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*Research article*

# Inequalities and bounds for the $p$-generalized trigonometric functions

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**Abstract:** In this paper, we mainly show some bounds and inequalities for the $p$-generalized trigonometric functions defined by Richter.

**Keywords:** $p$-generalized trigonometric functions; Turán type inequalities; inequality
**Mathematics Subject Classification:** 33B10

## 1. Introduction

It is well known that the level sets of the functions

$$T_p(x,y) = (|x|^p + |y|^p)^{\frac{1}{p}},\ (x,y) \in \mathbb{R}^2, p > 0$$

can be easily described by the equation $r_p = c$ if one makes use of polar or standard triangle coordinates in the cases $p = 2$ and $p = 1$, respectively. In [22], Richter considered $r_1$ and $r_2$ as special cases of the $p$-generalized radius coordinate $r_p = (|x|^p + |y|^p)^{\frac{1}{p}}, p > 0$ and the functions $\sin\varphi$, $\cos\varphi$ as special cases of certain $p$-generalized trigonometric functions. In fact, Richter dealt with $l_{n,p}$-spherical and simplicial coordinates. Here, we only discussed the $l_{2,p}$-generalized trigonometric functions(or called $p$-generalized trigonometric functions) as a generalization to classical trigonometric functions. Next, we show some definitions and formulas found by Richter. For the details, the reader may refer to references( [22, 23]).

**Definition 1.1.** *The p-generalized sine and cosine values of an angle $\varphi \in (0, 2\pi)$ between the directions of the positive x-axes and the line through the points $(0,0)$ and $(x,y) \in \mathbb{R}^2$ are defined for each $p > 0$ as*

$$\sin_p \varphi = \frac{y}{(|x|^p + |y|^p)^{\frac{1}{p}}}$$

*and*

$$\cos_p \varphi = \frac{x}{(|x|^p + |y|^p)^{\frac{1}{p}}}.$$

Obviously, it holds $\left|\sin_p \varphi\right| \leq 1$, $\left|\cos_p \varphi\right| \leq 1$ and

$$\left|\sin_p \varphi\right|^p + \left|\cos_p \varphi\right|^p = 1.$$

It is obvious to see that for each $p > 0, \varphi \in (0, 2\pi)$,

$$\sin_p \varphi = \frac{\sin \varphi}{N_p\varphi} \tag{1.1}$$

and

$$\cos_p \varphi = \frac{\cos \varphi}{N_p\varphi} \tag{1.2}$$

where $N_p\varphi = (|\sin\varphi|^p + |\cos\varphi|^p)^{\frac{1}{p}}$ . For $\varphi \neq \frac{k\pi}{2}, k = 1, 2, 3$, the first derivatives of $\sin_p$ and $\cos_p$ are

$$\sin'_p \varphi = \cos_p \varphi \frac{|\cos\varphi|^{p-2}}{\left(N_p\varphi\right)^p} \tag{1.3}$$

and

$$\cos'_p \varphi = -\sin_p \varphi \frac{|\sin\varphi|^{p-2}}{\left(N_p\varphi\right)^p}. \tag{1.4}$$

In particular, for $p > 0$ and $x \in \left(0, \frac{\pi}{2}\right)$, we have

$$\sin'_p x = \frac{(\cos x)^{p-1}}{((\sin x)^p + (\cos x)^p)^{\frac{p+1}{p}}}, \tag{1.5}$$

$$\cos'_p x = -\frac{(\sin x)^{p-1}}{((\sin x)^p + (\cos x)^p)^{\frac{p+1}{p}}} \tag{1.6}$$

and

$$\frac{\sin'_p x}{\cos'_p x} = -(\cot x)^{p-1}. \tag{1.7}$$

The generalized tangent function $\tan_p x$ is defined as

$$\tan_p x = \frac{\sin_p x}{\cos_p x}, \quad x \in \mathbb{R} \backslash \left\{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\right\}. \tag{1.8}$$

It is easy to see that $\tan_p x = \tan x$ for any $p > 0$. From 1.8, it follows that

$$\tan'_p x = 1 + |\tan x|^2, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{1.9}$$

It is clear that all these generalized functions coincide with the classical ones when $p = 2$.

