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Research article

Existence and subharmonicity of solutions for nonsmooth p-Laplacian systems

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Abstract: In this paper we study nonlinear periodic systems driven by the vectorial p-Laplacian with a nonsmooth locally Lipschitz potential function. Using variational methods based on nonsmooth critical point theory, some existence of periodic and subharmonic results are obtained, which improve and extend related works.

Keywords: p-Laplacian; locally Lipschitz continuous; nonsmooth saddle point theorem; subharmonic solution; periodic solution
Mathematics Subject Classification: 37J12, 37J46

1. Introduction

Consider the nonlinear nonsmooth periodic system with p-Laplacian

$$\begin{cases} -\dfrac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) \in \partial F(t,u(t)) \quad a.e.\ t\in[0,T], \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{cases} \tag{1}$$

where $T>0$, $p>1$, $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is locally Lipschitz continuous in the vectorial variable x and $\partial F(t,x)$ denotes the generalized subdifferential of F with respect to x in the sense of Clarke (see [1]).

When the potential function $F:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is T-period with respect to the first variable, problem (1) becomes the following Hamiltonian system with p-Laplacian

$$-\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) \in \partial F(t,u(t)),\ a.e.\ t\in\mathbb{R}. \tag{2}$$

A function is called subharmonic solution if it is kT-periodic solution for some positive integer k (see [2]).

When the potential functional is continuously differentiable (i.e., $F(t,\cdot) \in C^1(\mathbb{R}^N)$), the existence of periodic solutions and subharmonic solutions of Hamiltonian systems with p-Laplacian has been widely concerned by mathematical physicists because of its strong practical significance and theoretical research value (for example [3–12] and the references therein). In their papers, the following assumption is always required:

(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| + |\nabla F(t,x)| \leq a(|x|)b(t),$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, where $\mathbb{R}^+$ is the set of all nonnegative real number.

In recent years, extensive researches on problem (1) have been conducted, for example Gasin'ski and Papageorgiou [13] gave the existence and multiplicity of periodic solutions using the nonsmooth mountain lemma and saddle point theorem; Researchers studied the existence and multiplicity of solutions for nonlinear second-order periodic systems with one-dimensional p-Laplacian and nonsmooth potentials [8, 14, 15]; Zhang and Liu [16] obtained the existence of three solutions for the periodic eigenvalue problems driven by the p-Laplacian under a bounded interval for the parameter λ; In [17, 18], the authors discussed the existence of subharmonic solutions for problem (2).

Inspired by the above papers, we further investigate the periodic solutions and subharmonic solutions for p-Hamiltonian systems with nonsmooth potentials. Since the potential is nondifferentiable, the gradient is replaced by the subdifferential and the resulting problem is a quasilinear second order periodic differential inclusion, known as hemivariational inequality. Hemivariational inequalities arise in physical problems, when one wants to consider more realistic models with nonsmooth and non-convex energy functionals. The hemivariational inequalities formalism proved to be an efective analytical tool in the study of many complex mechanical structures, such as multilayered plates, Vonkarman plates in adhesive contact with rigid support, composite structures and others (see [7]).

Throughout this paper, we always suppose that $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ $(N \geq 1)$ satisfies the following assumption (A'):

(A') $F(t,x)$ is measurable in t over $[0,T]$ for each $x \in \mathbb{R}^N$ and is locally Lipschitz continuous in x for a.e. $t \in [0,T]$, $F(t,0) \in L^1(0,T)$.

We use nonsmooth critical point theories to prove the existence of periodic solutions for problem (1) and subharmonic solutions for problem (2). In the proof of the existence of periodic solutions, we prove that the energy functional satisfies the nonsmooth Cerami condition firstly, and then prove that it has saddle points. Finally we prove that the obtained critical point of the energy function φ is the weak solution of the problem (1) (See Theorem 3.1). In particular, in Theorem 4.1, we make use of a weaker condition and prove the existence of the subharmonic solutions for (2), generalizing a result contained in [12].

2. Basic definitions and preliminary results

We start with the subdifferential theory for locally Lipschitz functions. Let $(X, \|\cdot\|)$ be a real Banach space. Denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X and X^*. A functional $h : X \to \mathbb{R}$ is called locally Lipschitz continuous if for every $u \in X$ there corresponds a neighborhood V_u of u and a constant $L_u \geq 0$ such that

$$|h(z) - h(w)| \leq L_u\|z - w\|, \quad \forall z, w \in V_u.$$

If $u, v \in X$, we write $h^0(u; v)$ for the generalized directional derivative of h at the point u along the direction v, i.e.,

$$h^0(u; v) := \limsup_{w\to u, t\to 0^+} \frac{h(w + tv) - h(w)}{t}.$$

It is known that h^0 is upper semicontinuous on $X \times X$ (see [1, Proposition 2.1.1]).

For locally Lipschitz continuous functionals $h_1, h_2 : X \to \mathbb{R}$, we have

$$(h_1 + h_2)^0(u; v) \leq h_1^0(u; v) + h_2^0(u; v), \quad \forall u, v \in X.$$

The generalized gradient of the function h at u, denoted by $\partial h(u)$, is the set defined by

$$\partial h(u) := \{u^* \in X^* : \langle u^*, v\rangle \leq h^0(u; v), \ \forall v \in X\}.$$

In view of [1, Proposition 2.1.2] $\partial h(u)$ is a nonempty, convex in addition to weak* compact subset of X^*, thus the function $\lambda(x) = \min_{w\in\partial h(x)} \|w\|_{X^*}$ is well defined and lower semicontinuous, i.e., $\liminf_{x\to x_0} \lambda(x) \geq \lambda(x_0)$.

