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Research article

Implicative ideals of BCK-algebras based on MBJ-neutrosophic sets

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Abstract: The MBJ-neutrosophic set is applied to the implicative ideal of BCK-algebra to introduce the concept of implicative MBJ-neutrosophic ideal. Several properties are investigated. The relationship between implicative MBJ-neutrosophic ideal and each MBJ-neutrosophic subalgebra, (positive implicative, commutative) MBJ-neutrosophic ideal is established. Conditions for MBJ-neutrosophic subalgebra (resp., MBJ-neutrosophic ideal, positive implicative MBJ-neutrosophic ideal and commutative MBJ-neutrosophic ideal) to be implicative MBJ-neutrosophic ideal are provided. Characterizations of implicative MBJ-neutrosophic ideal are discussed.

Keywords: MBJ-neutrosophic set; MBJ-neutrosophic ideal; commutative MBJ-neutrosophic ideal; implicative MBJ-neutrosophic ideal; positive implicative MBJ-neutrosophic ideal; MBJ-neutrosophic subalgebra
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1. Introduction

A fuzzy set, which is introduced by Zadeh [1], is a nice tool to deal with uncertainties in several real applications. The notion of neutrosophic set, which is initiated by Smarandache ([2–4]), is a nice platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. So neutrosophic set is being applied in various fields. Neutrosophic set is composed of three types of fuzzy sets called truth membership function, indeterminate membership function and false membership function. Neutrosophic sets are well known for their wide application in various fields. Of course, it is also actively applied to BCK/BCI-algebras (see [5–7]). Mohseni Takallo et al. [8] used interval-valued fuzzy set, which is called indeterminate interval-valued membership function, instead of indeterminate membership function in introducing MBJ-neutrosophic set, and we can see that it applies to BCK/BCI-algebra, equality algebra, hyper BCK-algebra, and B-algebra etc.

(see [9–12]).

In this article, we introduce the concept of implicative MBJ-neutrosophic ideal and explore its properties by applying the MBJ-neutrosophic set to the implicative ideal of BCK-algebra. In consideration of A and B, which are described as follows:

(1) $\begin{cases} \text{A: implicative MBJ-neutrosophic ideal} \\ \text{B: MBJ-neutrosophic subalgebra} \end{cases}$

(2) $\begin{cases} \text{A: implicative MBJ-neutrosophic ideal} \\ \text{B: MBJ-neutrosophic ideal} \end{cases}$

(3) $\begin{cases} \text{A: implicative MBJ-neutrosophic ideal} \\ \text{B: commutative MBJ-neutrosophic ideal} \end{cases}$

(4) $\begin{cases} \text{A: implicative MBJ-neutrosophic ideal} \\ \text{B: positive implicative MBJ-neutrosophic ideal} \end{cases}$

we identify the relationship between A and B. We prove that A becomes B, and give examples of B not being A. We find and present the condition that B can be A.

2. Preliminaries

2.1. Default background for BCK-algebras

By a BCK-algebra (see [13]), we mean a set X with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $((\tilde{x} * \tilde{y}) * (\tilde{x} * \tilde{z})) * (\tilde{z} * \tilde{y}) = 0$,
(II) $(\tilde{x} * (\tilde{x} * \tilde{y})) * \tilde{y} = 0$,
(III) $\tilde{x} * \tilde{x} = 0$,
(IV) $\tilde{x} * \tilde{y} = 0,\ \tilde{y} * \tilde{x} = 0 \ \Rightarrow\ \tilde{x} = \tilde{y}$,
(V) $0 * \tilde{x} = 0$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in X$.

Every BCK-algebra X satisfies the following conditions (see [13]):

$$(\forall \tilde{x} \in X)\,(\tilde{x} * 0 = \tilde{x}), \tag{2.1}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\,(\tilde{x} \leq \tilde{y} \ \Rightarrow\ \tilde{x} * \tilde{z} \leq \tilde{y} * \tilde{z},\ \tilde{z} * \tilde{y} \leq \tilde{z} * \tilde{x}), \tag{2.2}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\,((\tilde{x} * \tilde{y}) * \tilde{z} = (\tilde{x} * \tilde{z}) * \tilde{y}), \tag{2.3}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\,((\tilde{x} * \tilde{z}) * (\tilde{y} * \tilde{z}) \leq \tilde{x} * \tilde{y}), \tag{2.4}$$
$$(\forall \tilde{x}, \tilde{y} \in X)\,(\tilde{x} * (\tilde{x} * \tilde{y}) \leq \tilde{x}) \tag{2.5}$$

where $\tilde{x} \leq \tilde{y}$ if and only if $\tilde{x} * \tilde{y} = 0$.

A BCK-algebra X is said to be

- positive implicative (see [13]) if it satisfies:

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\,((\tilde{x} * \tilde{z}) * (\tilde{y} * \tilde{z}) = (\tilde{x} * \tilde{y}) * \tilde{z}). \tag{2.6}$$

- commutative (see [13]) if it satisfies:

$$(\forall \tilde{x}, \tilde{y} \in X)\,(\tilde{x} * (\tilde{x} * \tilde{y}) = \tilde{y} * (\tilde{y} * \tilde{x})). \tag{2.7}$$

- implicative (see [13]) if it satisfies:

$$(\forall \tilde{x}, \tilde{y} \in X)\,(\tilde{x} * (\tilde{y} * \tilde{x}) = \tilde{x})\,. \tag{2.8}$$

