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Research article

The hybrid power mean of some special character sums of polynomials and two-term exponential sums modulo p

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Abstract: We consider the calculation problem of one kind hybrid power mean involving the character sums of polynomials and two-term exponential sums modulo p, an odd prime, and use the analytic method and the properties of classical Gauss sums to give some identities and asymptotic formulas for them.

Keywords: the character sums of polynomials; two-term exponential sums; hybrid power mean; analytic method; identity; asymptotic formula
Mathematics Subject Classification: 11L03, 11L40

1. Introduction

The so-called character sums of polynomials means that

$$S(\chi, f; q) = \sum_{a=N+1}^{N+M} \chi(f(a)),$$

where $q \geq 3$ is an integer, χ is any non-principal Dirichlet character modulo q, N and M are positive integers with $1 \leqslant M \leqslant q$, $f(x)$ is a rational coefficient polynomial of x with degree n.

Many classic problems in analytic number theory related to the upper bound estimation of $S(\chi, f; q)$. For example, the least quadratic non-residue and the primitive root, etc. Therefore, some experts and scholars in the number theory have studied the estimate problem of $S(\chi, f; q)$, and obtained many meaningful results. The first thing worth mentioning is Pólya and Vinogradov's classical work (see [1]: Theorem 8.21 and Theorem 13.15), they proved that for any non-principal character χ mod q, one has the estimate

$$\sum_{a=N+1}^{N+M} \chi(a) \ll q^{\frac{1}{2}} \ln q,$$

where the symbol $A \ll B$ denotes $|A| < cB$ for some constant c.

If $q = p$ is an odd prime, A. Weil [4] (or refer to D.A.Burgess [8]) obtained a general conclusion. That is, if χ is a k-th character mod p, polynomial $f(x)$ is not a perfect k-th power mod p, then we have the upper bound estimate

$$\left|\sum_{x=N+1}^{N+M} \chi\left(f(x)\right)\right| \ll p^{\frac{1}{2}} \ln p. \tag{1.1}$$

In fact, the estimate in (1.1) is the best one, it is impossible to improve the main term $p^{\frac{1}{2}}$ in (1.1). Even for the minor term $\ln p$ in (1.1), it is also difficult to improve, and it can not even be improved to $\ln^{\lambda} p$ for any fixed real number $0 < \lambda < 1$. Of course, there are many characters mod q and special polynomials $f(x)$ (see references [5,6] for details), they satisfy the identity

$$\left|\sum_{a=1}^{q} \chi\left(f(a)\right)\right| = \sqrt{q}.$$

For example, if p is an odd prime, for non-real character χ mod p, then from the separable Gauss sums (see the Theorem 8.19 in [1]) we have

$$\begin{aligned} \left|\sum_{a=0}^{p-1} \chi\left(a^2+1\right)\right| &= \left|1 + \sum_{a=1}^{p-1} \left(1+\chi_2(a)\right) \cdot \chi(a+1)\right| = \left|\sum_{a=1}^{p-1} \chi_2(a)\chi(a+1)\right| \\ &= \left|\frac{1}{\tau\left(\overline{\chi}\right)} \sum_{b=1}^{p-1} \overline{\chi}(b) \sum_{a=1}^{p-1} \chi_2(a) e\left(\frac{ba+b}{p}\right)\right| = \frac{1}{\sqrt{p}} \left|\tau\left(\chi_2\right)\tau\left(\overline{\chi}\chi_2\right)\right| = \sqrt{p}, \end{aligned}$$

where χ_2 denotes the Legendre symbol mod p.

For other papers related the character sums of polynomials, see [7–10], we will not list them all in here.

In addition, we will introduce the two-term exponential sums which are used in this paper. For integers m and n, the definition of the two-term exponential sums $G(m, n, h, k; q)$ is

$$G(m, n, h, k; q) = \sum_{a=0}^{q} e\left(\frac{ma^h + na^k}{q}\right),$$

where as usual, we abbreviate $e^{2\pi i y}$ to $e(y)$, and $h > k \geq 1$ are integers.

In the vast majority of cases, we only consider the case $h > 1$ and $k = 1$. If $q = p$ is an odd prime, these two-term exponential sums are closely related to Fourier analysis on finite fields. In this special case, W. P. Zhang and D. Han [11] obtained an identity for the sixth power mean of the two-term exponential sums. Some related papers can also be found in [12–15].

