry examines the set of lines in space with 3-dimension. The ambient space can be an Euclidean or a non-Euclidean space. It is one of the interesting subdivision of differential geometry since it is directly connected with the spatial motion (kinematics) and get therefore a lot of application in mechanism design [1–9].

A line congurence is a two-parameter set of lines in three-dimensional space- is a classical part of differential geometry, whose origins can be traced to Kummer in his memoir; Allgemeine Theorie der Gradlinigen Strahen system [1]. The normal vector field of a surface could be given as an example of line congruence. In this way the line congruence formed by the normal vector field of the surface constructs a special class, which is called normal line congruence. However, the line congruence does not usually need to consist only of normal vector field. The lines of the line congruence which pass through a curve on the surface form a one-parameter family of lines in the space or ruled surface (parameter ruled surface). Recently, line congruence has become relevant for practical applications (See for instance [2–7]).

As it is known, the most analytical tool in the study of three-dimensional kinematics and differential line geometry is based upon the so-called E. Study's map: The set of all oriented lines in

Euclidean 3-space $\mathbb{E}^3$ is in one-to-one correspondence with set of points of the dual unit sphere in the dual 3-space $\mathbb{D}^3$. It allows a complete generalization for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities. Hence, the ruled surfaces and line congruence obtained by the motion of a line, depending on one and two parameters, respectively. There exists a vast literature on the subject including several monographs, such as [6, 8–11].

This work develops the kinematic-geometry for line congruence in the space of lines by using the analogy with theory of surfaces. Then the well known formulae of J. Liouville, Hamilton, and Mannheim of surfaces theory are proved for the line congruence. Moreover, a new geometrical interpretation of J. Liouville formula has been defined for a given closed ruled surface in the line congruence. The Plücker conoid associated with the line congruence has been derived and it is shown that the principal axes of it are located at its center and at right angles. Finally, an example of application is investigated and explained in detail.

2. Preliminaries

In this section we list some notions, formulas of dual numbers and dual vectors (See for instance [1–6]). Therefore we start with recalling the use of appropriate line coordinates: An oriented line L in the Euclidean 3-space $\mathbb{E}^3$ can be determined by a point $\mathbf{p} \in L$ and a normalized direction vector $\mathbf{x}$ of L, i.e. $\|\mathbf{x}\| = 1$. To obtain components for L, one forms the moment vector

$$\mathbf{x}^* = \mathbf{p} \times \mathbf{x} \tag{2.1}$$

with respect to the origin point in $\mathbb{E}^3$. If $\mathbf{p}$ is substituted by any point

$$\mathbf{q} = \mathbf{p}+\mu\mathbf{x}, \ \mu \in \mathbb{R} \tag{2.2}$$

on L, then Eq (2.1) implies that $\mathbf{x}^*$ is independent of $\mathbf{p}$ on L. The two vectors $\mathbf{x}$ and $\mathbf{x}^*$ are not independent of one another; they satisfy the following relationships:

$$< \mathbf{x}, \mathbf{x} >=1, \ \ < \mathbf{x}^*, \mathbf{x} >=0. \tag{2.3}$$

The six components x_i, $x_i^*(i = 1, 2, 3)$ of $\mathbf{x}$ and $\mathbf{x}^*$are called the normalized Plűcker coordinates of the line L. Hence the two vectors $\mathbf{x}$ and $\mathbf{x}^*$ determine the oriented line L.

In line geometry, from a kinematic point of view, a line congruence (congruence for shortness) is a two-parameter set of lines in $\mathbb{E}^3$ generated by a straight line L moving along a surface. The various positions of the generating lines are called the rulings or generators of the congruence. This congruence holds a parameterization as form

$$Q : \mathbf{Y}(u_1, u_2, v) = \mathbf{y}(u_1, u_2) + v\mathbf{r}(u_1, u_2), \ v \in \mathbb{R}. \tag{2.4}$$

Here $\mathbf{y} = \mathbf{y}(u_1, u_2)$ is called the director surface, $\mathbf{r} = \mathbf{r}(u_1, u_2)$ is the unit vector giving the direction of generating line of the congruence, u_1, u_2 are the motion parameters, and v is the parameter of its points indicating the singed distance of the corresponding point on $\mathbf{y} = \mathbf{y}(u_1, u_2)$. The equations

$$u_1 = u_1(t), \ u_2 = u_2(t), \ u_1'^2 + u_2'^2 \neq 0, \ t \in \mathbb{R}, \tag{2.5}$$

define a ruled surface belonging to the congruence. It is a developable if and only if the determinate

$$(\mathbf{r}, d\mathbf{y}, d\mathbf{r}) = 0. \tag{2.6}$$

Ruled surfaces (such as cylinders and cones) contain rulings where the tangent plane touches the surface along the entire line. Such rulings are called torsal lines, to distinguish them from the common case of the non-torsal generators [6, 11–13]:

- Ruled surfaces with exclusively torsal generators are called developable surfaces;
- Ruled surfaces consisting largely of non-torsal generators are called skew ruled surfaces (or warped ruled surfaces);
- Cylinders, cones, and ruled surfaces that consist of tangents of a spatial curve are developable surfaces.

Definition 1. The singular surface points of a torsal generator is called its cuspidal point, and the tangent plane in its other direction is called torsal plane.

2.1. E. Study's map

The set of dual numbers is

$$\mathbb{D} = \{X = x + \varepsilon x^* \mid x,\ x^* \in \mathbb{R},\ \varepsilon \neq 0,\ \varepsilon^2 = 0\}. \tag{2.7}$$

This set is a commutative ring under addition and multiplication. This set cannot be a field under these operations, because $0 + \varepsilon x^*$ has no multiplication inverse in $\mathbb{D}$. But this ring has a unit element according to multiplication. A dual number $X = x + \varepsilon x^*$, is called proper if $x \neq 0$. An example of dual number is the dual angle subtended by two skew lines in the Euclidean 3-space $\mathbb{E}^3$ and defined as $\Theta = \vartheta + \varepsilon\vartheta^*$ in which ϑ and ϑ^* are, respectively, the projected angle and the minimal distance between the two lines.

