Mathematics

AIMS Mathematics, 6(10): 10964–10988.
DOI: 10.3934/math.2021637
Received: 26 March 2021
Accepted: 23 July 2021
Published: 29 July 2021

http://www.aimspress.com/journal/Math

Research article

New fuzzy-interval inequalities in fuzzy-interval fractional calculus by means of fuzzy order relation

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Abstract: It is well-known that interval analysis provides tools to deal with data uncertainty. In general, interval analysis is typically used to deal with the models whose data are composed of inaccuracies that may occur from certain kinds of measurements. In interval analysis and fuzzy-interval analysis, the inclusion relation ($\subseteq$) and fuzzy order relation ($\preccurlyeq$) both are two different concepts, respectively. In this article, with the help of fuzzy order relation, we introduce fractional Hermite-Hadamard inequality (HH-inequality) for h-convex fuzzy-interval-valued functions (h-convex-IVFs). Moreover, we also establish a strong relationship between h-convex fuzzy-IVFs and Hermite-Hadamard Fejér inequality (HH-Fejér inequality) via fuzzy Riemann Liouville fractional integral operator. It is also shown that our results include a wide class of new and known inequalities for h-convex fuzz-IVFs and their variant forms as special cases. Nontrivial examples are presented to illustrate the validity of the concept suggested in this review. This paper's techniques and approaches may serve as a springboard for further research in this field.

Keywords: h-convex fuzzy-interval-valued function; fuzzy-interval Riemann Liouville fractional integral operator; Hermite-Hadamard inequality; Hermite-Hadamard Fejér inequality
Mathematics Subject Classification: 26A33, 26A51, 26D10

1. Introduction

Hermite [1] and Hadamard [2] derived the familiar inequality is known as Hermite-Hadamard inequality (HH-inequality) and this inequality states that

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \leq \frac{1}{\nu-u}\int_u^\nu \mathcal{Q}(z)dz \leq \frac{\mathcal{Q}(u)+\mathcal{Q}(\nu)}{2}, \tag{1}$$

where $\mathcal{Q}: I \to \mathbb{R}$ is a convex function defined on a closed bounded interval $I \subseteq \mathbb{R}$ and $u, \nu \in I$ with $\nu > u$. If $\mathcal{Q}$ is a concave function, then both inequality symbols in (1) are reversed. Sine HH-inequalities are a useful technique for developing the qualitative and quantitative properties of convexity and nonconvexity. Because of diverse applications of these inequalities in different fields, there has been continuous growth of interest in such an area of research. Therefore many inequalities have been introduced as applications of convex functions and generalized convex function, see [3–6]. It is very important to mention that, Fejér [7] considered the major generalization of HH-inequality which is known as HH-Fejér inequality. It can be expressed as follows:
Let $\mathcal{Q}: \mathfrak{X} \to \mathbb{R}$ be a convex function on an interval $\mathfrak{X} = [u, \nu]$ and $u, \nu \in \mathfrak{X}$ with $u \leq \nu$. and let $\Omega: \mathfrak{X} = [u, \nu] \to \mathbb{R}, \Omega(z) \geq 0,$ be a integrable and symmetric with respect to $\frac{u+\nu}{2}$, and $\int_u^\nu \Omega(z)dz > 0$. Then, we have the following inequality.

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right).\int_u^\nu \Omega(z)dz \leq \int_u^\nu \mathcal{Q}(z)\Omega(z)dz \leq \frac{\mathcal{Q}(u)+\mathcal{Q}(\nu)}{2}.\int_u^\nu \Omega(z)dz. \tag{2}$$

If $\mathcal{Q}$ is a concave function, then inequality (2) is reversed. If $\Omega(z) = 1,$ then we obtain (1) from (2). It is also worthy to mention that Sarikaya et al. [8] provided the fractional version of inequality (1) and for convex function $\mathcal{Q}: \mathfrak{X} = [u, \nu] \to \mathbb{R}$, this inequality states that:

$$\mathcal{Q}\left(\frac{u+\nu}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha \mathcal{Q}(\nu) \tilde{+} \mathcal{J}_{\nu^-}^\alpha \mathcal{Q}(u)\right] \leq \frac{\mathcal{Q}(u)+\mathcal{Q}(\nu)}{2}, \tag{3}$$

where $\mathcal{Q}$ assumed to be a positive function on $[u, \nu]$, $\mathcal{Q} \in L_1([u, \nu])$ with $u \leq \nu$, and $\mathcal{J}_{u^+}^\alpha$ and $\mathcal{J}_{\nu^-}^\alpha$ are the left sided and right sided Riemann-Liouville fractional of order $0 \leq \alpha$, and respectively are defined as follows:

$$\mathcal{J}_{u^+}^\alpha \mathcal{Q}(z) = \frac{1}{\Gamma(\alpha)}\int_u^z (z-\tau)^{\alpha-1}\mathcal{Q}(\tau)\, d(\tau)(z > u), \tag{4}$$

$$\mathcal{J}_{\nu^-}^\alpha \mathcal{Q}(z) = \frac{1}{\Gamma(\alpha)}\int_z^\nu (\tau-z)^{\alpha-1}\mathcal{Q}(\tau)\, d(\tau)(z < \nu). \tag{5}$$

If $\alpha = 1,$ then from (3), we obtain (2). We can easily say that inequality (3) is generalization of inequality (2). Thereafter, many authors in the mathematical community have paid close attention in the view of inequality (3) and obtained several inequalities for different classes of convex and non-convex functions through various fractional integral; see [9–15].

On the other hand, it is well-known fact that interval-valued analysis was introduced as an attempt to overcome interval uncertainty that occurs in the computer or mathematical models of some deterministic real-word phenomena. A classic example of an interval closure is Archimedes' technique which is associated with the computation of the circumference of a circle. In 1966, Moore [16] given the concept of interval analysis in his book and discussed its applications in computational Mathematics. After that several authors have developed a strong relationship between inequalities and IVFs by means of inclusion relation via different integral operators, as one can see Costa [17], Costa and Roman-Flores [18], Roman-Flores et al. [19,20], and Chalco-Cano et al. [21,22], but also to more general set-valued maps by Nikodem et al. [23], and Matkowski and Nikodem [24]. In particular, Zhang et al. [25] derived the new version of Jensen's inequalities for set-valued and fuzzy set-valued functions by means of a pseudo order relation and proved that these Jensen's inequalities generalized form of Costa Jensen's inequalities [17]. After that, Budek [26] established fractional HH-inequality for convex-IVF through interval-valued fractional Riemann-Liouville fractional.

Our goal is to use the generalization of classical Riemann integral operator which is known as fuzzy Riemann-Liouville fractional integral operator. Recently, Allahviranloo et al. [27] introduced the following fuzzy-interval Riemann-Liouville fractional integral operator:

Let $\alpha > 0$ and $L([u, \nu], \mathbb{F}_0)$ be the collection of all Lebesgue measurable fuzzy-IVFs on$[u, \nu]$. Then, the fuzzy-interval left and right Riemann-Liouville fractional integral of $\tilde{Q} \in L([u, \nu], \mathbb{F}_0)$ with order $\alpha > 0$ are defined by

$$\mathcal{J}_{u^+}^{\alpha}\tilde{Q}(z) = \frac{1}{\Gamma(\alpha)}\int_u^z (z-\tau)^{\alpha-1}\tilde{Q}(\tau)\, d(\tau),\ (z > u), \tag{6}$$

and

$$\mathcal{J}_{\nu^-}^{\alpha}\tilde{Q}(z) = \frac{1}{\Gamma(\alpha)}\int_z^{\nu} (\tau - z)^{\alpha-1}\tilde{Q}(\tau)\, d(\tau),\ (z < \nu), \tag{7}$$

respectively, where $\Gamma(z) = \int_0^{\infty} \tau^{z-1}u^{-\tau}\, d(\tau)$ is the Euler gamma function. The fuzzy-interval left and right Riemann-Liouville fractional integral z based on left and right endpoint functions can be defined, that is

$$\begin{aligned}\left[\mathcal{J}_{u^+}^{\alpha}\tilde{Q}(z)\right]^{\gamma} &= \frac{1}{\Gamma(\alpha)}\int_u^z (z-\tau)^{\alpha-1}Q_{\gamma}(\tau)\, d(\tau)\\ &= \frac{1}{\Gamma(\alpha)}\int_u^z (z-\tau)^{\alpha-1}[Q_*(\tau,\gamma), Q^*(\tau,\gamma)]\, d(\tau), (z > u),\end{aligned} \tag{8}$$

where

$$\mathcal{J}_{u^+}^{\alpha}Q_*(z,\gamma) = \frac{1}{\Gamma(\alpha)}\int_u^z (z-\tau)^{\alpha-1}Q_*(\tau,\gamma)\, d(\tau), (z > u), \tag{9}$$

and

$$\mathcal{J}_{u^+}^{\alpha}Q^*(z,\gamma) = \frac{1}{\Gamma(\alpha)}\int_u^z (z-\tau)^{\alpha-1}Q^*(\tau,\gamma)\, d(\tau),\ (z > u). \tag{10}$$

Similarly, we can define the right Riemann-Liouville fractional integral $\tilde{Q}$ of z based on left and right endpoint functions.

Moreover, recently, Khan et al. [28] introduced the new class of convex fuzzy mappings is known as (h_1, h_2)-convex fuzzy-IVFs by means fuzzy order relation and presented the following new version of HH-type inequality for (h_1, h_2)-convex fuzzy-IVF involving fuzzy-interval Riemann integrals:

Theorem 1.1. Let $\tilde{Q}: [u, \nu] \to \mathbb{F}_0$ be a (h_1, h_2)-convex fuzzy-IVF with $h_1, h_2: [0, 1] \to \mathbb{R}^+$ and

$h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$, whose γ-levels define the family of IVFs $\mathcal{Q}_\gamma: [u,\nu] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $\mathcal{Q}_\gamma(z) = [\mathcal{Q}_*(z,\gamma), \mathcal{Q}^*(z,\gamma)]$ for all $z \in [u,\nu]$ and for all $\gamma \in [0,1]$. If $\tilde{\mathcal{Q}}$ is fuzzy-interval Riemann integrable (in sort, FR-integrable), then

$$\frac{1}{2\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}(FR)\int_u^\nu \tilde{\mathcal{Q}}(z)dz \preccurlyeq \left[\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(\nu)\right]\int_0^1 h_1(\tau)\,h_2(1-\tau)\,d\tau. \quad (11)$$

If $h_1(\tau) = \tau$ and $h_2(\tau) \equiv 1$, then from inequality (11), we obtain the following inequality:

$$\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}(FR)\int_u^\nu \tilde{\mathcal{Q}}(z)dz \preccurlyeq \frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(\nu)}{2}. \quad (12)$$

This inequality (12) is known as HH-inequality for convex fuzzy-IVF. We refer readers to [29–53] and the references therein for further review of literature on the applications and properties of fuzzy-interval, inequalities, and generalized convex fuzzy mappings.

Inspired by the ongoing research work, the new class of generalized convex fuzzy-IVFs is introduced which is known as h-convex fuzzy-IVF. With the help of h-convex fuzzy-IVF and fuzzy-interval Riemann fractional integral operator, we have introduced fuzzy fractional Hermite-Hadamard type inequalities by means of fuzzy order relation. Moreover, we have shown that our results include a wide class of new and known inequalities for h-convex fuzzy-IVFs and their variant forms as special cases. Some useful examples are also presented to verify the validity of our main results.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathcal{K}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\eta \in \mathcal{K}_C$ be defined by

$$\eta = [\eta_*, \eta^*] = \{z \in \mathbb{R} |\, \eta_* \leq z \leq \eta^*\},\ (\eta_*, \eta^* \in \mathbb{R}). \quad (13)$$

If $\eta_* = \eta^*$, then η is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\eta_* \geq 0$, then $[\eta_*, \eta^*]$ is called positive interval. The set of all positive interval is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{[\eta_*, \eta^*]: [\eta_*, \eta^*] \in \mathcal{K}_C \text{ and } \eta_* \geq 0\}$.