In this paper, we will consider the hybrid power mean involving character sums of polynomials and two-term exponential sums

$$\sum_{m=1}^{p-1} \left|\sum_{a=1}^{p-1} \chi\left(ma^h + a^k\right)\right|^2 \cdot \left|\sum_{b=0}^{p-1} e\left(\frac{mb^4+b}{p}\right)\right|^2. \tag{1.2}$$

For this kind hybrid power mean, if $p > 3$ is a prime with $(3, p-1) = 1$, and for non-principal even character χ mod p, X. Y. Du [14] provided the following identity

$$\begin{aligned}&\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2\\ =\ & 2p\left(p^2-p-1\right)-p\left(2+\left(\frac{3}{p}\right)\right)\sum_{u=1}^{p-1}\overline{\chi}(u)\sum_{a=1}^{p-1}\left(\frac{(a-1)(a^3-u^2)}{p}\right),\end{aligned}$$

where $\left(\frac{*}{p}\right) = \chi_2$ is the Legendre symbol mod p.

From Du's [14] we may immediately deduce the asymptotic formula:

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2 = 2p^3+O\left(p^2\right).$$

The mean values in Du's [14] or in (1.2) are meaningful, that is to say, from an average of sense, most values of these character sums of polynomials and two-term exponential sums are almost $\sqrt{p}$. In addition, the result in [14] shows that the size of values of these character sums of polynomials and two-term exponential sums are complementary, from a probabilistic point of view, their product are almost p. Of course, it would make more sense to give an exact computational formula for these kind mean values. This is our ultimate goal and we believe it is possible.

The main purpose of this article is to illustrate this point. That is, we used the analytic methods and the properties of the classical Gauss sums to obtain some identities and asymptotic formulas for (1.2) with $h = 5$ and $k = 1$, and proved the following results.

Theorem 1. Let p be an odd prime with $4 \nmid (p-1)$. Then for any non-principal even character χ mod p (i.e., $\chi(-1) = 1$), we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 = \begin{cases} 2p^2(p-1), & \text{if } \chi^5 \neq \chi_0,\\ p\left(p^2-1\right), & \text{if } \chi^5 = \chi_0,\end{cases}$$

where χ_0 denotes the principal character modulo p.

Theorem 2. If p is an odd prime with $4 \mid (p-1)$, then for any fourth character χ mod p (χ is the fourth character mod p if and only if there exists a character χ_1 mod p such that $\chi = \chi_1^4 \neq \chi_0$) with $\chi^5 \neq \chi_0$, we have the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 = 4p^3+O\left(p^2\right);$$

If χ is a fourth character mod p with $\chi^5 = \chi_0$, then we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 = 3p^3+O\left(p^2\right).$$

Some notes: We have not discussed the trivial cases in the above theorems. In fact, if χ is an odd character mod p in Theorem 1 or χ is not a fourth character mod p in Theorem 2, then we have the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2=0.$$

So in these cases, we have the trivial results:

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2=0.$$

If p is a prime with $4\mid(p-1)$, then for any non-principal even character χ mod p, whether there exists an exact computational formula for the mean value

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 ?$$

This is an open problem. We need to further study.

2. Some Lemmas

In this part, we introduce some elementary properties related to the classical Gauss sums $\tau(\chi)$ modulo q. As usual, the classical Gauss sums $\tau(\chi)$ are defined as

$$\tau(\chi)=\sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right).$$

Many of its properties can be found in analytic number theory textbooks, such as [1–3]. But there are two things we need to emphasize here. If χ is a primitive character mod q, then

$$\sum_{a=1}^{q}\chi(a)e\left(\frac{na}{q}\right)=\overline{\chi}(n)\tau(\chi)\text{ and }|\tau(\chi)|=\sqrt{q}.$$

By means of these properties, we can prove the following.