A nonempty subset S of a BCK/BCI-algebra X is called a subalgebra of X (see [13]) if $\tilde{x} * \tilde{y} \in S$ for all $\tilde{x}, \tilde{y} \in S$. A subset I of a BCK/BCI-algebra X is called an ideal of X (see [13]) if it satisfies:

$$0 \in I, \tag{2.9}$$

$$(\forall \tilde{x} \in X)\,(\forall \tilde{y} \in I)\,(\tilde{x} * \tilde{y} \in I \;\Rightarrow\; \tilde{x} \in I)\,. \tag{2.10}$$

A subset I of a BCK-algebra X is called

- a commutative ideal of X (see [13]) if it satisfies (2.9) and

$$(\forall \tilde{x}, \tilde{y} \in X)(\forall \tilde{z} \in I)\,((\tilde{x} * \tilde{y}) * \tilde{z} \in I \;\Rightarrow\; \tilde{x} * (y * (\tilde{y} * \tilde{x})) \in I)\,. \tag{2.11}$$

- a positive implicative ideal of X (see [13]) if it satisfies (2.9) and

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\,((\tilde{x} * \tilde{y}) * \tilde{z} \in I,\; \tilde{y} * \tilde{z} \in I \;\Rightarrow\; \tilde{x} * \tilde{z} \in I)\,. \tag{2.12}$$

- an implicative ideal of X (see [13]) if it satisfies (2.9) and

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\,((\tilde{x} * (\tilde{y} * \tilde{x})) * \tilde{z} \in I,\, \tilde{z} \in I \;\Rightarrow\; \tilde{x} \in I)\,. \tag{2.13}$$

2.2. Default background for interval-valued fuzzy sets

An interval number is defined to be a closed subinterval $\tilde{a} = [a^-, a^+]$ of $[0, 1]$, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $[[0, 1]]$ the set of all interval numbers. Let us define what is known as refined minimum (briefly, rmin) and refined maximum (briefly, rmax) of two elements in $[[0, 1]]$. We also define the symbols "$\geq$", "$\leq$", "$=$" in case of two elements in $[[0, 1]]$. Given two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$, we define

$$\begin{aligned}
&\text{rmin}\,\{\tilde{a}_1, \tilde{a}_2\} = [\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}]\,,\\
&\text{rmax}\,\{\tilde{a}_1, \tilde{a}_2\} = [\max\{a_1^-, a_2^-\}, \max\{a_1^+, a_2^+\}]\,,\\
&\tilde{a}_1 \geq \tilde{a}_2 \;\Leftrightarrow\; a_1^- \geq a_2^-,\; a_1^+ \geq a_2^+,
\end{aligned}$$

and similarly we may have $\tilde{a}_1 \leq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. Given interval numbers $\tilde{a}_i \in [[0, 1]]$ where $i \in \Lambda$, we define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad\text{and}\quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [[0, 1]]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[[0, 1]]^X$ stand for the set of all IVF sets in X. For every $A \in [[0, 1]]^X$ and $\tilde{x} \in X$, $A(\tilde{x}) = [A^-(\tilde{x}), A^+(\tilde{x})]$ is called the degree of membership of an element $\tilde{x}$ to A, where $A^- : X \to [0, 1]$ and $A^+ : X \to [0, 1]$ are fuzzy sets in X which are called a lower fuzzy set and an upper fuzzy set in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

2.3. Default background for MBJ-neutrosophic sets

Let X be a non-empty set. We consider three mappings $A_T : X \to [0,1]$, $A_I : X \to [0,1]$ and $A_F : X \to [0,1]$ which are called truth membership function, indeterminate membership function and false membership function, respectively. Then a neutrosophic set (NS) in X is defined to be a structure (see [3])

$$A := \{\langle X; A_T(\tilde{x}), A_I(\tilde{x}), A_F(\tilde{x})\rangle \mid \tilde{x} \in X\}. \tag{2.14}$$

Readers can refer to book [13] for more information on BCK algebra, and the site "http://fs.gallup.unm.edu/neutrosophy.htm" for more information on neutrosophic set theory.

Let X be a non-empty set. By an MBJ-neutrosophic set in X (see [8]), we mean a structure of the form:

$$\mathcal{A} := \{\langle X; A_M(\tilde{x}), A_{\tilde{B}}(\tilde{x}), A_J(\tilde{x})\rangle \mid \tilde{x} \in X\}$$

where A_M and A_J are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $A_{\tilde{B}}$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle X; A_M(\tilde{x}), A_{\tilde{B}}(\tilde{x}), A_J(\tilde{x})\rangle \mid \tilde{x} \in X\}.$$

The MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ in X can be represented as follows:

$$\begin{aligned} \mathcal{A} := (A_M, A_{\tilde{B}}, A_J) : X \to [0,1] \times [[0,1]] \times [0,1], \\ x \mapsto (A_M(x), A_{\tilde{B}}(x), A_J(x)) \end{aligned} \tag{2.15}$$

where $A_{\tilde{B}}(x) = [A^-_{\tilde{B}}(x), A^+_{\tilde{B}}(x)]$.