Let $\tau \in \mathbb{R}$ and $\tau\eta$ be defined by

$$\tau\eta = \begin{cases} [\tau\eta_*, \tau\eta^*] & \text{if } \tau > 0, \\ \{0\} & \text{if } \tau = 0 \\ [\tau\eta^*, \tau\eta_*] & \text{if } \tau < 0. \end{cases} \quad (14)$$

Then the Minkowski difference $-\eta$, addition $\eta + \zeta$ and $\eta \times \zeta$ for $\eta, \zeta \in \mathcal{K}_C$ are defined by

$$\begin{aligned} [\zeta_*, \zeta^*] - [\eta_*, \eta^*] &= [\zeta_* - \eta^*,\ \zeta^* - \eta_*], \\ [\zeta_*, \zeta^*] + [\eta_*, \eta^*] &= [\zeta_* + \eta_*,\ \zeta^* + \eta^*], \end{aligned} \quad (15)$$

and

$$[\zeta_*, \zeta^*] \times [\eta_*, \eta^*] = [min\{\zeta_*\eta_*, \zeta^*\eta_*, \zeta_*\eta^*, \zeta^*\eta^*\}, max\{\zeta_*\eta_*, \zeta^*\eta_*, \zeta_*\eta^*, \zeta^*\eta^*\}].$$

The inclusion " $\subseteq$ " means that

$$\zeta \subseteq \eta \text{ if and only if, } [\zeta_*, \zeta^*] \subseteq [\eta_*, \eta^*], \text{ if and only if } \eta_* \leq \zeta_*, \ \zeta^* \leq \eta^*. \quad (16)$$

Remark 2.1. [29] The relation " $\leq_I$ " defined on $\mathcal{K}_C$ by

$$[\zeta_*, \zeta^*] \leq_I [\eta_*, \eta^*] \text{ if and only if } \zeta_* \leq \eta_*, \ \zeta^* \leq \eta^*, \quad (17)$$

for all $[\zeta_*, \zeta^*]$, $[\eta_*, \eta^*] \in \mathcal{K}_C$, it is an order relation. For given $[\zeta_*, \zeta^*]$, $[\eta_*, \eta^*] \in \mathcal{K}_C$, we say that $[\zeta_*, \zeta^*] \leq_I [\eta_*, \eta^*]$ if and only if $\zeta_* \leq \eta_*$, $\zeta^* \leq \eta^*$.

For $[\zeta_*, \zeta^*]$, $[\eta_*, \eta^*] \in \mathcal{K}_C$, the Hausdorff-Pompeiu distance between intervals $[\zeta_*, \zeta^*]$ and $[\eta_*, \eta^*]$ is defined by

$$d([\zeta_*, \zeta^*], [\eta_*, \eta^*]) = max\{|\zeta_* - \eta_*|, |\zeta^* - \eta^*|\}.$$

It is familiar fact that $(\mathcal{K}_C, d)$ is a complete metric space.

A fuzzy subset A of $\mathbb{R}$ is characterize by a mapping $\tilde{\zeta}: \mathbb{R} \to [0,1]$ called the membership function, for each fuzzy set and if $\gamma \in (0, 1]$, then γ-level sets of $\tilde{\zeta}$ is denoted and defined as follows $\zeta_\gamma = \{u \in \mathbb{R} | \ \tilde{\zeta}(u) \geq \gamma\}$. If $\gamma = 0$, then $supp(\tilde{\zeta}) = \{z \in \mathbb{R} | \ \tilde{\zeta}(z) > 0\}$ is called support of $\tilde{\zeta}$. By $[\tilde{\zeta}]^0$ we define the closure of $supp(\tilde{\zeta})$.

Let $\mathbb{F}(\mathbb{R})$ be the family of all fuzzy sets and $\tilde{\zeta} \in \mathbb{F}(\mathbb{R})$ be a fuzzy set. Then, we define the following:

(1) $\tilde{\zeta}$ is said to be normal if there exists $z \in \mathbb{R}$ and $\tilde{\zeta}(z) = 1$;
(2) $\tilde{\zeta}$ is said to be upper semi continuous on $\mathbb{R}$ if for given $z \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\tilde{\zeta}(z) - \tilde{\zeta}(x) < \varepsilon$ for all $x \in \mathbb{R}$ with $|z - x| < \delta$;
(3) $\tilde{\zeta}$ is said to be fuzzy convex if ζ_γ is convex for every $\gamma \in [0, 1]$;
(4) $\tilde{\zeta}$ is compactly supported if $supp(\tilde{\zeta})$ is compact.

A fuzzy set is called a fuzzy number or fuzzy-interval if it has properties (1)–(4). We denote by $\mathbb{F}_0$ the family of all interval.

From these definitions, we have

$$[\tilde{\zeta}]^\gamma = [\zeta_*(\gamma), \zeta^*(\gamma)],$$

where

$$\zeta_*(\gamma) = inf\{z \in \mathbb{R} | \ \tilde{\zeta}(z) \geq \gamma\}, \ \zeta^*(\gamma) = sup\{z \in \mathbb{R} | \ \tilde{\zeta}(z) \geq \gamma\}.$$

Proposition 2.2. [18] If $\tilde{\zeta}, \tilde{\eta} \in \mathbb{F}_0$, then relation " $\preccurlyeq$ " defined on $\mathbb{F}_0$ by

$$\tilde{\zeta} \preccurlyeq \tilde{\eta} \text{ if and only if, } [\tilde{\zeta}]^\gamma \leq_I [\tilde{\eta}]^\gamma, \text{ for all } \gamma \in [0, 1], \quad (18)$$

this relation is known as partial order relation.

For $\tilde{\zeta}, \tilde{\eta} \in \mathbb{F}_0$ and $\tau \in \mathbb{R}$, the sum $\tilde{\zeta} \tilde{+} \tilde{\eta}$, product $\tilde{\zeta} \tilde{\times} \tilde{\eta}$, scalar product $\tau . \tilde{\zeta}$ and sum with scalar are defined by:

$$[\tilde{\zeta} \tilde{+} \tilde{\eta}]^\gamma = [\tilde{\zeta}]^\gamma + [\tilde{\eta}]^\gamma, \quad (19)$$

$$[\tilde{\zeta} \tilde{\times} \tilde{\eta}]^\gamma = [\tilde{\zeta}]^\gamma \times [\tilde{\eta}]^\gamma, \quad (20)$$

$$\left[\tau.\tilde{\zeta}\right]^{\gamma} = \tau.\left[\tilde{\zeta}\right]^{\gamma}, \tag{21}$$

$$\left[\tau\tilde{+}\tilde{\zeta}\right]^{\gamma} = \tau + \left[\tilde{\zeta}\right]^{\gamma}. \tag{22}$$

for all $\gamma \in [0,1]$. For $\tilde{\psi} \in \mathbb{F}_0$ such that $\tilde{\zeta} = \tilde{\eta}\tilde{+}\tilde{\psi}$, then by this result we have existence of Hukuhara difference of $\tilde{\zeta}$ and $\tilde{\eta}$, and we say that $\tilde{\psi}$ is the H-difference of $\tilde{\zeta}$ and $\tilde{\eta}$, and denoted by $\tilde{\zeta}\tilde{-}\tilde{\eta}$**.** If H-difference exists, then

$$(\psi)^*(\gamma) = (\zeta - \eta)^*(\gamma) = \zeta^*(\gamma) - \eta^*(\gamma),\ (\psi)_*(\gamma) = (\zeta - \eta)_*(\gamma) = \zeta_*(\gamma) - \eta_*(\gamma). \tag{23}$$

A partition of $[u, v]$ is any finite ordered subset P having the form

$$P = \{u = z_1 < z_2 < z_3 < z_4 < z_5 \dots\dots < z_k = v\}.$$

The mesh of a partition P is the maximum length of the subintervals containing P that is,

$$\text{mesh}(P) = max\{z_j - z_{j-1} : j = 1, 2, 3, \dots\dots k\}.$$

Let $\mathcal{P}(\delta, [u, v])$ be the set of all partitions P of $[u, v]$ such that mesh$(P) < \delta$. For each interval $[z_{j-1},\ z_j]$, where $1 \le j \le k$, choose an arbitrary point ξ_j and taking the sum

$$S(Q, P, \delta) = \sum_{j=1}^{k} Q(\xi_j)(z_j - z_{j-1}),$$

where $Q: [u, v] \to \mathcal{K}_C$. We call $S(Q, P, \delta)$ a Riemann sum of Q corresponding to $P \in \mathcal{P}(\delta, [u, v])$.

Definition 2.3. [30] A function $Q: [u, v] \to \mathcal{K}_C$ is called interval Riemann integrable (IR-integrable) on $[u, v]$ if there exists $B \in \mathcal{K}_C$ such that, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(S(Q, P, \delta), B) < \epsilon,$$

for every Riemann sum of Q corresponding to $P \in \mathcal{P}(\delta, [u, v])$ and for arbitrary choice of $\xi_j \in [z_{j-1},\ z_j]$ for $1 \le j \le k$. Then, we say that B is the IR-integral of Q on $[u, v]$ and is denote by $B = (IR)\int_u^v Q(z)dz$.

Moore [9] firstly proposed the concept of Riemann integral for IVF and it is defined as follow:

Theorem 2.4. [16] If $Q: [u, v] \subset \mathbb{R} \to \mathcal{K}_C$ is an IVF on such that $Q(z) = [Q_*, Q^*]$, then Q is Riemann integrable over $[u, v]$ if and only if, Q_* and Q^* both are Riemann integrable over $[u, v]$ such that

$$(IR)\int_u^v Q(z)dz = \left[(R)\int_u^v Q_*(z)dz\,, (R)\int_u^v Q^*(z)dz\right]. \tag{24}$$

Definition 2.5. [31] A fuzzy map $\tilde{Q}: [u, v] \to \mathbb{F}_0$ is called fuzzy-IVF. For each $\gamma \in [0,1]$, whose γ-levels define the family of IVFs $Q_\gamma: [u, v] \to \mathcal{K}_C$ are given by $Q_\gamma(z) = [Q_*(z, \gamma), Q^*(z, \gamma)]$ for all $z \in [u, v]$. Here, for each $\gamma \in [0,1]$, the left and right real valued functions $Q_*(z, \gamma), Q^*(z, \gamma): [u, v] \to \mathbb{R}$ are also called lower and upper functions of $\tilde{Q}$.

Remark 2.6. If $\tilde{Q}: [u, v] \subset \mathbb{R} \to \mathbb{F}_0$ is a fuzzy-IVF, then $\tilde{Q}(z)$ is called continuous function at $z \in [u, v]$, if for each $\gamma \in [0,1]$, both left and right real valued functions $Q_*(z, \gamma)$ and $Q^*(z, \gamma)$ are continuous at $z \in K$.

The following conclusion can be drawn from the above literature review, see [17, 31].

Definition 2.7. Let $\tilde{Q}: [u, v] \subset \mathbb{R} \to \mathbb{F}_0$ is called fuzzy-IVF. The fuzzy Riemann integral of $\tilde{Q}$ over $[u, v]$, denoted by $(FR)\int_u^v \tilde{Q}(z)dz$, it is defined level by level

$$\left[(FR)\int_u^v \tilde{Q}(z)dz\right]^{\gamma} = (IR)\int_u^v Q_{\gamma}(z)dz = \left\{\int_u^v Q(z,\gamma)dz : Q(z,\gamma) \in \mathcal{R}_{[u,v]}\right\}, \tag{25}$$

for all $\gamma \in [0, 1]$, where $\mathcal{R}_{[u,v]}$ contains the family of left and right functions of IVFs. $\tilde{Q}$ is (FR)-integrable over $[u, v]$ if $(FR)\int_u^v \tilde{Q}(z)dz \in \mathbb{F}_0$. Note that, if left and right real valued functions are Lebesgue-integrable, then $\tilde{Q}$ is fuzzy Aumann-integrable over $[u, v]$, denoted by $(FA)\int_u^v \tilde{Q}(z)dz$, see [31].