For all pairs $(\mathbf{x}, \mathbf{x}^*) \in \mathbb{E}^3 \times \mathbb{E}^3$ the set

$$\mathbb{D}^3 = \{\mathbf{X} = \mathbf{x} + \varepsilon\mathbf{x}^*,\ \varepsilon \neq 0,\ \varepsilon^2 = 0\}, \tag{2.8}$$

together with the scalar product

$$< \mathbf{X}, \mathbf{Y} >=< \mathbf{x}, \mathbf{y} > + \varepsilon(< \mathbf{y}, \mathbf{x}^* > + < \mathbf{y}^*, \mathbf{x} >), \tag{2.9}$$

forms the dual 3-space $\mathbb{D}^3$. Thereby a point $\mathbf{X} = (X_1, X_2, X_3)^t$ has dual coordinates $X_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. The norm is defined by

$$< \mathbf{X}, \mathbf{X} >^{\frac{1}{2}} := \|\mathbf{X}\| = \|\mathbf{x}\| \, (1 + \varepsilon \frac{< \mathbf{x}, \mathbf{x}^* >}{\|\mathbf{x}\|^2}). \tag{2.10}$$

In the dual 3-space $\mathbb{D}^3$ the dual unit sphere is defined by

$$\mathbb{K} = \{\mathbf{X} \in \mathbb{D}^3 \mid \|\mathbf{X}\|^2 = X_1^2 + X_2^2 + X_3^2 = 1\}. \tag{2.11}$$

Definition of dual unit sphere gives us that all points $\mathbf{X}$ of $\mathbb{K}$ must satisfy two equations

$$x_1^2 + x_2^2 + x_3^2 = 1, \quad x_1 x_1^* + x_2 x_2^* + x_3 x_3^* = 0. \tag{2.12}$$

Via this we have the following map (E. Study's map): The set of all oriented lines in the Euclidean 3-space $\mathbb{E}^3$ is in one-to-one correspondence with the set of points of dual unit sphere in the dual 3-space $\mathbb{D}^3$. By using this correspondence, one can derive the properties of the spatial motion of a line. Hence, the geometry of ruled surface is represented by the geometry of curves on the dual unit sphere in $\mathbb{D}^3$. Therefore, the dual curve and ruled surfaces are synonymous in this paper.

2.2. Line congruence as a dual region on $\mathbb{K}$

The E. Study's map admits us to revision Eq (2.4) using the dual vector function as

$$\mathbf{R}(u_1,u_2) = \mathbf{r}(u_1,u_2) + \varepsilon\mathbf{y}(u_1,u_2)\times\mathbf{r}(u_1,u_2) = \mathbf{r}(u_1,u_2) + \varepsilon\mathbf{r}^*(u_1,u_2), \tag{2.13}$$

where $\mathbf{r}^*$ is the moment of $\mathbf{r}$ about the origin in $\mathbb{E}^3$. Since the spherical image $\mathbf{r}(u_1,u_2)$ is a unit vector, then the dual vector $\mathbf{R}(u_1,u_2)$ also has unit magnitude as is seen from the computations:

$$<\mathbf{R},\mathbf{R}> = <\mathbf{r},\mathbf{r}> +2\varepsilon<\mathbf{r},\mathbf{y}\times\mathbf{r}> +\varepsilon^2<\mathbf{y}\times\mathbf{r},\mathbf{y}\times\mathbf{r}> =<\mathbf{r},\mathbf{r}>=1.$$

Thus the line congruence fills a domain on dual unit sphere $\mathbb{K}$ in $\mathbb{D}^3$. Hence, the line congruence can be viewed as a two-dimensional surface in $\mathbb{D}^3$-space. It follows that there are resemblances between theory of surface and theory of line congruence.

A relationship such as $f(u_1,u_2) = 0$ between the parameters u_1, u_2 reduces the congruence to a one-parameter set of lines (a ruled surface) in the congruence $\mathbf{R}(u_1,u_2)$. The equations $u_2 = c_2$ and $u_1 = c_1$(real constants) determines parameter ruled surfaces given by the parameter dual curves $\mathbf{R} = \mathbf{R}(u_1,c_2)$ and $\mathbf{R} = \mathbf{R}(c_1,u_2)$, respectively.

Definition 2. A line congruence is torsal, if the ruled surfaces defined by u_1 =const. are developable, and so are the ones defined by u_2 =const.

If L any function, scalar or vector, defined for the line congruence, we shall denote L_{u_1}, and L_{u_2} by $\partial L/\partial u_1$, and $\partial L/\partial u_2$, respectively. Thus $\mathbf{R}_{u_1}$ is a dual tangent vector in direction which u_1 alone varies, similarly for $\mathbf{R}_{u_2}$. Therefore,

$$\left.\begin{array}{l} <\mathbf{R}_{u_1},\mathbf{R}_{u_1}>=E = e+\varepsilon e^*, \\ <\mathbf{R}_{u_1},\mathbf{R}_{u_2}>=F = f+\varepsilon f^*, \\ <\mathbf{R}_{u_2},\mathbf{R}_{u_2}>=G = g+\varepsilon g^*. \end{array}\right\} \tag{2.14}$$

Thus, we arrive by means of the real and dual parts of Eq (2.14), at

$$\left.\begin{array}{l} e =<\mathbf{r}_{u_1},\mathbf{r}_{u_1}>,\ e^* = 2<\mathbf{r}_{u_1},\mathbf{r}^*_{u_1}>, \\ f =<\mathbf{r}_{u_1},\mathbf{r}_{u_2}>,\ f^* =<\mathbf{r}_{u_1},\mathbf{r}^*_{u_2}> + <\mathbf{r}^*_{u_1},\mathbf{r}_{u_2}>, \\ g =<\mathbf{r}_{u_2},\mathbf{r}_{u_2}>,\ g^* = 2<\mathbf{r}_{u_2},\mathbf{r}^*_{u_2}>. \end{array}\right\} \tag{2.15}$$