Given an MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ in X, we consider the following set:

$$\Omega(\mathcal{A}) := \left\{ (\tilde{x}, \tilde{y}) \;\middle|\; \begin{array}{l} A_M(\tilde{x}) \geq A_M(\tilde{y}) \\ A_{\tilde{B}}(\tilde{x}) \geq A_{\tilde{B}}(\tilde{y}) \\ A_J(\tilde{x}) \leq A_J(\tilde{y}) \end{array} \right\} \tag{2.16}$$

$$\mathcal{A}^{\min}_{\max} := \left\{ \tfrac{\tilde{x}}{\{\tilde{y}, \tilde{z}\}} \;\middle|\; \begin{array}{l} A_M(\tilde{x}) \geq \min\{A_M(\tilde{y}), A_M(\tilde{z})\} \\ A_J(\tilde{x}) \leq \max\{A_J(\tilde{y}), A_J(\tilde{z})\} \end{array} \right\} \tag{2.17}$$

Let X be a BCK-algebra. An MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ in X is called

- an MBJ-neutrosophic subalgebra of X (see [12]) if it satisfies:

$$(\forall \tilde{x}, \tilde{y} \in X)\left(\tfrac{x*y}{\{x,y\}} \in \mathcal{A}^{\min}_{\max}\right), \tag{2.18}$$

$$(\forall \tilde{x}, \tilde{y} \in X)\left(A_{\tilde{B}}(\tilde{x} * \tilde{y}) \geq \operatorname{rmin}\{A_{\tilde{B}}(\tilde{x}), A_{\tilde{B}}(\tilde{y})\}\right). \tag{2.19}$$

- an MBJ-neutrosophic ideal of X (see [12]) if it satisfies:

$$(\forall \tilde{x} \in X)\left((0, \tilde{x}) \in \Omega(\mathcal{A})\right), \tag{2.20}$$

$$(\forall \tilde{x}, \tilde{y} \in X)\left(\tfrac{\tilde{x}}{\{\tilde{x}*\tilde{y}, \tilde{y}\}} \in \mathcal{A}^{\min}_{\max}\right), \tag{2.21}$$

$$(\forall \tilde{x}, \tilde{y} \in X)\left(A_{\tilde{B}}(\tilde{x}) \geq \operatorname{rmin}\{A_{\tilde{B}}(\tilde{x} * \tilde{y}), A_{\tilde{B}}(\tilde{y})\}\right). \tag{2.22}$$

- a commutative MBJ-neutrosophic ideal of X (see [11]) if it satisfies (2.20) and

$$\frac{\tilde{x}*(\tilde{y}*(\tilde{y}*\tilde{x}))}{\{(\tilde{x}*\tilde{y})*\tilde{z},\, \tilde{z}\}} \in \mathcal{A}^{\min}_{\max}, \tag{2.23}$$

$$A_{\tilde{B}}(\tilde{x}*(\tilde{y}*(\tilde{y}*\tilde{x}))) \geq \operatorname{rmin}\{A_{\tilde{B}}((\tilde{x}*\tilde{y})*\tilde{z}), A_{\tilde{B}}(\tilde{z})\} \tag{2.24}$$

  for all $\tilde{x}, \tilde{y}, \tilde{z} \in X$.
- a positive implicative MBJ-neutrosophic ideal of X (see [10]) if it satisfies (2.20) and

$$\frac{\tilde{x}*\tilde{z}}{\{(\tilde{x}*\tilde{y})*\tilde{z},\, \tilde{y}*\tilde{z}\}} \in \mathcal{A}^{\min}_{\max}, \tag{2.25}$$

$$A_{\tilde{B}}(\tilde{x}*\tilde{z}) \geq \operatorname{rmin}\{A_{\tilde{B}}((\tilde{x}*\tilde{y})*\tilde{z}), A_{\tilde{B}}(\tilde{y}*\tilde{z})\} \tag{2.26}$$

  for all $\tilde{x}, \tilde{y}, \tilde{z} \in X$.

3. Implicative MBJ-neutrosophic ideals of BCK-algebras

In what follows, let X be a BCK-algebra unless otherwise specified.

Definition 3.1. An MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ in X is called an implicative MBJ-neutrosophic ideal of X if if it satisfies (2.20) and

$$\frac{x}{\{(x*(y*x))*z,\, z\}} \in \mathcal{A}^{\min}_{\max}, \tag{3.1}$$

$$A_{\tilde{B}}(x) \geq \operatorname{rmin}\{A_{\tilde{B}}((x*(y*x))*z), A_{\tilde{B}}(z)\}. \tag{3.2}$$

for all $x, y, z \in X$.

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$".

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic set in X defined by Table 2. It is routine to verify that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X.

We consider the relation between an implicative MBJ-neutrosophic ideal and an MBJ-neutrosophic ideal.

Theorem 3.3. *Every implicative MBJ-neutrosophic ideal is an MBJ-neutrosophic ideal.*

Proof. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an implicative MBJ-neutrosophic ideal of X. If we choose $y = 0$ in (3.1) and (3.2), then $\frac{x}{\{x*z,\, z\}} = \frac{x}{\{(x*(0*x))*z,\, z\}} \in \mathcal{A}^{\min}_{\max}$ and

$$A_{\tilde{B}}(x) \geq \operatorname{rmin}\{A_{\tilde{B}}((x*(0*x))*z), A_{\tilde{B}}(z)\} = \{A_{\tilde{B}}((x*z), A_{\tilde{B}}(z)\}$$

for all $x, z \in X$ by (V) and (2.1). Therefore $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an MBJ-neutrosophic ideal of X. □

Table 2. MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$.

X	$A_M(x)$	$A_{\tilde{B}}(x)$	$A_J(x)$
0	0.7	$[0.4, 0.9]$	0.2
1	0.5	$[0.2, 0.6]$	0.6
2	0.3	$[0.3, 0.7]$	0.5
3	0.3	$[0.2, 0.6]$	0.6

Corollary 3.4. *Every implicative MBJ-neutrosophic ideal is an MBJ-neutrosophic subalgebra.*

We can verify that any MBJ-neutrosophic subalgebra may not be an implicative MBJ-neutrosophic ideal by the following example.

Example 3.5. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 3.

Table 3. Cayley table for the binary operation "$*$".

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic set in X defined by Table 4.

Table 4. MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$.