Theorem 2.8. Let $\tilde{Q}: [u, v] \subset \mathbb{R} \to \mathbb{F}_0$ be a fuzzy-IVF, whose γ-levels obtain the collection of IVFs $Q_{\gamma}: [u, v] \subset \mathbb{R} \to \mathcal{K}_C$ are defined by $Q_{\gamma}(z) = [Q_*(z,\gamma), Q^*(z,\gamma)]$ for all $z \in [u, v]$ and for all $\gamma \in [0, 1]$. Then, $\tilde{Q}$ is (FR)-integrable over $[u, v]$ if and only if, $Q_*(z,\gamma)$ and $Q^*(z,\gamma)$ both are R-integrable over $[u, v]$. Moreover, if $\tilde{Q}$ is (FR)-integrable over $[u, v]$, then

$$\left[(FR)\int_u^v \tilde{Q}(z)dz\right]^{\gamma} = \left[(R)\int_u^v Q_*(z,\gamma)dz\,, (R)\int_u^v Q^*(z,\gamma)dz\right] = (IR)\int_u^v Q_{\gamma}(z)dz, \tag{26}$$

for all $\gamma \in [0, 1]$.

Definition 2.9. A real valued function $Q: [u, v] \to \mathbb{R}^+$ is called convex function if

$$Q(\tau x + (1-\tau)z) \leq \tau Q(x) + (1-\tau)Q(z), \tag{27}$$

for all $x, z \in [u, v], \tau \in [0, 1]$. If (27) is reversed, then Q is called concave.

Definition 2.10. [32] The fuzzy-IVF $\tilde{Q}: [u, v] \to \mathbb{F}_0$ is called convex fuzzy-IVF on $[u, v]$ if

$$\tilde{Q}(\tau x + (1-\tau)z\,) \preccurlyeq \tau\tilde{Q}(x)\tilde{+}(1-\tau)\tilde{Q}(z), \tag{28}$$

for all $x, z \in [u, v], \tau \in [0, 1]$, where $\tilde{Q}(z) \succcurlyeq \tilde{0}$ for all $z \in [u, v]$. If (28) is reversed, then $\tilde{Q}$ is called concave fuzzy-IVF on $[u, v]$. $\tilde{Q}$ is affine if and only if it is both convex and concave fuzzy-IVF.

Remark 2.11. If $Q_*(z,\gamma) = Q^*(z,\gamma)$ and $\gamma = 1$, then we obtain the inequality (1).

Definition 2.12. [28] Let $h_1, h_2: [0, 1] \subseteq [u, v] \to \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$. Then, fuzzy-IVF $\tilde{Q}: [u, v] \to \mathbb{F}_0$ is said to be (h_1, h_2)-convex fuzzy-IVF on $[u, v]$ if

$$\tilde{Q}(\tau x + (1-\tau)z) \preccurlyeq h_1(\tau)\, h_2(1-\tau)\, \tilde{Q}(x)\tilde{+}\, h_1(1-\tau)\, h_2(\tau)\, \tilde{Q}(z), \tag{29}$$

for all $x, z \in [u, v], \tau \in [0, 1]$, where $\tilde{Q}(x) \succcurlyeq \tilde{0}$. If $\tilde{Q}$ is (h_1, h_2)-concave on $[u, v]$, then inequality (29) is reversed.

Remark 2.13. [28] If $h_2(\tau) \equiv 1$, then (h_1, h_2)-convex fuzzy-IVF becomes h-convex fuzzy-IVF, that is

$$\tilde{Q}(\tau x + (1-\tau)z) \preccurlyeq h_1(\tau)\, \tilde{Q}(x)\tilde{+}\, h_1(1-\tau)\, \tilde{Q}(z), \forall\, x, z \in [u, v], \tau \in [0, 1]. \tag{30}$$

If $h_1(\tau)=\tau, h_2(\tau)\equiv 1$, then (h_1,h_2)-convex fuzzy-IVF becomes convex fuzzy-IVF, that is

$$\tilde{Q}(\tau x+(1-\tau)z) \preccurlyeq \tau\tilde{Q}(x)\tilde{+}(1-\tau)\tilde{Q}(z), \forall\, x,z\in[u,\nu], \tau\in[0,1]. \quad (31)$$

If $h_1(\tau)=h_2(\tau)\equiv 1$, then (h_1,h_2)-convex fuzzy-IVF becomes P-convex fuzzy-IVF, that is

$$\tilde{Q}(\tau x+(1-\tau)z) \preccurlyeq \tilde{Q}(x)\tilde{+}\tilde{Q}(z), \forall\ x,z\in[u,\nu], \tau\in[0,1]. \quad (32)$$

Theorem 2.14. Let $h:[0,1]\subseteq[u,\nu]\to\mathbb{R}$ be anon-negative real valued function such that $h\not\equiv 0$ and let $\tilde{Q}:[u,\nu]\to\mathbb{F}_0$ be a fuzzy-IVF, whose γ-levels define the family of IVFs $Q_\gamma:[u,\nu]\to\mathcal{K}_C{}^+\subset\mathcal{K}_C$ are given by

$$Q_\gamma(z)=[Q_*(z,\gamma),Q^*(z,\gamma)], \quad (33)$$

for all $z\in[u,\nu]$ and for all $\gamma\in[0,1]$. Then, $\tilde{Q}$ is h-convex fuzzy-IVF on $[u,\nu]$, if and only if, for all $\gamma\in[0,1]$, $Q_*(z,\gamma)$ and $Q^*(z,\gamma)$ are h-convex function.

Proof. The demonstration of proof of Theorem 2.14 is similar to the demonstration proof of Theorem 6 in [28].

Example 2.15. We consider $h(\tau)=\tau$, for $\tau\in[0,1]$ and the fuzzy-IVF $\tilde{Q}:[0,4]\to\mathbb{F}_0$ defined by

$$\tilde{Q}(z)(\sigma)=\begin{cases}\dfrac{\sigma}{2e^{z^2}} & \sigma\in\left[0,2e^{z^2}\right]\\ \dfrac{4e^{z^2}-\sigma}{2e^{z^2}} & \sigma\in(2e^{z^2},4e^{z^2}]\\ 0 & \text{otherwise,}\end{cases}$$

then, for each $\gamma\in[0,1]$, we have $Q_\gamma(z)=\left[2\gamma e^{z^2},2(2-\gamma)e^{z^2}\right]$. Since end point functions $Q_*(z,\gamma)$, $Q^*(z,\gamma)$ are h-convex functions for each $\gamma\in[0,1]$. Hence $\tilde{Q}(z)$ is h-convex fuzzy-IVF.

3. Fuzzy-interval fractional Hermite-Hadamard type inequalities

In this section, we will prove some new Hermite-Hadamard type inequalities for h-convex fuzzy-IVFs by means of fuzzy order relation via Riemann Liouville fractional integral operator. In what follows, we denote by $L([u,\nu],\mathbb{F}_0)$ the family of Lebesgue measureable fuzzy-IVFs.

Theorem 3.1. Let $\tilde{Q}:[u,\nu]\to\mathbb{F}_0$ be a h-convex fuzzy-IVF on $[u,\nu]$, whose γ-levels define the family of IVFs $Q_\gamma:[u,\nu]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $Q_\gamma(z)=[Q_*(z,\gamma),Q^*(z,\gamma)]$ for all $z\in[u,\nu]$ and for all $\gamma\in[0,1]$. If $\tilde{Q}\in L([u,\nu],\mathbb{F}_0)$, then

$$\frac{1}{\alpha\, h\left(\frac{1}{2}\right)}\tilde{Q}\left(\frac{u+\nu}{2}\right)\preccurlyeq\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha\tilde{Q}(\nu)\tilde{+}\mathcal{J}_{\nu^-}^\alpha\tilde{Q}(u)\right]\preccurlyeq\frac{\tilde{Q}(u)\tilde{+}\tilde{Q}(\nu)}{2}\int_0^1\tau^{\alpha-1}[h(\tau)-h(1-\tau)]d\tau. \quad (34)$$

If $\tilde{Q}(z)$ is concave fuzzy-IVF, then

$$\frac{1}{\alpha\, h\left(\frac{1}{2}\right)}\tilde{Q}\left(\frac{u+\nu}{2}\right)\succcurlyeq\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha\tilde{Q}(\nu)\tilde{+}\mathcal{J}_{\nu^-}^\alpha\tilde{Q}(u)\right]\succcurlyeq\frac{\tilde{Q}(u)\tilde{+}\tilde{Q}(\nu)}{2}\int_0^1\tau^{\alpha-1}[h(\tau)-h(1-\tau)]d\tau. \quad (35)$$

Proof. Let $\tilde{Q}:[u,\nu]\to\mathbb{F}_0$ be a h-convex fuzzy-IVF. Then, by hypothesis, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\tilde{Q}\left(\frac{u+\nu}{2}\right) \preccurlyeq \tilde{Q}(\tau u+(1-\tau)\nu)\tilde{+}\tilde{Q}\big((1-\tau)u+\tau\nu\big).$$

Therefore, for every $\gamma \in [0,1]$, we have

$$\frac{1}{h\left(\frac{1}{2}\right)}Q_*\left(\frac{u+\nu}{2},\gamma\right) \leq Q_*(\tau u+(1-\tau)\nu,\gamma)+Q_*\big((1-\tau)u+\tau\nu,\gamma\big),$$
$$\frac{1}{h\left(\frac{1}{2}\right)}Q^*\left(\frac{u+\nu}{2},\gamma\right) \leq Q^*(\tau u+(1-\tau)\nu,\gamma)+Q^*\big((1-\tau)u+\tau\nu,\gamma\big).$$

Multiplying both sides by $\tau^{\alpha-1}$ and integrating the obtained result with respect to τ over (0,1), we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1 \tau^{\alpha-1}Q_*\left(\frac{u+\nu}{2},\gamma\right)d\tau$$
$$\leq \int_0^1 \tau^{\alpha-1}Q_*(\tau u+(1-\tau)\nu,\gamma)d\tau+\int_0^1 \tau^{\alpha-1}Q_*\big((1-\tau)u+\tau\nu,\gamma\big)d\tau,$$
$$\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1 \tau^{\alpha-1}Q^*\left(\frac{u+\nu}{2},\gamma\right)d\tau$$
$$\leq \int_0^1 \tau^{\alpha-1}Q^*(\tau u+(1-\tau)\nu,\gamma)d\tau+\int_0^1 \tau^{\alpha-1}Q^*\big((1-\tau)u+\tau\nu,\gamma\big)d\tau.$$

Let $x=\tau u+(1-\tau)\nu$ and $z=(1-\tau)u+\tau\nu$. Then, we have

$$\frac{1}{\alpha\, h\left(\frac{1}{2}\right)}Q_*\left(\frac{u+\nu}{2},\gamma\right) \leq \frac{1}{(\nu-u)^\alpha}\int_u^\nu (\nu-x)^{\alpha-1}Q_*(x,\gamma)\,dx+\frac{1}{(\nu-u)^\alpha}\int_u^\nu (z-u)^{\alpha-1}Q_*(z,\gamma)\,dz$$
$$\frac{1}{\alpha\, h\left(\frac{1}{2}\right)}Q^*\left(\frac{u+\nu}{2},\gamma\right) \leq \frac{1}{(\nu-u)^\alpha}\int_u^\nu (\nu-x)^{\alpha-1}Q^*(x,\gamma)\,dx+\frac{1}{(\nu-u)^\alpha}\int_u^\nu (z-u)^{\alpha-1}Q^*(z,\gamma)\,dz,$$
$$\leq \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha Q_*(\nu,\gamma)+\mathcal{J}_{\nu^-}^\alpha Q_*(u,\gamma)\right]$$
$$\leq \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha Q^*(\nu,\gamma)+\mathcal{J}_{\nu^-}^\alpha Q^*(u,\gamma)\right].$$

That is

$$\frac{1}{\alpha\, h\left(\frac{1}{2}\right)}\left[Q_*\left(\frac{u+\nu}{2},\gamma\right),Q^*\left(\frac{u+\nu}{2},\gamma\right)\right]$$
$$\leq_I \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\Big[\left[\mathcal{J}_{u^+}^\alpha Q_*(\nu,\gamma)+\mathcal{J}_{\nu^-}^\alpha Q_*(u,\gamma)\right],\left[\mathcal{J}_{u^+}^\alpha Q^*(\nu,\gamma)+\mathcal{J}_{\nu^-}^\alpha Q^*(u,\gamma)\right]\Big],$$

thus,

$$\frac{1}{\alpha\, h\left(\frac{1}{2}\right)}Q_\gamma\left(\frac{u+\nu}{2}\right) \leq_I \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha Q_\gamma(\nu)+\mathcal{J}_{\nu^-}^\alpha Q_\gamma(u)\right]. \tag{36}$$

In a similar way as above, we have

$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_{\gamma}(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_{\gamma}(u)\right]\le_I\left[\mathcal{Q}_{\gamma}(u)+\mathcal{Q}_{\gamma}(\nu)\right]\int_0^1\tau^{\alpha-1}[h(\tau)-h(1-\tau)]d\tau. \quad (37)$$

Combining (36) and (37), we have

$$\frac{1}{\alpha h\left(\frac{1}{2}\right)}\mathcal{Q}_{\gamma}\left(\frac{u+\nu}{2}\right)\le_I\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_{\gamma}(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_{\gamma}(u)\right]$$

$$\le_I\left[\mathcal{Q}_{\gamma}(u)+\mathcal{Q}_{\gamma}(\nu)\right]\int_0^1\tau^{\alpha-1}[h(\tau)-h(1-\tau)]d\tau,$$

that is

$$\frac{1}{\alpha h\left(\frac{1}{2}\right)}\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right)\preccurlyeq\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\tilde{\mathcal{Q}}(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}\tilde{\mathcal{Q}}(u)\right]\preccurlyeq\left[\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(\nu)\right]\int_0^1\tau^{\alpha-1}[h(\tau)-h(1-\tau)]d\tau.$$

Hence, the required result.