If $h_1, h_2 : X \to \mathbb{R}$ are locally Lipschitz continuous, then

$$\partial(h_1 + h_2)(x) \subset \partial h_1(x) + \partial h_2(x), \ \forall x \in X.$$

A point $u \in X$ is said to be a critical point of h if $h^0(u; v) \geq 0, \ \forall v \in X$. In this framework, the functional h is said to satisfy the nonsmooth (PS) condition if any sequence $\{x_n\}$ in X such that $\{h(x_n)\}$ is bounded and $\lambda(x_n) \to 0$ possesses a convergent subsequence. Moreover the locally Lipschitz functional h is said to satisfy the nonsmooth Cerami condition if any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\{h(x_n)\}$ is bounded and $(1+\|x_n\|)\lambda(x_n) \to 0 \ (n \to \infty)$ possesses a convergent subsequence. For more details on this subject one could refer to [1, 19–24]. For convenience, in what follows we will denote various positive constants as $c_i, \ i = 1, 2, 3 \cdots$.

Finally, we shall make use of the following well known results.

Lemma 2.1. *([1, Theorem 2.3.7]) Let x and y be points in X, and suppose that f is Lipschitz on open set containing the line segment* $[x, y]$. *Then there exists a point u in* (x, y) *such that*

$$f(y) - f(x) \in \langle\partial f(u), y - x\rangle.$$

Lemma 2.2. *([25, Theorem 7]) If* $X = Y \oplus V$, *with* $\dim Y < \infty$, *there exists* $r > 0$ *such that*

$$\max[\phi(x) : x \in Y, \|x\| = r] \leq \inf[\phi(x) : x \in V]$$

and $\phi : X \to \mathbb{R}$ *satisfies the nonsmooth* $(C)_c$*-condition where* $c = \inf_{\gamma\in\Gamma} \max_{x\in E} \phi(\gamma(x))$ *with* $\Gamma = \{\gamma \in C(E, X) : \gamma|_{\partial E} = identity\}, E = \{x \in Y : \|x\| \leq r\}$ *and* $\partial E = \{x \in Y : \|x\| = r\}$, *then* $c \geq \inf_V \phi$ *and c is a crucial value of* ϕ. *Moreover, if* $c = \inf_V \phi$, *then*

$$V \cap K_c \neq \emptyset.$$

Lemma 2.3. *([22, Theorem 3.3]) Let X be a real Banach space, and let f be a locally Lipschitz function defined on X satisfying the nonsmooth (PS) condition. Suppose $X = X_1 \oplus X_2$ with a finite dimensional subspace X_1, and there exist contants $b_1 < b_2$ and a bounded neighborhood N of θ in X_1 such that*

$$f|_{X_2} \geq b_2, \quad f|_{\partial N} \leq b_1.$$

Then f has a critical point.

3. Existence theorems

In this section, we shall give the existence theorems for problem (1).

Theorem 3.1. *Suppose $F(t, x)$ satisfies assumption (A') and the following conditions:*

(l_1) *For every $r > 0$ there exists $a_r \in L^1([0, T])_+$ such that for a.e. $t \in [0, T]$, all $|x| \leq r$ and $\xi \in \partial F(t, x)$, we have $|\xi| \leq a_r(t)$;*

(l_2) *There exist $\mu \in (0, p)$ and $M > 0$ such that*

$$F^0(t, x; x) \leq \mu F(t, x)$$

for almost all $t \in [0, T]$, all $|x| \geq M$;

(l_3) $\int_0^T F(t, x)dt \to +\infty$ *as $|x| \to \infty$ uniformly for a.e. $t \in [0, T]$.*

Then problem (1) has at least one solution $u \in W_T^{1,p}$, where

$$W_T^{1,p} = \{u : [0, T] \to \mathbb{R}^N | u \text{ is absolutely continuous}, u(0) = u(T), \dot{u} \in L^p(0, T; \mathbb{R}^N)\}$$

is the reflexive Banach space with the norm

$$||u|| = \Big(\int_0^T |u(t)|^p dt + \int_0^T |\dot{u}(t)|^p dt\Big)^{1/p}.$$

Proof. We start by observing that, because of hypotheses (l_1), using the mean value theorem, we have that for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^N$ with $|x| \leq r$,

$$|F(t, x)| \leq b_r(t)\hat{a}(|x|), \tag{3}$$

where $b_r(t) = |F(t, 0)| + a_r(t)$ and

$$\hat{a}(s) = \begin{cases} 1, & 0 \leq s \leq 1, \\ s, & s > 1. \end{cases}$$

Let $\tilde{W}_T^{1,p} = \{u \in W_T^{1,p} : \int_0^T u(t)dt = 0\}$, then $W_T^{1,p} = \tilde{W}_T^{1,p} \oplus \mathbb{R}^N$, for every $u \in W_T^{1,p}$. Put $\bar{u} = \frac{1}{T}\int_0^T u(t)dt$, $\tilde{u}(t) = u(t) - \bar{u}$, then $\bar{u} \in \mathbb{R}^N$, $\tilde{u} \in \tilde{W}_T^{1,p}$ and the following inequalities hold:

$$||\tilde{u}||_\infty^p \leq C\int_0^T |\dot{u}(t)|^p dt, \text{ (Sobolev inequality)}$$

$$\int_0^T |\tilde{u}(t)|^p dt \leq C\int_0^T |\dot{u}(t)|^p dt, \text{ (Wirtinger's inequality)}$$

where $C > 0$ is a constant and $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.

Consider the energy functional $\varphi : W_T^{1,p} \to \mathbb{R}$ for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt - \int_0^T F(t, u(t))dt$$

for all $u \in W_T^{1,p}$.

It is straightforward to verify that φ is well defined and locally Lipschitz continuous on $W_T^{1,p}$ (see [1, page 83]) under assumption (A').