In this paper, we mainly show some bounds, classical inequalities and Turán type inequalities to parameters for the $p$-generalized trigonometric functions. This paper presents an attempt to study these functions, although many of the results are not perfect. We hope to provide some basic conclusions and research directions for interested researchers.

## 2. Turán type inequalities

**Lemma 2.1** ( [20, Lemma 1]). *The two variable power mean $M_p(a,b)$ is concave in $p$ for $p \geq 1$ and convex in $p$ for $p \leq -1$. That is, $\frac{\partial^2}{\partial p^2}\left[M_p(a,b)\right] \leq 0$ for $p \geq 1$ and $\frac{\partial^2}{\partial p^2}\left[M_p(a,b)\right] \geq 0$ for $p \leq -1$ with equality if and only if $a = b$.*

**Theorem 2.1.** *For fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $k > 1$, the function $p \mapsto 2^{\frac{k}{p}} \sin_p x$ is log-convex on $p \in [1, \infty)$. In particular, for fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $k > 1$, $p, q \geq 1$, we have*

$$\sin_{\frac{p+q}{2}} x \leq 2^{\frac{k}{2p}+\frac{k}{2q}-\frac{2k}{p+q}} \sin_p^{\frac{1}{2}} x \sin_q^{\frac{1}{2}} x. \tag{2.1}$$

*Proof.* By using the formula 1.1, we easily obtain

$$2^{\frac{k}{p}} \sin_p x = \frac{2^{\frac{k}{p}} \sin x}{(\sin^p x + \cos^p x)^{\frac{1}{p}}} = \frac{2^{\frac{k-1}{p}}}{\left(\frac{1+\cot^p x}{2}\right)^{\frac{1}{p}}} = \frac{2^{\frac{k-1}{p}}}{M_p(1,\lambda)}$$

and

$$\log\left(2^{\frac{k}{p}} \sin_p x\right) = \frac{k-1}{p} \log 2 - \log M_p(1,\lambda)$$

where $\lambda = \cot x$. On the one hand, we have

$$\frac{d}{dp}\left(\frac{k-1}{p} \log 2\right) = -\frac{k-1}{p^2} \log 2$$

and

$$\frac{d^2}{dp^2}\left(\frac{k-1}{p} \log 2\right) = \frac{2(k-1)}{p^3} \log 2 > 0.$$

This implies the function $2^{\frac{k-1}{p}}$ is strictly log-convex for $p \geq 1$ and $k > 1$. On the other hand, simple computation yields

$$\left[M_{pt}(1,\lambda)\right]^t = M_p(1,\lambda^t),$$

and

$$t \log\left(M_{pt}(1,\lambda)\right) = \log\left(M_p(1,\lambda^t)\right). \tag{2.2}$$

Differentiating (2.2) with respect to $p$ while holding $\lambda$ fixed, we get

$$t^2 \left[\log\left(M_{pt}(1,\lambda)\right)\right]' = \left[\log\left(M_p(1,\lambda^t)\right)\right]',$$

and

$$t^3 \left[\log\left(M_{pt}(1,\lambda)\right)\right]'' = \left[\log\left(M_p(1,\lambda^t)\right)\right]''.$$

Furthermore, we have

$$t^3 \left[\log\left(M_t(1,\lambda)\right)\right]'' \leq 0$$

by setting $p = 1$, using Lemma 2.1 and the fact that the concave function meant log-concave function. So it is easy to see that the function $2^{\frac{k}{p}} \sin_p x$ is log-convex on $p \in [1, \infty)$. This completes the proof. □