X	$A_M(x)$	$A_{\tilde{B}}(x)$	$A_J(x)$
0	0.65	$[0.42, 0.91]$	0.27
1	0.33	$[0.33, 0.76]$	0.63
2	0.56	$[0.31, 0.71]$	0.54
3	0.47	$[0.29, 0.63]$	0.76

It is routine to verify that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an MBJ-neutrosophic subalgebra of X. But it is not an implicative MBJ-neutrosophic ideal of X since $\frac{1}{\{(1*(2*1))*3,\, 3\}} = \frac{1}{\{0,3\}} \notin \mathcal{A}^{\min}_{\max}$ and/or

$$A_{\tilde{B}}(2) = [0.31, 0.71] \not\succeq [0.33, 0.76] = \operatorname{rmin}\{A_{\tilde{B}}((2 * (3 * 2)) * 1), A_{\tilde{B}}(1)\}.$$

We look for the condition that MBJ-neutrosophic subalgebra can be implicative MBJ-neutrosophic ideal.

Lemma 3.6 ([12]). *Every MBJ-neutrosophic ideal $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ of X satisfies:*

$$x * y \leq z \;\Rightarrow \begin{cases} \frac{x}{\{y,z\}} \in \mathcal{A}^{\min}_{\max}, \\ A_{\tilde{B}}(x) \succeq \operatorname{rmin}\{A_{\tilde{B}}(y), A_{\tilde{B}}(z)\} \end{cases} \tag{3.3}$$

for all $x, y, z \in X$.

Theorem 3.7. *Let* $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ *be an MBJ-neutrosophic subalgebra of* X. *Then it is an implicative MBJ-neutrosophic ideal of* X *if and only if it satisfies*

$$(x * (y * x)) * z \leq a \;\Rightarrow \begin{cases} \frac{x}{\{a,z\}} \in \mathcal{A}^{\min}_{\max}, \\ A_{\tilde{B}}(x) \succeq \text{rmin}\{A_{\tilde{B}}(a), A_{\tilde{B}}(z)\} \end{cases} \tag{3.4}$$

for all $x, y, z, a \in X$.

Proof. Assume that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X and let $x, y, z, a \in X$ be such that $(x * (y * x)) * z \leq a$. Then $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an MBJ-neutrosophic ideal of X by Theorem 3.3, and so $\frac{x}{\{a,z\}} = \frac{x*(y*x)}{\{a,z\}} \in \mathcal{A}^{\min}_{\max}$ and $A_{\tilde{B}}(x) = A_{\tilde{B}}(x * (y * x)) \succeq \text{rmin}\{A_{\tilde{B}}(a), A_{\tilde{B}}(z)\}$ by Lemma 3.6.

Conversely, let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic subalgebra of X that satisfies (3.4). If we take $x = 0$ and $z = a = x$ in (3.4), then we can obtain $(0, x) \in \Omega(\mathcal{A})$. Since $(x*(y*x))*((x*(y*x))*z) \leq z$, it follows from (3.4) that $\frac{x}{\{(x*(y*x))*z,\, z\}} \in \mathcal{A}^{\min}_{\max}$ and $A_{\tilde{B}}(x) \succeq \text{rmin}\{A_{\tilde{B}}((x * (y * x)) * z), A_{\tilde{B}}(z)\}$. Therefore $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X. □

The next example verify that the converse of Theorem 3.3 is not true in general.

Example 3.8. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 5.

Table 5. Cayley table for the binary operation "$*$".

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic set in X defined by Table 6.

Table 6. MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$.

X	$A_M(x)$	$A_{\tilde{B}}(x)$	$A_J(x)$
0	0.7	$[0.4, 0.9]$	0.2
1	0.5	$[0.3, 0.7]$	0.3
2	0.6	$[0.3, 0.7]$	0.4
3	0.4	$[0.2, 0.5]$	0.5
4	0.4	$[0.2, 0.5]$	0.5

It is routine to verify that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an MBJ-neutrosophic ideal of X. But it is not an implicative MBJ-neutrosophic ideal of X since $\frac{1}{\{(1*(3*1))*0,\, 0\}} = \frac{1}{\{0,\, 0\}} \notin \mathcal{A}^{\min}_{\max}$ and/or

$$A_{\tilde{B}}(1) = [0.3, 0.7] \not\succeq [0.4, 0.9] = \text{rmin}\{A_{\tilde{B}}(0), A_{\tilde{B}}(0)\} = \text{rmin}\{A_{\tilde{B}}((1 * (3 * 1)) * 0), A_{\tilde{B}}(0)\}.$$

Proposition 3.9. *Every implicative MBJ-neutrosophic ideal $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ of X satisfies:*

$$(\forall x, y \in X)\,((x, x * (y * x)) \in \Omega(\mathcal{A}))\,. \tag{3.5}$$

Proof. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an implicative MBJ-neutrosophic ideal of X. If we take $z = 0$ in (3.1) and (3.2) and use (2.1), then

$$\frac{x}{\{x*(y*x), 0\}} = \frac{x}{\{(x*(y*x))*0, 0\}} \in \mathcal{A}_{\max}^{\min}$$

and

$$\begin{aligned} A_{\tilde{B}}(x) &\geq \mathrm{rmin}\{A_{\tilde{B}}((x * (y * x)) * 0), A_{\tilde{B}}(0)\} \\ &= \mathrm{rmin}\{A_{\tilde{B}}(x * (y * x)), A_{\tilde{B}}(0)\}. \end{aligned}$$

It follows from (2.20) that $(x, x * (y * x)) \in \Omega(\mathcal{A})$ for all $x, y \in X$. □

The following example verify that any MBJ-neutrosophic ideal $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ of X does not satisfy (3.5).

Example 3.10. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$ which is given in Table 7.

Table 7. Cayley table for the binary operation "$*$".

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic set in X defined by Table 8.