Claim 1. φ satisfies the nonsmooth Cerami condition.

Let$\{u_n\} \subset W_T^{1,p}$ be a sequence such that $|\varphi(u_n)| \le M_1$ for some $M_1 > 0$, $\forall\, n \ge 1$ and $(1+\|u_n\|)\lambda(u_n) \to 0$ as $n \to \infty$.

Since $\partial\varphi(u_n) \subset (W_T^{1,p})^*$ is nonempty, weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, by Weierstrass theorem we can find $u_n^* \in \partial\varphi(u_n)$ such that $\lambda(u_n) = \|u_n^*\|_{X^*}$, $n \ge 1$, then (see [1, page 76]) there exist $\xi_n \in L^1(0, T)$, $\xi_n(t) \in \partial F(t, u_n(t))$ a.e. on $[0, T]$ such that

$$\langle u_n^*, v\rangle = \int_0^T |\dot{u}_n(t)|^{p-2}(\dot{u}_n(t), \dot{v}(t))dt - \int_0^T (\xi_n(t), v(t))dt, \quad \forall v \in W_T^{1,p}.$$

From the choice of the sequence $\{u_n\} \subset W_T^{1,p}$, we have

$$\begin{aligned}\langle u_n^*, u_n\rangle &= \int_0^T |\dot{u}_n(t)|^p dt - \int_0^T (\xi_n(t), u_n(t))dt\\ &\le (1 + \|u_n\|)\lambda(u_n) \le \epsilon_n,\end{aligned}$$

where $\epsilon_n \downarrow 0$. Since

$$\int_0^T F^0(t, u_n(t); u_n(t))dt \ge \int_0^T (\xi_n(t), u_n(t))dt,$$

we deduce

$$\int_0^T |\dot{u}_n(t)|^p dt - \int_0^T F^0(t, u_n(t); u_n(t))dt \le \epsilon_n, \tag{4}$$

and since $|\varphi(u_n)| \le M_1$, we obtain

$$-\frac{\mu}{p}\int_0^T |\dot{u}_n(t)|^p dt + \int_0^T \mu F(t, u_n(t))dt \le \mu M_1, \tag{5}$$

for all $n \ge 1$. Adding (4) and (5),we have then

$$(1 - \frac{\mu}{p})\int_0^T |\dot{u}_n(t)|^p dt + \int_0^T [\mu F(t, u_n(t)) - F^0(t, u_n(t); u_n(t))]dt \le \epsilon_n + \mu M_1.$$

Now set $A_n = \Big\{t \in [0, T] \,|\, |u_n(t)| < M\Big\}$ and $B_n = \Big\{t \in [0, T] \,|\, |u_n(t)| \ge M\Big\}$. From (l_1), (l_2) and the properties of F^0 (see [7, page 545]) we obtain

$$\int_{B_n} (\mu F(t, u_n(t)) - F^0(t, u_n(t); u_n(t)))dt \ge 0,$$

while

$$\begin{aligned}&|\int_{A_n}(\mu F(t,u_n(t))-F^0(t,u_n(t);u_n(t)))dt|\\&\leq\int_{A_n}[\mu(|F(t,0)|+a_M(t)M)+c_1M]dt\leq c_2.\end{aligned}$$

It follows that

$$(1-\frac{\mu}{p})\|\dot{u}_n\|_p^p\leq\varepsilon_n+\mu M_1+c_2,\quad\forall n\geq 1,$$

then by Poincare-Wirtinger inequality, $\{\tilde{u}_n\}$ is bounded in $W_T^{1,p}$ and by Sobolev inequality, we get $\|\tilde{u}_n\|_\infty$ is bounded.

We claim that the sequence $\{\bar{u}_n\}$ is bounded, otherwise, there is a subsequence, again denoted by $\{\bar{u}_n\}$, such that $|\bar{u}_n|\to\infty$ as $n\to\infty$.

Thus

$$|u_n(t)|=|\tilde{u}_n(t)+\bar{u}_n|\geq|\bar{u}_n|-\|\tilde{u}_n\|_\infty\to\infty,\ \text{ as } n\to\infty,$$

for all $t\in[0,T]$. From the condition (l_3), we have

$$\varphi(u_n)=\frac{1}{p}\int_0^T|\dot{u}_n(t)|^pdt-\int_0^T F(t,u_n(t))dt\to-\infty\text{ as }n\to\infty,$$

which contradicts the choice of $\{u_n\}$. Hence $\{u_n\}$ is bounded in $W_T^{1,p}$. By the compactness of the embedding $W_T^{1,p}\subset C(0,T;\mathbb{R}^N)$, the sequence $\{u_n\}$ has a subsequence, denoted by $\{u_n\}$ again, such that $u_n\rightharpoonup u$ weakly in $W_T^{1,p}$ and $u_n\to u$ strongly in $C(0,T;\mathbb{R}^N)$.

Note that

$$\langle u_n^*,u_n-u\rangle=\int_0^T|\dot{u}_n(t)|^{p-2}(\dot{u}_n(t),\dot{u}_n(t)-\dot{u}(t))dt-\int_0^T(\xi_n(t),u_n(t)-u(t))dt,$$

and

$$\langle u_n^*,u_n-u\rangle\to 0\ \text{ as }\ n\to\infty.$$

Since $\{u_n\}$ is bounded and

$$|u_n(t)|=|\tilde{u}_n(t)+\bar{u}_n|\leq|\bar{u}_n|+\|\tilde{u}_n\|_\infty\leq|\bar{u}_n|+C^{\frac{1}{p}}\|\dot{u}_n\|_p,$$

there exists $M_2>0$ such that $|u_n(t)|\leq M_2$ for a.e. $t\in[0,T]$ and $n\geq 1$. By (l_2), we have

$$|\int_0^T(\xi_n(t),u_n(t)-u(t))dt|\leq\int_0^T a_{M_2}(t)|u_n(t)-u(t)|dt\leq c_3\|u_n-u\|_\infty$$

for some positive constants c_3.