**Corollary 2.1.** *For fixed $x \in \left(0, \frac{\pi}{2}\right)$, the function $p \mapsto 2^{\frac{1}{p}} \sin_p x$ is log-convex on $p \in [1, \infty)$. In particular, for fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $p, q \geq 1$, we have*

$$\sin_{\frac{p+q}{2}} x \leq 2^{\frac{1}{2p}+\frac{1}{2q}-\frac{2}{p+q}} \sin_p^{\frac{1}{2}} x \sin_q^{\frac{1}{2}} x. \tag{2.3}$$

*Proof.* By taking $k \mapsto 1$, we easily complete the proof. For the sake of simplicity, we omit the details. □

Similar to proof of Theorem 2.1, we easily obtain the following Theorem 2.2.

**Theorem 2.2.** *For fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $k > 1$, the function $p \mapsto 2^{\frac{k}{p}} \cos_p x$ is log-convex on $p \in [1, \infty)$. In particular, for fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $k > 1, p, q \geq 1$, we have*

$$\cos_{\frac{p+q}{2}} x \leq 2^{\frac{k}{2p}+\frac{k}{2q}-\frac{2k}{p+q}} \cos_p^{\frac{1}{2}} x \cos_q^{\frac{1}{2}} x. \tag{2.4}$$

**Corollary 2.2.** *For fixed $x \in \left(0, \frac{\pi}{2}\right)$, the function $p \mapsto 2^{\frac{1}{p}} \cos_p x$ is log-convex on $p \in [1, \infty)$. In particular, for fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $p, q \geq 1$, we have*

$$\cos_{\frac{p+q}{2}} x \leq 2^{\frac{1}{2p}+\frac{1}{2q}-\frac{2}{p+q}} \cos_p^{\frac{1}{2}} x \cos_q^{\frac{1}{2}} x. \tag{2.5}$$

## 3. Bounds for the $p$-generalized trigonometric functions

**Lemma 3.1** ( [18]). *Let $x \in \left(0, \frac{\pi}{2}\right)$ and $n \geq 2$. Then*

$$\sin^n x + \cos^n x \leq 1. \tag{3.1}$$

**Lemma 3.2** ( [18, $c_p$-inequality]). *Let $a, b \in \mathbb{R}$ and $p > 0$. Then*

$$(|a| + |b|)^p \leq c_p \left(|a|^p + |b|^p\right) \tag{3.2}$$

*where* $c_p = \begin{cases} 1 \ , & 0 < p \leq 1, \\ 2^{p-1} \ , & p > 1. \end{cases}$

**Lemma 3.3** ( [18]). *For $x \in (0, 1)$, we have*

$$x < \tan x < \frac{x}{\sqrt{1 - x^2}}.$$

**Theorem 3.1.** *For fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $p \geq 2$, the following inequalities hold true:*

*(i)* $\sin x \leq \sin_p x \leq 2^{\frac{p-1}{p}} \sin_1 x$;

*(ii)* $\cos x \leq \cos_p x \leq 2^{\frac{p-1}{p}} \cos_1 x$.

*Proof.* By using Lemma 3.1, Lemma 3.2 and definitions of the functions $\sin_p x$, $\cos_p x$, we have

$$\sin x \leq \sin_p x \leq \frac{\sin x}{2^{\frac{1-p}{p}}(\sin x + \cos x)} = 2^{\frac{p-1}{p}} \sin_1 x$$

and

$$\cos x \leq \cos_p x \leq \frac{\cos x}{2^{\frac{1-p}{p}}(\sin x + \cos x)} = 2^{\frac{p-1}{p}} \cos_1 x.$$

□

**Theorem 3.2.** *For fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $p \geq 2$, the following inequalities hold true:*
*(i)* $(\cos x)^{p-1} \leq \sin'_p x \leq 2^{\frac{p^2-1}{p}} \frac{(\cos x)^{p-1}}{(\sin x+\cos x)^{p+1}}$;
*(ii)* $-2^{\frac{p^2-1}{p}} \frac{(\sin x)^{p-1}}{(\sin x+\cos x)^{p+1}} \leq \cos'_p x \leq -(\sin x)^{p-1}$.