Now we consider two neighboring dual points, with position vectors $\mathbf{R}$ and $\mathbf{R}+d\mathbf{R}$, from (u_1,u_2) to (u_1+du_1,u_2+du_2), respectively. Then

$$d\mathbf{R} = \mathbf{R}_{u_1}du_1 + \mathbf{R}_{u_2}du_2. \tag{2.16}$$

Since the two dual points are adjacent points on a dual curve passing through them, the dual arc length $dS(= ds+\varepsilon ds^*)$ is

$$dS^2 =<d\mathbf{R},d\mathbf{R}>=<\mathbf{R}_{u_1}du_1 + \mathbf{R}_{u_2}du_2, \mathbf{R}_{u_1}du_1 + \mathbf{R}_{u_2}du_2>,$$

or finally,

$$dS^2 = Edu_1^2 + 2Fdu_1du_2 + Gdu_2^2. \tag{2.17}$$

By separating the real and dual parts of this equation, respectively, we get

$$I(u_1, u_2) := ds^2 =< dr,dr> = edu_1^2 + 2fdu_1du_2 + gdu_2^2, \tag{2.18}$$

and

$$II(u_1, u_2) := 2dsds^* =< dr,dr^*> = e^*du_1^2 + 2f^*du_1du_2 + g^*du_2^2. \tag{2.19}$$

The forms I and II are called the first and second fundamental forms of the line congruence, respectively. The distribution parameter of a ruled surface belong to the congruence is given by [10, 11]

$$\lambda = \frac{1}{2}\left(\frac{II}{I}\right) = \frac{e^*du_1^2 + 2f^*du_1du_2 + g^*du_2^2}{edu_1^2 + 2fdu_1du_2 + gdu_2^2}. \tag{2.20}$$

A ruled surface belong to the congruence for which

$$e^*du_1^2 + 2f^*du_1du_2 + g^*du_2^2 = 0 \tag{2.21}$$

is a developable surface. If we let $\eta = du_1/du_2$, we can write

$$e^*\eta^2 + 2f^*\eta + g^* = 0. \tag{2.22}$$

For developable surfaces of the congruence, we equate the coefficients of the Eq (2.22) to zero identically, and we can write

$$\left.\begin{array}{l} e^* = 2 < \mathbf{r}_{u_1}, \mathbf{r}^*_{u_1} >=0, \\ f^* =< \mathbf{r}_{u_1}, \mathbf{r}^*_{u_2} > + < \mathbf{r}^*_{u_1}, \mathbf{r}_{u_2} >=0, \\ g^* = 2 < \mathbf{r}_{u_2}, \mathbf{r}^*_{u_2} >=0. \end{array}\right\} \tag{2.23}$$

2.3. Blaschke moving frames

Through every line in the congruence there pass two principal surfaces whose images on the dual unit sphere we take as parameter curves. For this special system we assume that u_1, and u_2 dual curves of $\mathbf{R}(u_1, u_2)$ are principal ruled surfaces, i.e., the elements f and f^* of the first and second fundamental forms vanish identically ($f = f^* = 0$). So, according to Eq (2.17), the dual arc length of the dual curves $u_2 = c_2$(real const.), and $u_1 = c_1$(real const.), respectively are $dS_1 = ds_1 + \varepsilon ds_1^* = \sqrt{E}du_1$ and $dS_2 = ds_2 + \varepsilon ds_2^* = \sqrt{G}du_2$.

In order to examine the geometrical properties of the congruence, we set up a moving frame coincident with the point on the sphere. If we label the point on the sphere, the generator as $\mathbf{R}(u_1, u_2)$, then the Blaschke frame can be set up [10, 11, 15]

$$\mathbf{R}_{12} = \frac{\mathbf{R}_{u_1}}{\left\|\mathbf{R}_{u_1}\right\|} = \frac{\mathbf{R}_{u_1}}{\sqrt{E}},\ \mathbf{R}_{22} = \frac{\mathbf{R}_{u_2}}{\left\|\mathbf{R}_{u_2}\right\|} = \frac{\mathbf{R}_{u_1}}{\sqrt{G}},\ \mathbf{R} = \mathbf{R}_{12} \times \mathbf{R}_{22}, \tag{2.24}$$

which are invariants vector functions on the congruence Q; we fix det($\mathbf{R}$, $\mathbf{R}_{12}, \mathbf{R}_{22}$) $= +1$ and consequently

$$\lambda = \frac{1}{2}(\frac{e^*du_1^2 + g^*du_2^2}{edu_1^2 + gdu_2^2}) = \frac{e^*}{2}(\frac{du_1}{ds})^2 + \frac{g^*}{2}(\frac{du_2}{ds})^2. \tag{2.25}$$

The extreme values of the distribution parameter, corresponding to the principal surfaces $\mathbf{R}(u_1, c_2)$, and $\mathbf{R}(c_1, u_2)$, respectively, are obtained by

$$\lambda_1 := \frac{ds_1^*}{ds_1} = \frac{e^*}{2e}, \ \lambda_2 := \frac{ds_2^*}{ds_2} = \frac{g^*}{2g}. \tag{2.26}$$

According to the elements of spherical kinematics, the motion of the frame $\{\mathbf{O}; \mathbf{R}, \mathbf{R}_{12}, \mathbf{R}_{22}\}$ at any instant is a rotation around the Darboux vector of this frame. Hence, by means of the derivatives with respect to the dual arc-length parameter of the dual curves $u_2 = c_2$ with tangent $\mathbf{R}_{12}$, the derivative formula is [10, 15]

$$\frac{\partial}{\partial S_1}\begin{pmatrix} \mathbf{R} \\ \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \Sigma_1 \\ 0 & -\Sigma_1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{R} \\ \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix} = \mathbf{\Omega}_1 \times \begin{pmatrix} \mathbf{R} \\ \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix}, \tag{2.27}$$

where

$$\mathbf{\Omega}_1 = \Sigma_1 \mathbf{R} + \mathbf{R}_{22}. \tag{2.28}$$

$\Sigma_1 = \sigma_1 + \varepsilon\sigma_1^* = -\frac{E_{u_2}}{2E\sqrt{G}}$ is the geodesic curvature of the dual curves $\mathbf{R}(u_1, c_2)$. Similarly, the derivative formula of the Blaschke frame of the dual curves $\mathbf{R}(c_1, u_2)$, with tangent $\mathbf{R}_{22}$ is

$$\frac{\partial}{\partial S_2}\begin{pmatrix} \mathbf{R} \\ \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \Sigma_2 \\ -1 & -\Sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{R} \\ \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix} = \mathbf{\Omega}_2 \times \begin{pmatrix} \mathbf{R} \\ \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix}, \tag{2.29}$$

where

$$\mathbf{\Omega}_2 = \mathbf{\Sigma}_2 \mathbf{R} - \mathbf{R}_{12}. \tag{2.30}$$