Hence one has

$$\int_0^T|\dot{u}_n(t)|^{p-2}(\dot{u}_n(t),\dot{u}_n(t)-\dot{u}(t))dt\to 0\text{ as }n\to\infty.$$

Besides it is easy to drive from $u_n\to u$ strongly in $C(0,T;\mathbb{R}^N)$ that

$$\int_0^T|u_n(t)|^{p-2}(u_n(t),u_n(t)-u(t))dt\to 0\text{ as }n\to\infty.$$

Since $||u||^p = ||u||_p^p + ||\dot{u}||_p^p$ and the norm in a Banach space is weakly lower semicontinuous, we have

$$||\dot{u}||_p^p \le \liminf_{n\to\infty} ||\dot{u}_n||_p^p.$$

Using the Hölder inequality, we have

$$\begin{aligned} 0 &\le (||u_n||^{p-1} - ||u||^{p-1})(||u_n|| - ||u||) \\ &\le \int_0^T |u_n(t)|^{p-2}(u_n(t), u_n(t) - u(t))dt + \int_0^T |\dot{u}_n(t)|^{p-2}(\dot{u}_n(t), \dot{u}_n(t) - \dot{u}(t))dt \to 0, \end{aligned}$$

which yields $||u_n|| \to ||u||$. Since $u_n \rightharpoonup u$ weakly in $W_T^{1,p}$ and $\dot{u}_n \rightharpoonup \dot{u}$ in $L^p(T, \mathbb{R}^N)$ and the latter space is uniformly convex, by the Kadec-Klee property, we have $\dot{u}_n \to \dot{u}$ in $L^p(T, \mathbb{R}^N)$. Therefore $u_n \to u$ in $W_T^{1,p}$ and φ satisfies the nonsmooth Cerami condition.

Claim 2. φ is coercive on $\tilde{W}_T^{1,p}$.

For every $u \in \tilde{W}_T^{1,p}$ we have

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt - \int_0^T F(t, u(t))dt \\ &= \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt - \int_A F(t, u(t))dt - \int_B F(t, u(t))dt, \end{aligned}$$

where $A = \{t \in [0, T] : |u(t)| < M\}$ and $B = \{t \in [0, T] : |u(t)| \ge M\}$. Note that from the mean value theorem and (l_1), for $|x| \le M$ and a.e. $t \in [0, T]$, it is possible to find $r \in [0, 1]$ and $\xi \in \partial F(t, rx)$ such that

$$|F(t, x)| \le |F(t, 0)| + |\langle \xi, x\rangle| \le |F(t, 0)| + a_M(t)M.$$

Therefore we can see that for all $|x| \le M$ and a.e. $t \in [0, T]$

$$F(t, x) \le \beta_M(t),$$

where $\beta_M(t) \in L^1(0, T)_+$. Immediately we have

$$\int_A F(t, u(t))dt \le ||\beta_M||_1. \tag{6}$$

By (l_2), for a.e. $t \in [0, T]$, all $|x| \ge M$ and all $s \ge 1$, one has $F(t, sx) \le s^\mu F(t, x)$ (see [26, Theorem 3.14]), then

$$\int_B F(t, u(t))dt \le \int_B \frac{|u(t)|^\mu}{M^\mu} F(t, \frac{Mu(t)}{|u(t)|})dt \le \frac{||u||_\infty^\mu}{M^\mu}\int_B F(t, \frac{Mu(t)}{|u(t)|})dt,$$

and so

$$\int_B F(t, u(t))dt \le \frac{||u||_\infty^\mu}{M^\mu}||\beta_M||_1. \tag{7}$$

Now from (6) and (7), using the Poincaré-Wirtinger inequality again, we obtain

$$\varphi(u) \ge \frac{1}{p}||\dot{u}||_p^p - c_4||\dot{u}||_p^\mu - c_5.$$

Since $\mu < p$, we conclude that φ is coercive on $\tilde{W}_T^{1,p}$ as claimed.

Claim 3. φ is anticoercive on $\mathbb{R}^N$.

Since for $x \in \mathbb{R}^N$, $\varphi(x) = -\int_0^T F(t,x)dt$, the claim is a direct consequence of hypothesis (l_3).

From the claims proved we are in the position of applying Lemma 2.2 and obtaining the existence of a $u \in W_T^{1,p}$ such that $\theta \in \partial\varphi(u)$. Moreover, there exists $\xi(t) \in \partial F(t,u(t))$ a.e. $t \in [0,T]$ such that

$$0 = \int_0^T |\dot{u}(t)|^{p-2}(\dot{u}(t),\dot{v}(t))dt - \int_0^T (\xi(t),v(t))dt, \quad \forall v \in W_T^{1,p},$$

which implies

$$\int_0^T |\dot{u}(t)|^{p-2}(\dot{u}(t),\dot{v}(t))dt = -\int_0^T (\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)),v(t))dt = \int_0^T (\xi(t),v(t))dt,$$

thus

$$-\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) \in \partial F(t,u(t)) \text{ a.e. on } [0,T].$$

So $u \in W_T^{1,p}$ is a solution of problem (1). □

Example 3.2. *Let $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ be defined as*

$$F(t,x) = \begin{cases} 2 - |x|, & |x| \le 1, \\ x^\mu, & x > 1, \\ |x|^\mu + 1, & x < -1. \end{cases}$$

where $\mu \in (0,p)$. Then

$$\partial F(t,x) = \begin{cases} -\frac{x}{|x|}, & 0 < |x| < 1, \\ [-1,1], & x = 0, \\ [-1,\mu], & x = 1, \\ [-\mu,-1], & x = -1, \\ \mu x^{\mu-1}, & x > 1, \\ \mu|x|^{\mu-2}x, & x < -1. \end{cases}$$

It is easy to verify that $F(t,x)$ satisfies the condition of theorem 3.1.