*Proof.* By applying Lemma 3.1, Lemma 3.2 and the formulas (1.5), (1.6), we easily complete the proof. □

**Theorem 3.3.** *For $x \in \left(0, \frac{\pi}{2}\right)$ and $k, l > 0$, we have*

$$0 \leq \sin_p^k x \cos_p^l x \leq \left(\frac{k^k l^l}{(k+l)^{k+l}}\right)^{\frac{1}{p}}. \tag{3.3}$$

*Proof.* Since the function $\ln x$ is strictly concave on $(0, \infty)$, we take $\alpha = \frac{k}{l+k}, \beta = \frac{l}{l+k}$, $x_1 = \frac{1}{k} \sin_p^p x$ and $x_2 = \frac{1}{l} \cos_p^l x$ in Jessen inequality

$$\alpha \ln x_1 + \beta \ln x_2 \leq \ln(\alpha x_1 + \beta x_2).$$

This implies the inequality (3.3). □

**Theorem 3.4.** *For $x \in \left(0, \frac{\pi}{2}\right)$, the following inequalities hold true:*
*(i)* $\left(\sin_p x\right)^{\cos_p x} \leq \left(\cos_p x\right)^{\sin_p x}$, *for* $x \in \left(0, \frac{\pi}{4}\right)$;
*(ii)* $\left(\cos_p x\right)^{\sin_p x} \leq \left(\sin_p x\right)^{\cos_p x}$, *for* $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$;
*(iii)* $\left(\sin_p x\right)^{\sin_p x} \leq \left(\cos_p x\right)^{\cos_p x}$, *for* $x \in \left(0, \frac{\pi}{4}\right)$.

*Proof.* Considering to $\frac{\ln x}{x}$ is strictly increasing on $(0, \infty)$ and the inequalities

$$x \in \left(0, \frac{\pi}{4}\right), \sin_p x \leq \cos_p x,$$

$$x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right), \cos_p x \leq \sin_p x,$$

we easily obtain proofs of (i) and (ii).

Due to (iii), the formula $(x \ln x)' = 1 + \ln x > 0$ for $x \in \left(0, \frac{\sqrt{2}}{2}\right)$ implies that the function $x \ln x$ is strictly increasing. So, we have

$$\sin_p x \ln(\sin_p x) \leq \cos_p x \ln(\cos_p x).$$

The proof is complete. □

**Theorem 3.5.** *For fixed $x \in \left(0, \frac{\pi}{2}\right)$ and $p \geq 0$, we have*

$$\left(\sin_p x\right)^{p \sin_p^p x} + \left(\cos_p x\right)^{p \cos_p^p x} \geq \sqrt{2}. \tag{3.4}$$

*Proof.* The convexity of function $x^x$ implies the above inequality. we omit the detail for the sake of simplicity. □

**Theorem 3.6.** *For fixed $x \in (0,1)$ and $p \geq 0$, we have*

$$\frac{\sqrt{1-x^2}}{\left[x^p + \left(\sqrt{1-x^2}\right)^p\right]^{\frac{1}{p}}} < \cos_p x < \frac{1}{(1+x^p)^{\frac{1}{p}}}. \tag{3.5}$$

*Proof.* By using Lemma 3.3 and the formula $\frac{1}{\cos_p^p x} - 1 = \tan^p x$, we easily complete the proof. □

## 4. Classical inequalities

**Lemma 4.1** ( [4, Lemma 3, p246]). *Let us consider the function $f : (a, \infty) \to \mathbb{R}$, where $a \geqslant 0$. If the function $g$, defined by $g(x) = \frac{1}{x}(f(x) - 1)$, is increasing on $(a, \infty)$, then for the function $h$, defined by $h(x) = f(x^2)$, we have the following Grünbaum type inequality*

$$1 + h(z) \geqslant h(x) + h(y), \tag{4.1}$$

*where $x, y \geqslant a$ and $z^2 = x^2 + y^2$. If the function $g$ is decreasing, then the inequality (2.1) is reversed.*