$\Sigma_2 = \sigma_2 + \varepsilon\sigma_2^* = \frac{G_{u_1}}{2G\sqrt{E}}$ has the same meaning as in Eq (2.27). In other words, dual geodesic curvatures of the dual curves $\mathbf{R}(u_1, c_2)$, and $\mathbf{R}(c_1, u_2)$, respectively, are

$$\Sigma_i = \sigma_i + \varepsilon\sigma_i^* = \det\left(\mathbf{R}, \frac{d\mathbf{R}}{dS_i}, \frac{d^2\mathbf{R}}{dS_i^2}\right), \ (i = 1, 2). \tag{2.31}$$

3. Main results

Consider $u_i = u_i(t)$, $(i = 1, 2)$ as functions of real parameter $t \in \mathbb{R}$. Then $\mathbf{R} = \mathbf{R}(u_1(t), u_2(t))$ represents a ruled surface in the congruence Q. The dual vector $\mathbf{R}_t = (\partial\mathbf{R}/\partial t)$ is tangent to this dual curve;

$$\mathbf{R}_t = \mathbf{R}_{u_1}\frac{du_1}{dt} + \mathbf{R}_{u_2}\frac{du_2}{dt}. \tag{3.1}$$

If $< \mathbf{R}_t, \mathbf{R}_t > \neq 0$, then we have a dual unit vector

$$\mathbf{R}_2 = \frac{\mathbf{R}_t}{\|\mathbf{R}_t\|} = \frac{1}{P}\left(\mathbf{R}_{u_1}\frac{du_1}{dt} + \mathbf{R}_{u_2}\frac{du_2}{dt}\right), \tag{3.2}$$

where $P = p + \varepsilon p^* = \left\|\mathbf{R}_{u_1}\frac{du_1}{dt} + \mathbf{R}_{u_2}\frac{du_2}{dt}\right\| = \sqrt{EG}$. Hence, the dual arc length of the dual curve $\mathbf{R} = \mathbf{R}(u_1(t), u_2(t))$ is given by

$$dS = ds + \varepsilon ds^* = Pdt. \tag{3.3}$$

In order to research the properties of $\mathbf{R} = \mathbf{R}(u_1(t), u_2(t))$, the Blaschke frame relative to $\mathbf{R}(t)$ will be defined as the frame of which this line and the central normal $\mathbf{R}_2$ to the ruled surface at the central point of $\mathbf{R}$ are two edges. The third edge $\mathbf{R}_3$ is the central tangent to the ruled surface $\mathbf{R}(t)$. Likwise, the frame $\{\mathbf{R}_1 = \mathbf{R}(t), \mathbf{R}_2 = \frac{\mathbf{R}_t}{\|\mathbf{R}_t\|}, \mathbf{R}_3(t) = \mathbf{R}_1 \times \mathbf{R}_2\}$ is called Blaschke frame. Then we have

$$\frac{d}{dS}\begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \Sigma \\ 0 & -\Sigma & 0 \end{pmatrix}\begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix} = \mathbf{\Omega} \times \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix}, \tag{3.4}$$

where

$$\mathbf{\Omega} = \Sigma\mathbf{R}_1 + \mathbf{R}_3, \tag{3.5}$$

$\Sigma = \sigma + \varepsilon\sigma^*$ is the dual geodesic curvature of the dual curve $\mathbf{R}(t) = \mathbf{R}(u_1(t), u_2(t))$. It is easily seen from the last two equations that

$$\Sigma = \sigma + \varepsilon\sigma^* = \frac{\det(\mathbf{R}, \mathbf{R}_t, \mathbf{R}_{tt})}{\|\mathbf{R}_t\|^2} = \det\left(\mathbf{R}, \frac{d\mathbf{R}}{dS}, \frac{d^2\mathbf{R}}{dS^2}\right). \tag{3.6}$$

Referring to the congruence Q, since $\mathbf{R}_t$ is tangent of the dual curve $\mathbf{R}(t)$, then Eq (3.2) rewritten as

$$\mathbf{R}_2 = \frac{dS_1}{dS}\mathbf{R}_{12} + \frac{dS_2}{dS}\mathbf{R}_{22}. \tag{3.7}$$

So, we can find a dual angle $\Theta = \vartheta + \varepsilon\vartheta^*$ such that (Figure 1)

$$\begin{pmatrix} \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}\begin{pmatrix} \mathbf{R}_{12} \\ \mathbf{R}_{22} \end{pmatrix}, \tag{3.8}$$

where

$$\left.\begin{array}{l} dS^2 = dS_1^2 + dS_2^2, \\ \cos\Theta = \frac{dS_1}{dS} = \sqrt{E}\frac{du_1}{dS}, \ \sin\Theta = \frac{dS_2}{dS} = \sqrt{G}\frac{du_2}{dS}. \end{array}\right\} \tag{3.9}$$

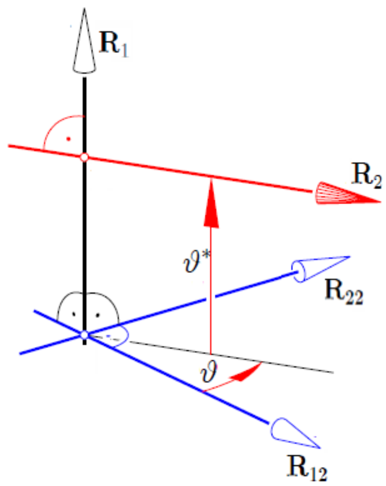

Figure 1. $\mathbf{R}_2 = \cos\Theta\mathbf{R}_{12} + \sin\Theta\mathbf{R}_{22}$.