4. Subharmonic solutions

Consider problem (2)

$$-\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) \in \partial F(t,u(t)) \text{ a.e. } t \in \mathbb{R},$$

where $p > 1$ and $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic ($T > 0$) in its first variable for all $x \in \mathbb{R}^N$.

Theorem 4.1. *Suppose $F(t,x)$ satisfies the assumption (A') and the following conditions:*

(h_1) *There exist* $a \in C(\mathbb{R}^+, \mathbb{R}^+), b \in L^1(0, T; \mathbb{R}^+)$ *such that*

$$|F(t, x)| \leq a(|x|)b(t),$$

for all $x \in R^N$ *and a.e.* $t \in [0, T]$;

(h_2) *There exist constants* $C^* > 0,\ K_1 > 0,\ K_2 > 0,\ \alpha \in [0, p-1)$ *and a positive function* $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with the properties:*

(i) $h(s) \leq h(t)$ *for all* $s \leq t,\ s, t \in \mathbb{R}^+$,
(ii) $h(s + t) \leq C^*(h(s) + h(t))$ *for all* $s, t \in \mathbb{R}^+$,
(iii) $0 < h(t) \leq K_1 t^\alpha + K_2$ *for all* $t \in \mathbb{R}^+$,
(iv) $h(t) \to +\infty$ *as* $t \to +\infty$.

Moreover, there exist $f, g \in L^1(0, T; \mathbb{R}^+)$ *such that for a.e.* $t \in [0, T]$, *all* $x \in \mathbb{R}^N$ *and* $\xi \in \partial F(t, x)$, *one has*

$$|\xi| \leq f(t)h(|x|) + g(t);$$

(h_3)

$$\frac{1}{h^q(|x|)} \int_0^T F(t, x)dt \to +\infty \quad as \ |x| \to \infty,$$

uniformly for a.e. $t \in [0, T]$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Then problem (2) has kT-periodic solution $u_k \in W^{1,p}_{kT}$ *for every positive integer k such that* $\|u_k\|_\infty \to +\infty$ *as* $k \to +\infty$, *where* $\|u_k\|_\infty = max_{0 \leq t \leq kT}|u_k(t)|$ *and*

$$W^{1,p}_{kT} = \{u : [0, kT] \to \mathbb{R}^N | u \text{ is absolutely continuous}, u(0) = u(kT), \dot{u} \in L^p(0, kT; \mathbb{R}^N)\}$$

is the reflexive Banach space with the norm

$$\|u\| = (\int_0^{kT} |u(t)|^p dt + \int_0^{kT} |\dot{u}(t)|^p dt)^{1/p}.$$

Proof. For $u \in W^{1,p}_{kT}$, set $\bar{u} = \frac{1}{kT} \int_0^{kT} u(t)dt,\ \ \tilde{u}(t) = u(t) - \bar{u}$ and $\tilde{W}^{1,p}_{kT} = \{u \in W^{1,p}_{kT} | \int_0^{kT} u(t)dt = 0\}$, then $W^{1,p}_{kT} = \tilde{W}^{1,p}_{kT} \oplus \mathbb{R}^N$. By [2, Proposition 1.1], there exists a constant $C_k > 0$ such that

$$\|u\|^p_\infty \leq C_k \int_0^{kT} |\dot{u}(t)|^p dt,$$

and

$$\int_0^{kT} |\tilde{u}(t)|^p dt \leq C_k \int_0^{kT} |\dot{u}(t)|^p dt,$$

for every $u \in W^{1,p}_{kT}$. Hence

$$\int_0^{kT} |\dot{u}(t)|^p dt \leq \|u\|^p \leq (1 + C_k) \int_0^{kT} |\dot{u}(t)|^p dt,\ \forall u \in \tilde{W}^{1,p}_{kT}.$$

By assumption (A'), the corresponding energy functional $\varphi_k : W^{1,p}_{kT} \to \mathbb{R}$ of problem (2) defined by

$$\varphi_k(u) = \frac{1}{p} \int_0^{kT} |\dot{u}(t)|^p dt - \int_0^{kT} F(t, u(t))dt,\ \ u \in W^{1,p}_{kT},$$

is locally Lipschitz continuous on $W^{1,p}_{kT}$ and for every $u \in W^{1,p}_{kT}$ and $u^* \in \partial\varphi_k(u)$ there exists $\xi \in L^1(0,kT)$, $\xi(t) \in \partial F(t,u(t))$ a.e. on $[0,kT]$ such that

$$\langle u^*, v\rangle = \int_0^{kT} |\dot{u}(t)|^{p-2}(\dot{u}(t),\dot{v}(t))dt - \int_0^{kT} (\xi(t), v(t))dt, \quad \forall v \in W^{1,p}_{kT}.$$

First, we prove that φ_k satisfies the nonsmooth (PS) condition on $W^{1,p}_{kT}$.

Let$\{u_n\} \subset W^{1,p}_{kT}$ be a sequence such that $\{\varphi_k(u_n)\}$ is bounded and $\lambda_k(u_n) \to 0$ as $n \to \infty$, where $\lambda_k(x) = \min_{w\in\partial\varphi_k(x)} \|w\|$.