Table 8. MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$.

X	$A_M(x)$	$A_{\tilde{B}}(x)$	$A_J(x)$
0	0.7	$[0.4, 0.9]$	0.2
1	0.5	$[0.3, 0.7]$	0.3
2	0.6	$[0.3, 0.6]$	0.6
3	0.4	$[0.2, 0.4]$	0.5
4	0.3	$[0.2, 0.5]$	0.7

It is routine to verify that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an MBJ-neutrosophic ideal of X. Since

$$(2, 2 * (4 * 2)) = (2, 0) \notin \Omega(\mathcal{A}),$$

$\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ does not satisfy (3.5).

We provide conditions for an MBJ-neutrosophic ideal to be an implicative MBJ-neutrosophic ideal.

Theorem 3.11. *If an MBJ-neutrosophic ideal* $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ *of* X *satisfies* (3.5)*, then it is an implicative MBJ-neutrosophic ideal of* X.

Proof. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic ideal of X that satisfies (3.5). Using (2.21) and (2.22), we know that $\frac{x*(y*x)}{\{(x*(y*x))*z,\, z\}} \in \mathcal{A}_{\max}^{\min}$ and

$$A_{\tilde{B}}(x * (y * x)) \geq \mathrm{rmin}\{A_{\tilde{B}}((x * (y * x)) * z), A_{\tilde{B}}(z)\}.$$

Combining them with (3.5) induces $\frac{x}{\{(x*(y*x))*z,\, z\}} \in \mathcal{A}_{\max}^{\min}$ and

$$A_{\tilde{B}}(x) \geq \mathrm{rmin}\{A_{\tilde{B}}((x * (y * x)) * z), A_{\tilde{B}}(z)\}.$$

Therefore $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X. □

Theorem 3.12. *In an implicative BCK-algebra, every MBJ-neutrosophic ideal is an implicative MBJ-neutrosophic ideal.*

Proof. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic ideal of an implicative BCK-algebra X. The combination of (III), (2.3) and (2.8) induces

$$\begin{aligned}(x * ((x * (y * x)) * z)) * z &= (x * z) * ((x * (y * x)) * z)\\ &= ((x * (y * x)) * z) * ((x * (y * x)) * z)\\ &= 0,\end{aligned}$$

i.e., $x * ((x * (y * x)) * z) \leq z$ for all $x, y, z \in X$. It follows from Lemma 3.6 that $\frac{x}{\{(x*(y*x))*z,\, z\}} \in \mathcal{A}_{\max}^{\min}$ and $A_{\tilde{B}}(x) \geq \mathrm{rmin}\{A_{\tilde{B}}((x * (y * x)) * z), A_{\tilde{B}}(z)\}$ for all $x, y, z \in X$. Consequently, $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X. □

Corollary 3.13. *If a BCK-algebra* X *satisfies any one of the following assertions:*

$$(\forall x, y \in X)((x * (x * y)) * (x * y) = y * (y * x)), \tag{3.6}$$

$$(\forall x, y \in X)((x * (x * y)) * (x * y) = (y * (y * x)) * (y * x)), \tag{3.7}$$

then every MBJ-neutrosophic ideal is an implicative MBJ-neutrosophic ideal.

Corollary 3.14. *Let* X *be an implicative BCK-algebra. If* $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ *is an MBJ-neutrosophic set in* X *which satisfies* (3.1) *and* (3.3)*, then it is an implicative MBJ-neutrosophic ideal of* X.

Lemma 3.15 ([13])**.** *A BCK-algebra* X *is positive implicative if and only if it satisfies:*

$$(\forall x, y \in X)(x * y = (x * y) * y).$$

Lemma 3.16 ([13])**.** *If a BCK-algebra is both commutative and positive implicative, then it is an implicative BCK-algebra.*

Theorem 3.17. *Let* X *be a commutative BCK-algebra such that for any* $y, z \in X$*, the set*

$$\circ(y, z) := \{x \in X \mid x * y \leq z\} \tag{3.8}$$

has the greatest element. For every $x, y \in X$ *with* $x \leq y$*, if the greast element of* $\circ(x, y)$ *is* y*, then every MBJ-neutrosophic ideal is an implicative MBJ-neutrosophic ideal.*

Proof. Let X be a commutative BCK-algebra and assume that the set $\circ(x, y)$ has the greatest element for every $x, y \in X$. We observe that the greatest element of $\circ(x, y)$ is the same as the greatest element of $\circ(y, x)$, and we denote it by $x|y$. Let $x, y, z \in X$. If $x \leq y$, then $(x|z) * y \leq (x|z) * x \leq z$ and so $x|z \leq y|z$. Since $x * (x * y) \leq y$, we get $x \leq (x * y)|y$. Hence

$$x|z \leq ((x * y)|y)|z = (x * y)|(y|z),$$

and thus

$$(x|z) * (z|y) = (x|z) * (y|z) \leq ((x * y)|(y|z)) * (y|z) \leq x * y \leq x = x|0 \leq x|y.$$