**Theorem 4.1.** *For fixed $x, y, z \in \left(0, \frac{\pi}{2}\right)$ and $x^2 + y^2 = z^2$, we have*

$$z^2 \sin_p\left(z^2\right) \geq x^2 \sin_p\left(x^2\right) + y^2 \sin_p\left(y^2\right) \tag{4.2}$$

*and*

$$z^2 \cos_p\left(z^2\right) \geq x^2 \cos_p\left(x^2\right) + y^2 \cos_p\left(y^2\right). \tag{4.3}$$

*Proof.* We only prove the inequality (4.2). In fact, putting $f(x) = x\sin_p(x) + 1$, we get the function $g(x) = \frac{f(x)-1}{x} = \sin_p(x)$ is increasing on $\left(0, \frac{\pi}{2}\right)$. by using Lemma (4.1), we easily obtain inequality (4.2). Similarly, the inequality (4.3) follows from the Lemma (4.1).

□

**Lemma 4.2** ( [21, Mitrinović-Adamović inequality]). *Let $x \in \left(0, \frac{\pi}{2}\right)$. Then*

$$\left(\frac{\sin x}{x}\right)^3 \geq \cos x.$$

**Theorem 4.2.** *For $p > 0$ and $x \in (0, \frac{\pi}{2})$, we have*

$$\frac{\sin_p^3 x}{x^3} \geq \frac{\cos_p^3 x}{\cos^2 x}. \tag{4.4}$$

*Proof.* Using the definition of $\sin_p x, \cos_p x$ and Lemma 4.1, we have

$$\begin{aligned}\frac{\sin_p^3 x}{x^3} &= \frac{\sin^3 x}{x^3(\sin^p x + \cos^p x)^{\frac{3}{p}}} \\ &\geq \left[\frac{\cos x}{(\sin^p x + \cos^p x)^{\frac{1}{p}}}\right]^3 \frac{1}{\cos^2 x} = \frac{\cos_p^3 x}{\cos^2 x}.\end{aligned}$$

This completes the proof. □

The above inequality in the Theorem 4.2 is the so called Mitrinović-Adamović inequality. The next theorem 4.3 shows the famous Huygens-type inequality for the $p$-generalized trigonometric functions.

**Theorem 4.3.** *For $p > 0$ and $x \in (0, \frac{\pi}{2})$, we have*

$$\frac{3 \sin_p x}{x} + \frac{\cos^2 x}{\cos_p^3 x} > 4. \tag{4.5}$$

*Proof.* The weighted AG inequality shows that

$$ta + (1-t)b > a^t b^{1-t}$$

for $a > 0, b > 0$ and $0 < t < 1$. Taking $t = \frac{3}{4}, a = \frac{\sin_p x}{x}$ and $b = \frac{\cos^2 x}{\cos_p^3 x}$ and applying Theorem 4.3, we get

$$\frac{3}{4}\frac{\sin_p x}{x} + \frac{1}{4}\frac{\cos^2 x}{\cos_p^3 x} > \left(\frac{\sin_p x}{x}\right)^{\frac{3}{4}} \left(\frac{\cos^2 x}{\cos_p^3 x}\right)^{\frac{1}{4}} \geq 1.$$