Since $\partial\varphi_k(u_n) \subset (W^{1,p}_{kT})^*$ is nonempty, weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, by Weierstrass theorem we can find $u_n^* \in \partial\varphi_k(u_n)$ such that $\lambda_k(u_n) = \|u_n^*\|_{X^*}$, $n \geq 1$, then there exists $\xi_n(t) \in \partial F(t,u_n(t))$ such that

$$\langle u_n^*, v\rangle = \int_0^{kT} |\dot{u}_n(t)|^{p-2}(\dot{u}_n(t),\dot{v}(t))dt - \int_0^{kT} (\xi_n(t), v(t))dt, \quad \forall v \in W^{1,p}_{kT}.$$

Since $\varphi_k(u_n)$ is bounded and $\lambda_k(u_n) \to 0$ as $n \to \infty$, there exists $M_3 > 0$ such that $|\varphi_k(u_n)| \leq M_3$ and $\|u_n^*\| \leq 1$ when n is large enough, hence $|\langle u_n^*, v\rangle| \leq \|v\|$ for large n.

By condition (h_2), Sobolev inequality and Young inequality, we have

$$\begin{aligned}
&|\int_0^{kT} (\xi_n(t), \tilde{u}_n(t))dt| \\
\leq& \int_0^{kT} (f(t)h(|\bar{u}_n + s\tilde{u}_n(t)|) + g(t))|\tilde{u}_n(t)|dt \\
\leq& \int_0^{kT} f(t)C^*(h(|\bar{u}_n|) + h(|\tilde{u}_n(t)|))|\tilde{u}_n(t)|dt + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t)dt \\
\leq& C^*(h(|\bar{u}_n|) + h(|\tilde{u}_n(t)|))\|\tilde{u}_n\|_\infty \int_0^{kT} f(t)dt + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t)dt \\
\leq& C^*[\frac{1}{2pC^*C_k^p}\|\tilde{u}_n\|_\infty^p + (2pC^*C_k^p)^{\frac{1}{p-1}} h^q(|\bar{u}_n|)(\int_0^{kT} f(t)dt)^q] + \|\tilde{u}_n\|_\infty \int_0^{kT} g(t)dt \\
&+ C^*h(\|\tilde{u}_n\|_\infty)\|\tilde{u}_n\|_\infty \int_0^{kT} f(t)dt \\
\leq& \frac{1}{2p}\int_0^{kT} |\dot{u}_n(t)|^p dt + c_6 h^q(|\bar{u}_n|) + C^*(K_1\|\tilde{u}_n\|_\infty^\alpha + K_2)\|\tilde{u}_n\|_\infty \int_0^{kT} f(t)dt \\
&+ \|\tilde{u}_n\|_\infty \int_0^{kT} g(t)dt \\
\leq& \frac{1}{2p}\int_0^{kT} |\dot{u}_n(t)|^p dt + c_6 h^q(|\bar{u}_n|) + c_7(\int_0^{kT} |\dot{u}_n(t)|^p dt)^{\frac{\alpha+1}{p}} + c_8(\int_0^{kT} |\dot{u}_n(t)|^p dt)^{\frac{1}{p}}.
\end{aligned}$$

Hence,

$$\begin{aligned}\|\tilde{u}_n\| \geq & \langle u_n^*, \tilde{u}_n\rangle = \int_0^{kT} |\dot{u}_n(t)|^p dt - \int_0^{kT} (\xi_n(t), \tilde{u}_n(t)) dt \\ \geq & (1 - \frac{1}{2p}) \int_0^{kT} |\dot{u}_n(t)|^p dt - c_6 h^q(|\bar{u}_n|) - c_7 (\int_0^{kT} |\dot{u}_n(t)|^p dt)^{\frac{\alpha+1}{p}} \\ & - c_8 (\int_0^{kT} |\dot{u}_n(t)|^p dt)^{\frac{1}{p}}\end{aligned}$$

for large n.

Since $\alpha < p - 1$ and by Wirtinger inequality

$$\|\tilde{u}_n\| \leq (1 + C_k)^{\frac{1}{p}} (\int_0^{kT} |\dot{u}_n(t)|^p dt)^{\frac{1}{p}},$$

we obtain

$$c_9 h^q(|\bar{u}_n|) \geq \int_0^{kT} |\dot{u}_n(t)|^p dt - c_{10}, \tag{8}$$

for all large n, which implies that

$$\begin{aligned}\|\tilde{u}_n\|_\infty & \leq (C_k \int_0^{kT} |\dot{u}_n(t)|^p dt)^{\frac{1}{p}} \leq C_k (c_9 h^q(|\bar{u}_n|) + c_{10}) \\ & \leq c_{11} (|\bar{u}_n|^{q\alpha} + 1)^{\frac{1}{p}}.\end{aligned}$$

Then

$$|u_n(t)| = |\tilde{u}_n(t) + \bar{u}_n| \geq |\bar{u}_n| - \|\tilde{u}_n\|_\infty \geq |\bar{u}_n| - c_{11} (|\bar{u}_n|^{q\alpha} + 1)^{\frac{1}{p}} \tag{9}$$

for all large n and every $t \in [0, kT]$.