It follows that $(x|z) * (x * y) \leq z|y$. Thus

$$((x|z) * y) * (x * y) = ((x|z) * (x * y)) * y \leq (z|y) * y = (x|y) * (y|y) \leq z * y,$$

which implies that $(x|z) * y \leq (x * y)|(z * y)$. On the other hand, since $x \leq x|z$ and $z \leq x|z$, we get $x * y \leq (x|z) * y$ and $z * y \leq (x|z) * y$. Hence

$$(x * y)|(z * y) \leq ((x|z) * y)|(z * y) \leq ((x|z) * y)|((x|z) * y) = (x|z) * y,$$

and therefore $(x|z) * y = (x * y)|(z * y)$. Suppose $x|y = y$ for all $x, y \in X$ with $x \leq y$. Then $x|x = x$ for all $x \in X$. Since $z \leq x|(z * x)$, we get

$$x|z \leq x|(x|(z * x)) = (x|x)|(z * x) = x|(z * x).$$

It is clear that $x|(z * x) \leq x|z$. Thus $x|z = x|(z * x)$. Since $(x * z) * y = x * (z|y)$, it follows that

$$(x * z) * z = x * (z|z) = x * (z|(z * z)) = x * (z|0) = x * z.$$

This shows that X is a positive implicative BCK-algebra by Lemma 3.15, and so X is an implicative BCK-algebra by Lemma 3.16. Consequently, every MBJ-neutrosophic ideal is an implicative MBJ-neutrosophic ideal by Theorem 3.12. □

We discuss relationship between implicative MBJ-neutrosophic ideal and positive implicative MBJ-neutrosophic ideal.

Lemma 3.18. *Every MBJ-neutrosophic ideal $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ of X satisfies:*

$$(\forall x, y \in X)(x \leq y \Rightarrow (x, y) \in \Omega(\mathcal{A})). \tag{3.9}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$ and so $\frac{x}{\{0, y\}} = \frac{x}{\{x*y, y\}} \in \mathcal{A}^{\min}_{\max}$ and

$$A_{\tilde{B}}(x) \succeq \text{rmin}\{A_{\tilde{B}}(x * y), A_{\tilde{B}}(y)\} = \text{rmin}\{A_{\tilde{B}}(0), A_{\tilde{B}}(y)\}$$

by (2.21) and (2.22). It follows from (2.20) that $(x, y) \in \Omega(\mathcal{A})$. □

Theorem 3.19. *Every implicative MBJ-neutrosophic ideal is a positive implicative MBJ-neutrosophic ideal.*

Proof. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an implicative MBJ-neutrosophic ideal of X. Since $((x*z)*z)*(y*z) \leq (x*y)*z$ for all $x, y, z \in X$, we get

$$\frac{x*z}{\{y*z,\,(x*y)*z\}} = \frac{(x*z)*(x*(x*z))}{\{y*z,\,(x*y)*z\}} = \frac{(x*z)*z}{\{y*z,\,(x*y)*z\}} \in \mathcal{A}^{\min}_{\max}$$

and

$$\begin{aligned} A_{\tilde{B}}(x*z) &= A_{\tilde{B}}((x*z)*(x*(x*z))) \\ &= A_{\tilde{B}}((x*z)*z) \\ &\succeq \operatorname{rmin}\{A_{\tilde{B}}(y*z), A_{\tilde{B}}((x*y)*z)\} \end{aligned}$$

by Lemma 3.6. Therefore $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is a positive implicative MBJ-neutrosophic ideal of X. □

The following example shows that the converse of Theorem 3.19 is not true.

Example 3.20. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 9.

Table 9. Cayley table for the binary operation "$*$".

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic set in X defined by Table 10.

Table 10. MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$.

X	$A_M(x)$	$A_{\tilde{B}}(x)$	$A_J(x)$
0	0.7	$[0.4, 0.9]$	0.2
1	0.5	$[0.3, 0.6]$	0.3
2	0.2	$[0.2, 0.6]$	0.5
3	0.5	$[0.3, 0.6]$	0.3
4	0.2	$[0.1, 0.5]$	0.6

It is routine to verify that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is a positive implicative MBJ-neutrosophic ideal of X. But it is not an implicative MBJ-neutrosophic ideal of X since $\frac{2}{\{(2*(4*2))*3,\,3\}} = \frac{2}{\{0,3\}} \notin \mathcal{A}^{\min}_{\max}$ and/or

$$A_{\tilde{B}}(2) = [0.2, 0.6] \not\succeq [0.3, 0.6] = \operatorname{rmin}\{A_{\tilde{B}}((2*(4*2))*3), A_{\tilde{B}}(3)\}.$$

Proposition 3.21. *Every implicative MBJ-neutrosophic ideal $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ of X satisfies:*

$$(\forall x, y \in X)\,((x*(x*y)),\ y*(y*x)) \in \Omega(\mathcal{A})\,. \tag{3.10}$$

Proof. Assume that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X. The combination of (I), (2.2), (2.3) and (2.5) induces

$$(x * (x * y)) * (y * (x * (x * y))) \leq y * (y * x)$$

for all $x, y \in X$. It follows from Lemma 3.18 that

$$((x * (x * y)) * (y * (x * (x * y))),\ y * (y * x)) \in \Omega(\mathcal{A}). \tag{3.11}$$

Using (2.1), (3.1) and (3.2), we get

$$\frac{x*(x*y)}{\{(x*(x*y))*(y*(x*(x*y))), 0\}} = \frac{x*(x*y)}{\{((x*(x*y))*(y*(x*(x*y))))*0, 0\}} \in \mathcal{A}^{\min}_{\max}$$

and

$$\begin{aligned} A_{\tilde{B}}(x * (x * y)) &\geq \operatorname{rmin}\{A_{\tilde{B}}(((x * (x * y)) * (y * (x * (x * y)))) * 0), A_{\tilde{B}}(0)\} \\ &= \operatorname{rmin}\{A_{\tilde{B}}((x * (x * y)) * (y * (x * (x * y)))), A_{\tilde{B}}(0)\} \end{aligned}$$

These combine with (3.11) to induce $\frac{x*(x*y)}{\{y*(y*x), 0\}} \in \mathcal{A}^{\min}_{\max}$ and

$$A_{\tilde{B}}(x * (x * y)) \geq \operatorname{rmin}\{A_{\tilde{B}}(y * (y * x)), A_{\tilde{B}}(0)\}.$$

It follows from (2.20) that $(x * (x * y), y * (y * x)) \in \Omega(\mathcal{A})$. □

We provide conditions for a positive implicative MBJ-neutrosophic ideal to be an implicative MBJ-neutrosophic ideal.