So, we complete the proof. $\square$

**Theorem 4.4.** *For $t, p > 0$ and $x \in (0, \frac{\pi}{2})$, the following inequalities hold true:*
*(i)* $\left[1 + \left(\frac{\sin_p x}{x}\right)^{2t}\right]\left[1 + \left(\frac{\tan_p x}{x}\right)^{t}\right] > 4;$
*(ii)* $\left(\frac{\sin_p x}{x}\right)^{2t} + \left(\frac{\tan_p x}{x}\right)^{t} \geq 2\sqrt{\left[1 + \left(\frac{\sin_p x}{x}\right)^{2t}\right]\left[1 + \left(\frac{\tan_p x}{x}\right)^{t}\right]} - 2 > 2;$
*(iii)* $\left[1 + \left(\frac{\sin_p x}{x}\right)^{t}\right]^2\left[1 + \left(\frac{\tan_p x}{x}\right)^{t}\right] > 8;$
*(iv)* $2\left(\frac{\sin_p x}{x}\right)^{t} + \left(\frac{\tan_p x}{x}\right)^{t} \geq 3\sqrt[3]{\left[1 + \left(\frac{\sin_p x}{x}\right)^{t}\right]\left[1 + \left(\frac{\tan_p x}{x}\right)^{t}\right]} - 3 > 3.$

*Proof.* By using the known inequality (See the reference [24])

$$a + b \geq 2\sqrt{(1+a)(1+b)} - 2 > 2\sqrt{ab}$$

and taking $a = \left(\frac{\sin_p x}{x}\right)^{2t}, b = \left(\frac{\tan_p x}{x}\right)^{t}$ and applying Theorem 3.1, Theorem 4.2, we can obtain (i) and (ii).

For (iii) and (iv), by using formula (See the reference [24])

$$2a + b \geq 3\sqrt[3]{(1+a)^2(1+b)} - 3 > 3\sqrt[3]{a^2 b}$$

and putting $a = \left(\frac{\sin_p x}{x}\right)^{2t}, b = \left(\frac{\tan_p x}{x}\right)^{t}$, we easily obtain the expected results (iii) and (iv). $\square$

*Remark* 4.1. Inequality (i) of Theorem 4.4 could be called multiplicative Wilker inequality, and likewise inequality (iii) of Theorem 4.4 as a multiplicative Huygens inequality.

## 5. Comments and open problems

The another generalized trigonometric and hyperbolic functions depending on a parameter $p > 1$ were studied by P. Lindqvist in a highly cited paper (see [19]). Motivated by this work, many authors have studied the equalities and inequalities related to generalized trigonometric and hyperbolic functions in [5, 6, 10]. Recently, S. Takeuchi [25] has investigated the $(p, q)$-trigonometric functions depending on two parameters and in which the case of $p = q$ coincides with the $p$-function of Lindqvist, and for $p = q = 2$ they coincide with familiar elementary functions. These functions are differently defined from the above $p$-trigonometric functions , they are all defined by integration. These functions have been thoroughly studied, such as the multiple angle formula, classical inequalities, monotonicity of parameters, relations with hypergeometric functions, and generalized elliptic integrals defined by these functions. The reader may refer references [1–3, 11–13, 16, 17, 26–36]. As an example, we introduce generalized trigonometric function with one parameter to illustrate the difference between them.

For $1 < q < \infty$ and $0 \leq x \leq 1$, the arcsine may be generalized as

$$\arcsin_q x = \int_0^x \frac{1}{(1-t^q)^{1/q}} dt$$

and

$$\frac{\pi_q}{2} = \arcsin_q 1 = \int_0^1 \frac{1}{(1-t^q)^{1/q}} dt.$$

The inverse of $\arcsin_q$ on $[0, \frac{\pi_q}{2}]$ is called the generalized sine function, denoted by $\sin_q$ and may be extended to $(-\infty, \infty)$. See [7, 10] and closely related references therein.