By simple verification, it follows that

$$\frac{d\mathbf{R}_2}{dS} = \frac{\partial \mathbf{R}_2}{\partial S_1}\frac{dS_1}{dS} + \frac{\partial \mathbf{R}_2}{\partial S_2}\frac{dS_2}{dS}, \tag{3.10}$$

we get

$$\begin{aligned}\frac{d\mathbf{R}_2}{dS} &= \left(\frac{\partial \mathbf{R}_{12}}{\partial S_2} + \frac{\partial \mathbf{R}_{22}}{\partial S_1}\right)\sin\Theta\cos\Theta + \frac{\partial \mathbf{R}_{12}}{\partial S_1}\cos^2\Theta + \frac{\partial \mathbf{R}_{22}}{\partial S_2}\sin^2\Theta \\ &\quad + (-\mathbf{R}_{12}\sin\Theta + \mathbf{R}_{22}\cos\Theta)\frac{d\Theta}{dS}. \end{aligned} \tag{3.11}$$

From Eqs (2.27), (2.29) and (3.8) it follows that

$$\frac{d\mathbf{R}_2}{dS} = \left(\Sigma_1\cos\Theta + \Sigma_2\sin\Theta + \frac{d\Theta}{dS}\right)\mathbf{R}_3 - \mathbf{R}_1. \tag{3.12}$$

Thus, from Eqs (3.4) and (3.12), one finds that

$$\Sigma{=}\Sigma_1\cos\Theta + \Sigma_2\sin\Theta + \frac{d\Theta}{dS}. \tag{3.13}$$

This formula has the same nature of J. Liouville's formula in surface theory [1, 6, 9].

3.1. The dual angle of pitch

In this subsection we give a geometric characterization of the of J. Liouvile's formula. For this purpose, if $\mathbf{R}_1(t) = \mathbf{R}_1(t+2\pi)$ then $\mathbf{R}_1(t)$ is called a closed differentiable curve. According to E. Study's map, this corresponds to $\mathbf{R}_1(t)$-closed ruled surface belong to the line congruence. Also, let $\mathbf{R}_2$ generates a developable ruled surface (torse) along the orthogonal trajectory of the $\mathbf{R}_1(t)$-closed ruled surface. From Eq (3.12), this is expressed by

$$\frac{d\mathbf{R}_2}{dS}\times\mathbf{R}_1 = \mathbf{0} \Rightarrow \Sigma_1\cos\Theta + \Sigma_2\sin\Theta + \frac{d\Theta}{dS} = 0. \tag{3.14}$$

Then we call the total differential of Θ as the dual angle of pitch of the $\mathbf{R}_1(t)$-closed ruled surface. Thus, if it is denoted the dual angle of pitch of the $\mathbf{R}_1(t)$-closed ruled surface by the symbol Λ_1, then it can be written

$$\Lambda_1 = \lambda_1 - \varepsilon L_1 := \oint d\Theta, \tag{3.15}$$

where λ_r is the angel of pitch and L_r is the pitch of the $\mathbf{R}_1(t)$-closed ruled surface. The pitch and the angle of pitch are well-known real integral invariants of a closed ruled surface [10, 14–16]. Equations (3.14) and (3.15) shown that

$$\Lambda_1 = -\oint(\Sigma_1\cos\Theta + \Sigma_2\sin\Theta)\,dS. \tag{3.16}$$

We found by application of Green's formula, that

$$\Lambda_1 = -\iint\left(\frac{\partial}{\partial u_1}\left(\Sigma_2\sqrt{G}\right) - \frac{\partial}{\partial u_2}\left(\Sigma_1\sqrt{E}\right)\right)du_1du_2. \tag{3.17}$$

Since the Gaussian curvature of the dual unit sphere can be written by

$$1=\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u_1}\left(\Sigma_2\sqrt{G}\right)-\frac{\partial}{\partial u_2}\left(\Sigma_1\sqrt{E}\right)\right). \tag{3.18}$$

So that, we obtain as a result the formula

$$\Lambda_1=-\iint dA_1, \tag{3.19}$$

where $dA_1=\sqrt{EG}du_1du_2$ is the dual area on the dual unit sphere enclosed by the closed dual curve $\mathbf{R}_1(t)=\mathbf{R}_1(t+2\pi)$. Hence the following theorem is proved.

Theorem 1. For a closed ruled surface in the Euclidean 3-space $\mathbb{E}^3$. The dual angel of pitch is equal to minus the total dual spherical area of its dual image.

In fact from Eqs (3.9) and (3.19), we have that

$$\Lambda_1:=-\iint dS_1dS_2=-\iint(1+\varepsilon\lambda_1)(1+\varepsilon\lambda_2)\,ds_1ds_2. \tag{3.20}$$

If we separate the real and dual parts of Eq (3.20), then we find

$$\lambda_1=-a_1,\text{ and } L_1=\iint(\lambda_1+\lambda_2)\,ds_1ds_2, \tag{3.21}$$

where a_1 is the element of area on real unit sphere enclosed by the real spherical curve $\mathbf{r}_1(t)=\mathbf{r}_1(t+2\pi)$. More explicitly, we can rewrite Eq (3.16) as

$$\Lambda_1=-\oint<\frac{\partial\mathbf{R}_{12}}{\partial S_1},\mathbf{R}_{22}>dS_1+\oint<\frac{\partial\mathbf{R}_{22}}{\partial S_1},\mathbf{R}_{12}>dS_2, \tag{3.22}$$

and shown that

$$\Lambda_1=\Lambda_{r_1}-\Lambda_{r_2}, \tag{3.23}$$

where Λ_{r_1}, and Λ_{r_2} are the dual angel of pitches of the principal ruled surfaces $\mathbf{R}=\mathbf{R}(u_1,c_2)$ and $\mathbf{R}=\mathbf{R}(c_1,u_2)$, respectively.

3.2. Plücker conoid and Dupin's indicatrix

By separating the real and dual parts of Eq (3.9), bearing in mind Eqs (2.25) and (2.26), we get

$$\left.\begin{array}{c}\lambda=\lambda_1\cos^2\vartheta+\lambda_2\sin^2\vartheta,\\ \vartheta^*=(\frac{\lambda_2-\lambda_1}{2})\sin 2\vartheta.\end{array}\right\} \tag{3.24}$$

These formulas are Hamilton and Mannhiem formulae of surfaces theory in Euclidean 3-space, respectively [1]. The surface described by ϑ^* in Eq (3.24) is the Plücker conoid. The Plücker conoid is a smooth regular ruled surface sometimes also called the cylindroid [1,6,9].