Lemma 3.22 ([10]). *An MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ in X is a positive implicative MBJ-neutrosophic ideal of X if and only if it is an MBJ-neutrosophic ideal of X that satisfies:*

$$(\forall x, y \in X)((x * y,\ (x * y) * y) \in \Omega(\mathcal{A})). \tag{3.12}$$

Theorem 3.23. *If a positive implicative MBJ-neutrosophic ideal $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ of X satisfies* (3.10)*, then it is an implicative MBJ-neutrosophic ideal of X.*

Proof. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be a positive implicative MBJ-neutrosophic ideal of X that satisfies (3.10). Then $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an MBJ-neutrosophic ideal of X (see [10]). Let $x, y, z \in X$. Using (2.21) and (2.22), we have

$$\frac{x*(y*x)}{\{(x*(y*x))*z, z\}} \in \mathcal{A}^{\min}_{\max}, \tag{3.13}$$

$$A_{\tilde{B}}(x * (y * x)) \geq \operatorname{rmin}\{A_{\tilde{B}}((x * (y * x)) * z), A_{\tilde{B}}(z)\}. \tag{3.14}$$

Since $(y * (y * x)) * (y * x) \leq x * (y * x)$, we get

$$((y * (y * x)) * (y * x),\ x * (y * x)) \in \Omega(\mathcal{A})$$

by Lemma 3.18. Hence

$$(y * (y * x),\ x * (y * x)) \in \Omega(\mathcal{A})$$

by Lemma 3.22, and so

$$(x * (x * y),\ x * (y * x)) \in \Omega(\mathcal{A})$$

by (3.10). Since $(x * y) * z \leq x * y \leq x * (y * x)$, we have $((x * y) * z,\ x * (y * x)) \in \Omega(\mathcal{A})$ by Lemma 3.18. It follows from (2.21), (2.22), (3.13) and (3.14) that $\frac{x*z}{\{(x*(y*x))*z,\, z\}} \in \mathcal{A}^{\min}_{\max}$ and

$$A_{\tilde{B}}(x * z) \succeq \text{rmin}\{A_{\tilde{B}}((x * (y * x)) * z), A_{\tilde{B}}(z)\}.$$

Combining these with (2.21) and (2.22) induces $\frac{x}{\{(x*(y*x))*z,\, z\}} \in \mathcal{A}^{\min}_{\max}$ and

$$A_{\tilde{B}}(x) \succeq \text{rmin}\{A_{\tilde{B}}((x * (y * x)) * z), A_{\tilde{B}}(z)\}.$$

Therefore $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X. □

We discuss relationship between implicative MBJ-neutrosophic ideal and commutative MBJ-neutrosophic ideal.

Lemma 3.24 ([11]). *An MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ in X is a commutative MBJ-neutrosophic ideal of X if and only if it is an MBJ-neutrosophic ideal of X that satisfies:*

$$(\forall x, y \in X)(x * (y * (y * x)),\ x * y) \in \Omega(\mathcal{A})). \tag{3.15}$$

Theorem 3.25. *Every implicative MBJ-neutrosophic ideal is a commutative MBJ-neutrosophic ideal.*

Proof. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an implicative MBJ-neutrosophic ideal of X. Then $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an MBJ-neutrosophic ideal of X by Theorem 3.3. For every $x, y \in X$, we obtain

$$(x * (y * (y * x))) * (y * (x * (y * (y * x)))) \leq x * y.$$

It follows from Lemma 3.18 that

$$((x * (y * (y * x))) * (y * (x * (y * (y * x)))),\ x * y) \in \Omega(\mathcal{A}).$$

The combination of this with Proposition 3.9 induces

$$(x * (y * (y * x)),\ x * y) \in \Omega(\mathcal{A}).$$

Consequently, $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is a commutative MBJ-neutrosophic ideal of X by Lemma 3.24. □

The converse of Theorem 3.25 is not true in general as seen in the next example.

Example 3.26. Consider the BCK-algebra $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.20. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic set in X defined by Table 11. It is routine to verify that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is a commutative MBJ-neutrosophic ideal of X. Since $(1, 1*(3*1)) = (1, 0) \notin \Omega(\mathcal{A})$, we know from Theorem 3.11 that it is not an implicative MBJ-neutrosophic ideal of X.

We explore the conditions under which commutative MBJ-neutrosophic ideal can be implicative MBJ-neutrosophic ideal.

Table 11. MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$.

X	$A_M(x)$	$A_{\tilde{B}}(x)$	$A_J(x)$
0	0.7	[0.4, 0.8]	0.2
1	0.4	[0.3, 0.6]	0.6
2	0.7	[0.4, 0.8]	0.2
3	0.4	[0.3, 0.6]	0.6
4	0.5	[0.2, 0.5]	0.5

Theorem 3.27. *If a commutative MBJ-neutrosophic ideal $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ of X satisfies* (3.12)*, then it is an implicative MBJ-neutrosophic ideal of X.*