We claim that $|\bar{u}_n|$ is bounded, if not, without loss of generality we may assume that

$$|\bar{u}_n| \to \infty \text{ as } n \to \infty. \tag{10}$$

Since $0 \leq \alpha < p - 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have $\frac{\alpha q}{p} < 1$. From (9), one has

$$|u_n(t)| \geq \frac{1}{2} |\bar{u}_n|,$$

for all large n and every $t \in [0, kT]$. Then we have

$$h(|\bar{u}_n|) \leq h(2|u_n(t)|) \leq 2C^* h(|u_n(t)|).$$

In virtue of (h_3) and the T-periodicity of $F(t, x)$, for every $\beta > 0$, there exists $M_4 \geq 1$ such that

$$\frac{1}{h^q(|x|)} \int_0^{kT} F(t, x) dt = \frac{k}{h^q(|x|)} \int_0^T F(t, x) dt \geq k\beta, \tag{11}$$

for all $|x| \geq M_4$. By (9) and (10), when n is large enough, one has

$$|u_n(t)| \geq M_4 \ \ a.e.\ t \in [0, kT].$$

Thus

$$
\begin{aligned}
\varphi_k(u_n) &= \frac{1}{p}\int_0^{kT} |\dot{u}_n(t)|^p dt - \int_0^{kT} F(t, u_n(t))dt \\
&\le \frac{1}{p}(c_9 h^q(|\bar{u}_n|) + c_{10}) - k\beta h^q(|u_n(t)|) \\
&\le \frac{1}{p}(c_9 h^q(|\bar{u}_n|) + c_{10}) - \frac{k\beta}{2C^*} h^q(|\bar{u}_n|),
\end{aligned}
$$

for all large n. So by the arbitrariness of β, one has

$$
\limsup_{n\to+\infty} \frac{1}{h^q(|\bar{u}_n|)}\varphi_k(u_n) = -\infty.
$$

Since $|\bar{u}_n| \to \infty$, by (iv) of (h_2) and (h_3), $h(|\bar{u}_n|) \to +\infty$ as $n \to \infty$, thus $\varphi_k(u_n) = -\infty$, which contradicts the boundedness of $\varphi_k(u_n)$. Hence $\{|\bar{u}_n|\}$ is bounded. Furthermore, by (8) and (iii) of (h_2), we know $\{u_n\}$ is bounded. Arguing then as the proof of Theorem 3.1, we conclude that φ_k satisfies nonsmooth (PS) condition.

Next we verify the following condition:

(Π_1) $\varphi_k(u) \to +\infty$ as $\|u\| \to \infty$ in $\tilde{W}^{1,p}_{kT}$;
(Π_2) $\varphi_k(x + e_k(t)) \to -\infty$ as $|x| \to \infty$ in $\mathbb{R}^N$, where $e_k(t) = kcos(k^{-1}\omega t)x_0 \in \tilde{W}^{1,p}_{kT}$, $x_0 \in \mathbb{R}^N$, $|x_0| = 1$ and $\omega = \frac{2\pi}{T}$.

For every $u \in \tilde{W}^{1,p}_{kT}$, it follows from the Sobolev inequality that there exist $s \in [0, 1]$ and $\xi(t) \in \partial F(t, su(t))$ such that

$$
\begin{aligned}
\varphi_k(u) &= \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt - \int_0^{kT} [F(t, u(t)) - F(t, 0)]dt - \int_0^{kT} F(t, 0)dt \\
&\ge \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt - \int_0^{kT} |F(t, u(t)) - F(t, 0)|dt - \int_0^{kT} F(t, 0)dt \\
&\ge \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt - \int_0^{kT} |(\xi(t), u(t))|dt - \int_0^{kT} F(t, 0)dt \\
&\ge \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt - \int_0^{kT} f(t)h(su(t))|u(t)|dt - \int_0^{kT} g(t)|u(t)|dt - \int_0^{kT} F(t, 0)dt \\
&\ge \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt - \int_0^{kT} f(t)(K_1|u(t)|^\alpha + K_2)|u(t)|dt \\
&\quad - \|u\|_\infty \int_0^{kT} g(t)dt - \int_0^{kT} F(t, 0)dt \\
&\ge \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt - K_1\|u\|_\infty^\alpha \int_0^{kT} f(t)dt - K_2\|u\|_\infty \int_0^{kT} f(t)dt \\
&\quad - \|u\|_\infty \int_0^{kT} g(t)dt - \int_0^{kT} F(t, 0)dt \\
&\ge \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt - c_{12}\Big(\int_0^{kT} |\dot{u}(t)|^p dt\Big)^{\frac{\alpha+1}{p}} - c_{13}\Big(\int_0^{kT} |\dot{u}(t)|^p dt\Big)^{\frac{1}{p}} - c_{14}.
\end{aligned}
$$

Since $p > 1$ and $\alpha < p-1$, then $\varphi_k(u) \to +\infty$ as $\|u\| \to \infty$ in $\tilde{W}^{1,p}_{kT}$, which proves (Π_1).

For all $x \in \mathbb{R}^N$, it follows from (11) that

$$\begin{aligned}\varphi_k(x+e_k(t)) &= \frac{1}{p}\int_0^{kT} |\dot{e}_k(t)|^p dt - \int_0^{kT} F(t_1 x + kcos(k^{-1}\omega t)x_0)dt \\ &\leq \frac{1}{p}\int_0^{kT} |\omega(sink^{-1}\omega t)x_0|^p dt - \beta k h^q(|x + kcos(k^{-1}\omega t)x_0|) \\ &\leq c_{15}k - k\beta h^q(M_3),\end{aligned}$$

for all $|x| \geq M_3 + k$. By the arbitrariness of β, one has

$$\varphi_k(x+e_k(t)) \to -\infty \ \text{ as } \ |x| \to \infty \ \text{ in } \mathbb{R}^N.$$

Thus (Π_2) is satisfied. By (Π_1), (Π_2) and the nonsmooth saddle point theorem, there exists a critical point $u_k \in \tilde{W}^{1,p}_{kT}$ for φ_k such that

$$-\infty < \inf_{\tilde{W}^{1,p}_{kT}} \varphi_k \leq \varphi_k(u_k) \leq \sup_{\mathbb{R}^N + e_k} \varphi_k.$$

