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Research article

The law of iterated logarithm for a class of random variables satisfying Rosenthal type inequality

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Abstract: Let $\{Y_n, n \geq 1\}$ be sequence of random variables with $EY_n = 0$ and $\sup_n E|Y_n|^p < \infty$ for each $p > 2$ satisfying Rosenthal type inequality. In this paper, the law of the iterated logarithm for a class of random variable sequence with non-identical distributions is established by the Rosenthal type inequality and Berry-Esseen bounds. The results extend the known ones from i.i.d and NA cases to a class of random variable satisfying Rosenthal type inequality.

Keywords: law of the iterated logarithm; Rosenthal type inequality; Berry-Esseen bounds
Mathematics Subject Classification: 60F05, 60F15

1. Introduction and main results

We first introduce the definition of the Rosenthal type maximal inequality, which is one of the most interesting inequalities in probability theory and mathematical statistics. Suppose that $\{Y_n, n \geq 1\}$ is a sequence of random variables satisfying $E|Y_i|^r < \infty$ for $r \geq 2$, then there exists a positive constant $C(r)$ depending only on r such that

$$\begin{aligned} & E\max_{1\leq j\leq n}|\sum_{k=1}^{j}(Y_k - EY_k)|^r \\ \leq\ & C(r)[\sum_{k=1}^{n}E|Y_k - EY_k|^r + (\sum_{k=1}^{n}E|Y_k - EY_k|^2)^{r/2}] \\ \leq\ & 2C(r)n^{r/2}\sup_n E|Y_n - EY_n|^r. \end{aligned} \tag{1.1}$$

(1.1) can be satisfied by many dependent or mixing sequences. Peligrad [1], Zhou [2], Wang and Lu [3], Utev and Peligrad [4] established the above inequality for $\rho-$mixing sequence, $\varphi-$mixing sequence, $\rho^{-}-$mixing sequence and $\widetilde{\rho}-$mixing sequence, respectively. We also refer to Shao [5], Stoica [6], Shen [7], Yuan and An [8], Shen et al. [9] and Merlevéde and Peligrad [10] for negatively associated sequence (NA), martingale difference sequence, extend negatively dependent sequence (END), asymptotically almost negatively associated random sequence (AANA), negatively superadditive dependent (NSD), stationary processes, respectively.

The law of iterated logarithm (LIL, for short) is an important aspect in probability theory because it can describe the precise convergence rates. Petrov [11] established the following LIL for independent random variables.

Theorem A. Let $\{X_n, n \geq 1\}$ be independent random variables sequences with $EX_n = 0, \sigma_n^2 = EX_n^2 < \infty$, $B'_n = \sum_{k=1}^n \sigma_k^2$, $S_n = \sum_{k=1}^n X_k$. If the following assumptions are satisfied:

(i) $B'_n \to \infty$, when $n \to \infty$,

(ii) $B'_{n+1}/B'_n \to 1$, when $n \to \infty$,

(iii) $\Delta_n = \sup_x |P(S_n < x\sqrt{B'_n}) - \Phi(x)| = O[(\log B'_n)^{-1-\delta}]$, $\delta > 0$, here and in the sequel $\Phi(\cdot)$ is a standard normal distribution function, hold.

Then

$$\limsup_{n\to\infty} \frac{S_n}{(2B'_n \log\log B'_n)^{1/2}} = 1 \quad a.s.$$

Later on, Cai and Wu [12] established the following LIL for NA random variables.

Theorem B. Let $\{X_n, n \geq 1\}$ be NA random variables sequence with $EX_n = 0$ and $\sup_n EX_n^2(\log|X_n|)^{1+\delta} < \infty$ for some $\delta > 0$. Let $S_n = \sum_{k=1}^n X_k$, $B_n = Var(S_n) > 0$, $B'_n = \sum_{k=1}^n EX_k^2$. $\Delta_n = \sup_x |P(S_n < x\sqrt{B'_n}) - \Phi(x)|$. If

(i) $B_n = O(n)$,

(ii) $B_{n+1}/B_n \to 1$, when $n \to \infty$,

(iii) $\Delta_n = O[(\log B_n)^{-1}]$,

(iv) $B_n/B'_n \to 1$, when $n \to \infty$,

hold, then

$$\limsup_{n\to\infty} \frac{S_n}{(2B_n \log\log B_n)^{1/2}} = 1 \quad a.s.$$

In this paper, the main purpose is to establish the law of the iterated logarithm for a class of random variables satisfying Rosenthal type maximal inequality with non-identical distributions. The following is the main result.