Remark 3.2 From Theorem 3.1 we clearly see that:
If $\alpha=1$, then Theorem 3.1 reduces to the result for h-convex fuzzy-IVF:

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-u}(FR)\int_u^{\nu}\tilde{\mathcal{Q}}(z)dz\preccurlyeq\left[\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(\nu)\right]\int_0^1 h(\tau)\,d\tau. \quad (38)$$

If $h(\tau)=\tau$, then Theorem 3.1 reduces to the result for convex fuzzy-IVF:

$$\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right)\preccurlyeq\frac{\Gamma(\alpha+1)}{2(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\tilde{\mathcal{Q}}(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}\tilde{\mathcal{Q}}(u)\right]\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(\nu)}{2}. \quad (39)$$

Let $\alpha=1$ and $h(\tau)=\tau$. Then, Theorem 3.1 reduces to the result for convex-IVF given in [28]:

$$\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-u}(FR)\int_u^{\nu}\tilde{\mathcal{Q}}(z)dz\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(\nu)}{2}. \quad (40)$$

If $\mathcal{Q}_*(z,\gamma)=\mathcal{Q}^*(z,\gamma)$ and $\gamma=1$, then, from Theorem 3.1 we get following inequality given in [12]:

$$\frac{1}{\alpha\,h\left(\frac{1}{2}\right)}\mathcal{Q}\left(\frac{u+\nu}{2}\right)\le\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}(u)\right]\le[\mathcal{Q}(u)+\mathcal{Q}(\nu)]\int_0^1\tau^{\alpha-1}[h(\tau)-h(1-\tau)]d\tau. \quad (41)$$

Let $\alpha=1=\gamma$ and $\mathcal{Q}_*(z,\gamma)=\mathcal{Q}^*(z,\gamma)$. Then, from Theorem 3.1 we obtain following inequality given in [2]:

$$\frac{1}{2\,h\left(\frac{1}{2}\right)}\mathcal{Q}\left(\frac{u+\nu}{2}\right)\le\frac{1}{\nu-u}(R)\int_u^{\nu}\mathcal{Q}(z)dz\le[\mathcal{Q}(u)+\mathcal{Q}(\nu)]\int_0^1 h(\tau)\,d\tau. \quad (42)$$

Example 3.3. Let $=\frac{1}{2}$, $h(\tau)=\tau$, for all $\tau\in[0,1]$and the fuzzy-IVF $\tilde{\mathcal{Q}}:[u,\nu]=[2,3]\to\mathbb{F}_0$, defined by

$$\tilde{Q}(z)(\theta)=\begin{cases}\dfrac{\theta}{2-z^{\frac{1}{2}}}, & \theta\in\left[0,2-z^{\frac{1}{2}}\right]\\ \dfrac{2\left(2-z^{\frac{1}{2}}\right)-\theta}{2-z^{\frac{1}{2}}}, & \theta\in(2-z^{\frac{1}{2}},2\left(2-z^{\frac{1}{2}}\right)]\\ 0, & \text{otherwise.}\end{cases}$$

Then, for each $\gamma\in[0,1]$, we have $Q_{\gamma}(z)=\left[\gamma\left(2-z^{\frac{1}{2}}\right),(2-\gamma)\left(2-z^{\frac{1}{2}}\right)\right]$. Since left and right end point functions $Q_{*}(z,\gamma)=\gamma\left(2-z^{\frac{1}{2}}\right)$, $Q^{*}(z,\gamma)=(2-\gamma)\left(2-z^{\frac{1}{2}}\right)$, are h-convex functions for each $\gamma\in[0,1]$, then $\tilde{Q}(z)$ is h-convex fuzzy-IVF. We clearly see that $\tilde{Q}\in L([u,\nu],\mathbb{F}_0)$ and

$$\frac{1}{\alpha\,h\left(\frac{1}{2}\right)}Q_{*}\left(\frac{u+\nu}{2},\gamma\right)=Q_{*}\left(\frac{5}{2},\gamma\right)=\gamma\frac{4-\sqrt{10}}{8}$$

$$\frac{1}{\alpha\,h\left(\frac{1}{2}\right)}Q^{*}\left(\frac{u+\nu}{2},\gamma\right)=Q^{*}\left(\frac{5}{2},\gamma\right)=(2-\gamma)\frac{4-\sqrt{10}}{8},$$

$$\frac{Q_{*}(u,\gamma)+Q_{*}(\nu,\gamma)}{2}\int_{0}^{1}\tau^{\alpha-1}[h(\tau)-h(1-\tau)]d\tau=\gamma\left(4-\sqrt{2}-\sqrt{3}\right)$$

$$\frac{Q^{*}(u,\gamma)+Q^{*}(\nu,\gamma)}{2}\int_{0}^{1}\tau^{\alpha-1}[h(\tau)-h(1-\tau)]d\tau=(2-\gamma)\left(4-\sqrt{2}-\sqrt{3}\right).$$

Note that

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^{+}}^{\alpha}Q_{*}(\nu,\gamma)+\mathcal{J}_{\nu^{-}}^{\alpha}Q_{*}(u,\gamma)\right]\\&=\frac{\Gamma(\frac{1}{2})}{2}\frac{1}{\sqrt{\pi}}\int_{2}^{3}(3-z)^{\frac{-1}{2}}.\gamma\left(2-z^{\frac{1}{2}}\right)dz\\&+\frac{\Gamma(\frac{1}{2})}{2}\frac{1}{\sqrt{\pi}}\int_{2}^{3}(z-2)^{\frac{-1}{2}}.\gamma\left(2-z^{\frac{1}{2}}\right)dz\\&=\frac{1}{2}\gamma\left[\frac{7393}{10{,}000}+\frac{9501}{10{,}000}\right]\\&=\gamma\frac{8447}{20{,}000}.\end{aligned}$$

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^{+}}^{\alpha}Q^{*}(\nu,\gamma)+\mathcal{J}_{\nu^{-}}^{\alpha}Q^{*}(u,\gamma)\right]\\&=\frac{\Gamma(\frac{1}{2})}{2}\frac{1}{\sqrt{\pi}}\int_{2}^{3}(3-z)^{\frac{-1}{2}}.(2-\gamma)\left(2-z^{\frac{1}{2}}\right)dz\\&+\frac{\Gamma(\frac{1}{2})}{2}\frac{1}{\sqrt{\pi}}\int_{2}^{3}(z-2)^{\frac{-1}{2}}.(2-\gamma)\left(2-z^{\frac{1}{2}}\right)dz\end{aligned}$$

$$= \frac{1}{2}(2-\gamma)\left[\frac{7393}{10{,}000} + \frac{9501}{10{,}000}\right]$$
$$= (2-\gamma)\frac{8447}{20{,}000}.$$

Therefore

$$\left[\gamma\frac{4-\sqrt{10}}{8}, (2-\gamma)\frac{4-\sqrt{10}}{8}\right] \leq_I \left[\gamma\frac{8447}{20{,}000}, (2-\gamma)\frac{8447}{20{,}000}\right]$$
$$\leq_I \left[\gamma\left(4-\sqrt{2}-\sqrt{3}\right), (2-\gamma)\left(4-\sqrt{2}-\sqrt{3}\right)\right],$$

and Theorem 3.1 is verified.

From Theorem 3.4 and Theorem 3.5, we obtain some fuzzy-interval fractional integral inequalities related to fuzzy-interval fractional HH-inequalities

Theorem 3.4. Let $\tilde{\mathcal{Q}}, \tilde{\mathcal{P}} : [u, \nu] \to \mathbb{F}_0$ be h_1-convex and h_2-convex fuzzy-IVFs on $[u, \nu]$, respectively, whose γ-levels $\mathcal{Q}_\gamma$, $\mathcal{P}_\gamma : [u, \nu] \subset \mathbb{R} \to \mathcal{K}_C^+$ are defined by $\mathcal{Q}_\gamma(z) = [\mathcal{Q}_*(z, \gamma), \mathcal{Q}^*(z, \gamma)]$ and $\mathcal{P}_\gamma(z) = [\mathcal{P}_*(z, \gamma), \mathcal{P}^*(z, \gamma)]$ for all $z \in [u, \nu]$ and for all $\gamma \in [0, 1]$. If $\tilde{\mathcal{Q}} \,\tilde{\times}\, \tilde{\mathcal{P}} \in L([u, \nu], \mathbb{F}_0)$, then

$$\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^{\alpha}\tilde{\mathcal{Q}}(\nu) \,\tilde{\times}\, \tilde{\mathcal{P}}(\nu) + \mathcal{J}_{\nu^-}^{\alpha}\tilde{\mathcal{Q}}(u) \,\tilde{\times}\, \tilde{\mathcal{P}}(u)\right]$$
$$\preccurlyeq \tilde{\Delta}(u, \nu)\int_0^1 \tau^{\alpha-1}[h_1(\tau)h_2(\tau) + h_1(1-\tau)h_2(1-\tau)]d\tau$$
$$+ \tilde{\nabla}(u, \nu)\int_0^1 \tau^{\alpha-1}[h_1(\tau)h_2(1-\tau) + h_1(1-\tau)h_2(\tau)]d\tau.$$

Where $\tilde{\Delta}(u, \nu) = \tilde{\mathcal{Q}}(u) \,\tilde{\times}\, \tilde{\mathcal{P}}(u) \,\tilde{+}\, \tilde{\mathcal{Q}}(\nu) \,\tilde{\times}\, \tilde{\mathcal{P}}(\nu)$, $\tilde{\nabla}(u, \nu) = \tilde{\mathcal{Q}}(u) \,\tilde{\times}\, \tilde{\mathcal{P}}(\nu) \,\tilde{+}\, \tilde{\mathcal{Q}}(\nu) \,\tilde{\times}\, \tilde{\mathcal{P}}(u)$, and $\Delta_\gamma(u, \nu) = \left[\Delta_*\left((u, \nu), \gamma\right), \Delta^*\left((u, \nu), \gamma\right)\right]$ and $\nabla_\gamma(u, \nu) = \left[\nabla_*\left((u, \nu), \gamma\right), \nabla^*\left((u, \nu), \gamma\right)\right]$.