Lemma 1. Let p be an odd prime with $p\equiv 1 \pmod 4$, m be any integer with $(m,p)=1$, $\chi_2=\left(\frac{*}{p}\right)$ denote the Legendre symbol mod p, and β be a four-order character mod p. That is, $\beta^4=\chi_2^2=\chi_0$. If χ is not a fourth character mod p, then we have

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2=0;$$

If χ is a fourth character mod p and $\chi^5\neq\chi_0$, then we have the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 = 4p+\chi_2(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)$$

$$+\beta(-m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\beta}\left(b^5a^4-1\right)\beta(b-1)$$
$$+\overline{\beta}(-m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\beta\left(b^5a^4-1\right)\overline{\beta}(b-1);$$

If χ is a fourth character mod p and $\chi^5=\chi_0$, then we have the identity

$$\begin{aligned}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 &= 3p+1+\chi_2(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\\ &+\beta(-m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\beta}\left(b^5a^4-1\right)\beta(b-1)\\ &+\overline{\beta}(-m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\beta\left(b^5a^4-1\right)\overline{\beta}(b-1).\end{aligned}$$

Proof. If χ is not a fourth character mod p, then there exists an integer r such that $r^4\equiv 1(\bmod p)$ and $\chi(r)\neq 1$. So from the properties of the reduced residue system mod p we have

$$\begin{aligned}&\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)=\sum_{a=1}^{p-1}\chi\left((ra)^5+mra\right)\\ =\ &\chi(r)\sum_{a=1}^{p-1}\chi\left(r^4a^5+ma\right)=\chi(r)\sum_{a=1}^{p-1}\chi\left(a^5+ma\right).\end{aligned}$$

Therefore, we have the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2=0. \tag{2.1}$$

If χ is a fourth character mod p, then note that for any integer m with $(m,p)=1$ we have the identity

$$\begin{aligned}\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right) &= 1+\sum_{a=1}^{p-1}\left(1+\beta(a)+\beta^2(a)+\overline{\beta}(a)\right)e\left(\frac{ma}{p}\right)\\ &= \overline{\beta}(m)\tau(\beta)+\chi_2(m)\tau(\chi_2)+\beta(m)\tau\left(\overline{\beta}\right).\end{aligned} \tag{2.2}$$

So if $\chi^5\neq\chi_0$, then from the properties of Gauss sums we have

$$\begin{aligned}&\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2=\left|\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p-1}\overline{\chi}(b)\sum_{a=1}^{p-1}e\left(\frac{b\left(a^5+ma\right)}{p}\right)\right|^2\\ =\ &\frac{1}{p}\left|\sum_{b=1}^{p-1}\overline{\chi}(b\overline{a})\sum_{a=1}^{p-1}e\left(\frac{b\overline{a}\left(a^5+ma\right)}{p}\right)\right|^2\end{aligned}$$

$$= \frac{1}{p}\left|\sum_{b=1}^{p-1}\overline{\chi}(b)\sum_{a=1}^{p-1}\chi(a)e\left(\frac{b\left(a^4+m\right)}{p}\right)\right|^2,$$

then expand the square and use the properties of the reduced residue system mod p we obtain

$$\begin{aligned}
&\frac{1}{p}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\overline{\chi}(b\overline{d})\sum_{a=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\overline{c})e\left(\frac{b\left(a^4+m\right)-d\left(c^4+m\right)}{p}\right)\\
=\ &\frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{dc^4\left(ba^4-1\right)+md\left(b-1\right)}{p}\right)\\
=\ &\frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{c=0}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{dc^4\left(ba^4-1\right)+md\left(b-1\right)}{p}\right),
\end{aligned}$$

from the formula (2.2), the above formula is

$$\begin{aligned}
&\sum_{\substack{a=1\\ ba^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{d=1}^{p-1}e\left(\frac{md\left(b-1\right)}{p}\right)\\
&+\frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{d=1}^{p-1}\beta\left(d(ba^4-1)\right)\tau\left(\overline{\beta}\right)e\left(\frac{md\left(b-1\right)}{p}\right)\\
&+\frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{d=1}^{p-1}\overline{\beta}\left(d(ba^4-1)\right)\tau(\beta)e\left(\frac{md\left(b-1\right)}{p}\right)\\
&+\frac{\tau(\chi_2)}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{d=1}^{p-1}\chi_2\left(d(ba^4-1)\right)e\left(\frac{md\left(b-1\right)}{p}\right)\\
=\ &\sum_{\substack{a=1\\ ba^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{d=0}^{p-1}e\left(\frac{md\left(b-1\right)}{p}\right)-\sum_{\substack{a=1\\ ba^4\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\\
&+\frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{d=1}^{p-1}\beta\left(d(ba^4-1)\right)\tau\left(\overline{\beta}\right)e\left(\frac{md\left(b-1\right)}{p}\right)\\
&+\frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{d=1}^{p-1}\overline{\beta}\left(d(ba^4-1)\right)\tau(\beta)e\left(\frac{md\left(b-1\right)}{p}\right)\\
&+\frac{\tau(\chi_2)}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{d=1}^{p-1}\chi_2\left(d(ba^4-1)\right)e\left(\frac{md\left(b-1\right)}{p}\right)\\
=\ &p\sum_{\substack{a=1\\ a^4\equiv 1 \bmod p}}^{p-1}\chi(a)-\sum_{a=1}^{p-1}\chi^5(a)+\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\chi_2\left(ba^4-1\right)\chi_2(m(b-1))
\end{aligned}$$