Theorem 1.1. *Let $\{Y_n, n \geq 1\}$ be a sequence of random variables with $EY_n = 0$ and $\sup_n E|Y_n|^p < \infty$ for each $p > 2$ satisfying (1.1), denote $S_n = \sum_{k=1}^n Y_k$, $B_n = Var(S_n) > 0$. If*

(i) $B_n = O(n)$, $B_{n+1}/B_n \to 1$, when $n \to \infty$,

(ii) $\Delta_{n,m} = \sup_x |P(S_{n+m} - S_m < x\sqrt{B_{n+m} - B_m}) - \Phi(x)| = O[(\log(B_{n+m} - B_m))^{-1-\delta}]$ for some $\delta > 0$ and any $m \geq 0$, $n \geq 1$, where $S_0 = 0$, $B_0 = 0$,

hold, then

$$\limsup_{n\to\infty} \frac{S_n}{(2B_n \log\log B_n)^{1/2}} = 1 \quad a.s. \tag{1.2}$$

Corollary 1.2. *Let $\{Y_n, n \geq 1\}$ be a strictly stationary sequence of random variables with $EY_1 = 0$ and $E|Y_1|^p < \infty$ for each $p > 2$ satisfying (1.1), denote $S_n = \sum_{k=1}^n Y_k$, $B_n = Var(S_n) > 0$. If* *(i)* $0 < \sigma^2 =: EY_1^2 + \sum_{k=1}^{\infty} EY_1Y_{1+k} < \infty$,

(ii) $\Delta_n = \sup_x |P(S_n < x\sqrt{B_n}) - \Phi(x)| = O[(\log B_n)^{-1-\delta}]$ *for some* $\delta > 0$,

hold, then

$$\limsup_{n\to\infty} \frac{S_n}{(2n\sigma^2 \log\log n)^{1/2}} = 1 \quad a.s. \tag{1.3}$$

Remark 1.3. *The assumption (ii) in Theorem 1.1, that is the Berry-Esseen bounds, can be satisfied by many sequence, such as independent sequence, NA sequence with convergence rate* $n^{-\alpha}$, $0 < \alpha \le 1/2$.

Throughout the sequel, C represents a positive constant although its value may change from one appearance to the next, $I\{A\}$ denotes the indicator function of the set A, $[x]$ denotes the integer part of x, $\log x = \ln\max\{e, x\}$.

2. Proof

Some lemmas which will be useful to prove the main results are given firstly.

Lemma 2.1. *(Wittmann [13]) Let* $\{a_n\}$ *be a sequence of strictly positive real numbers with* $\lim_{n\to\infty} a_n = \infty$. *Then for any* $M > 1$, *there exists a subsequence* $\{n_k, k \ge 1\} \in \mathbb{N} = \{1, 2, \cdots\}$, *such that*

$$Ma_{n_k} \le a_{n_{k+1}} \le M^3 a_{n_k+1}.$$

Lemma 2.2. *Under the assumptions of Theorem 1.1, let* $\{g(n)\}$ *be a nondecreasing sequence of positive numbers and* $\{n_k\}$ *be a nondecreasing sequence of positive integers such that* $\sum_{k=1}^{\infty} \frac{1}{(\log n_k)^{1+\delta}} < \infty$. *Then the following statements are equivalent*

(A) $\sum_{k=1}^{\infty} P(S_{n_k} > g(n_k)\sqrt{B_{n_k}}) < \infty$,

(B) $\sum_{k=1}^{\infty} \frac{1}{g(n_k)} \exp\{-\frac{1}{2}g^2(n_k)\} < \infty$.