The parametric form can also be given in terms of point coordinates. We may choose $\mathbf{R}_1$ is coincident with the $z-$axis of a fixed frame $(\mathbf{o}xyz)$, while the position of the dual unit vector $\mathbf{R}_2$ is given by angle ϑ and distance ϑ^* along the positive $z-$axis. The oriented lines $\mathbf{R}_{12}$ and $\mathbf{R}_{22}$ can be selected in sense of x and $y-$axes, respectively, as depicted in Figure 2. Clearly, Hamilton's formula shows that the angle ϑ varies from 0 to $\frac{\pi}{2}$ as λ varies from λ_1 to λ_2, the principal surfaces being perpendicular to each other. This shows that the oriented lines $\mathbf{R}_{12}$ and $\mathbf{R}_{22}$ are the principal axes and together with the oriented line $\mathbf{R}_1$ create the fundamental coordinate system of the Plücker's conoid. Let ζ denote a point on this surface, it is possible to have the following point coordinates:

$$\zeta(\vartheta, \vartheta^*, \upsilon)=(0,0,\vartheta^*) + \upsilon(\cos\vartheta, \sin\vartheta, 0), \quad \upsilon \in \mathbb{R}. \tag{3.25}$$

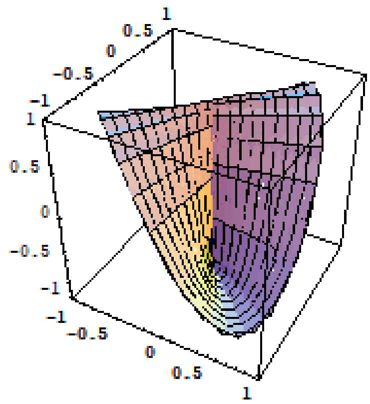

Figure 2. Plücker conoid.

Thus, the $z-$axis acts as base curve and the circle $\vartheta \to (\cos\vartheta, \sin\vartheta, 0)$ as director curve for the parametrization. Using this parametrization, the rulings are clearly visible passing through the $z-$axis, namely,

$$x = \upsilon\cos\vartheta, \quad y = \upsilon\sin\vartheta \ \text{ and } \vartheta^* := z = \frac{(\lambda_2 - \lambda_1)}{2}\sin 2\vartheta, \tag{3.26}$$

which gives us the intersection point of the principal axes $\mathbf{R}_{12}$ and $\mathbf{R}_{22}$ lies at a half of the conoid height ϑ^*. It can easily be verified by direct computations that

$$\left(x^2 + y^2\right) z + (\lambda_1 - \lambda_2)\, xy = 0, \tag{3.27}$$

The Eq (3.27) shows that the Plücker's conoid has two integral invariants of the first order, λ_1, and λ_2. It is assumed by convention that $\lambda_1 > \lambda_2$. The Plücker's conoid depends only on their difference; $\lambda_1 - \lambda_2 = 1$, $\vartheta \in [0, 2\pi]$, $\upsilon \in [-1, 1]$ (Figure 2). This surface has two torsal planes π_1, π_2 and each of which contains one torsal line L as follow: Solving for $\frac{y}{x}$, one obtains a second-order algebraic equation, whose roots are:

$$\frac{y}{x} = \frac{1}{2z}\left[\lambda_2 - \lambda_1 \pm \sqrt{(\lambda_2 - \lambda_1)^2 - 4z^2}\right], \tag{3.28}$$

The limit points of the Plücker's conoid can be determined by the vanishing of the discriminant of Eq (3.19), which leads to the two extreme positions, that is,

$$2z = \pm(\lambda_2 - \lambda_1). \tag{3.29}$$

According to the value of λ in Eq (3.28), the geometric properties of the Plücker's conoid are discussed as follows:

A- If the parameter ruled surface $\mathbf{R}(t) = \mathbf{R}(u_1(t), u_2(t))$ is a non-developable surface, i.e. $\lambda \neq 0$, then there are two real generators passing through the point $(0, 0, z)$ only if $z < (\lambda_2 - \lambda_1)/2$; for the two limit points $z = \pm(\lambda_2 - \lambda_1)/2$ they coincide with the principal axes $\mathbf{R}_{12}$ and $\mathbf{R}_{22}$.

B- If the parameter ruled surface $\mathbf{R}(t) = \mathbf{R}(u_1(t), u_2(t))$ is a developable surface, i.e. $\lambda = 0$, then their two torsal lines L_1, L_2 are represented by

$$L_1,\ L_2 : \frac{y}{x} = \tan\vartheta = \pm\sqrt{-\frac{\lambda_1}{\lambda_2}},\ z = \pm(\lambda_2 - \lambda_1)/2, \tag{3.30}$$

Thus the two torsal lines L_1, and L_2 are perpendicular each other, however, only real if $\lambda_1\lambda_2 \leq 0$. Hence, in this special case, if λ_1 and λ_2 have the same value, it follows that the Plücker's conoid degenerates into the pencil of lines through "**o**" in the torsal plane $z = 0$ (and the two isotropic planes through $z-$axis). Such a line congruence is called an elliptic line congruence. If, however, λ_1 and λ_2 have opposite signs, the lines L_1 and L_2 are real and they coincide with the generating lines $\mathbf{R}_{12}$ and $\mathbf{R}_{22}$. Such a line congruence is a hyperbolic line congruence. If either λ_1 or λ_2 is zero the line congruence is a parabolic line congruence, the lines L_1 and L_2 both coincide with $x-$axis; for $\lambda_1 \neq 0$, and $\lambda_2 = 0$ they coincide with $y-$axis.

Transition from polar coordinates to Cartesian coordinates could be performed by substituting

$$x = \frac{\cos\vartheta}{\sqrt{\lambda}}, y = \frac{\sin\vartheta}{\sqrt{\lambda}}, \tag{3.31}$$

into Hamilton's formula, one obtain the equation

$$D : |\lambda_1|\, x^2 + |\lambda_2|\, y^2 = 1, \tag{3.32}$$

of a conic section. As for theory of surfaces, this conic section is the Dupin's indicatrix of the line congruence. We now examine three cases in detail:

(1) If λ_1, and λ_2 are both positive, the Dupin's indicatrix is an ellipse has the principal semi-axes are $\frac{1}{\sqrt{\lambda_1}}$, and $\frac{1}{\sqrt{\lambda_2}}$. The lines through the center intersects the ellipse in the points

$$x = \pm\frac{\cos\vartheta}{\sqrt{\lambda}}, y = \pm\frac{\sin\vartheta}{\sqrt{\lambda}}. \tag{3.33}$$

The distance intercepted by the ellipse on the line $\frac{y}{x} = \tan\vartheta$ is (See Figure 3):

$$\sqrt{x^2 + y^2} = \frac{1}{\sqrt{\lambda}}. \tag{3.34}$$

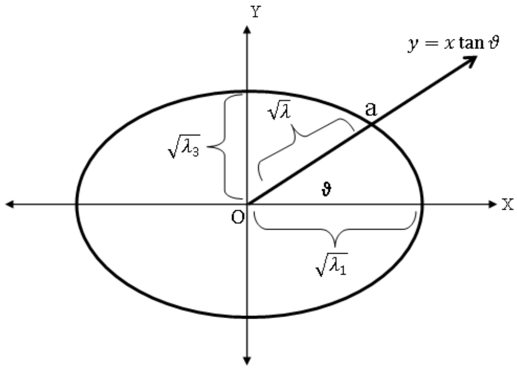

Figure 3. Dupin's indicatrix with λ_1 and λ_2 are both positive.