Proof. Since $\tilde{\mathcal{Q}}, \tilde{\mathcal{P}}$ both are h_1-convex and h_2-convex fuzzy-IVFs then, for each $\gamma \in [0, 1]$ we have

$$\mathcal{Q}_*(\tau u + (1-\tau)\nu, \gamma) \leq h_1(\tau)\mathcal{Q}_*(u, \gamma) + h_1(1-\tau)\mathcal{Q}_*(\nu, \gamma)$$
$$\mathcal{Q}^*(\tau u + (1-\tau)\nu, \gamma) \leq h_1(\tau)\mathcal{Q}^*(u, \gamma) + h_1(1-\tau)\mathcal{Q}^*(\nu, \gamma).$$

and

$$\mathcal{P}_*(\tau u + (1-\tau)\nu, \gamma) \leq h_2(\tau)\mathcal{P}_*(u, \gamma) + h_2(1-\tau)\mathcal{P}_*(\nu, \gamma)$$
$$\mathcal{P}^*(\tau u + (1-\tau)\nu, \gamma) \leq h_2(\tau)\mathcal{P}^*(u, \gamma) + h_2(1-\tau)\mathcal{P}^*(\nu, \gamma).$$

From the definition of h-convex fuzzy-IVFs it follows that $\tilde{0} \preccurlyeq \tilde{\mathcal{Q}}(z)$ and $\tilde{0} \preccurlyeq \tilde{\mathcal{P}}(z)$, so

$$\begin{gathered}\mathcal{Q}_*(\tau u+(1-\tau)\nu,\gamma)\times\mathcal{P}_*(\tau u+(1-\tau)\nu,\gamma)\\ \leq h_1(\tau)h_2(\tau)\mathcal{Q}_*(u,\gamma)\times\mathcal{P}_*(u,\gamma)+h_1(1-\tau)h_2(1-\tau)\mathcal{Q}_*(\nu,\gamma)\times\mathcal{P}_*(\nu,\gamma)\\ +h_1(\tau)\,h_2(1-\tau)\,\mathcal{Q}_*(u,\gamma)\times\mathcal{P}_*(\nu,\gamma)+h_1(1-\tau)\,h_2(\tau)\,\mathcal{Q}_*(\nu,\gamma)\times\mathcal{P}_*(u,\gamma)\\ \mathcal{Q}^*(\tau u+(1-\tau)\nu,\gamma)\times\mathcal{P}^*(\tau u+(1-\tau)\nu,\gamma)\\ \leq h_1(\tau)h_2(\tau)\mathcal{Q}^*(u,\gamma)\times\mathcal{P}^*(u,\gamma)+h_1(1-\tau)h_2(1-\tau)\mathcal{Q}^*(\nu,\gamma)\times\mathcal{P}^*(\nu,\gamma)\\ +h_1(\tau)h_2(1-\tau)\mathcal{Q}^*(u,\gamma)\times\mathcal{P}^*(\nu,\gamma)+h_1(1-\tau)h_2(\tau)\mathcal{Q}^*(\nu,\gamma)\times\mathcal{P}^*(u,\gamma).\end{gathered}\tag{43}$$

Analogously, we have

$$\begin{gathered}\mathcal{Q}_*\big((1-\tau)u+\tau\nu,\gamma\big)\mathcal{P}_*\big((1-\tau)u+\tau\nu,\gamma\big)\\ \leq h_1(1-\tau)h_2(1-\tau)\mathcal{Q}_*(u,\gamma)\times\mathcal{P}_*(u,\gamma)+h_1(\tau)h_2(\tau)\mathcal{Q}_*(\nu,\gamma)\times\mathcal{P}_*(\nu,\gamma)\\ +h_1(1-\tau)h_2(\tau)\mathcal{Q}_*(u,\gamma)\times\mathcal{P}_*(\nu,\gamma)+h_1(\tau)h_2(1-\tau)\mathcal{Q}_*(\nu,\gamma)\times\mathcal{P}_*(u,\gamma)\\ \mathcal{Q}^*\big((1-\tau)u+\tau\nu,\gamma\big)\times\mathcal{P}^*\big((1-\tau)u+\tau\nu,\gamma\big)\\ \leq h_1(1-\tau)h_2(1-\tau)\mathcal{Q}^*(u,\gamma)\times\mathcal{P}^*(u,\gamma)+h_1(\tau)h_2(\tau)\mathcal{Q}^*(\nu,\gamma)\times\mathcal{P}^*(\nu,\gamma)\\ +h_1(1-\tau)h_2(\tau)\mathcal{Q}^*(u,\gamma)\times\mathcal{P}^*(\nu,\gamma)+h_1(\tau)h_2(1-\tau)\mathcal{Q}^*(\nu,\gamma)\times\mathcal{P}^*(u,\gamma).\end{gathered}\tag{44}$$

Adding (43) and (44), we have

$$\begin{gathered}\mathcal{Q}_*(\tau u+(1-\tau)\nu,\gamma)\times\mathcal{P}_*(\tau u+(1-\tau)\nu,\gamma)\\ +\mathcal{Q}_*\big((1-\tau)u+\tau\nu,\gamma\big)\times\mathcal{P}_*\big((1-\tau)u+\tau\nu,\gamma\big)\\ \leq[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)][\mathcal{Q}_*(u,\gamma)\times\mathcal{P}_*(u,\gamma)+\mathcal{Q}_*(\nu,\gamma)\times\mathcal{P}_*(\nu,\gamma)]\\ +[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)][\mathcal{Q}_*(\nu,\gamma)\times\mathcal{P}_*(u,\gamma)+\mathcal{Q}_*(u,\gamma)\times\mathcal{P}_*(\nu,\gamma)]\\ \mathcal{Q}^*(\tau u+(1-\tau)\nu,\gamma)\times\mathcal{P}^*(\tau u+(1-\tau)\nu,\gamma)\\ +\mathcal{Q}^*\big((1-\tau)u+\tau\nu,\gamma\big)\times\mathcal{P}^*\big((1-\tau)u+\tau\nu,\gamma\big)\\ \leq[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)][\mathcal{Q}^*(u,\gamma)\times\mathcal{P}^*(u,\gamma)+\mathcal{Q}^*(\nu,\gamma)\times\mathcal{P}^*(\nu,\gamma)]\\ +[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)][\mathcal{Q}^*(\nu,\gamma)\times\mathcal{P}^*(u,\gamma)+\mathcal{Q}^*(u,\gamma)\times\mathcal{P}^*(\nu,\gamma)].\end{gathered}\tag{45}$$

Taking multiplication of (45) with $\tau^{\alpha-1}$ and integrating the obtained result with respect to τ over (0,1), we have

$$\begin{gathered}\int_0^1\tau^{\alpha-1}\mathcal{Q}_*(\tau u+(1-\tau)\nu,\gamma)\times\mathcal{P}_*(\tau u+(1-\tau)\nu,\gamma)\\ +\tau^{\alpha-1}\mathcal{Q}_*\big((1-\tau)u+\tau\nu,\gamma\big)\times\mathcal{P}_*\big((1-\tau)u+\tau\nu,\gamma\big)d\tau\\ \leq\Delta_*\big((u,\nu),\gamma\big)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)]\,d\tau\\ +\nabla_*\big((u,\nu),\gamma\big)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)]d\tau\\ \int_0^1\tau^{\alpha-1}\mathcal{Q}^*(\tau u+(1-\tau)\nu,\gamma)\times\mathcal{P}^*(\tau u+(1-\tau)\nu,\gamma)\\ +\tau^{\alpha-1}\mathcal{Q}^*\big((1-\tau)u+\tau\nu,\gamma\big)\times\mathcal{P}^*\big((1-\tau)u+\tau\nu,\gamma\big)d\tau\\ \leq\Delta^*\big((u,\nu),\gamma\big)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)]d\tau\\ +\nabla^*\big((u,\nu),\gamma\big)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)]d\tau.\end{gathered}$$

It follows that,

$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\,\mathcal{Q}_*(\nu,\gamma)\times\mathcal{P}_*(\nu,\gamma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathcal{Q}_*(u,\gamma)\times\mathcal{P}_*(u,\gamma)\right]$$
$$\leq \Delta_*\big((u,\nu),\gamma\big)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)]d\tau$$
$$+\nabla_*\big((u,\nu),\gamma\big)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)]d\tau.$$
$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\,\mathcal{Q}^*(\nu,\gamma)\times\mathcal{P}^*(\nu,\gamma)+\mathcal{J}_{\nu^-}^{\alpha}\,\mathcal{Q}^*(u,\gamma)\times\mathcal{P}^*(u,\gamma)\right]$$
$$\leq \Delta^*\big((u,\nu),\gamma\big)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)]d\tau$$
$$+\nabla^*\big((u,\nu),\gamma\big)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)]d\tau.$$

It follows that

$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\Big[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_*(\nu,\gamma)\times\mathcal{P}_*(\nu,\gamma)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_*(u,\gamma)\times\mathcal{P}_*(u,\gamma),$$
$$\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}^*(\nu,\gamma)\times\mathcal{P}^*(\nu,\gamma)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}^*(u,\gamma)\times\mathcal{P}^*(u,\gamma)\Big]$$
$$\leq_I \left[\Delta_*\big((u,\nu),\gamma\big),\Delta^*\big((u,\nu),\gamma\big)\right]\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)]d\tau$$
$$+\left[\nabla_*\big((u,\nu),\gamma\big),\nabla^*\big((u,\nu),\gamma\big)\right]\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)]d\tau,$$

that is

$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_\gamma(\nu)\times\mathcal{P}_\gamma(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_\gamma(u)\times\mathcal{P}_\gamma(u)\right]$$
$$\leq_I \Delta_\gamma(u,\nu)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)]d\tau$$
$$+\nabla_\gamma(u,\nu)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)]d\tau.$$

Thus,

$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\tilde{\mathcal{Q}}(\nu)\,\tilde{\times}\,\tilde{\mathcal{P}}(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\tilde{\mathcal{Q}}(u)\,\tilde{\times}\,\tilde{\mathcal{P}}(u)\right]$$
$$\preccurlyeq \tilde{\Delta}(u,\nu)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)]d\tau$$
$$+\,\tilde{\nabla}(u,\nu)\int_0^1\tau^{\alpha-1}[h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)]d\tau.$$

and the theorem has been established.

Theorem 3.5. Let $\tilde{\mathcal{Q}}, \tilde{\mathcal{P}} : [u, \nu] \to \mathbb{F}_0$ be two h_1-convex and h_2-convex fuzzy-IVFs, respectively, whose γ-levels define the family of IVFs $\mathcal{Q}_\gamma$, $\mathcal{P}_\gamma : [u, \nu] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $\mathcal{Q}_\gamma(z) = [\mathcal{Q}_*(z, \gamma), \mathcal{Q}^*(z, \gamma)]$ and $\mathcal{P}_\gamma(z) = [\mathcal{P}_*(z, \gamma), \mathcal{P}^*(z, \gamma)]$ for all $z \in [u, \nu]$ and for all $\gamma \in [0, 1]$. If $\tilde{\mathcal{Q}} \,\tilde{\times}\, \tilde{\mathcal{P}} \in L([u, \nu], \mathbb{F}_0)$, then

$$\frac{1}{\alpha\, h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)} \tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right) \tilde{\times}\, \tilde{\mathcal{P}}\left(\frac{u+\nu}{2}\right)$$
$$\preccurlyeq \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^{\alpha} \tilde{\mathcal{Q}}(\nu) \,\tilde{\times}\, \tilde{\mathcal{P}}(\nu) \,\tilde{+}\, \mathcal{J}_{\nu^-}^{\alpha} \tilde{\mathcal{Q}}(u) \,\tilde{\times}\, \tilde{\mathcal{P}}(u)\right] \tilde{+}\, \tilde{\nabla}(u, \nu) \int_0^1 [\tau^{\alpha-1} + (1-\tau)^{\alpha-1}] h_1(\tau) h_2(1$$
$$-\tau) d\tau \,\tilde{+}\, \tilde{\Delta}(u, \nu) \int_0^1 [\tau^{\alpha-1} + (1-\tau)^{\alpha-1}] h_1(1-\tau) h_2(1-\tau) d\tau.$$

Where $\tilde{\Delta}(u, \nu) = \tilde{\mathcal{Q}}(u) \,\tilde{\times}\, \tilde{\mathcal{P}}(u) \,\tilde{+}\, \tilde{\mathcal{Q}}(\nu) \,\tilde{\times}\, \tilde{\mathcal{P}}(\nu)$, $\tilde{\nabla}(u, \nu) = \tilde{\mathcal{Q}}(u) \,\tilde{\times}\, \tilde{\mathcal{P}}(\nu) \,\tilde{+}\, \tilde{\mathcal{Q}}(\nu) \,\tilde{\times}\, \tilde{\mathcal{P}}(u)$, and $\Delta_\gamma(u, \nu) = \left[\Delta_*\left((u, \nu), \gamma\right), \Delta^*\left((u, \nu), \gamma\right)\right]$ and $\nabla_\gamma(u, \nu) = \left[\nabla_*\left((u, \nu), \gamma\right), \nabla^*\left((u, \nu), \gamma\right)\right]$.