$$+\frac{\tau(\beta)\tau\left(\overline{\beta}\right)}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\beta\left(ba^4-1\right)\overline{\beta}\left(m(b-1)\right)$$
$$+\frac{\tau(\beta)\tau\left(\overline{\beta}\right)}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\overline{\beta}\left(ba^4-1\right)\beta\left(m(b-1)\right). \tag{2.3}$$

Since $\chi^5 \neq \chi_0$, then we have $\sum_{a=1}^{p-1}\chi^5(a) = 0$, from (2.3) we obtain

$$\begin{aligned}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 &= 4p+\chi_2(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\\ &\quad+\beta(-m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\beta}\left(b^5a^4-1\right)\beta(b-1)\\ &\quad+\overline{\beta}(-m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\beta\left(b^5a^4-1\right)\overline{\beta}(b-1),\end{aligned}$$

where we have used the identity $\tau(\beta)\tau\left(\overline{\beta}\right) = \beta(-1)\cdot p$.

If $\chi^5 = \chi_0$, then

$$\sum_{a=1}^{p-1}\chi^5(a) = \sum_{a=1}^{p-1}1 = p-1,$$

from (2.3) we also have the identity

$$\begin{aligned}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 &= 3p+1+\chi_2(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\\ &\quad+\beta(-m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\beta}\left(b^5a^4-1\right)\beta(b-1)\\ &\quad+\overline{\beta}(-m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\beta\left(b^5a^4-1\right)\overline{\beta}(b-1).\end{aligned} \tag{2.4}$$

This completes the proof of Lemma 1.

Lemma 2. Let p be an odd prime with $p \equiv 3 \pmod 4$, m be any integer with $(m,p)=1$. If χ is not an even character mod p, then we have

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 = 0;$$

If χ is an even character mod p and $\chi^5 \neq \chi_0$, then we have the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 = 2p-\chi_2(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1);$$

If χ is an even character mod p and $\chi^5 = \chi_0$, then we have the identity

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 = p+1-\chi_2(m)\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1).$$

Proof. If $p \equiv 3 \pmod 4$, then $\chi_2(-1) = -1$, so for any odd character χ mod p, we have

$$\begin{aligned}\sum_{a=1}^{p-1}\chi\left(a^5+ma\right) &= \sum_{a=1}^{p-1}\chi\left((-a)^5-ma\right)\\ &= \chi(-1)\sum_{a=1}^{p-1}\chi\left(a^5+ma\right) = -\sum_{a=1}^{p-1}\chi\left(a^5+ma\right).\end{aligned}$$

That is,

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 = 0. \tag{2.5}$$

If $4 \nmid (p-1)$ and χ is an even character mod p with $\chi^5 \neq \chi_0$, then for any integer m with $(m,p)=1$, note that

$$\begin{aligned}\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right) &= 1+\sum_{a=1}^{p-1}\left(1+\chi_2(a)\right)e\left(\frac{ma^2}{p}\right)\\ &= 1+\sum_{a=1}^{p-1}e\left(\frac{ma^2}{p}\right)+\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma^2}{p}\right).\end{aligned}$$

Since $4 \nmid (p-1)$, $\chi_2(-1) = -1$, then we have

$$\begin{aligned}\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma^2}{p}\right) &= \sum_{a=1}^{p-1}\chi_2(-a)e\left(\frac{ma^2}{p}\right)\\ &= -\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma^2}{p}\right) = 0.\end{aligned}$$