For fixed $x \in \mathbb{R}^N$, set

$$A_k = \{t \in [0, kT] \big| |x + kcos(k^{-1}\omega t)x_0| \leq M_3\}.$$

Then we have $measA_k \leq \frac{kT}{2}$ for all large k. In fact if $measA_k > \frac{kT}{2}$, there exists $t_1 \in A_k$ such that

$$\frac{kT}{8} \leq t_1 \leq \frac{3kT}{8},$$

or

$$\frac{5kT}{8} \leq t_1 \leq \frac{7kT}{8}.$$

Moreover, there exists $t_2 \in A_k$ such that

$$|t_2 - t_1| \geq \frac{kT}{8}, \tag{12}$$

and

$$|t_2 - (kT - t_1)| \geq \frac{kT}{8}.$$

It follows that

$$|\frac{1}{2}(k^{-1}t_1 + k^{-1}t_2) - \frac{1}{2}T| \geq \frac{1}{16}T, \tag{13}$$

and

$$\frac{1}{16}T \leq \frac{1}{2}(k^{-1}t_1 + k^{-1}t_2) \leq \frac{15}{16}T. \tag{14}$$

From (13) and (14) we obtain

$$|\sin(\frac{1}{2}(k^{-1}t_1 + k^{-1}t_2)\omega)| \geq \sin(\frac{\pi}{8}).$$

Furthermore, by (12) we have

$$\begin{aligned}
&|\cos(k^{-1}\omega t_1) - \cos(k^{-1}\omega t_2)|\\
=&2|\sin(\frac{1}{2}(k^{-1}t_1 + k^{-1}t_2)\omega)||\sin(\frac{1}{2}(k^{-1}t_1 - k^{-1}t_2)\omega)|\\
\geq&2\sin^2(\frac{\pi}{8}) > 0.
\end{aligned}$$

But due to $t_1,\ t_2 \in A_k$, one has

$$\begin{aligned}
&|\cos(k^{-1}\omega t_1) - \cos(k^{-1}\omega t_2)|\\
=&\frac{1}{k}|x + k(\cos(k^{-1}\omega t_1))x_0 - (x + k(\cos(k^{-1}\omega t_2))x_0)|\\
\leq&\frac{2M}{k} \to 0 \ \text{ as } k \to \infty,
\end{aligned}$$

which is a contradiction for large k. Hence

$$meas([0, kT] \setminus A_k) \geq \frac{1}{2}kT > 0$$

for large k. From (h_1) and (h_3), we have

$$\begin{aligned}
k^{-1}\varphi_k(x + e_k(t)) &= \frac{1}{p}\int_0^{kT} |\dot{e}_k(t)|^p dt - \int_{A_k} F(t, x + e_k(t))dt - \int_{[0,kT]\setminus A_k} F(t, x + e_k(t))dt\\
&\leq c_{16} - \beta h^q(M_3),
\end{aligned}$$

for every $x \in \mathbb{R}^N$ and all large k, which implies that

$$\limsup_{k\to+\infty} \sup_{x\in\mathbb{R}^N} k^{-1}\varphi_k(x + e_k(t)) \leq c_{16} - \beta h^q(M_3).$$

By the arbitrariness of β, we obtain

$$\limsup_{k\to+\infty} \sup_{x\in\mathbb{R}^N} k^{-1}\varphi_k(x + e_k) = -\infty,$$

which follows that

$$\limsup_{k\to+\infty} k^{-1}\varphi_k(u_k) = -\infty. \tag{15}$$

Now we prove that $\|u_k\|_\infty \to +\infty$ as $k \to \infty$. If not, going to a subsequence if necessary, we may assume that

$$\|u_k\|_\infty \leq c_{17},$$

for all $k \in N$. Hence, by (h_1) we have

$$k^{-1}\varphi_k(u_k) \geq -k^{-1}\int_0^{kT} F(t,u_k(t))dt \geq -k^{-1}\max_{0\leq s\leq c_{17}} a(s)\int_0^{kT} b(t)dt$$
$$= -\max_{0\leq s\leq c_{17}} a(s)\int_0^{T} b(t)dt,$$

it follows that

$$\liminf_{k\to+\infty} k^{-1}\varphi_k(u_k) > -\infty,$$

which contradicts (15), therefore by Lemma 2.3 the proof is completed. □

Remark 4.2. *Theorem* 4.1 *generalizes [12, Theorem 1.2] and the conclusion in the document [17]. There exists function F satisfying the conditions in Theorem 4.1 but not satisfying conditions in [12,17]. For example, let*

$$F(t,x) = \sin[(1+|x|^2)^{\frac{1}{2}}\ln^{\frac{1}{2}}(e+|x|^2)] + |\sin\omega t|\ln^{\frac{3}{2}}(e+|x|^2) + |x|,$$

for all $x \in \mathbb{R}^N$ *and* $t \in \mathbb{R}$, *where* $\omega = \frac{2\pi}{T}$. *It is clear that F is locally Lipschitz continuous in x and*

$$|\xi| \leq \ln^{\frac{1}{2}}(e+|x|^2) + 11, \quad \forall \xi \in \partial F(t,x).$$

Moreover, one has

$$\frac{1}{|x|^{2\alpha}}F(t,x) \to 0 \ \text{ as } |x| \to +\infty,$$

for any $\alpha \in (\frac{1}{2}, 1)$ *and* $t \in \mathbb{R}$. *Hence this example can not be solved by the results in [11, 15, 27, 28] even when* $p = 2$.

5. Conclusions

In this paper we investigate the existence and subharmonicity of solutions for two nonsmooth p-Laplacian systems. We use nonsmooth critical point theories to prove the existence of periodic solutions for problem (1) and subharmonic solutions for problem (2). Since the potential is nondifferentiable, the gradient is replaced by the subdifferential and the resulting problem is a quasilinear second order periodic differential inclusion, known as hemivariational inequality. In particular, we make use of a weaker condition and prove the existence of the subharmonic solutions for (2), generalizing the results of the reference. Thus the results we obtain could be applied more widely.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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