Proof. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be a commutative MBJ-neutrosophic ideal of X that satisfies (3.12). Then $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an MBJ-neutrosophic ideal of X satisfying the condition (3.12). Since $(y * (y * x)) * (y * x) \leq x * (y * x)$ for all $x, y \in X$, we get $((y * (y * x)) * (y * x),\ x * (y * x)) \in \Omega(\mathcal{A})$ by Lemma 3.18 which implies from (3.12) that

$$(y * (y * x),\ x * (y * x)) \in \Omega(\mathcal{A}). \tag{3.16}$$

The combination of the inequality $x * y \leq x * (y * x)$ with Lemma 3.18 induces $(x * y,\ x * (y * x)) \in \Omega(\mathcal{A})$. It follows from Lemma 3.24 that

$$(x * (y * (y * x)),\ x * (y * x)) \in \Omega(\mathcal{A}). \tag{3.17}$$

Using (2.21) and (2.22), we have

$$\begin{aligned} &\frac{x}{\{x*(y*(y*x)),\, y*(y*x)\}} \in \mathcal{A}^{\min}_{\max}, \\ &A_{\tilde{B}}(x) \succeq \operatorname{rmin}\{A_{\tilde{B}}(x * (y * (y * x))), A_{\tilde{B}}(y * (y * x))\}. \end{aligned} \tag{3.18}$$

The combination of (3.16)–(3.18) induces $(x,\ x * (y * x)) \in \Omega(\mathcal{A})$, and therefore $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X by Theorem 3.11. □

The two concepts commutative MBJ-neutrosophic ideal and positive implicative MBJ-neutrosophic ideal are independent of each other as seen in the next example.

Example 3.28. (1) Consider the BCK-algebra $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.10. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be an MBJ-neutrosophic set in X defined by Table 12. It is routine to verify that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is a positive implicative MBJ-neutrosophic ideal of X. Since

$$(1 * (2 * (2 * 1)), 1 * 2) = (1, 0) \notin \Omega(\mathcal{A}),$$

it follows from Lemma 3.24 that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is a not a commutative MBJ-neutrosophic ideal of X.

Table 12. MBJ-neutrosophic set $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$.

X	$A_M(x)$	$A_{\tilde{B}}(x)$	$A_J(x)$
0	0.8	$[0.4, 0.9]$	0.2
1	0.6	$[0.3, 0.7]$	0.3
2	0.5	$[0.3, 0.6]$	0.6
3	0.7	$[0.1, 0.4]$	0.5
4	0.4	$[0.2, 0.5]$	0.7

(2) Consider the commutative MBJ-neutrosophic ideal $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ of X which is described in Example 3.26. Since $(3 * 1, (3 * 1) * 1) = (1, 0) \notin \Omega(\mathcal{A})$, we know that $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is not a positive implicative MBJ-neutrosophic ideal of X by Lemma 3.22.

Theorem 3.29. *An MBJ-neutrosophic set in X is an implicative MBJ-neutrosophic ideal of X if and only if it is both a commutative MBJ-neutrosophic ideal and a positive implicative MBJ-neutrosophic ideal of X.*

Proof. The necessity is stated in Theorems 3.19 and 3.25. Let $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ be both a commutative MBJ-neutrosophic ideal and a positive implicative MBJ-neutrosophic ideal of X. For any $x, y \in X$, we have $(y * (y * x)) * (y * x) \leq x * (y * x)$, and so

$$(y * (y * x),\ x * (y * x)) = ((y * (y * x)) * (y * x),\ x * (y * x)) \in \Omega(\mathcal{A}) \tag{3.19}$$

by Lemmas 3.18 and 3.22. On the other hand, the inequality $x*y \leq x*(y*x)$ implies from Lemmas 3.18 and 3.24 that

$$(x * (y * (y * x)),\ x * (y * x)) = (x * y,\ x * (y * x)) \in \Omega(\mathcal{A}). \tag{3.20}$$

Using (2.21) and (2.22), we obtain $\frac{x}{\{x*(y*(y*x)),\, y*(y*x)\}} \in \mathcal{A}^{\min}_{\max}$ and

$$A_{\tilde{B}}(x) \succeq \text{rmin}\{A_{\tilde{B}}(x * (y * (y * x))), A_{\tilde{B}}(y * (y * x))\}.$$

Combining these with (3.19) and (3.20) induces $(x, x * (y * x)) \in \Omega(\mathcal{A})$. Consequently, $\mathcal{A} := (A_M, A_{\tilde{B}}, A_J)$ is an implicative MBJ-neutrosophic ideal of X by Theorem 3.11. □

4. Conclusions

By applying the MBJ-neutrosophic set to the implicative ideal of BCK-algebra, we have introduced the concept of implicative MBJ-neutrosophic ideal and have explored its properties. We have identified a relationship between implicative MBJ-neutrosophic ideal and each MBJ-neutrosophic subalgebra, MBJ-neutrosophic ideal, positive implicative MBJ-neutrosophic ideal, commutative MBJ-neutrosophic ideal. We have provided examples to show that any MBJ-neutrosophic subalgebra (resp., MBJ-neutrosophic ideal, positive implicative MBJ-neutrosophic ideal and commutative MBJ-neutrosophic ideal) is not an implicative MBJ-neutrosophic ideal. We have discussed conditions under which MBJ-neutrosophic subalgebra (resp., MBJ-neutrosophic ideal, positive implicative MBJ-neutrosophic ideal and commutative MBJ-neutrosophic ideal) can be implicative MBJ-neutrosophic ideal. We have established characterizations of implicative MBJ-neutrosophic ideal.

In future, based on the ideas and results of this paper, we will study MBJ-neutrosophic structures in several algebraic systems associated with BCK/BCI-algebra, such as MV-algebra, BL-algebra, EQ-algebra, and equality algebra etc. We will also study MBJ-neutrosophic structures in hyperalgebraic structures. As a further study in the future, we will extend the proposed methods to Pythagorean fuzzy uncertain environments, sch as Pythagorean fuzzy interactive Hamacher power aggregation operators for assessment of express service quality with entropy weight.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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