Proof. Consider $\tilde{\mathcal{Q}}, \tilde{\mathcal{P}} : [u, \nu] \to \mathbb{F}_0$ are h_1-convex and h_2-convex fuzzy-IVFs. Then, by hypothesis, for each $\gamma \in [0, 1]$, we have

$$\mathcal{Q}_*\left(\frac{u+\nu}{2}, \gamma\right) \times \mathcal{P}_*\left(\frac{u+\nu}{2}, \gamma\right)$$
$$\mathcal{Q}^*\left(\frac{u+\nu}{2}, \gamma\right) \times \mathcal{P}^*\left(\frac{u+\nu}{2}, \gamma\right)$$
$$\leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \begin{bmatrix} \mathcal{Q}_*(\tau u + (1-\tau)\nu, \gamma) \times \mathcal{P}_*(\tau u + (1-\tau)\nu, \gamma) \\ + \mathcal{Q}_*(\tau u + (1-\tau)\nu, \gamma) \times \mathcal{P}_*\left((1-\tau)u + \tau\nu, \gamma\right) \end{bmatrix}$$
$$+ h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \begin{bmatrix} \mathcal{Q}_*\left((1-\tau)u + \tau\nu, \gamma\right) \times \mathcal{P}_*(\tau u + (1-\tau)\nu, \gamma) \\ + \mathcal{Q}_*\left((1-\tau)u + \tau\nu, \gamma\right) \times \mathcal{P}_*\left((1-\tau)u + \tau\nu, \gamma\right) \end{bmatrix}$$
$$\leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \begin{bmatrix} \mathcal{Q}^*(\tau u + (1-\tau)\nu, \gamma) \times \mathcal{P}^*(\tau u + (1-\tau)\nu, \gamma) \\ + \mathcal{Q}^*(\tau u + (1-\tau)\nu, \gamma) \times \mathcal{P}^*\left((1-\tau)u + \tau\nu, \gamma\right) \end{bmatrix}$$
$$+ h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \begin{bmatrix} \mathcal{Q}^*\left((1-\tau)u + \tau\nu, \gamma\right) \times \mathcal{P}^*(\tau u + (1-\tau)\nu, \gamma) \\ + \mathcal{Q}^*\left((1-\tau)u + \tau\nu, \gamma\right) \times \mathcal{P}^*\left((1-\tau)u + \tau\nu, \gamma\right) \end{bmatrix},$$
$$\leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \begin{bmatrix} \mathcal{Q}_*(\tau u + (1-\tau)\nu, \gamma) \times \mathcal{P}_*(\tau u + (1-\tau)\nu, \gamma) \\ + \mathcal{Q}_*\left((1-\tau)u + \tau\nu, \gamma\right) \times \mathcal{P}_*\left((1-\tau)u + \tau\nu, \gamma\right) \end{bmatrix}$$
$$+ h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \begin{bmatrix} \left(\tau\mathcal{Q}_*(u, \gamma) + (1-\tau)\mathcal{Q}_*(\nu, \gamma)\right) \\ \times \left((1-\tau)\mathcal{P}_*(u, \gamma) + \tau\mathcal{P}_*(\nu, \gamma)\right) \\ + \left((1-\tau)\mathcal{Q}_*(u, \gamma) + \tau\mathcal{Q}_*(\nu, \gamma)\right) \\ \times \left(\tau\mathcal{P}_*(u, \gamma) + (1-\tau)\mathcal{P}_*(\nu, \gamma)\right) \end{bmatrix}$$
$$\leq h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \begin{bmatrix} \mathcal{Q}^*(\tau u + (1-\tau)\nu, \gamma) \times \mathcal{P}^*(\tau u + (1-\tau)\nu, \gamma) \\ + \mathcal{Q}^*\left((1-\tau)u + \tau\nu, \gamma\right) \times \mathcal{P}^*\left((1-\tau)u + \tau\nu, \gamma\right) \end{bmatrix}$$
$$+ h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \begin{bmatrix} \left(\tau\mathcal{Q}^*(u, \gamma) + (1-\tau)\mathcal{Q}^*(\nu, \gamma)\right) \\ \times \left((1-\tau)\mathcal{P}^*(u, \gamma) + \tau\mathcal{P}^*(\nu, \gamma)\right) \\ + \left((1-\tau)\mathcal{Q}^*(u, \gamma) + \tau\mathcal{Q}^*(\nu, \gamma)\right) \\ \times \left(\tau\mathcal{P}^*(u, \gamma) + (1-\tau)\mathcal{P}^*(\nu, \gamma)\right) \end{bmatrix},$$

$$\begin{aligned}
&= h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \mathcal{Q}_*(\tau u+(1-\tau)\nu,\gamma)\times\mathcal{P}_*(\tau u+(1-\tau)\nu,\gamma) \\ +\mathcal{Q}_*\big((1-\tau)u+\tau\nu,\gamma\big)\times\mathcal{P}_*\big((1-\tau)u+\tau\nu,\gamma\big)\end{bmatrix} \\
&+ h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)\}\nabla_*\big((u,\nu),\gamma\big) \\ +\{h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)\}\Delta_*\big((u,\nu),\gamma\big)\end{bmatrix} \\
&= h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \mathcal{Q}^*(\tau u+(1-\tau)\nu,\gamma)\times\mathcal{P}^*(\tau u+(1-\tau)\nu,\gamma) \\ +\mathcal{Q}^*\big((1-\tau)u+\tau\nu,\gamma\big)\times\mathcal{P}^*\big((1-\tau)u+\tau\nu,\gamma\big)\end{bmatrix} \\
&+ h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\begin{bmatrix} \{h_1(\tau)h_2(1-\tau)+h_1(1-\tau)h_2(\tau)\}\nabla^*\big((u,\nu),\gamma\big) \\ +\{h_1(\tau)h_2(\tau)+h_1(1-\tau)h_2(1-\tau)\}\Delta^*\big((u,\nu),\gamma\big)\end{bmatrix}.
\end{aligned} \tag{46}$$

Taking multiplication of (46) with $\tau^{\alpha-1}$ and integrating over $(0,1)$, we get

$$\begin{aligned}
&\frac{1}{\alpha\, h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathcal{Q}_*\left(\frac{u+\nu}{2},\gamma\right)\times\mathcal{P}_*\left(\frac{u+\nu}{2},\gamma\right) \\
&\le \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_*(\nu)\times\mathcal{P}_*(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_*(u)\times\mathcal{P}_*(u)\right] \\
&+\nabla_*\big((u,\nu),\gamma\big)\int_0^1[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}]h_1(\tau)h_2(1-\tau)d\tau \\
&+\Delta_*\big((u,\nu),\gamma\big)\int_0^1[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}]h_1(1-\tau)h_2(1-\tau)d\tau \\
&\frac{1}{\alpha h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathcal{Q}^*\left(\frac{u+\nu}{2},\gamma\right)\times\mathcal{P}^*\left(\frac{u+\nu}{2},\gamma\right) \\
&\le \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}^*(\nu)\times\mathcal{P}^*(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}^*(u)\times\mathcal{P}^*(u)\right] \\
&+\nabla^*\big((u,\nu),\gamma\big)\int_0^1[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}]h_1(\tau)h_2(1-\tau)d\tau \\
&+\Delta^*\big((u,\nu),\gamma\big)\int_0^1[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}]h_1(1-\tau)h_2(1-\tau)d\tau,
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{\alpha\, h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathcal{Q}_\gamma\left(\frac{u+\nu}{2}\right)\times\mathcal{P}_\gamma\left(\frac{u+\nu}{2}\right) \\
&\qquad\le_I \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_\gamma(\nu)\times\mathcal{P}_\gamma(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_\gamma(u)\times\mathcal{P}_\gamma(u)\right] \\
&\qquad+\nabla_\gamma(u,\nu)\int_0^1[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}]h_1(1-\tau)h_2(1-\tau)d\tau \\
&\qquad+\Delta_\gamma(u,\nu)\int_0^1[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}]h_1(1-\tau)h_2(1-\tau)d\tau,
\end{aligned}$$

that is

$$\frac{1}{\alpha\, h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)}\tilde{Q}\left(\frac{u+\nu}{2}\right) \tilde{\times}\, \tilde{\mathcal{P}}\left(\frac{u+\nu}{2}\right)$$
$$\preccurlyeq \frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\tilde{Q}(\nu)\,\tilde{\times}\,\tilde{\mathcal{P}}(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}\tilde{Q}(u)\,\tilde{\times}\,\tilde{\mathcal{P}}(u)\right]\tilde{+}\tilde{\nabla}(u,\nu)\int_0^1[\tau^{\alpha-1}$$
$$+(1-\tau)^{\alpha-1}]h_1(1-\tau)h_2(1-\tau)d\tau\,\tilde{+}\tilde{\Delta}(u,\nu)\int_0^1[\tau^{\alpha-1}+(1-\tau)^{\alpha-1}]h_1(1$$
$$-\tau)h_2(1-\tau)d\tau.$$

Hence, the required result.

The Theorem 3.6 and Theorem 3.7 are directly connected with right and left part of classical HH-Fejér inequality, respectively. Now firstly, we obtain the right part of classical HH-Fejér inequality through fuzzy Riemann Liouville fractional integral is known as second fuzzy fractional HH-Fejér inequality.

Theorem 3.6. (Second fuzzy fractional HH-Fejér inequality) Let $\tilde{Q}: [u,\nu] \to \mathbb{F}_0$ be a h-convex fuzzy-IVF with $u < \nu$, whose γ-levels define the family of IVFs $Q_\gamma: [u,\nu] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $Q_\gamma(z) = [Q_*(z,\gamma), Q^*(z,\gamma)]$ for all $z \in [u,\nu]$ and for all $\gamma \in [0,1]$. If $\tilde{Q} \in L([u,\nu],\mathbb{F}_0)$ and $\Omega: [u,\nu] \to \mathbb{R}, \Omega(z) \geq 0$, symmetric with respect to $\frac{u+\nu}{2}$, then

$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\tilde{Q}\Omega(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}\tilde{Q}\Omega(u)\right] \preccurlyeq \frac{\tilde{Q}(u)\tilde{+}\tilde{Q}(\nu)}{2}\int_0^1 \tau^{\alpha-1}[h(\tau)+h(1-\tau)]\Omega\big((1-\tau)u+\tau\nu\big)d\tau. \quad (47)$$

If $\tilde{Q}$ is concave fuzzy-IVF, then inequality (47) is reversed.