Combine the above two formulas we get

$$\begin{aligned}\sum_{a=0}^{p-1}e\left(\frac{ma^4}{p}\right) &= \sum_{a=0}^{p-1}e\left(\frac{ma^2}{p}\right)\\ &= 1+\sum_{a=1}^{p-1}\left(1+\chi_2(a)\right)e\left(\frac{ma}{p}\right) = \chi_2(m)\cdot\tau(\chi_2),\end{aligned} \tag{2.6}$$

from the method of proving Lemma 1 we obtain

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2 = \sum_{\substack{a=1\\ ba^4\equiv 1 \pmod p}}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\bar{b}\right)\sum_{d=0}^{p-1}e\left(\frac{md(b-1)}{p}\right)-\sum_{\substack{a=1\\ ba^4\equiv 1 \pmod p}}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\bar{b}\right)$$

$$+\frac{1}{p}\sum_{\substack{a=1\\ p\nmid\left(ba^4-1\right)}}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\sum_{c=0}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{dc^4\left(ba^4-1\right)+md\left(b-1\right)}{p}\right).$$

In the above formula, use equation (2.6) for $\sum_{c=0}^{p-1}e\left(\frac{d(ba^4-1)c^4}{p}\right)$ we have

$$\begin{aligned}&\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\\ =\;& 2p-\sum_{a=1}^{p-1}\chi\left(a^5\right)+\frac{\tau(\chi_2)}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\overline{b}\right)\chi_2\left(ba^4-1\right)\sum_{d=1}^{p-1}\chi_2(d)e\left(\frac{md\left(b-1\right)}{p}\right)\\ =\;& 2p-\chi_2(m)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\right)\chi_2\left(b^5a^4-1\right)\chi_2(b-1), \qquad (2.7)\end{aligned}$$

where we have used $\tau^2\left(\chi_2\right)=-p$, if $p\equiv 3 \pmod 4$.

Similarly, if χ is an even character mod p with $\chi^5=\chi_0$, then from the method of proving (2.7) we have

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2=p+1-\chi_2(m)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a\right)\chi_2\left(b^5a^4-1\right)\chi_2(b-1). \qquad (2.8)$$

According to (2.7) and (2.8) we deduce Lemma 2.

Lemma 3. Let p be an odd prime, then we have the identity

$$\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2=\begin{cases} p(p-3), & \text{if } 4\mid(p-1),\\ p(p-1), & \text{if } 4\nmid(p-1).\end{cases}$$

Proof. In fact, for any positive integer $q>1$, from the trigonometric identity

$$\sum_{m=1}^{q}e\left(\frac{nm}{q}\right)=\begin{cases} q, & \text{if } q\mid n,\\ 0, & \text{if } q\nmid n\end{cases}$$

and the properties of the reduced residue system mod p we have

$$\begin{aligned}&\sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2=\sum_{m=0}^{p-1}\left|1+\sum_{a=1}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\\ =\;& p+\sum_{a=1}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)+\sum_{a=1}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{-ma^4-a}{p}\right)\\ &+\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{mb^4\left(a^4-1\right)+b(a-1)}{p}\right)\end{aligned}$$

$$= \quad p + p \sum_{\substack{a=1 \\ a^4 \equiv 1 \ (\bmod \ p)}}^{p-1} \sum_{b=1}^{p-1} e\left(\frac{b(a-1)}{p}\right) = p + p(p-1) - p \sum_{\substack{a=2 \\ a^4 \equiv 1 \ (\bmod \ p)}}^{p-1} 1$$

$$= \quad \begin{cases} p(p-3), & \text{if } 4 \mid (p-1), \\ p(p-1), & \text{if } 4 \nmid (p-1). \end{cases}$$

This proves Lemma 3.

Lemma 4. Let p be an odd prime with $p \equiv 1 \pmod 4$. Then for any four-order character λ mod p, we have the identity

$$\sum_{m=1}^{p-1} \lambda(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = \begin{cases} -2 \cdot \tau(\chi_2) \cdot \tau\left(\overline{\lambda}\right), & \text{if } 8 \mid (p-1), \\ 0, & \text{if } 8 \nmid (p-1). \end{cases}$$

Proof. Note that $\lambda^4 = \chi_0$, the principal character mod p. Then expand the square we have

$$\sum_{m=1}^{p-1} \lambda(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = \sum_{m=1}^{p-1} \lambda(m) + \sum_{a=1}^{p-1} \sum_{m=1}^{p-1} \lambda(m) e\left(\frac{ma^4 + a}{p}\right)$$
$$+ \sum_{a=1}^{p-1} \sum_{m=1}^{p-1} \lambda(m) e\left(\frac{-ma^4 - a}{p}\right) + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} \lambda(m) e\left(\frac{mb^4\left(a^4 - 1\right) + b(a-1)}{p}\right),$$