Proof. Noting $B_n = O(n)$,

$$\sum_{k=1}^{\infty} \Delta_{n_k,0} \le \sum_{k=1}^{\infty} \frac{C}{(\log B_{n_k})^{1+\delta}} \le \sum_{k=1}^{\infty} \frac{C}{(\log n_k)^{1+\delta}} < \infty. \tag{2.1}$$

Thanks to (2.1), condition (A) is equivalent to

$$\sum_{k=1}^{\infty} (1 - \Phi(g(n_k))) < \infty.$$

If $g(n_k) \nrightarrow \infty$, it is easy to see that conditions (A) and (B) can not be satisfied. So we can assume that $g(n_k) \to \infty$, then noting that $\frac{1}{x}\varphi(x) \sim 1 - \Phi(x)$ for x large enough, where $\varphi(x)$ is the density function of the standard normal, one gets

$$\sum_{k=1}^{\infty} (1 - \Phi(g(n_k))) < \infty \Longleftrightarrow \sum_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}g(n_k)} \exp\{-\frac{1}{2}g^2(n_k)\} < \infty.$$

Thus, the proof of this lemma is completed. □

Lemma 2.3. *Under the conditions of Theorem 1.1, one gets*

$$\limsup_{n\to\infty} \frac{|S_n|}{(2B_n \log\log B_n)^{1/2}} \le 1 \quad a.s. \tag{2.2}$$

Proof. For any $0 < \varepsilon < 1/3$, let $n_k = [e^{k^\alpha}]$, $k \ge 1$ with $\max\{\frac{1}{(1+\varepsilon)^2}, \frac{1}{1+\delta}\} < \alpha < 1$ and $g(n_k) = (1+\varepsilon)(2\log\log B_{n_k})^{1/2}$. Noting that $B_n = O(n)$, one can get

$$\sum_{k=1}^{\infty} \Delta_{n_k,0} \le \sum_{k=1}^{\infty} \frac{C}{(\log n_k)^{1+\delta}} \le \sum_{k=1}^{\infty} \frac{C}{k^{\alpha(1+\delta)}} < \infty,$$

and

$$\begin{aligned}
& \sum_{k=1}^{\infty} \frac{1}{g(n_k)} \exp\{-\frac{1}{2} g(n_k)\} \\
= & \sum_{k=1}^{\infty} \frac{1}{(1+\varepsilon)(2\log\log B_{n_k})^{1/2}} \exp\{-\frac{1}{2}(1+\varepsilon)^2 (2\log\log B_{n_k})\} \\
\le & \sum_{k=1}^{\infty} \frac{C}{(\log B_{n_k})^{(1+\varepsilon)^2} (\log\log B_{n_k})^{1/2}} \\
\le & \sum_{k=1}^{\infty} \frac{C}{k^{\alpha(1+\varepsilon)^2} (\log k^\alpha)^{1/2}} < \infty.
\end{aligned}$$

Then by Lemma 2.2, one can obtain

$$\sum_{k=1}^{\infty} P(S_{n_k} > (1+\varepsilon)(2B_{n_k} \log\log B_{n_k})^{1/2}) < \infty.$$

By Broel-Cantelli lemma and the arbitrariness of ε, we have

$$\limsup_{k\to\infty} \frac{|S_{n_k}|}{(2B_{n_k} \log\log B_{n_k})^{1/2}} = \lim_{\varepsilon \searrow 0} \limsup_{k\to\infty} \frac{|S_{n_k}|}{(2B_{n_k} \log\log B_{n_k})^{1/2}} \le 1 \quad a.s. \tag{2.3}$$

For given α, choose $p > 2$ such that $p(1-\alpha) > 2$, then by (1.1), $B_n = O(n)$ and $\sup_n E|Y_n|^p < \infty$, we get

$$\begin{aligned}
& \sum_{k=1}^{\infty} P(\max_{n_k \le n < n_{k+1}} |S_n - S_{n_k}| > \varepsilon (2B_{n_k} \log\log B_{n_k})^{1/2}) \\
\le & \sum_{k=1}^{\infty} \frac{E \max_{n_k \le n < n_{k+1}} |S_n - S_{n_k}|^p}{(\varepsilon)^p (2B_{n_k} \log\log B_{n_k})^{p/2}} \\
\le & \sum_{k=1}^{\infty} C \frac{(n_{k+1} - n_k)^{p/2}}{(n_k \log\log n_k)^{p/2}} \\
\le & \sum_{k=1}^{\infty} C \frac{1}{k^{p(1-\alpha)/2} (\log k^\alpha)^{p/2}} < \infty
\end{aligned}$$

Thus by Broel-Cantelli lemma and the arbitrariness of ε, one has

$$\limsup_{k\to\infty}\frac{\max_{n_k\le n<n_{k+1}}|S_n-S_{n_k}|}{(2B_{n_k}\log\log B_{n_k})^{1/2}}=0\quad a.s. \tag{2.4}$$