Proof. Let $\tilde{Q}$ be a h-convex fuzzy-IVF and $\tau^{\alpha-1}\Omega(\tau u+(1-\tau)\nu) \geq 0$. Then, for each $\gamma \in [0,1]$, we have

$$\tau^{\alpha-1}Q_*(\tau u+(1-\tau)\nu,\gamma)\Omega(\tau u+(1-\tau)\nu)$$
$$\leq \tau^{\alpha-1}\big(h(\tau)Q_*(u,\gamma)+h(1-\tau)Q_*(\nu,\gamma)\big)\Omega(\tau u+(1-\tau)\nu)$$
$$\tau^{\alpha-1}Q^*(\tau u+(1-\tau)\nu,\gamma)\Omega(\tau u+(1-\tau)\nu)$$
$$\leq \tau^{\alpha-1}\big(h(\tau)Q^*(u,\gamma)+h(1-\tau)Q^*(\nu,\gamma)\big)\Omega(\tau u+(1-\tau)\nu), \quad (48)$$

And

$$\tau^{\alpha-1}Q_*\big((1-\tau)u+\tau\nu,\gamma\big)\Omega\big((1-\tau)u+\tau\nu\big)$$
$$\leq \tau^{\alpha-1}\big(h(1-\tau)Q_*(u,\gamma)+h(\tau)Q_*(\nu,\gamma)\big)\Omega\big((1-\tau)u+\tau\nu\big)$$
$$\tau^{\alpha-1}Q^*\big((1-\tau)u+\tau\nu,\gamma\big)\Omega\big((1-\tau)u+\tau\nu\big)$$
$$\leq \tau^{\alpha-1}\big(h(1-\tau)Q^*(u,\gamma)+h(\tau)Q^*(\nu,\gamma)\big)\Omega\big((1-\tau)u+\tau\nu\big). \quad (49)$$

After adding (48) and (49), and integrating over $[0, 1]$, we get

$$\begin{aligned}
&\int_0^1 \tau^{\alpha-1} \mathcal{Q}_*(\tau u + (1-\tau)\nu, \gamma)\Omega(\tau u + (1-\tau)\nu)\, d\tau \\
&+ \int_0^1 \tau^{\alpha-1} \mathcal{Q}_*\big((1-\tau)u + \tau\nu, \gamma\big)\Omega\big((1-\tau)u + \tau\nu\big)\, d\tau \\
&\leq \int_0^1 \begin{bmatrix} \tau^{\alpha-1}\mathcal{Q}_*(u,\gamma)\{h(\tau)\Omega(\tau u + (1-\tau)\nu) + h(1-\tau)\Omega\big((1-\tau)u + \tau\nu\big)\} \\ +\tau^{\alpha-1}\mathcal{Q}_*(\nu,\gamma)\{h(1-\tau)\Omega(\tau u + (1-\tau)\nu) + h(\tau)\Omega\big((1-\tau)u + \tau\nu\big)\} \end{bmatrix} d\tau, \\
&= \mathcal{Q}_*(u,\gamma)\int_0^1 \tau^{\alpha-1}[h(\tau) + h(1-\tau)]\Omega(\tau u + (1-\tau)\nu)\, d\tau \\
&+\mathcal{Q}_*(\nu,\gamma)\int_0^1 \tau^{\alpha-1}[h(\tau) + h(1-\tau)]\Omega\big((1-\tau)u + \tau\nu\big)\, d\tau, \\
&\int_0^1 \tau^{\alpha-1} \mathcal{Q}^*\big((1-\tau)u + \tau\nu, \gamma\big)\Omega\big((1-\tau)u + \tau\nu\big)\, d\tau \\
&+ \int_0^1 \tau^{\alpha-1} \mathcal{Q}^*(\tau u + (1-\tau)\nu, \gamma)\Omega(\tau u + (1-\tau)\nu)\, d\tau \\
&\leq \int_0^1 \begin{bmatrix} \tau^{\alpha-1}\mathcal{Q}^*(u,\gamma)\{h(\tau)\Omega(\tau u + (1-\tau)\nu) + h(1-\tau)\Omega\big((1-\tau)u + \tau\nu\big)\} \\ +\tau^{\alpha-1}\mathcal{Q}^*(\nu,\gamma)\{h(1-\tau)\Omega(\tau u + (1-\tau)\nu) + h(\tau)\Omega\big((1-\tau)u + \tau\nu\big)\} \end{bmatrix} d\tau, \\
&= \mathcal{Q}^*(u,\gamma)\int_0^1 \tau^{\alpha-1}[h(\tau) + h(1-\tau)]\Omega(\tau u + (1-\tau)\nu)\, d\tau \\
&+\mathcal{Q}^*(\nu,\gamma)\int_0^1 \tau^{\alpha-1}[h(\tau) + h(1-\tau)]\Omega\big((1-\tau)u + \tau\nu\big)\, d\tau.
\end{aligned} \tag{50}$$

Taking right hand side of inequality (50), we have

$$\begin{aligned}
&\int_0^1 \tau^{\alpha-1} \mathcal{Q}_*(\tau u + (1-\tau)\nu, \gamma)\Omega\big((1-\tau)u + \tau\nu\big)\, d\tau \\
&+ \int_0^1 \tau^{\alpha-1} \mathcal{Q}_*\big((1-\tau)u + \tau\nu, \gamma\big)\Omega\big((1-\tau)u + \tau\nu\big)\, d\tau \\
&= \frac{1}{(\nu-u)^\alpha}\int_u^\nu (z-u)^{\alpha-1}\mathcal{Q}_*(u-\nu-z,\gamma)\Omega(z)dz \\
&+ \frac{1}{(\nu-u)^\alpha}\int_u^\nu (z-u)^{\alpha-1}\mathcal{Q}_*(z,\gamma)\Omega(z)dz \\
&= \frac{1}{(\nu-u)^\alpha}\int_u^\nu (\nu-u)^{\alpha-1}\mathcal{Q}_*(z,\gamma)\Omega(u-\nu-z)dz \\
&+ \frac{1}{(\nu-u)^\alpha}\int_u^\nu (z-u)^{\alpha-1}\mathcal{Q}_*(z,\gamma)\Omega(z)dz \\
&= \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha \mathcal{Q}_*\Omega(\nu) + \mathcal{J}_{\nu^-}^\alpha \mathcal{Q}_*\Omega(u)\right], \\
&\int_0^1 \tau^{\alpha-1} \mathcal{Q}^*(\tau u + (1-\tau)\nu, \gamma)\Omega\big((1-\tau)u + \tau\nu\big)\, d\tau \\
&+ \int_0^1 \tau^{\alpha-1} \mathcal{Q}^*\big((1-\tau)u + \tau\nu, \gamma\big)\Omega\big((1-\tau)u + \tau\nu\big)\, d\tau \\
&= \frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha \mathcal{Q}^*\Omega(\nu) + \mathcal{J}_{\nu^-}^\alpha \mathcal{Q}^*\Omega(u)\right].
\end{aligned} \tag{51}$$

From (51), we have

$$\frac{\Gamma(\alpha)}{(\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha \mathcal{Q}_*\Omega(\nu) + \mathcal{J}_{\nu^-}^\alpha \mathcal{Q}_*\Omega(u)\right] \leq \frac{\mathcal{Q}_*(u,\gamma) + \mathcal{Q}_*(\nu,\gamma)}{2}\int_0^1 \tau^{\alpha-1}[h(\tau) + h(1-\tau)]\Omega\big((1-\tau)u + \tau\nu\big)$$

$$\frac{\Gamma(\alpha)}{\nu-u)^\alpha}\left[\mathcal{J}_{u^+}^\alpha \mathcal{Q}^*\Omega(\nu) + \mathcal{J}_{\nu^-}^\alpha \mathcal{Q}^*\Omega(u)\right] \leq \frac{\mathcal{Q}^*(u,\gamma) + \mathcal{Q}^*(\nu,\gamma)}{2}\int_0^1 \tau^{\alpha-1}[h(\tau) + h(1-\tau)]\Omega\big((1-\tau)u + \tau\nu\big),$$

that is

$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_*\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_*\Omega(u),\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}^*\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}^*\Omega(u)\right]$$
$$\leq_I\left[\frac{\mathcal{Q}_*(u,\gamma)+\mathcal{Q}_*(\nu,\gamma)}{2},\frac{\mathcal{Q}^*(u,\gamma)+\mathcal{Q}^*(\nu,\gamma)}{2}\right]\int_0^1\tau^{\alpha-1}[h(\tau)+h(1-\tau)]\Omega\big((1-\tau)u+\tau\nu\big)d\tau,$$

hence

$$\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\tilde{\mathcal{Q}}\Omega(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}\tilde{\mathcal{Q}}\Omega(u)\right]\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(\nu)}{2}\int_0^1\tau^{\alpha-1}[h(\tau)+h(1-\tau)]\Omega\big((1-\tau)u+\tau\nu\big)d\tau.$$

Now, we obtain the following result connected with left part of classical HH-Fejér inequality for h-convex fuzzy-IVF through fuzzy order relation which is known as first fuzzy fractional HH-Fejér inequality.

Theorem 3.7. (First fuzzy fractional HH-Fejér inequality) Let $\tilde{\mathcal{Q}}:[u,\nu]\to\mathbb{F}_0$ be a h-convex fuzzy-IVF with $u<\nu$, whose γ-levels define the family of IVFs $\mathcal{Q}_\gamma:[u,\nu]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathcal{Q}_\gamma(z)=[\mathcal{Q}_*(z,\gamma),\mathcal{Q}^*(z,\gamma)]$ for all $z\in[u,\nu]$ and for all $\gamma\in[0,1]$. If $\tilde{\mathcal{Q}}\in L([u,\nu],\mathbb{F}_0)$ and $\Omega:[u,\nu]\to\mathbb{R},\Omega(z)\geq 0$, symmetric with respect to $\frac{u+\nu}{2}$, then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right)\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right]\preccurlyeq\left[\mathcal{J}_{u^+}^{\alpha}\tilde{\mathcal{Q}}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\tilde{\mathcal{Q}}\Omega(u)\right].\quad(52)$$

If $\tilde{\mathcal{Q}}$ is concave fuzzy-IVF, then inequality (52) is reversed.

Proof. Since $\tilde{\mathcal{Q}}$ is a h-convex fuzzy-IVF, then for $\gamma\in[0,1]$, we have

$$\begin{aligned}\mathcal{Q}_*\left(\frac{u+\nu}{2},\gamma\right)&\leq h\left(\frac{1}{2}\right)\Big(\mathcal{Q}_*(\tau u+(1-\tau)\nu,\gamma)+\mathcal{Q}_*\big((1-\tau)u+\tau\nu,\gamma\big)\Big)\\ \mathcal{Q}^*\left(\frac{u+\nu}{2},\gamma\right)&\leq h\left(\frac{1}{2}\right)\Big(\mathcal{Q}^*(\tau u+(1-\tau)\nu,\gamma)+\mathcal{Q}^*\big((1-\tau)u+\tau\nu,\gamma\big)\Big).\end{aligned}\quad(53)$$

Since $\Omega(\tau u+(1-\tau)\nu)=\Omega\big((1-\tau)u+\tau\nu\big)$, then by multiplying (53) by $\tau^{\alpha-1}\Omega\big((1-\tau)u+\tau\nu\big)$ and integrate it with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}&\mathcal{Q}_*\left(\frac{u+\nu}{2},\gamma\right)\int_0^1\tau^{\alpha-1}\Omega\big((1-\tau)u+\tau\nu\big)d\tau\\ &\quad\leq h\left(\frac{1}{2}\right)\begin{pmatrix}\int_0^1\tau^{\alpha-1}\mathcal{Q}_*(\tau u+(1-\tau)\nu,\gamma)\Omega\big((1-\tau)u+\tau\nu\big)d\tau\\+\int_0^1\tau^{\alpha-1}\mathcal{Q}_*\big((1-\tau)u+\tau\nu,\gamma\big)\Omega\big((1-\tau)u+\tau\nu\big)d\tau\end{pmatrix},\\ &\mathcal{Q}^*\left(\frac{u+\nu}{2},\gamma\right)\int_0^1\tau^{\alpha-1}\Omega\big((1-\tau)u+\tau\nu\big)d\tau\\ &\quad\leq h\left(\frac{1}{2}\right)\begin{pmatrix}\int_0^1\tau^{\alpha-1}\mathcal{Q}^*(\tau u+(1-\tau)\nu,\gamma)\Omega\big((1-\tau)u+\tau\nu\big)d\tau\\+\int_0^1\tau^{\alpha-1}\mathcal{Q}^*\big((1-\tau)u+\tau\nu,\gamma\big)\Omega\big((1-\tau)u+\tau\nu\big)d\tau\end{pmatrix}.\end{aligned}\quad(54)$$