(2) If λ_1, and λ_2 have opposite signs the Dupin's indicatrix is set of conjugate hyperbolas

$$D : \lambda_1 x^2 \mp \lambda_2 y^2 = \pm 1, \tag{3.35}$$

shown in Figure 4. The two asymptotic directions of the hyperbolas represent the torsal lines at which $\lambda = 0$.

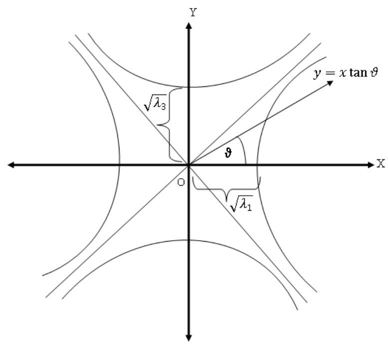

Figure 4. Dupin's indicatrix with λ_1 and λ_2 have opposite signs.

(3) If either λ_1 or λ_2 is zero the Dupin's indicatrix is a set of parallels lines corresponding to one of the forms

$$y^2 = \left|\frac{1}{\lambda_2}\right| \text{ with } \lambda_1 = 0, \text{ or } x^2 = \left|\frac{1}{\lambda_1}\right| \text{ with } \lambda_2 = 0.$$

3.3. Developable ruled surfaces

The parametric equation of the dual unit sphere $\mathbb{K} = \{\mathbf{R} \in \mathbb{D}^3 \mid R_1^2 + R_2^2 + R_3^2 = 1\}$, may be given by the equations

$$R_1 = \cos\Theta\sin\Phi,\ R_2 = \sin\Theta\sin\Phi,\ R_3 = \cos\Phi, \tag{3.36}$$

where $\Theta = \vartheta + \varepsilon\vartheta^*$, and $\Phi = \varphi + \varepsilon\varphi^*$ are dual angles with $-\pi \leq \vartheta \leq \pi$, and $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$. Separating Eq (3.36) into real and dual parts we obtain

$$r_1 = \cos\vartheta\sin\varphi,\ r_2 = \sin\vartheta\sin\varphi,\ r_3 = \cos\varphi, \tag{3.37}$$

and

$$\left.\begin{array}{c} r_1^* = -\vartheta^* \sin\vartheta \sin\varphi + \varphi^* \cos\varphi \cos\vartheta, \\ r_2^* = \vartheta^* \cos\vartheta \sin\varphi + \varphi^* \cos\varphi \sin\vartheta, \\ r_3^* = -\varphi^* \sin\varphi. \end{array}\right\} \tag{3.38}$$

These coordinates represent the four parameter family of lines in $\mathbb{E}^3$. To form line congruence, which describes a two-real parameter motion on the dual unit sphere, we may assume that $\vartheta^* = \vartheta^*(\vartheta, \varphi)$, and $\varphi^* = \varphi^*(\vartheta, \varphi)$. Thus, the line congruence is given by

$$\begin{array}{rcl} \mathbf{R}(\vartheta, \varphi) & = & (\cos\vartheta \sin\varphi, \sin\vartheta \sin\varphi, \cos\varphi) \\ & & +\varepsilon \left[\begin{array}{c} \vartheta^* (-\sin\vartheta \sin\varphi, \cos\vartheta \sin\varphi, 0) \\ +\varphi^* (\cos\vartheta \cos\varphi, \sin\vartheta \cos\varphi, -\sin\varphi) \end{array} \right]. \end{array} \tag{3.39}$$

According to Eqs (2.15) and (2.23), we obtain that

$$\left.\begin{array}{c} e = \sin^2\varphi, \; f = 0, \; g = 1, \; e^* = 2(\frac{\partial\vartheta^*}{\partial\vartheta} \sin\varphi + \varphi^* \cos\varphi) \sin\varphi, \\ \\ f^* = (\frac{\partial\vartheta^*}{\partial\varphi} \sin^2\varphi + \frac{\partial\varphi^*}{\partial\vartheta}), \; g^* = 2\frac{\partial\varphi^*}{\partial\varphi}. \end{array}\right\} \tag{3.40}$$

Thus,

$$\lambda = \frac{(\frac{\partial\vartheta^*}{\partial\vartheta} \sin\varphi + \varphi^* \cos\varphi) \sin\varphi d\vartheta^2 + (\frac{\partial\vartheta^*}{\partial\varphi} \sin^2\varphi + \frac{\partial\varphi^*}{\partial\vartheta}) d\vartheta d\varphi + \frac{\partial\varphi^*}{\partial\varphi} d\varphi^2}{\sin^2\varphi d\vartheta^2 + d\varphi^2}. \tag{3.41}$$

In order to identify the principal ruled surfaces of the line congruence, from Eq (3.40), we have

$$F = f + \varepsilon f^* = 0 \Leftrightarrow \varphi^* = \cos\vartheta, \text{ and } \vartheta^* = -\sin\vartheta \cot\varphi, \tag{3.42}$$

which yields $\lambda = e^* = g^* = f^* = 0$. Therefore, the principal ruled surfaces are developable ruled surfaces of the line congruence.