For $x \in [0, \frac{\pi_q}{2}]$, the generalized cosine function $\cos_q x$ is defined by

$$\cos_q x = \frac{\mathrm{d} \sin_q x}{\mathrm{d}x}.$$

It is easy to see that

$$\cos_q x = (1 - \sin_q^q x)^{1/q}$$

and

$$\frac{\mathrm{d} \cos_q x}{\mathrm{d}x} = -\cos_q^{2-q} x \sin_q^{q-1} x.$$

Very naturally, we can define the following generalized elliptic integrals. We repeat the definition of complete $q-$elliptic integrals of the first kind $K_q(k)$ and of the second kind $E_q(k)$: for $k \in (0, 1)$

$$K_q(k) := \int_0^{\frac{\pi_q}{2}} \frac{d\theta}{(1 - k^q \sin_q^q \theta)^{1-\frac{1}{q}}} = \int_0^1 \frac{dt}{(1-t^q)^{\frac{1}{q}}(1-k^q t^q)^{1-\frac{1}{q}}},$$

$$E_q(k) := \int_0^{\frac{\pi_q}{2}} (1 - k^q \sin_q^q \theta)^{\frac{1}{q}} d\theta = \int_0^1 \left(\frac{1-k^q t^q}{1-t^q}\right)^{\frac{1}{q}} dt.$$

Takeuchi made a detailed, in-depth study of integrals of this kind. He showed Legendre's relation for $K_q(k)$ and $E_q(k)$ and established relationship between the complete p-elliptic integrals and the Gaussian hyperbolic functions. As applications of complete $p$-elliptic, he also gave a computation formula of $\pi_q$ with $q = 3$ and an elementary proof of Ramanujan's cubic transformation. The reader can see [28] and closely related references therein.

*Remark* 5.1. Since the functions of the inverse of $\arcsin_q$ and two complete $q$-elliptic integrals are represented by rather some complicated integrals, they can be approximated by some numerical quadrature techniques such as [8,9]. Moreover, they can be bounded and controlled by some improved and generalized inequalities such as [14,15].

In contrast to Lindqvist's trigonometric functions and Takeuchi's generalized elliptic integrals, similar elliptic integrals can be defined using the definition of $p$-trigonometric functions. But unfortunately, this is a very difficult subject to study. Here are some questions for further study.

**Open Problem 5.1.** *(i) Discuss the complete monotonicity, concavity or convexity of these functions* $\sin_p x, \cos_p x$ *and their inverse functions;*
*(ii) Discuss the complete monotonicity of these functions* $\sin_p x, \cos_p x$ *and their inverse to parameter* $p$*;*
*(iii) Establish some classical inequalities of these functions* $\sin_p x, \cos_p x$ *and their inverse, such as Wilker type inequality, Cusa-Huygens type inequality, Kober inequality, Lazerević inequality et. al.;*
*(iv) Establish series representations, integral representations of* $\sin_p x, \cos_p x$ *and their inverse to parameter* $p$*;*
*(v) Establish representations of hypergeometric function of* $\sin_p x, \cos_p x$ *and their inverse functions to parameter* $p$*.*

By using the definition of $p$-trigonometric functions, we may define similar $p$-elliptic integrals $K_p(k)$ and $E_p(k)$. For these kinds of integral, we pose the following interesting question.

**Open Problem 5.2.** *(i) Study Legendre identity of p-elliptic integrals* $K_p(k)$ *and* $E_p(k)$*;*
*(ii) Study multiple angle formula of p-elliptic integrals* $K_p(k)$ *and* $E_p(k)$*;*
*(iii) Establish the complete monotonicity, concavity or convexity of p-elliptic integrals* $K_p(k)$ *and* $E_p(k)$ *to parameter* $p$*;*
*(iv) Establish representations of hypergeometric function of p-elliptic integrals* $K_p(k)$ *and* $E_p(k)$*.*

## 6. Conclusions

In this work, we study the $p$-generalized trigonometric functions defined by Richter. Some new Turán type inequalities and classical inequalities such as Mitrinović-Adamović type inequality and Huygens-type inequality are obtained. Further, we establish some bounds for the $p$-generalized trigonometric functions. Finally, several comments and open problems are posed.

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## Conflict of interest

The authors declare no conflict of interest.

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