From the separability of Gauss sums and the trigonometric identity, the above formula is

$$\tau(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}^4(a) e\left(\frac{a}{p}\right) + \lambda(-1)\tau(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}^4(a) e\left(\frac{-a}{p}\right)$$
$$+\tau(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}\left(a^4 - 1\right) \sum_{b=1}^{p-1} \overline{\lambda}^4(b) e\left(\frac{b(a-1)}{p}\right)$$
$$= \quad -\tau(\lambda) - \lambda(-1)\tau(\lambda) - \tau(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}\left(a^4 - 1\right),$$

Decrease the power of a in the above formula we get

$$\sum_{m=1}^{p-1} \lambda(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2$$
$$= \quad -\tau(\lambda) - \lambda(-1)\tau(\lambda) - \tau(\lambda) \sum_{a=1}^{p-1} \left(1 + \lambda(a) + \chi_2(a) + \overline{\lambda}(a)\right) \overline{\lambda}(a-1)$$
$$= \quad -\tau(\lambda) \sum_{a=0}^{p-1} \overline{\lambda}(a-1) - \tau(\lambda) \sum_{a=0}^{p-1} \overline{\lambda}(1-a) - \tau(\lambda) \sum_{a=1}^{p-1} \chi_2(a) \overline{\lambda}(a-1)$$
$$-\tau(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}(a) \overline{\lambda}(a-1)$$

$$
\begin{aligned}
&= -\sum_{b=1}^{p-1}\lambda(b)\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{b(a-1)}{p}\right)-\sum_{b=1}^{p-1}\lambda(b)\sum_{a=1}^{p-1}\overline{\lambda}(a)e\left(\frac{b(a-1)}{p}\right)\\
&= -\lambda(-1)\tau(\chi_2)\tau\left(\overline{\lambda}\right)-\tau(\chi_2)\tau\left(\overline{\lambda}\right).
\end{aligned}
\tag{2.9}
$$

Note that $\lambda(-1) = -1$, if $4 \mid (p-1)$ and $8 \nmid (p-1)$; $\lambda(-1) = 1$, if $8 \mid (p-1)$. From (2.9) we may immediately deduce Lemma 4.

Lemma 5. Let p be an odd prime with $p \equiv 1 \pmod 4$, then we have

$$
\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 = -\lambda(-1)\left(\tau^2(\lambda)+\tau^2\left(\overline{\lambda}\right)\right),
$$

where λ is any four-order character mod p.

Proof. Note that $\chi_2(-1) = 1$, from the method of proving Lemma 4 we have

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 = \sum_{m=1}^{p-1}\chi_2(m)+\sum_{a=1}^{p-1}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{-ma^4-a}{p}\right)\\
&+\sum_{a=1}^{p-1}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{ma^4+a}{p}\right)+\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{mb^4\left(a^4-1\right)+b(a-1)}{p}\right)\\
=\ & -2\tau(\chi_2)+\tau(\chi_2)\sum_{a=1}^{p-1}\chi_2\left(a^4-1\right)\sum_{b=1}^{p-1}e\left(\frac{b(a-1)}{p}\right)\\
=\ & -2\tau(\chi_2)-\tau(\chi_2)\sum_{a=1}^{p-1}\chi_2\left(a^4-1\right)\\
=\ & -2\tau(\chi_2)-\tau(\chi_2)\sum_{a=1}^{p-1}\left(1+\lambda(a)+\chi_2(a)+\overline{\lambda}(a)\right)\chi_2(a-1)\\
=\ & -2\tau(\chi_2)-\tau(\chi_2)\left(\sum_{a=1}^{p-1}\chi_2(a-1)+\sum_{a=1}^{p-1}\lambda(a)\chi_2(a-1)+\sum_{a=1}^{p-1}\chi_2(1-a)+\sum_{a=1}^{p-1}\overline{\lambda}(a)\chi_2(a-1)\right)\\
=\ & -\lambda(-1)\left(\tau^2(\lambda)+\tau^2\left(\overline{\lambda}\right)\right),
\end{aligned}
$$

where we have used

$$
\sum_{a=1}^{p-1}\chi_2(a-1)+\sum_{a=1}^{p-1}\chi_2(1-a) = -\chi_2(-1)-\chi_2(1) = -2,
$$