Now we may calculate the equation of the developable ruled surfaces of the congruence $\mathbf{R} = \mathbf{R}(\vartheta, \varphi)$ in terms of the Plücker coordinates as follows: Since $\mathbf{y} \times \mathbf{r} = \mathbf{r}^*$ we have the system of linear equations in y_1, y_2 and y_3 (y_{is} are the coordinates of $\mathbf{y}$)

$$\left.\begin{array}{c} y_2 \cos\varphi - y_3 \sin\vartheta \sin\varphi = \cos\varphi, \\ -y_1 \cos\varphi + y_3 \cos\vartheta \sin\varphi = 0, \\ y_1 \sin\vartheta \sin\varphi - y_2 \cos\vartheta \sin\varphi = -\cos\vartheta \sin\varphi. \end{array}\right\} \tag{3.43}$$

The matrix of coefficients of unknowns y, y_2 and y_3 is the skew-symmetric matrix

$$\begin{pmatrix} 0 & \cos\varphi & -\sin\vartheta \sin\varphi \\ -\cos\varphi & 0 & \cos\vartheta \sin\varphi \\ \sin\vartheta \sin\varphi & -\cos\vartheta \sin\varphi & 0 \end{pmatrix},$$

and therefore its rank is 2 with $\varphi \neq 0$, and $\vartheta \neq 0$. Also the rank of the augmented matrix

$$\begin{pmatrix} 0 & \cos\varphi & -\sin\vartheta \sin\varphi & \cos\varphi \\ -\cos\varphi & 0 & \cos\vartheta \sin\varphi & 0 \\ \sin\vartheta \sin\varphi & -\cos\vartheta \sin\varphi & 0 & -\cos\vartheta \sin\varphi \end{pmatrix},$$

is 2. Hence this system has infinite solutions given by

$$y_1 = y_3 \tan\varphi \cos\vartheta,\ y_2 = y_3 \tan\varphi \sin\vartheta + 1,\ y_3 = y_3(\vartheta, \varphi). \tag{3.44}$$

Since $y_3(\vartheta, \varphi)$ can be chosen arbitrarily, then we may take $y_3 - \vartheta^* = 0$. In this case, Eq (3.44) reduces to

$$y_1(\vartheta, \varphi) = -\cos\vartheta \sin\vartheta,\ y_2(\vartheta, \varphi) = \cos^2\vartheta, , y_3(\vartheta, \varphi) = -\sin\vartheta \cot\varphi. \tag{3.45}$$

According to Eqs (3.37) and (3.45) we obtain

$$\mathbf{Y}(\vartheta, \varphi, v) = (-\cos\vartheta \sin\vartheta + v\cos\vartheta \sin\varphi, \cos^2\vartheta + v\sin\vartheta \sin\varphi, -\sin\vartheta \cot\varphi + v\cos\varphi).$$

It is very clear that if the functions ϑ, and φ are given, then the following developable can be determined. The developable of the congruence is obtained for $\vartheta(t) = \varphi(t) = t \in \mathbb{R}$, $\vartheta = const$, and $\varphi = const$, see Figure 5.

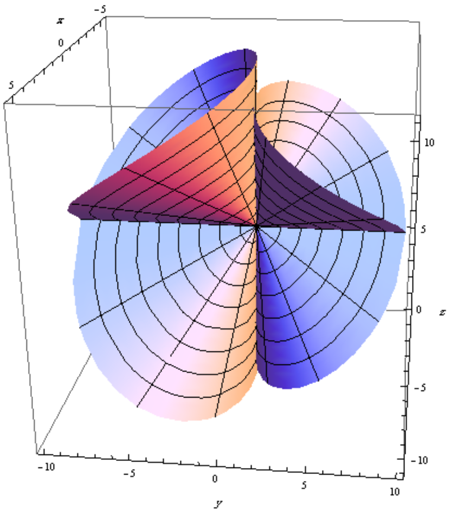

Figure 5. Developable surface of the congruence.

4. Conclusions

This work mainly deals with the dual representation of line congruence and explains the resemblance between theory of surfaces and theory of line congruences. In terms of this, several some new and well-known formulae of line congruence in the Euclidean 3-space have been introduced. Furthermore, we determine kinematic-geometry of the Plucker conoid and its characterization. In addition, the degenerated cases of the Plücker conoid are discussed according to the Dupin's indicatrix having specific trajectories. We hope that this work will lead to a wider usage in the differential line geometry and rational design of space mechanisms.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. L. P. Eisenhart, *A treatise in differential geometry of curves and surfaces*, London and Boston: Ginn, 1909.
2. B. Jüttler, K. Rittenschober, Using line congruences for parametrizing special algebraic surfaces, In: *The mathematics of surfaces*, Lecture Notes in Computer Science, Berlin: Springer, **2768** (2003), 223–243.
3. J. A. Schaaf, B. Ravani, Geometric continuity of ruled surfaces, *Comput. Aided Geom. Des.*, **15** (1998), 289–310.
4. M. D. Shepherd, Line congruences as surfaces in the space of lines, *Differ. Geom. Appl.*, **10** (1999), 1–26.
5. B. Odehnal, H. Pottmann, *Computing with discrete models of ruled surfaces and line congruences*, Proceedings of the 2nd workshop on computational kinematics, Seoul, 2001.
6. H. Pottmann, J. Wallner, *Computational line geometry*, Springer Science & Business Media, 2009.
7. B. Odehnal, *Geometric optimization methods for line congruences*, Ph. D. Thesis, Vienna University of Technology, 2002.
8. O. Bottema, B. Roth, *Theoretical kinematic*, North-Holland Series in Applied Mathematics and Mechanics, North-Holland Publishing Company, 1979.
9. A. Karger, J. Novak, *Space kinematics and Lie groups*, New York: Routledge, 1985.
10. R. A. Abdel-Baky, The relation among Darboux vectors of ruled surfaces in a line congruence, *Riv. Mat. Univ. Parma*, **5** (1997), 201–211.
11. W. Blaschke, *Vorlesungen über differential geometrie I*, Springer-Verlag Berlin Heidelberg, 1945.
12. Y. L. Li, S. Y. Liu, Z. G. Wang, Tangent developables and Darboux developables of framed curves, *Topol. Appl.*, (2020), 107526.
13. Y. L. Li, Z. G. Wang, T. H. Zhao, Geometric algebra of singular ruled surfaces, *Adv. Appl. Clifford Algebras*, **31** (2021), 19.
14. O. Gursoy, The dual angle of pitch of a closed ruled surface, *Mech. Mach. Theory*, **25** (1990), 131–140.
15. R. A. Abdel-Baky, On a line congruence which has the parameter ruled surfaces as principal ruled surfaces, *Appl. Math. Comput.*, **151** (2004), 849–862.
16. R. A. Abdel-Baky, A. J. Al-Bokhary, A new approach for describing instantaneous line congruence, *Arch. Math.*, **44** (2008), 223–236.