Let $z=(1-\tau)u+\tau\nu$. Then, right hand side of inequality (54), we have

$$\begin{aligned}
&\int_0^1 \tau^{\alpha-1}\mathcal{Q}_*(\tau u+(1-\tau)\nu,\gamma)\Omega\big((1-\tau)u+\tau\nu\big)\,d\tau \\
&\quad+\int_0^1 \tau^{\alpha-1}\mathcal{Q}_*\big((1-\tau)u+\tau\nu,\gamma\big)\Omega\big((1-\tau)u+\tau\nu\big)\,d\tau \\
&\quad=\frac{1}{(\nu-u)^{\alpha}}\int_u^{\nu}(z-u)^{\alpha-1}\mathcal{Q}_*(u-\nu-z,\gamma)\Omega(z)dz \\
&\quad+\frac{1}{(\nu-u)^{\alpha}}\int_u^{\nu}(z-u)^{\alpha-1}\mathcal{Q}_*(z,\gamma)\Omega(z)dz \\
&\quad=\frac{1}{(\nu-u)^{\alpha}}\int_u^{\nu}(\nu-u)^{\alpha-1}\mathcal{Q}_*(z,\gamma)\Omega(u-\nu-z)dz \\
&\quad+\frac{1}{(\nu-u)^{\alpha}}\int_u^{\nu}(z-u)^{\alpha-1}\mathcal{Q}_*(z,\gamma)\Omega(z)dz \\
&\quad=\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\,\mathcal{Q}_*\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_*\Omega(u)\right], \\
&\int_0^1 \tau^{\alpha-1}\mathcal{Q}^*(\tau u+(1-\tau)\nu,\gamma)\Omega\big((1-\tau)u+\tau\nu\big)\,d\tau \\
&\quad+\int_0^1 \tau^{\alpha-1}\mathcal{Q}^*\big((1-\tau)u+\tau\nu,\gamma\big)\Omega\big((1-\tau)u+\tau\nu\big)\,d\tau \\
&\quad=\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}\,\mathcal{Q}^*\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}^*\Omega(u)\right].
\end{aligned} \tag{55}$$

Then from (55), we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathcal{Q}_*\left(\frac{u+\nu}{2},\gamma\right)\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right]\le\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_*\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_*\Omega(u)\right]$$
$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathcal{Q}^*\left(\frac{u+\nu}{2},\gamma\right)\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right]\le\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}^*\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}^*\Omega(u)\right],$$

from which, we have

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)}\left[\mathcal{Q}_*\left(\frac{u+\nu}{2},\gamma\right),\ \mathcal{Q}^*\left(\frac{u+\nu}{2},\gamma\right)\right]\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right] \\
&\le_I\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_*\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_*\Omega(u),\ \mathcal{J}_{u^+}^{\alpha}\mathcal{Q}^*\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}^*\Omega(u)\right],
\end{aligned}$$

it follows that

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathcal{Q}_{\gamma}\left(\frac{u+\nu}{2}\right)\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right]\le_I\left[\mathcal{J}_{u^+}^{\alpha}\mathcal{Q}_{\gamma}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\mathcal{Q}_{\gamma}\Omega(u)\right],$$

that is

$$\frac{1}{2h\left(\frac{1}{2}\right)}\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right)\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right]\preccurlyeq\left[\mathcal{J}_{u^+}^{\alpha}\tilde{\mathcal{Q}}\Omega(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}\tilde{\mathcal{Q}}\Omega(u)\right].$$

This completes the proof.

Remark 3.8. If $\Omega(z)=1$, then from Theorem 3.6 and Theorem 3.7, we get Theorem 3.1.
If $h(\tau)=\tau$, then from Theorem 3.6 and Theorem 3.7, we get following factional HH-Fejér inequality:

$$\tilde{\mathcal{Q}}\left(\frac{u+\nu}{2}\right)\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right]\preccurlyeq\left[\mathcal{J}_{u^+}^{\alpha}\tilde{\mathcal{Q}}\Omega(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}\tilde{\mathcal{Q}}\Omega(u)\right]\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(\nu)}{2}\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right]. \tag{56}$$

Let $h(\tau)=\tau$ and $\alpha=1$. Then, from Theorem 3.6 and Theorem 3.7, we obtain following

HH-Fejér inequality for convex fuzzy-IVF which is also new one.

$$\tilde{Q}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\int_u^\nu \Omega(z)dz}(FR)\int_u^\nu \tilde{Q}(z)\Omega(z)dz \preccurlyeq \frac{\tilde{Q}(u)\tilde{+}\tilde{Q}(\nu)}{2}. \tag{57}$$

Let $h(\tau)=\tau$ and $\alpha=1=\Omega(z)$. Then, from Theorem 3.6 and Theorem 3.7, we obtain following HH-inequality for convex fuzzy-IVF given in [28]:

$$\tilde{Q}\left(\frac{u+\nu}{2}\right) \preccurlyeq (FR)\int_u^\nu \tilde{Q}(z)dz \preccurlyeq \frac{\tilde{Q}(u)\tilde{+}\tilde{Q}(\nu)}{2}. \tag{58}$$

If $Q_*(z,\gamma)=Q^*(z,\gamma)$ and $1=\gamma$ and $h(\tau)=\tau$, then from Theorem 3.6 and Theorem 3.7, following HH-Fejér inequality for classical function following inequality given in [9]:

$$Q\left(\frac{u+\nu}{2}\right)\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right] \le \left[\mathcal{J}_{u^+}^{\alpha}Q\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}Q\Omega(u)\right] \le \frac{Q(u)+Q(\nu)}{2}\left[\mathcal{J}_{u^+}^{\alpha}\Omega(\nu)+\mathcal{J}_{\nu^-}^{\alpha}\Omega(u)\right]. \tag{59}$$

If $Q_*(z,\gamma)=Q^*(z,\gamma)$ and $\alpha=1=\gamma$ and $h(\tau)=\tau$, then from Theorem 3.6 and Theorem 3.7, we obtain the classical HH-Fejér inequality (2).

If $Q_*(z,\gamma)=Q^*(z,\gamma)$ and $\Omega(z)=\alpha=1=\gamma$ and $h(\tau)=\tau$, then from Theorem 3.6 and Theorem 3.7, we get the classical HH-inequality (1).

Example 3.9. We consider the fuzzy-IVF $\tilde{Q}\colon[0,2]\to\mathbb{F}_0$ defined by,

$$\tilde{Q}(z)(\sigma)=\begin{cases}\dfrac{\sigma}{2-\sqrt{z}}, & \sigma\in[0,2-\sqrt{z}],\\ \dfrac{2(2-\sqrt{z})-\sigma}{2-\sqrt{z}}, & \sigma\in(2-\sqrt{z},2(2-\sqrt{z})],\\ 0, & otherwise.\end{cases}$$

Then, for each $\gamma\in[0,1]$, we have $Q_\gamma(z)=\left[\gamma(2-\sqrt{z}),(2-\gamma)(2-\sqrt{z})\right]$. Since end point functions $Q_*(z,\gamma)$, $Q^*(z,\gamma)$ are h-convex functions for each $\gamma\in[0,1]$, then $\tilde{Q}(z)$ is h-convex fuzzy-IVF. If

$$\Omega(z)=\begin{cases}\sqrt{z}, & \sigma\in[0,1],\\ \sqrt{2-z}, & \sigma\in(1,2],\end{cases}$$

then $\Omega(2-z)=\Omega(z)\ge 0$, for all $z\in[0,2]$. Since $Q_*(z,\gamma)=\gamma(2-\sqrt{z})$ and $Q^*(z,\gamma)=(2-\gamma)(2-\sqrt{z})$. If $h(\tau)=\tau$ and $\alpha=\frac{1}{2}$, then we compute the following:

$$\begin{aligned}\frac{Q_*(u,\gamma)+Q_*(\nu,\gamma)}{2}\int_0^1 \tau^{\alpha-1}[h(\tau)+h(1-\tau)]\Omega((1-\tau)u+\tau\nu)&=\frac{\pi}{\sqrt{2}}\gamma\left(\frac{4-\sqrt{2}}{2}\right),\\ \frac{Q^*(u,\gamma)+Q^*(\nu,\gamma)}{2}\int_0^1 \tau^{\alpha-1}[h(\tau)+h(1-\tau)]\Omega((1-\tau)u+\tau\nu)&=\frac{\pi}{\sqrt{2}}(2-\gamma)\left(\frac{4-\sqrt{2}}{2}\right),\end{aligned} \tag{60}$$

$$\begin{aligned}\frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}Q_*\Omega(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}Q_*\Omega(u)\right]&=\frac{1}{\sqrt{\pi}}\gamma\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),\\ \frac{\Gamma(\alpha)}{(\nu-u)^{\alpha}}\left[\mathcal{J}_{u^+}^{\alpha}Q^*\Omega(\nu)\tilde{+}\mathcal{J}_{\nu^-}^{\alpha}Q^*\Omega(u)\right]&=\frac{1}{\sqrt{\pi}}(2-\gamma)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right).\end{aligned} \tag{61}$$

From (61) and (62), we have

$$\frac{1}{\sqrt{\pi}}\left[\gamma\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),(2-\gamma)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\right]\leq_I\frac{\pi}{\sqrt{2}}\left[\gamma\left(\frac{4-\sqrt{2}}{2}\right),(2-\gamma)\left(\frac{4-\sqrt{2}}{2}\right)\right],\ \text{for each }\ \gamma\in[0,1].$$

Hence, Theorem 10 is verified.
For Theorem 11, we have

$$\begin{aligned}
&\mathcal{J}^{\alpha}_{u^+}\mathcal{Q}_*\Omega(v)+\mathcal{J}^{\alpha}_{v^-}\mathcal{Q}_*\Omega(u)\\
&=\frac{1}{\sqrt{\pi}}\int_0^2(2-z)^{\frac{-1}{2}}\Omega(z)\left(\gamma\left(2-\sqrt{z}\right)\right)dz+\frac{1}{\sqrt{\pi}}\int_0^2(z)^{\frac{-1}{2}}\Omega(z)\left(\gamma\left(2-\sqrt{z}\right)\right)dz\\
&=\frac{1}{\sqrt{\pi}}\gamma\left(\pi+\frac{8-8\sqrt{2}}{3}\right)+\frac{1}{\sqrt{\pi}}\gamma\left(\pi-\frac{4}{3}\right)=\frac{1}{\sqrt{\pi}}\gamma\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\\
&\mathcal{J}^{\alpha}_{u^+}\mathcal{Q}^*\Omega(v)+\mathcal{J}^{\alpha}_{v^-}\mathcal{Q}^*\Omega(u)\\
&=\frac{1}{\sqrt{\pi}}\int_0^2(2-z)^{\frac{-1}{2}}\Omega(z)\left((2-\gamma)\left(2-\sqrt{z}\right)\right)dz+\frac{1}{\sqrt{\pi}}\int_0^2(z)^{\frac{-1}{2}}\Omega(z)\left((2-\gamma)\left(2-\sqrt{z}\right)\right)dz\\
&=\frac{1}{\sqrt{\pi}}(2-\gamma)\left(\pi+\frac{8-8\sqrt{2}}{3}\right)+\frac{1}{\sqrt{\pi}}(2-\gamma)\left(\pi-\frac{4}{3}\right)=\frac{1}{\sqrt{\pi}}(2-\gamma)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right).
\end{aligned}\tag{62}$$

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)}\mathcal{Q}_*\left(\frac{u+v}{2},\gamma\right)\left[\mathcal{J}^{\alpha}_{u^+}\Omega(v)+\mathcal{J}^{\alpha}_{v^-}\Omega(u)\right]=\gamma\sqrt{\pi},\\
&\frac{1}{2h\left(\frac{1}{2}\right)}\mathcal{Q}^*\left(\frac{u+v}{2},\gamma\right)\left[\mathcal{J}^{\alpha}_{u^+}\Omega(v)+\mathcal{J}^{\alpha}_{v^-}\Omega(u)\right]=(2-\gamma)\sqrt{\pi}.
\end{aligned}\tag{63}$$

From (63) and (63), we have

$$\sqrt{\pi}[\gamma,(2-\gamma)]\leq_I\frac{1}{\sqrt{\pi}}\left[\gamma\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),(2-\gamma)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\right],\ \text{for each }\ \gamma\in[0,1].$$

4. Conclusions

In this study, we used fuzzy-interval Riemann-Liouville fractional integrals to prove some new Hermite-Hadamard inequalities for h-convex fuzzy IVFs. The results are consistent with those found in [1,2,7,16,26,28]. Furthermore, these results could be expanded in the future for different types of convexities and fractional integrals.

Acknowledgments

This work was supported by the Taif University Researchers Supporting Project Number (TURSP-2020/96), Taif University, Taif, Saudi Arabia.

Conflict of interest

The authors declare that they have no competing interests.

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