and

$$
\begin{aligned}
&\sum_{a=1}^{p-1}\lambda(a)\chi_2(a-1) = \frac{1}{\tau(\chi_2)}\sum_{a=1}^{p-1}\lambda(a)\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b(a-1)}{p}\right)\\
=\ & \frac{1}{\tau(\chi_2)}\sum_{b=1}^{p-1}\chi_2(b)\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ba}{p}\right)e\left(\frac{-b}{p}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\tau(\lambda)}{\tau(\chi_2)} \sum_{b=1}^{p-1} \chi_2(b)\overline{\lambda}(b) e\left(\frac{-b}{p}\right) \\
&= \frac{\tau(\lambda)}{\tau(\chi_2)} \cdot \lambda(-1)\tau(\chi_2\overline{\lambda}) \\
&= \lambda(-1)\frac{\tau^2(\lambda)}{\tau(\chi_2)}.
\end{aligned}
$$

This proves Lemma 5.

Lemma 6. Let p be an odd prime with $p \equiv 3 \pmod 4$, then we have the identity

$$
\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = 0.
$$

Proof. Note that $\chi_2(-1) = -1$, from the properties of the complete residue system mod p we have

$$
\begin{aligned}
\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 &= \sum_{m=1}^{p-1} \chi_2(-m) \left| \sum_{a=0}^{p-1} e\left(\frac{-m(-a)^4 + (-a)}{p}\right) \right|^2 \\
= -\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{-ma^4 - a}{p}\right) \right|^2 &= -\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2.
\end{aligned}
$$

So that we have

$$
\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = 0.
$$

The lemma 6 is proved.

Lemma 7. Let p be an odd prime with $p \equiv 1 \pmod 4$. Then for any non-principal fourth character χ mod p, we have the estimate

$$
\left| \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \chi_2\left(b^5a^4 - 1\right)\chi_2(b-1) \right|^2 + \left| \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \overline{\lambda}\left(b^5a^4 - 1\right)\lambda(b-1) \right|^2 \ll p^2,
$$

where λ is any four-order character mod p.

Proof. Since $4 \mid (p-1)$, so it is clear that there exists an integer $1 < r < p-1$ such that $r^4 \equiv 1 \pmod p$ and $1 + \lambda(r) + \lambda\left(r^2\right) + \lambda\left(r^3\right) = 0$. For any integer m with $(m, p) = 1$, note that $\lambda(m) + \lambda(mr) + \lambda\left(r^2m\right) + \lambda\left(r^3m\right) = \lambda(m)\left(1 + \lambda(r) + \lambda(r^2) + \lambda\left(r^3\right)\right) = 0$, from Lemma 1 we know that for any fourth character χ mod p with $\chi^5 \neq \chi_0$, we have the identity

$$
\begin{aligned}
&\left| \sum_{a=1}^{p-1} \chi\left(a^5 + ma\right) \right|^2 + \left| \sum_{a=1}^{p-1} \chi\left(a^5 + mra\right) \right|^2 + \left| \sum_{a=1}^{p-1} \chi\left(a^5 + mr^2a\right) \right|^2 \\
&\quad + \left| \sum_{a=1}^{p-1} \chi\left(a^5 + mr^3a\right) \right|^2 = 16p, \qquad (2.10)
\end{aligned}
$$

where we have used the following identity

$$1+\chi_2(r)+\chi_2(r^2)+\chi_2(r^3)=2+2\chi_2(r)=0,$$

which is because that

$$\begin{aligned} & 1+\lambda(r)+\lambda\left(r^2\right)+\lambda\left(r^3\right) \\ = & 1+\lambda(r)+\chi_2(r)+\chi_2(r)\lambda(r) \\ = & (1+\lambda(r))(1+\chi_2(r))=0, \end{aligned}$$

where $\lambda(r)=1$, then $\chi_2(r)=-1$.