Thanks to (2.3) and (2.4) and $B_{n_{k+1}}/B_{n_k}\to 1$, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{|S_n|}{(2B_n\log\log B_n)^{1/2}}\le\limsup_{k\to\infty}\frac{\max_{n_k\le n<n_{k+1}}|S_n|}{(2B_{n_k}\log\log B_{n_k})^{1/2}}\\
\le\ &\limsup_{k\to\infty}\frac{|S_{n_k}|}{(2B_{n_k}\log\log B_{n_k})^{1/2}}+\limsup_{k\to\infty}\frac{\max_{n_k\le n<n_{k+1}}|S_n-S_{n_k}|}{(2B_{n_k}\log\log B_{n_k})^{1/2}}\\
\le\ &1,
\end{aligned}$$

thus, the proof of Lemma 2.3 is completed. □

Lemma 2.4. *Under the conditions of Theorem 1.1, one gets*

$$\limsup_{n\to\infty}\frac{|S_n|}{(2B_n\log\log B_n)^{1/2}}\ge 1\quad a.s. \tag{2.5}$$

Proof. Noting that $B_n=O(n)$, $B_{n+1}/B_n\to 1$, by Lemma 2.1, for any $\tau>0$, there exists a nondecreasing sequence of positive integers $\{n'_k, k\ge 1\}$, such that for $k\to\infty$, we have

$$n'_k\to\infty,\ \ and\ \ B_{n'_{k-1}}\le(1+\tau)^k<B_{n'_k},\ \ k=1,2,\cdots \tag{2.6}$$

Let

$$\chi(n'_k)=(2B_{n'_k}\log\log B_{n'_k})^{1/2}\ and\ \ \psi(n'_k)=(2(B_{n'_k}-B_{n'_{k-1}})\log\log(B_{n'_k}-B_{n'_{k-1}}))^{1/2}.$$

From (2.6), it is easy to check that

$$(1-\theta)\psi(n'_k)-2\chi(n'_{k-1})\sim[(1-\theta)\tau^{1/2}(1+\tau)^{-1/2}-2(1+\tau)^{-1/2}]\chi(n'_k),\ \ k\to\infty, \tag{2.7}$$

For given $0<\varepsilon<1$, one can choose $0<\theta<1$ and $\tau>0$, such that

$$(1-\theta)\tau^{1/2}(1+\tau)^{-1/2}-2(1+\tau)^{-1/2}>1-\varepsilon.$$

Let $g(n'_k)=(1+\varepsilon)(2\log\log B_{n'_k})^{1/2}$. Noting that $B_n=O(n)$, one can get

$$\sum_{k=1}^{\infty}\Delta_{n'_k,0}\le\sum_{k=1}^{\infty}\frac{C}{(\log n'_k)^{1+\delta}}\le\sum_{k=1}^{\infty}\frac{C}{k^{1+\delta}}<\infty,$$

and

$$\sum_{k=1}^{\infty}\frac{1}{g(n'_k)}\exp\{-\frac{1}{2}g(n'_k)\}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{1}{(1+\varepsilon)(2\log\log B_{n'_k})^{1/2}} \exp\{-\frac{1}{2}(1+\varepsilon)^2(2\log\log B_{n'_k})\} \\
&\le \sum_{k=1}^{\infty} \frac{C}{(\log B_{n'_k})^{(1+\varepsilon)^2}(\log\log B_{n'_k})^{1/2}} \\
&\le \sum_{k=1}^{\infty} \frac{C}{k^{(1+\varepsilon)^2}(\log k)^{1/2}} < \infty.
\end{aligned}
$$

Then by Lemma 2.2, one can obtain

$$\sum_{k=1}^{\infty} P(S_{n'_k} > (1+\varepsilon)(2B_{n'_k}\log\log B_{n'_k})^{1/2}) < \infty.$$

By Broel-Cantelli lemma and $0 < \varepsilon < 1$, we have

$$|S_{n'_{k-1}}| \le 2(2B_{n'_{k-1}}\log\log B_{n'_{k-1}})^{1/2} = 2\chi(n'_{k-1}) \quad a.s. \tag{2.8}$$

In order to prove (2.5), it is sufficient to show that

$$\limsup_{k\to\infty} \frac{|S_{n'_k}|}{(2B_{n'_k}\log\log B_{n'_k})^{1/2}} \ge 1 \quad a.s. \tag{2.9}$$