Therefore, for $i=0,\ 1,\ 2,\ 3$, applying (2.10) we have the estimate

$$\left|\sum_{a=1}^{p-1}\chi\left(a^5+mr^ia\right)\right|^2\leq 16p. \tag{2.11}$$

On the other hand, note that $\lambda^3(m)=\overline{\lambda}(m)$ and $\lambda^2(m)=\chi_2(m)$, applying Lemma 1 and the identity $1+\lambda(r)+\lambda\left(r^2\right)+\lambda\left(r^3\right)=0$ we also have

$$\begin{aligned} & \left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^4+\left|\sum_{a=1}^{p-1}\chi\left(a^5+mra\right)\right|^4 \\ & \quad +\left|\sum_{a=1}^{p-1}\chi\left(a^5+mr^2a\right)\right|^4+\left|\sum_{a=1}^{p-1}\chi\left(a^5+mr^3a\right)\right|^4 \\ = & \ 64p^2+4\left|\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\right|^2 \\ & \quad +8\left|\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\lambda}\left(b^5a^4-1\right)\lambda(b-1)\right|^2. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12) we can get the estimate

$$\left|\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\right|^2+\left|\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\lambda}\left(b^5a^4-1\right)\lambda(b-1)\right|^2\ll p^2.$$

This proves Lemma 7.

3. Proofs of the Theorems

Now we shall prove our main results. First we prove Theorem 1. If $4\nmid(p-1)$, then for any even character χ mod p with $\chi^5\neq\chi_0$, from Lemma 2, Lemma 3 and Lemma 6 we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2=2p\cdot\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2$$

$$
\begin{aligned}
&-\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\right)\\
=\;& 2p^2(p-1). \qquad (3.1)
\end{aligned}
$$

Similarly, if $\chi(-1)=1$ and $\chi^5=\chi_0$, then from Lemma 2, Lemma 3 and Lemma 6 we also have

$$
\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2=p\left(p^2-1\right). \qquad (3.2)
$$

Combining (3.1) and (3.2) we proved Theorem 1.

Proof of Theorem 2. If $4\mid(p-1)$, then for fourth character χ mod p with $\chi^5\neq\chi_0$, from Lemma 1 we have

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2=4p\cdot\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\\
&+\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\right)\\
&+\sum_{m=1}^{p-1}\lambda(-m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\lambda}\left(b^5a^4-1\right)\lambda(b-1)\right)\\
&+\sum_{m=1}^{p-1}\overline{\lambda}(-m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\lambda\left(b^5a^4-1\right)\overline{\lambda}(b-1)\right) \qquad (3.3)
\end{aligned}
$$

If 8—(p-1), put the results of the Lemma 3, Lemma 4, Lemma 5 and Lemma 7 into (3.3) we obtain

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\\
=\;& 4p\cdot p(p-3)-\left(\tau^2(\lambda)+\tau^2(\overline{\lambda})\right)\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\right)\\
&-2\tau(\chi_2)\tau(\overline{\lambda})\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\overline{\lambda}\left(b^5a^4-1\right)\lambda(b-1)\right)\\
&-2\tau(\chi_2)\tau(\lambda)\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\lambda\left(b^5a^4-1\right)\overline{\lambda}(b-1)\right)\\
=\;& 4p^2(p-3)+O\left(p^2\right).
\end{aligned}
$$

If $8\nmid(p-1)$, put the results of the Lemma 3, Lemma 4, Lemma 5 and Lemma 7 into (3.3) we obtain

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\\
= &\ 4p\cdot p(p-3)+\left(\tau^2(\lambda)+\tau^2(\overline{\lambda})\right)\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\chi_2\left(b^5a^4-1\right)\chi_2(b-1)\right).
\end{aligned}$$

Combining the above two formulas is easy to get

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\\
= &\ 4p^2(p-3)+O\left(p^2\right)=4p^3+O\left(p^2\right).
\end{aligned}\tag{3.4}$$

If χ is a fourth character mod p with $\chi^5=\chi_0$, then from Lemma 1 and the method of proving (3.4) we also have the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^5+ma\right)\right|^2\cdot\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2=3p^3+O\left(p^2\right).\tag{3.5}$$

It is clear that Theorem 2 follows from (3.4) and (3.5).

This completes the proofs of our all results.

4. Conclusions

The main results of this paper are two theorems. Let p be an odd prime with $4\nmid(p-1)$, for any non-principal even character χ mod p, the Theorem 1 gives exact calculating formulas for (1.2) with $h=5$ and $k=1$. Let p be an odd prime with $4\mid(p-1)$, for any fourth character χ mod p, the Theorem 2 gives asymptotic formulas for (1.2) with $h=5$ and $k=1$. As a supplement, the value of (1.2) with $h=5$ and $k=1$ in the trivial cases are given in the Some notes.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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