Noting $(1-\theta)\tau^{1/2}(1+\tau)^{-1/2} - 2(1+\tau)^{-1/2} > 1-\varepsilon$, then by (2.6) and (2.8) and $P(AB) \ge P(A) - P(\overline{B})$, it is easy to prove

$$
\begin{aligned}
&P(S_{n'_k} > (1-\varepsilon)\chi(n'_k)\ i.o.) \ge P(S_{n'_k} > (1-\theta)\psi(n'_k) - 2\chi(n'_{k-1})\ i.o.) \\
\ge\ & P(S_{n'_k} - S_{n'_{k-1}} > (1-\theta)\psi(n'_k)\ i.o.) - P(|S_{n'_{k-1}}| \ge 2\chi(n'_{k-1})\ i.o.) \\
=\ & P(S_{n'_k} - S_{n'_{k-1}} > (1-\theta)\psi(n'_k)\ i.o.).
\end{aligned} \tag{2.10}
$$

Thus by (2.10), in order to prove (2.9), it suffices to prove

$$P(S_{n'_k} - S_{n'_{k-1}} > (1-\theta)\psi(n'_k)\ \ i.o.) = 1. \tag{2.11}$$

Noting $\Delta_{n,m} = \sup_x |P(S_{n+m} - S_m < x\sqrt{B_{n+m} - B_m}) - \Phi(x)| = O[(\log(B_{n+m} - B_m))^{-1-\delta}]$ and $\frac{1}{x}\varphi(x) \le 1 - \Phi(x)$ for $x \ge 1$, where $\varphi(x)$ is the density function of the standard normal random variables, recall $\psi(n'_k) = (2(B_{n'_k} - B_{n'_{k-1}})\log\log(B_{n'_k} - B_{n'_{k-1}}))^{1/2}$ and $B_{n'_k} \sim (1+\tau)^k$, one can deduce

$$
\begin{aligned}
&\sum_{k=1}^{\infty} P(S_{n'_k} - S_{n'_{k-1}} > (1-\theta)\psi(n'_k)) \\
\ge\ & \sum_{k=1}^{\infty} [1 - \Phi((1-\theta)(2\log\log(B_{n'_k} - B_{n'_{k-1}}))^{1/2}) - \Delta_{n'_k - n'_{k-1}, n'_{k-1}}] \\
\ge\ & \sum_{k=1}^{\infty} [\frac{1}{\sqrt{2\pi}(1-\theta)(2\log\log(B_{n'_k} - B_{n'_{k-1}}))^{1/2}} \cdot e^{-\frac{(1-\theta)^2(2\log\log(B_{n'_k} - B_{n'_{k-1}}))}{2}} - \frac{C}{(\log(B_{n'_k} - B_{n'_{k-1}}))^{1+\delta}}]
\end{aligned}
$$

$$\geq \sum_{k=1}^{\infty} \frac{C}{(\log(B_{n'_k} - B_{n'_{k-1}}))^{(1-\theta)^2}(\log\log(B_{n'_k} - B_{n'_{k-1}}))^{1/2}}$$
$$\geq \sum_{k=1}^{\infty} \frac{C}{k^{(1-\theta)^2}(\log k)^{1/2}} = \infty. \tag{2.12}$$

Hence, by the generalized Borel-Cantelli lemma (see, e.g., Kochen and Stone [14]), (2.12) yields (2.11), the proof is completed. □

Proof of Theorem 1.1. Theorem 1.1 can be obtained by combining Lemma 2.3 with Lemma 2.4 directly.

Proof of Corollary 1.2. By the strictly stationary and $0 < \sigma^2 =: EY_1^2 + \sum_{k=1}^{\infty} EY_1Y_{1+k} < \infty$, it is easy to see

$$\lim_{n\to\infty} \frac{B_n}{n} = \lim_{n\to\infty} \frac{Var(S_n)}{n} = \sigma^2.$$

Then Corollary 1.2 follows from Theorem 1.1.

3. Conclusions

In this paper, using the Rosenthal type maximal inequality and Berry-Esseen bounds, the law of the iterated logarithm for a class of random variables is established, this extends the results of Cai and Wu [12] from NA case to general case, because that END and NSD random variables are much weaker than independent random variables and NA random variables thanking to Shen [7] and Shen et al. [9] for details.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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